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Using the Post–Widder formula to compute the Earth's viscoelastic Love numbers

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SUMMARY

The post-glacial or post-seismic relaxation of a Maxwell viscoelastic earth, 1-D or slightly laterally heterogeneous, can be calculated in a normal-mode approach, based on an application of the propagator technique. This semi-analytical approach, widely documented in the literature, allows to compute the response of an earth model whose rheological parameters vary quite strongly with depth, at least as accurately and efficiently as by 1-D numerical integration (Runge–Kutta). Its main drawback resides in the need to identify the roots of a secular polynomial, introduced after reformulating the problem in the Laplace domain, and required to transform the solution back to the time domain. Root finding becomes increasingly difficult, and ultimately unaffordable, as the complexity of rheological profiles grows: the secular polynomial gradually gets more ill behaved, and a larger number of more and more closely spaced roots is to be found. Here, we apply the propagator method to solve the Earth's viscoelastic momentum equation, like in the above-mentioned normal-mode framework, but bypass root finding, using, instead, the Post–Widder formula to transform the solution, found again in the Laplace domain, back to the time domain. We test our method against earlier normal-mode results, and prove its effectiveness in modelling the relaxation of earth models with extremely complex rheological profiles.

Key words: earth rheology, Love numbers, viscoelasticity.

1 INTRODUCTION

The solution of the equilibrium equations of a spherically symmetric earth subject to a surface load is usually expressed in terms of the so called load-deformation coefficients, also improperly referred to as Love numbers with the terminology in use for the tidal problem (e.g. Munk & MacDonald 1960). If the mantle layers behave like a viscoelastic solid, the Love numbers are commonly determined by the normal-mode method introduced by Peltier (1974) that implies a layer-by-layer propagation of the fundamental solution of the equilibrium equation in the Laplace domain, and the solution of the secular equation whose roots determine the spectrum of relaxation of the Earth. In the last two decades, the normal-mode method has been successfully employed to compute the response of layered earth models with various degrees of complexity. The reader is referred to Vermeersen & Sabadini (1997) and to references therein for a review, and to Sabadini *et al.* (1982) and Spada *et al.* (1992) for specific applications concerning models with a small number of layers, based on the analytical expression of the propagator matrix. A brief introduction to the normal-mode method is also given in Section 2.1 below.

In more recent years, the validity of the normal-mode approach has been questioned in the case when the rheological parameters vary continuously with radius (Fang & Hager 1995), and the existence of 'non-modal' contributions to the relaxation spectrum has been proposed (Hanyk *et al.* 1995). These latter studies that were based upon direct integration schemes in the time domain, have stimulated new analyses in which the normal-mode method has been applied to finely layered models (i.e. models that consist of many homogeneous layers). Vermeersen *et al.* (1996a) and Vermeersen & Sabadini (1997) have not found any shortcomings in classical normal-mode theory when many layers are included in modelling. In particular, they have suggested that non-modal contributions simply reflect numerical inaccuracies in the time-domain methods, and that the normal-mode technique indeed allows to compute the response of an earth model whose rheological parameters are effectively continuous with depth (but it remains to be established to what extent a fine stratification is a good analogue for continuous variations with depth). Wu & Ni (1996) and Boschi *et al.* (1999) pointed to the existence of a non physical singular factor in the numerically determined secular equation, a problem entirely bypassed in the normal-mode approach. Last, the possibility of applying the normal-mode approach to laterally heterogeneous earth models has been explored by Tromp & Mitrovica (2000).

Despite its success, the normal-mode method is characterized by several shortcomings, mostly associated with the secular equation

and with the determination of the amplitude of the viscoelastic modes. Namely,

(i) for an incompressible model, the number of roots of the secular equation increases linearly with increasing number of layers (Wu & Ni 1996; Spada *et al.* 2004), while a denumerably infinite number of modes appear even for a homogeneous earth if compressibility is accounted for (Vermeersen *et al.* 1996b);

(ii) especially at large harmonic degrees, the roots become closely spaced and sometimes difficult to resolve, even for incompressible models (Vermeersen & Sabadini 1997);

(iii) there is no way to determine a priori where the most significant roots (i.e. those associated with the residues of largest strength) are placed along the real negative axis, and neglecting some roots leads to a loss of precision;

(iv) one wrongly identified root may cause errors in the computation of the residue that may alter the whole time dependence of the Love numbers;

(v) some aspects of the numerical implementation of the normal-mode method are complicated and cumbersome, since it is necessary to factorize the various powers of the complex Laplace variable in order to construct the secular equation and to compute the viscoelastic residues (Spada *et al.* 1992);

(vi) for a viscous lithosphere, the solution of the secular equation is particularly challenging, since the normal-mode method may fail due to the small amplitude of the buoyancy modes (Vermeersen & Sabadini 1997).

As shown by Vermeersen *et al.* (1996a) and Vermeersen & Sabadini (1997), some of these difficulties can be overcome through high-precision numerical procedures, or by a careful tuning of the root-finding routines. However, as we show in the following, the whole issue of root finding can be bypassed without giving up the elegant analytical form of the propagators technique that characterizes the normal-mode approach, and without invoking purely numerical time-domain schemes such as Runge–Kutta (Fang & Hager 1995) or the method of lines (Hanyk 1999). The opportunity is provided by the Post–Widder formula (Post 1930; Widder 1934, 1946). While most of the available Laplace inversion techniques demand the explicit discretization of the (possibly deformed) Bromwich path (Davies & Martin 1979; Abate & Valkó 2004), the Post–Widder method requires the evaluation of the s-derivative of the Laplace transform at specific points along the real positive axis, hence the name of 'real' formula (Tuan & Duc 2002). For numerical applications, the derivatives can be discretized using the forward difference operator, and an explicit recursive algorithm for the computation of the resulting Gaver functionals can be easily constructed and implemented (Valkó & Abate 2004). Within the study of multilayered earth models, the main advantages of this Post–Widder–Gaver (PWG) approach are that

(i) the numerical solution of the secular equation, with the implied difficulties (e.g. root finding), is no longer necessary,

(ii) the Love numbers are directly computed in the time domain for any time history,

(iii) the algebraic structure of the numerical codes is greatly simplified and

(iv) the method can be extended to (possibly compressible) finely layered models in a straightforward manner, as well as to arbitrary linear viscoelastic rheologies.

As discussed by Abate & Valkó (2004), the main theoretical shortcomings of the PWG method are (i) its slow logarithmic convergence and (ii) the propagation of round-off errors that can make the Gaver recurrence unstable by catastrophic cancellation. Since nowadays these two difficulties can be overcome by powerful computing resources and by multiprecision algorithms, it is only in recent years that the usefulness of the Post–Widder formula has been fully recognized (Abate & Valkó 2004). As we will discuss in the following, the use of convergence accelerators allows us to reduce considerably the cost of multiprecision computations that can be efficiently programmed using publicly available Fortran 90 libraries. Another main objection against the PWG approach may be that the individual viscoelastic modes cannot be directly retrieved. Rather, the outcome of the procedure is the time-dependent Love numbers corresponding to a specific load time history. Beside a loss of elegance of the method, this implies a loss of generality, since the availability of the classical normal modes relaxation times and amplitudes would allow to compute the response for any load time history (e.g. Spada 1992). This difficulty can be overcome following Hanyk (1999), who has shown how to retrieve the most significant normal modes from the solution determined by his time-domain integration scheme.

The manuscript is organized as follows. In the first section we give the essentials of the background theory, that focuses on a description of the normal-mode and PWG methods of implementation. Next, we address the problems of the convergence of the PWG method and of its acceleration studying the case of a simple earth model for which the normal-mode solution is well established and universally accepted. In the third section we explore the case of multilayered incompressible models, leaving other potentially interesting applications (compressible models, internal dislocations sources, comparisons between the PWG and the normal-modes method in terms of computational efficiency) to a companion paper.

2 THEORY

2.1 Viscoelastic normal modes

The essential features of the normal-mode method have been illustrated in a number of papers. Here we largely follow Spada (1992) and Vermeersen *et al.* (1996a), although we employ a slightly different notation.

The most important outcome of the normal-mode approach is constituted by the set of the three Love numbers h, l, and k, defined as

$$h(s) = \frac{m_e}{a} u(a, s) \tag{1}$$

$$l(s) = \frac{m_e}{a} v(a, s) \tag{2}$$

$$k(s) = -1 - \frac{m_e}{a g_a} \phi(a, s) \tag{3}$$

where dependence on the harmonic degree is implicit, s is the complex Laplace variable, m_e is the mass of the Earth, a is its radius, g_a is the surface gravity acceleration, $u(a, s)$, $v(a, s)$, and $\phi(a, s)$ represent the vertical and horizontal components of the displacement and the incremental potential at the Earth surface, respectively.

Within the normal-mode method, the Love numbers can be obtained solving a boundary value problem in the Laplace domain that involves the use of the matrix propagation technique (e.g. Spada *et al.* 1992). Similarly to Vermeersen *et al.* (1996a), here we consider the case of an incompressible model earth that includes an elastic lithosphere on top of a stack of L homogeneous viscoelastic layers, and a homogeneous, inviscid core. In the following, with

$r_i (i = 0, 1, \ldots, L + 2)$ we denote the radii of the interfaces, where $r_0 = 0, r_1$ corresponds to the core-mantle boundary (CMB) and r_{L+2} to the free surface of the Earth. Within each layer, the solution vector containing vertical and horizontal displacements and stresses, the incremental potential and its gradient can be expressed as

$$\vec{y}(r, s) = Y_k(r, s)\vec{c}_k(s), \quad r_k \le r \le r_{k+1}, \quad k = 1, \ldots, L + 1, \tag{4}$$

where Y_k is the 6×6 fundamental matrix pertaining to the kth layer, and the elastic lithosphere is labelled by $k = L + 1$. The reader is referred to Spada *et al.* (1992) and to the corrections of Vermeersen *et al.* (1996a) for the analytical form of the Y matrix for an incompressible earth and of its inverse. In agreement with the correspondence principle of linear viscoelasticity (see e.g. Fung 1965), Y depends on the variable s through a complex shear modulus that for a Maxwell viscoelastic rheology reads

$$\mu(s) = \frac{\mu s}{s + \frac{\mu}{\eta}}, \tag{5}$$

where μ is the usual (elastic) shear modulus, and η is the Newtonian viscosity of the layer. The fundamental matrix for a purely elastic layer does not depend on s since for large values of η the complex shear modulus approaches the elastic limit μ.

Imposing continuity conditions for all of the field variables across the mantle boundaries and the lithosphere-mantle boundary and appropriate boundary conditions at the CMB (Wu & Ni 1996) yields the solution vector at the Earth's surface

$$\vec{y}(a, s) = WJ\vec{K}, \tag{6}$$

where the s-dependence is implicit at the right-hand side, J is an interface CMB matrix (Sabadini *et al.* 1982), and W is the propagator

$$W = \prod_{j=L+1}^{1} Y_j(r_{j+1})Y_j^{-1}(r_j), \tag{7}$$

where $\vec{K}$ is to be determined imposing the surface boundary conditions. The latter can be imposed via a projection matrix P_b that extracts from $\vec{y}(a, s)$ the three known components at the surface (vertical and horizontal stress, and gradient of the geopotential):

$$P_b\vec{y}(a, s) = \vec{b}f(s), \tag{8}$$

where $\vec{b}$ is given by Sabadini *et al.* (1982) and $f(s)$ is the Laplace transform of the time-history of the point-like surface load. Introducing a further projection matrix such that

$$P_x\vec{y}(a, s) = \vec{x}(a, s) \equiv [u, v, \phi]^t(a, s), \tag{9}$$

and using eqs (9) and (8) in eq. (6) we obtain

$$\vec{x}(a, s) = P_x WJ(P_b WJ)^{-1} f(s)\vec{b}, \tag{10}$$

that, using eqs (1)–(3) and the residues theorem, leads to the the standard spectral form of the Love numbers

$$\hat{\ell}(s) = \left(\ell_e + \sum_{k=1}^{M} \frac{\ell_k}{s - s_k}\right) f(s), \tag{11}$$

where $\hat{\ell}$ is the Laplace transform of any of the three Love numbers and M is the number of viscoelastic modes. For an incompressible and self-gravitating earth including an elastic lithosphere and an inviscid fluid core, the secular equation is an algebraic equation of degree $M = 4L$, where L is the number of distinct mantle viscoelastic layers that characterize the earth model (see Spada *et al.* 2004). The terms s_k in eq. (11) are the roots of the secular equation

$$|P_b WJ| = 0, \tag{12}$$

and $\ell_k (k = 1, \ldots, M)$ are the associated residues, with ℓ_e representing the limit for large s values in the case of an impulsive forcing ($f(s) = 1$). For a stably stratified earth, the roots of the secular equation are all placed along the real negative axis. From eq. (11) we observe that the Love numbers have a multiexponential form in the time domain.

2.2 The PWG method

The most attractive features of the Post–Widder method are that (i) it provides an analytical expression for the inverse transform without invoking contour integrals (see e.g. Davies & Martin 1979), and that (ii) it only depends on the values taken by the transform along the real positive axis; in the words of Gaver (1966) *the Laplace transform can be interpreted directly—and perhaps usefully—without the necessity of direct inversion.* With respect to other methods that are not based on the Bromwich inversion integral, such as the Laguerre or the Fourier series methods (Abate & Valkó 2004), the one discussed here is formally simpler and easy to program. Judging from extensive comparison studies (Davies & Martin 1979) the PWG method and its variants are less time consuming than other inversion methods even if it is difficult to judge their overall performance from a limited number of theoretical case studies. In the following the normal-mode and PWG techniques will be tested and compared for the first time in the context of a real geophysical problem.

Paraphrasing Gaver (1966), the Post–Widder formula reads

$$f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \hat{f}^{(n)}\left(\frac{n}{t}\right), \tag{13}$$

where t is time, f is a bounded, continuous real-valued function for $t \ge 0$, $\hat{f}$ represents its Laplace transform, and $\hat{f}^{(n)}$ is the nth s-derivative of $\hat{f}$ (Abate & Valkó 2004). Other real formulas exist for the inversion of the Laplace transforms, that do not involve the nth derivative, but are characterized by other complexities. The most important ones are briefly reviewed by Tuan & Duc (2002).

For numerical applications of the PWG method, a closed-form expression for $\hat{f}(s)$ is not generally available, so that $\hat{f}^{(n)}$ must be computed numerically. In the realm of spherical earth models, analytical forms for $\hat{f}(s)$ are only available for a homogeneous incompressible sphere (the so-called Kelvin model, see e.g. Lambeck 1980), and for a two-layer non-self-gravitating earth (Wu & Ni 1996). These solutions can be transformed into the time domain in a straightforward manner. Thus, in general, for practical applications Gaver (1966) has shown that $f(t)$ can be approximated by a sequence of functions as

$$f_n(t) = (-1)^n \frac{n\alpha}{t} \binom{2n}{n} \Delta^n \hat{f}\left(\frac{n\alpha}{t}\right), \quad n = 1, 2, \ldots, N, \tag{14}$$

where $\binom{n}{k}$ is the binomial coefficient, N is the maximum order of the sequence, $\alpha = \ln 2$, and Δ is the forward difference operator, with $\Delta f(nx) = f((n + 1)x) - f(nx)$. Expanding the Δ^n term in eq. (14), Abate & Valkó (2004) have shown that it is possible to write

$$f_n(t) = G_n^n, \tag{15}$$

with the recursive algorithm

$$G_0^k = \frac{k\alpha}{t} \hat{f}\left(\frac{k\alpha}{t}\right), \ 1 \le k \le 2N, \tag{16}$$

$$G_n^k = \left(1 + \frac{k}{n}\right) G_{n-1}^k - \left(\frac{k}{n}\right) G_{n-1}^{k+1}, \ 1 \le n \le N; 1 \le k \le 2N - n, \tag{17}$$

where the G_n^k are the so-called Gaver functionals. Eqs (15), (16) and (17) represent the basis of the PWG method.

The PWG approach is of practical use in the present context, because eqs (15)–(17) allow to compute directly the Laplace inverse of $\vec{x}(a, s)$ and to derive the Love numbers for any value of t. Since for a stably stratified incompressible earth all of the poles of $\vec{x}(a, s)$ are located on the negative axis (Vermeersen & Mitrovica 2000), in the PWG approach the Laplace transform $\hat{f}$ is sampled in a singularity-free region. This approach allows us to bypass both the analytical inversion of the array $P_b WJ$ in eq. (10) and the root-finding procedures that are needed to solve the secular eq. (12), with subsequent simplification of the numerical codes. If the above scheme is adopted, the standard normal-mode spectral form of the Love numbers given by eq. (11) is no longer available, so that the individual normal modes (s_i, ℓ_i) and the elastic term ℓ_e must be obtained by non-linear regression.

As discussed by Abate & Valkó (2004), the simplicity of the PWG method comes at a cost. First, the convergence of the sequence (14) is logarithmic, so that a large number of terms must be computed to obtain a sufficiently large number of significant digits in the inverse function. Since the evaluation of $f_n(t)$ demands the computation of $\hat{f}$, that is, the numerical evaluation of eq. (7) at specific real and positive s-values, the PWG approach, like the normal-mode one, should become more inefficient as the number of viscoelastic layers grows. Secondly, Gaver recurrence is known to be numerically unstable and to lead to catastrophic cancellation above some finite, threshold value of N; computing precision must then grow with the order of Gaver functionals (Abate & Valkó 2004), and this in turn requires that computations be conducted in a multiprecision environment. The resulting expense in CPU time is limited by means of sequence accelerators, as discussed by Valkó & Abate (2004). Among the available transformations for logarithmically converging sequences, here we have adopted the one based on the Salzer weights, given by eqs (6) and (7) of Valkó & Abate (2004).

3 RESULTS

We apply the PWG method to the study of time-dependent Love numbers for two specific cases: (i) a simple, 3-layer model including a homogeneous mantle, a lithosphere, and a core, and (ii) a suite of multilayered 1-D models in which the viscosity profile of the mantle exhibits various degrees of complexity. The results are reported in Sections 3.1 and 3.2, respectively. In Section 3.3 we will address the problem of reproducing the response of a finely layered PREM model (Dziewonski & Anderson 1981) by means of a model characterized by a limited number of layers.

In our discussion, we will analyse our results in the spectral (spherical-harmonic) domain. The results are readily extended to finite-size surface loads (Spada *et al.* 2004). In particular, we shall focus on the so-called Heaviside Love numbers, associated with the choice $f(s) = 1/s$. The computation of a Heaviside response instead of the impulsive response does not imply any loss of generality, and has the advantage of being easier to interpret physically. Furthermore, according to Hanyk (1999), it allows a simple numerical determination of the relaxation spectrum and residues of the Love numbers. In all the PWG computations that follow, we have used the Fortran 90 multiprecision library FMZM90 (Smith 1989), freely available from the web site http://myweb.lmu.edu/dmsmith/FMLIB.html. Our PWG code is compiled using the IBM XL Fortran compiler and runs on a 1.5 GHz Macintosh PowerPC G4.

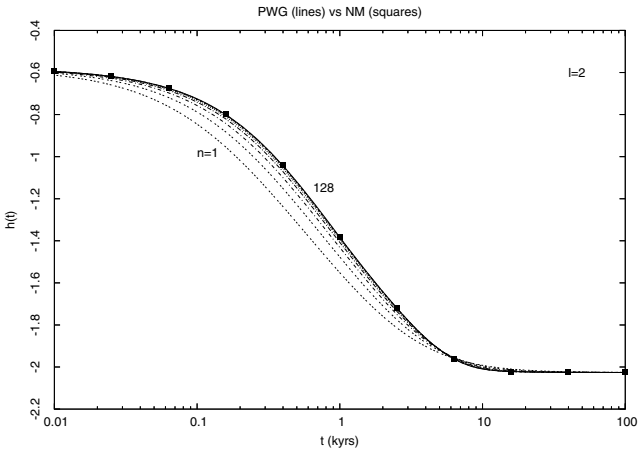

Figure 1. Comparison between the h Love numbers of degree 2 obtained by the normal-mode method (NM, squares), and various iterations based on the PWG approach, from $n = 1$ to $n = 128$ (lines). The three-layer earth model employed here and in Fig. 2 is characterized by an elastic lithosphere with density of 3300 kg m^{-3}, rigidity 0.28×10^{11} Pa, and thickness of 100 km. The mantle has a density of 4518 kg m^{-3}, a rigidity of 1.45×10^{11} Pa, and a Newtonian viscosity of 10^{21} Pa s. The radius of the inviscid core is 3480 km and its density is 10977 kg m^{-3}.

3.1 A simple three-layer test model

We test and calibrate the PWG method on a simplified earth model with $L = 1$. The model parameters are given in the caption of Fig. 1 that shows a comparison between the degree 2 h Love number computed by the normal-mode method (squares) and by Gaver functionals of increasing degree n (dashed curves). The solution of largest degree ($n = N = 128$) corresponds to the solid curve. To avoid the failure of eq. (17) by catastrophic cancellation, we have carried out our computations using $D = 128$ digits, even though this largely exceeds the minimum precision required to safely compute the terms of the sequence, for a wide range of multilayered models (see Section 3.2 below).

Substituting the transformed time history $f(s) = 1/s$ into eq. (11), it is straightforward to see that, for a given harmonic degree, the normal-mode results of Fig. 1 obey the simple formula

$$h(t) = h_e - \sum_{i=1}^{M} \frac{h_i}{s_i}(1 - \exp(s_i t)), \tag{18}$$

where short-term (elastic) and long-term (fluid) asymptotes are given by h_e and

$$h_f = h_e - \sum_{i=1}^{M} \frac{h_i}{s_i}, \tag{19}$$

respectively. We note that in the PWG approach, the computation of Love numbers pertaining to different time histories is straightforward, as it is sufficient to specify $f(s)$ in the Gaver sequence (16); in the normal-mode method, the further step of a time convolution is needed to construct the Love numbers by eq. (11) (see e. g. Spada *et al.* 2004).

The rate of convergence shown by the Love number h is not dissimilar to that shown by the Heaviside l and k Love numbers at the same harmonic degree 2 (top and bottom frames of Fig. 2, respectively). The time evolution of these Love numbers follows the same form (eq. 18), where the s_i terms are unchanged, but different residues appear. We also remark that l has not the monotonous behaviour shown by h and k, as it is often observed also for more

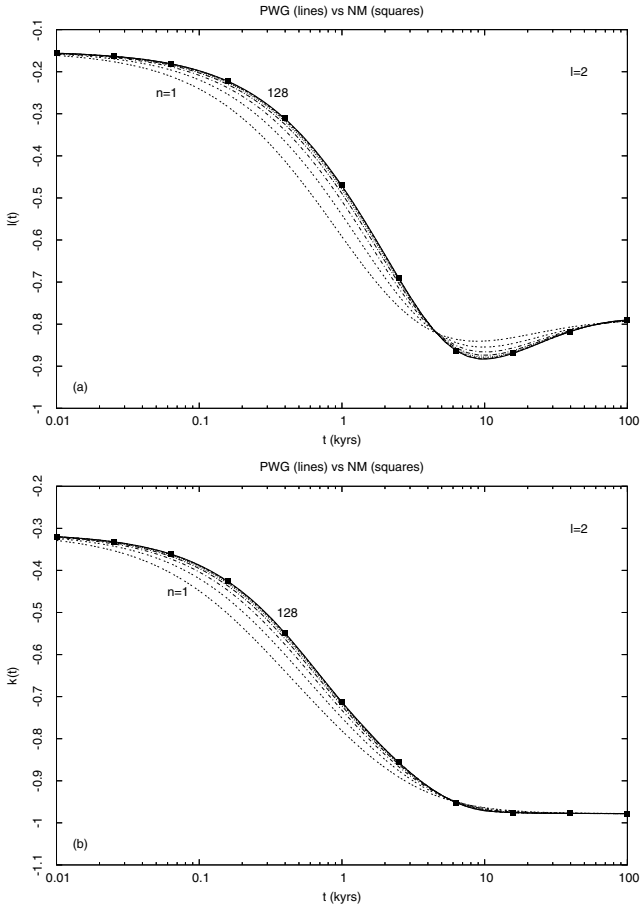

Figure 2. Degree 2 l (a) and k (b) Love numbers for the same three-layer model employed in Fig. 1. The normal-mode (NM) and the PWG results are shown by squares and dashed lines, respectively.

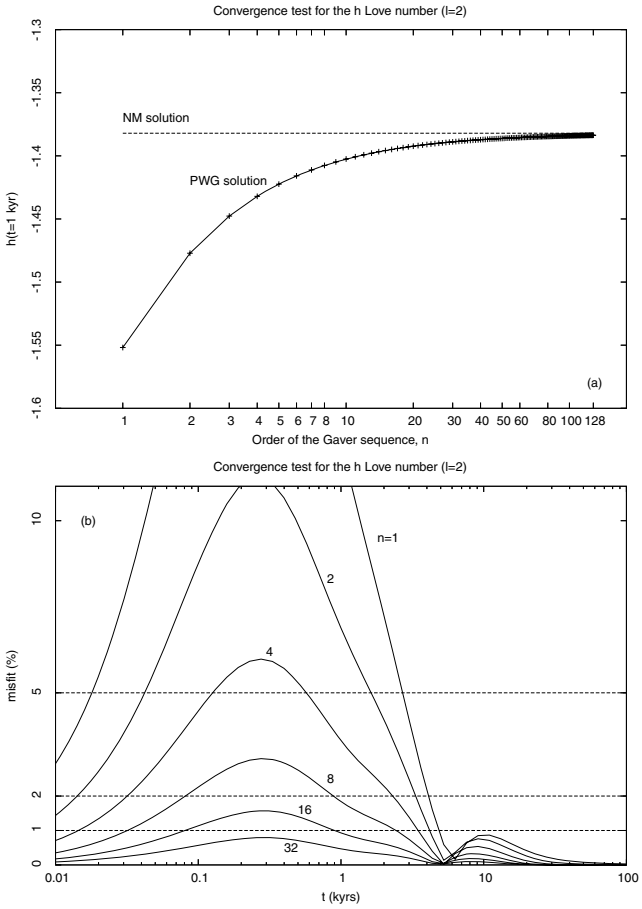

Figure 3. Frame (a) shows the convergence of the degree 2 h Love number obtained by the PWG method towards the normal-mode (NM) value, for $t = 1$ kyr. In frame (b) we show the misfit between PWG and normal-mode solutions as a function of time, for the same Love number considered in (a). The labels indicate the order of the Gaver solution. Dashed lines show the confidence levels of 5, 2, and 1 per cent.

complicated models at different harmonic degrees (Vermeersen & Sabadini 1997), and that the long-term asymptote of k does not exactly attain the value -1, as a consequence of the presence of the lithosphere that supports elastically the long-wavelength loads even for $t \mapsto \infty$ (e.g. Spada 1992).

While the first-order Gaver approximation ($n = 1$) reproduces well the two asymptotes, and predicts a transient behaviour that qualitatively agrees with that found from the normal-mode analysis, Figs 1 and 2 show that to accurately fit the normal-mode predictions a suitable number of Gaver functionals need to be computed. While this does not affect significantly the efficiency of PWG in simple models, for multistratified models appropriate sequence accelerators are needed, as discussed in Section 3.2. The issue of convergence is addressed in detail in Fig. 3, whose top frame shows how the PWG solution approaches the normal-mode solution $h = -1.3820$ as the number of the terms of the Gaver sequence grows (these computations have been done at fixed time $t = 1$ kyr). When a normal-mode solution is available, as is the case here, a useful stopping criterion is that of interrupting the sequence when a previously established agreement with the reference normal-mode solution is attained. For the h Love number, the bottom frame of Fig. 3 shows that the PWG solution departs from the normal-mode solution by less than 1 per cent for any value of time as far as 32 terms of the Gaver sequence are retained. A considerably smaller number of terms is needed if 2–5 per cent accuracy is considered acceptable. It is worth to remark that the misfit between the two solutions is strongly time dependent, with the largest misfit found at values of t associated with the largest slope of h, as it can be easily guessed from Fig. 1. Since increasing the number of layers does not change the exponential nature of the Heaviside responses (Wu & Ni 1996), nor it substantially alters the smoothness of the relaxation curves (Vermeersen & Sabadini 1997), we can expect that the convergence to the normal-mode solution will be still comparable to that of Fig. 3(b), when more complicated models will be employed. However, since the time dependence of $l(t)$ is oscillatory (see Fig. 2a), we also expect that a larger number of iterations will be needed to accurately represent this Love number.

3.2 Multilayered models

The top and middle frames of Fig. 4 show the density and rigidity profiles of the multilayered ($L = 28$) model of Vermeersen *et al.* (1996a), corresponding to PREM-averaged values; the viscosity profile is shown in the bottom frame. Note that layer thickness is not constant, and that the region across the 670 km discontinuity is more finely discretized than the others. The lithosphere is uniform and characterized by a viscosity of 10^{50} Pa s, that is, elastic on the timescales of glacial isostasy. As in the $L = 1$ models above, the core is homogeneous and inviscid with density 10 925 kg m^{-3}. The particular depth-dependence of viscosity is inspired by the work of

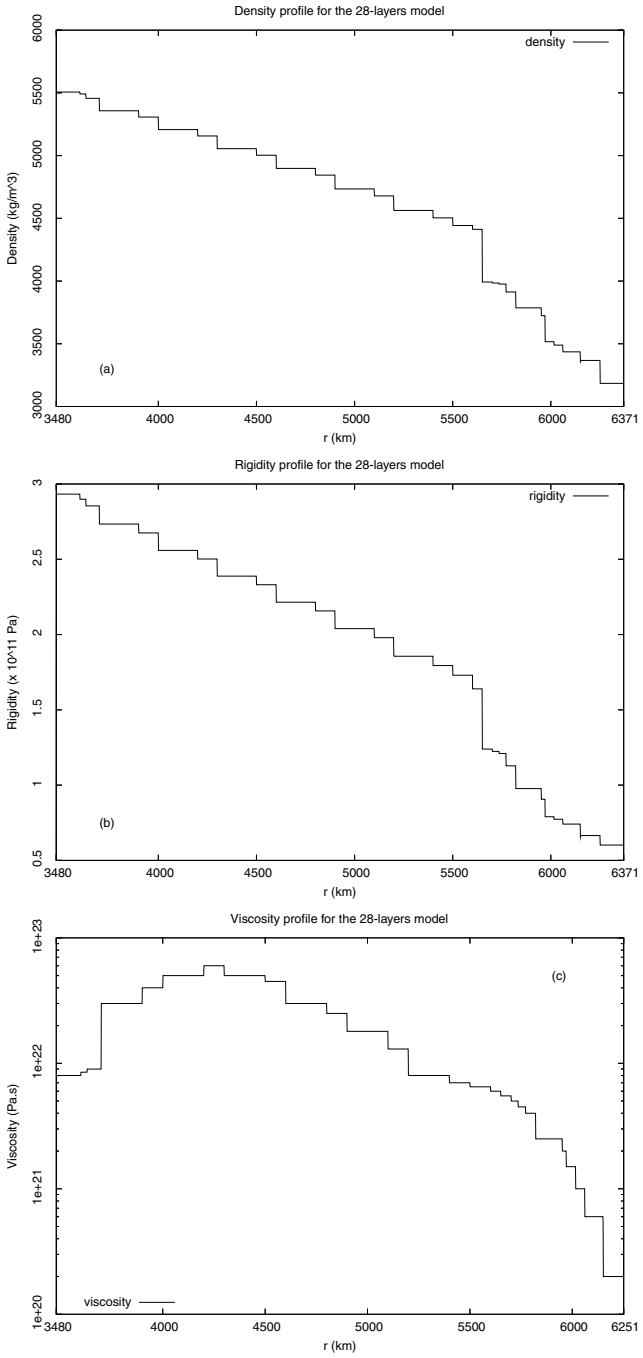

Figure 4. Density (a), rigidity (b), and viscosity profiles (c) for the multilayered ($L = 28$) model introduced by Vermeersen *et al.* (1996a). The earth parameters in (a) and (b) are obtained by volume-averaging the PREM model of Dziewonski & Anderson (1981). The viscosity profile shown in (c) is characterized by a 120-km thick virtually elastic lithosphere, with a viscosity of 10^{50} Pa s.

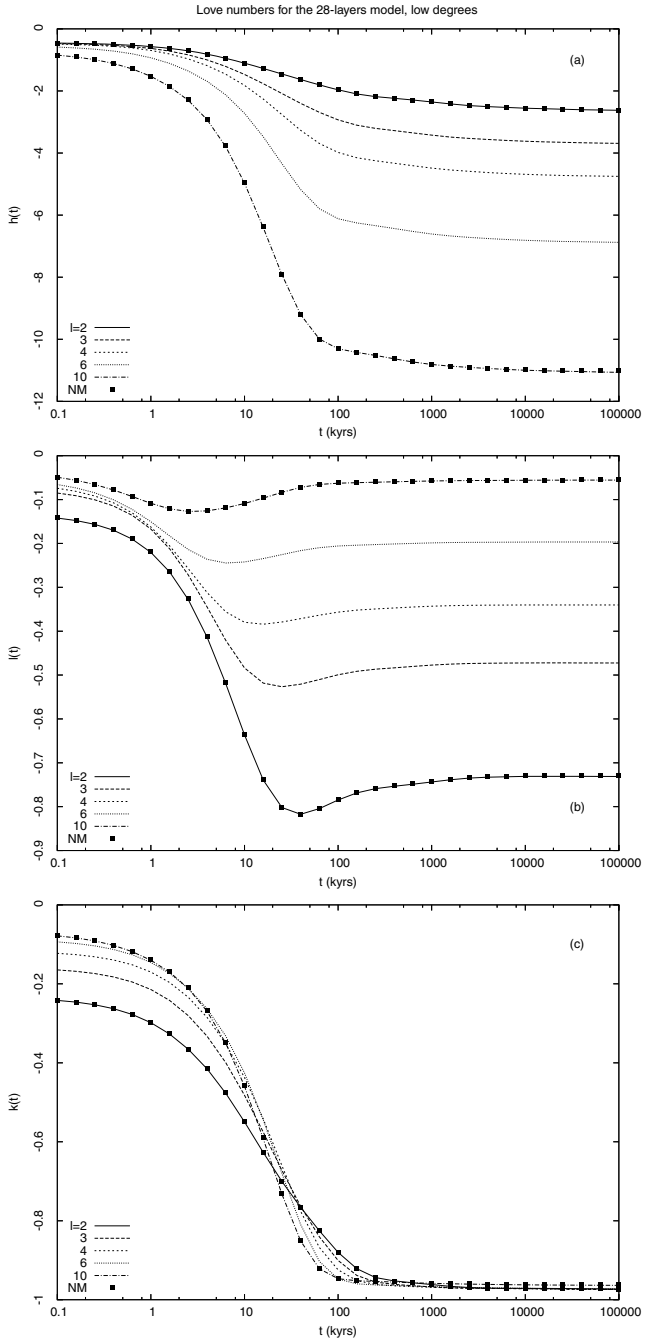

Figure 5. Low degree ($2 \leq l \leq 10$) viscoelastic Love numbers for the rheological parameters shown in Fig. 4. Frames (a), (b), and (c) show the h, l, and k Love numbers, respectively. The PWG predictions are depicted by lines; the black squares show independent normal-mode (NM) results (J. X. Mitrovica, private communication, 2005).

Ricard & Wuming (1991), and is characterized by a convex profile where the asthenosphere and the D″ layer correspond to viscosity minima. The density profile contains a barely visible density inversion at ∼200 km depth, with a $(\rho_{28} - \rho_{27})/\rho_{28} \sim 0.22$ per cent, giving rise to no Rayleigh–Taylor instabilities (Plag & Jüttner 1995) on the timescales of concern here.

Some low-degree Love numbers for the model of Fig. 4 are shown by dashed lines in Fig. 5, with upper, middle, and lower frames pertaining to h, l, and k, respectively. As in Figs 1 and 2, here we have applied the PWG method with ($N = 128$, $D = 128$). Love numbers show a quite complex time dependence, and, with the exception of $k(t)$, they reach asymptotic values on timescales far in excess of those that characterize the rebound process. The Love number $l(t)$ does not evolve monotonously with time, similarly to what we have observed for the $L = 1$ model (see Fig. 2b). However, due to the presence of a low-viscosity asthenosphere, the relaxation timescale is considerably reduced in the 28-layer model. A non monotonous

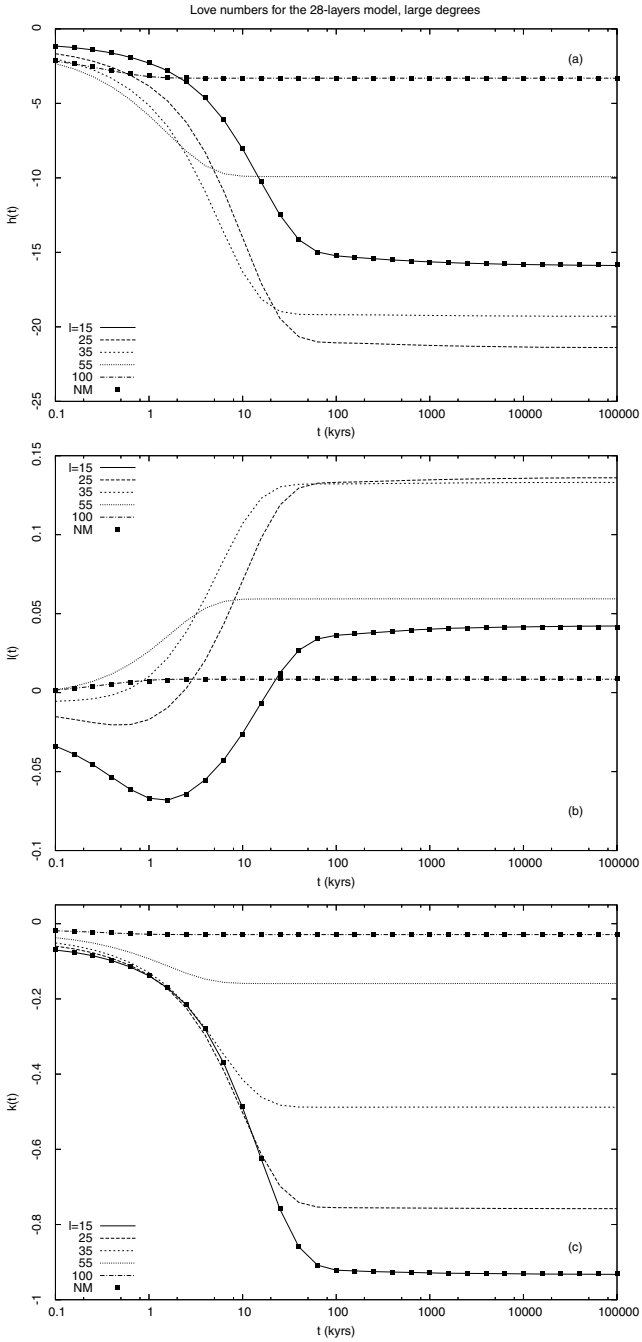

Figure 6. Large degree ($15 \leq l \leq 100$) viscoelastic Love numbers for the rheological parameters shown in Fig. 4. Frames (a), (b), and (c) show the h, l, and k Love numbers, respectively. The PWG predictions are depicted by lines; the black squares show independent normal-mode (NM) results (J. X. Mitrovica, private communication, 2005).

behaviour with increasing harmonic degree is particularly evident for $k(t)$ that does not reach the long-term value of -1 since the high-viscosity lithosphere prevents isostasy on these timescales.

Fig. 6 shows the results obtained in the range of degrees $15 \leq l \leq 100$. Clearly, with increasing harmonic degree the long-term asymptote is reached at progressively shorter times. The reason is that short-wavelength deformations tend to be localized close to the surface, and are rapidly relaxed because asthenospheric viscosity is low (Fig. 4c).

The squares in Figs 5 and 6 show results obtained from the classical normal-mode theory (J. X. Mitrovica, personal communication, 2005). In general, normal-mode and PWG solutions are in very close agreement. However, a detailed inspection of very large-degree Love numbers ($l = 100$ in Fig. 6) reveals some mismatch between the two, that differ by a few percent. It is unlikely that, in geodynamical applications, these discrepancies result in misinterpretations, but it is nevertheless important to discuss their possible origin. Since the normal modes implementation by Mitrovica only provides a subset of the whole family of 112 modes that we theoretically would expect for this 28-layer model, and the number of effectively computed modes diminishes with increasing harmonic degree, we speculate that the small detected discrepancies are due to these missing modes. It is well known from practical implementations of normal-mode theory that with increasing degree the roots of the secular polynomial tend to coalesce (e.g. Spada *et al.* 1992), so that their numerical determination becomes an increasingly difficult task. Since in the PWG approach the problem of root finding (and residues finding) is completely bypassed, this may perhaps indicate a better performance of PWG with respect to normal modes in the range of short-wavelength deformations. Of course, there is no standard way to numerically implement and tune normal-mode theory, and different implementations can give different results. A close inspection of the results by Vermeersen *et al.* (1996a), based on the same model considered here, reveals that discrepancies between PWG and normal modes are clearly visible also at relatively low harmonic degrees (e.g. compare the $l = 6$ curve of the l Love number of Fig. 2 by Vermeersen *et al.* (1996a) with that of Fig. 5 here). As above, our conjecture is that this disagreement may be caused by missing normal modes.

The curves of Fig. 7 show the degree 2 Love numbers for a 28-layer model similar to that employed above, in which the elastic lithosphere has been substituted by a viscoelastic layer with viscosity of 10^{21} Pa s. The figure shows the results obtained by three independent implementations of the normal-mode method. The first, labelled V96, is reproduced from the paper of Vermeersen *et al.* (1996a); the second and third show recent computations by Hugo Schotman (H05) and Jerry Mitrovica (M05) (personal communications, 2005). These normal-mode solutions are compared with our PWG solution (squares), obtained using the same settings as above. It is apparent that there is a disagreement between the various implementations of the normal-mode method, that is particularly severe for the l Love number (frame b). The mismatch is less pronounced for h (a), while for k (c) the four solutions are almost superimposed. Since the numerical codes used for these computations are not available, it is difficult to ascertain the exact reason of the discrepancies among the various normal-mode solutions. However, it is known from the literature that finding the roots of the secular equation is particularly challenging in the presence of a viscoelastic lithosphere, due to the instabilities that may occur for small values of s (Vermeersen & Sabadini 1997). Arguably, differences we find reflect distinct settings of the root finders used to solve the secular equation in the three cases shown, rather than a fundamental theoretical problem. If one is concerned with the response of the Earth on the timescales of post-glacial deformations ($\sim 10^2$ to $\sim 10^4$ kyr), the various normal-mode solutions can be considered as basically equivalent. The results of Fig. 7 confirm the observation of Vermeersen & Sabadini (1997) (page 535, lower right column) that for low harmonic degrees the condition $k_f \neq -1$ is sufficient for the results to be wrong, while $k_f = -1$ is not sufficient for them to be right. The PWG solutions of Fig. 7 show a very good agreement with the normal-modes solution H05 over the whole considered time

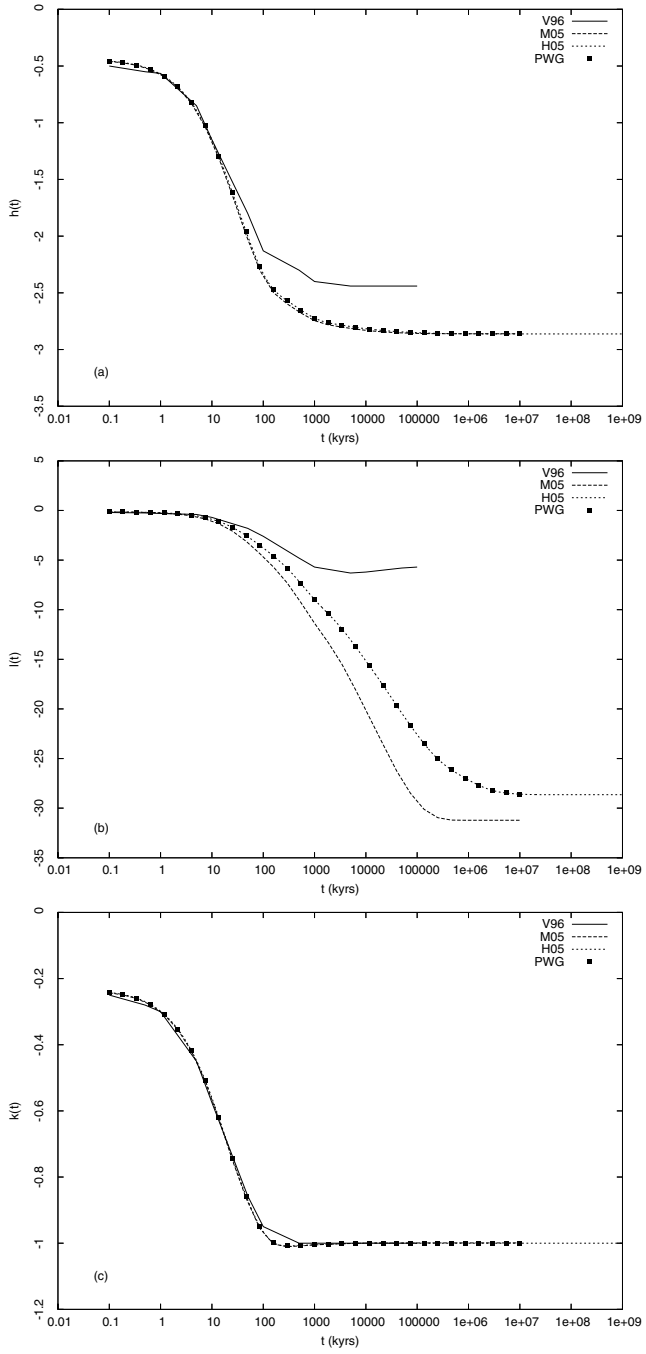

Figure 7. Degree 2 response of the 28-layer model in the case of a viscoelastic lithosphere with viscosity 10^{21} Pa s, according to independent normal-mode computations (V96, M05, H05), and to the PWG method (squares).

interval spanning from a few days to more than the age of the Earth. Of course, by no means can we infer that PWG and H05 are correct, and the other normal-mode solutions are not. Rather, we can only state that in the presence of a viscoelastic lithosphere, different tunings of the normal-mode method can lead to significantly different results, especially for the horizontal Love number and for long timescales. Our application of the PWG method may concur to discriminate among competing normal-mode implementations.

Up to now, the convergence of PWG has been attained computing a relatively large number of Gaver functionals ($N = 128$), and employing a sufficient number of digits ($D = 128$) to avoid the catastrophic cancellation inherent to eq. (17). However, with growing L, the CPU cost may become a serious limitation. For $L = 28$ (and $N = 128$, $D = 128$), the CPU time needed to compute all Love numbers at a fixed degree and time is $t_{\rm cpu} \simeq 25$s. The CPU cost is found to be basically independent of the harmonic degree. In order to tune the PWG method with the purpose of saving CPU time, we have first verified that the maximum misfit between the solution with ($N = 128, D = 128$) and that with ($N = 129, D = 128$) is of the order of 10^{-3} per cent. Thus, the high-order solutions of Figs 5 and 6 are stable if the order of the Gaver sequence is increased. To better illustrate this point, the solid curve in Fig. 8(a) shows how the solution ($N = 128$, $D = 128$) for the l Love number at degree 2 converges with increasing n, for a fixed value of time ($t = 1$ kyrs). Starting from this reference solution, we have implemented the Salzer acceleration scheme described by Valkó & Abate (2004) and we have examined the convergence rate for a suite of smaller N and D values. The dashed line shows that the relatively expensive reference solution can be approximated to within 0.5 per cent by the low-order and low-precision solution ($N = 8, D = 8$). Since the latter only demands $t_{\rm cpu} \simeq 0.75$s, the Salzer acceleration scheme increases the efficiency of our previous implementation of PWG by more than a factor of 30. Our computations have shown that $t_{\rm cpu}$ scales linearly with both N and D, but the former parameter affects the efficiency significantly more that the latter, since $\Delta t_{\rm cpu}/\Delta N \simeq 1$ and $\Delta t_{\rm cpu}/\Delta D \simeq 10^{-2}$. In our next computations with multilayered models, we will always employ accelerated Gaver sequences with ($N = 8, D = 64$), for which $t_{\rm cpu} \sim 1$ s. We found this to be a good compromise between efficiency and stability on the whole range of harmonic degrees ($2 \leq l \leq 100$) and number of layers ($1 \leq L \leq 200$) considered in this study. The agreement between the reference solution ($N = 128$, $D = 128$) and other (accelerated) solutions can be visually appreciated in Fig. 8(b), where the four curves are virtually indistinguishable.

3.3 Fitting the PREM results

A major issue concerns the number of layers that are needed to accurately describe the relaxation process of a real earth model. A partial answer has been provided by Vermeersen *et al.* (1996b), who have shown that solutions with L ranging between 30 and 40 are stable with respect to an increase in the number of layers, on all timescales and up to large degrees, regardless of the viscosity profile. Since the PWG technique constitutes a very simple and inexpensive numerical tool, here we have a chance to substantiate Vermeersen *et al.* (1996b)'s results, using as target solutions Love numbers computed by a 200-layer PREM profile. Density and rigidity of PREM are shown by solid lines in Figs 9(a) and (b), respectively; these profiles are virtually continuous away from major seismic discontinuities. With dashed lines we also show a set of volume-averaged approximations of PREM. All the layers have the same thickness, and the lithospheric thickness is kept constant to 120 km. Assuming the ocean-less structure of PREM, the average density and rigidity of the lithosphere are 3275.61 kg m^{-3} and 0.6084×10^{11} Pa, respectively. The fluid core is uniform and inviscid, with a density of 10980.54 kg m^{-3}.

Since it will be obviously impossible to establish results valid for *any* viscosity structure, our following computations are based on the relatively complex profile shown by the solid curve of Fig. 9(c), defined by

$$\frac{\eta(r)}{\eta_0} = \left[1 + \left(\frac{\eta_1 - \eta_0}{\eta_0}\right)\left(\frac{r - r_c}{r_l - r_c}\right)\left(\frac{r - r_m}{r_l - r_m}\right)^2\right], \qquad (20)$$

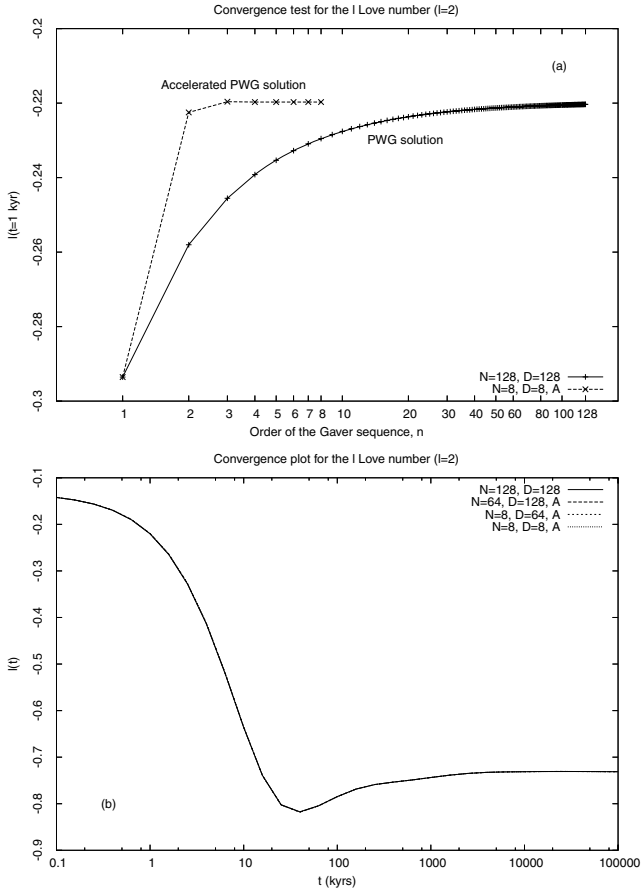

Figure 8. For the l Love number at degree 2 computed at $t = 1$ kyr, frame (a) compares the standard PWG solution ($N = 128$, $D = 128$) with a Salzer-accelerated solution ($N = 8$, $D = 8$, A) that considerably reduces the CPU requirements, as a function of the Gaver order n. Only eight iterations are necessary to fit the usual solution to within 1 per cent. In frame (b) three accelerated PWG solutions for the same Love number are compared to the reference solution, and shown as a function of time.

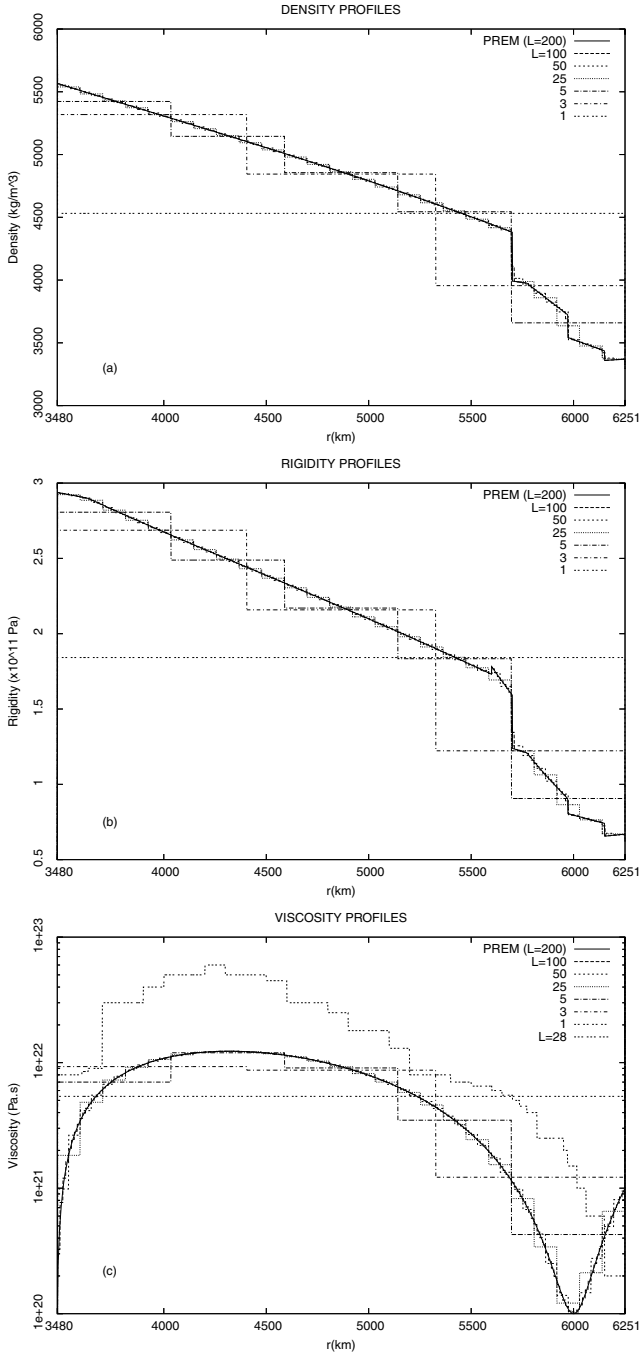

Figure 9. Density (a), rigidity (b), and viscosity profiles (c) for various multilayered models with L ranging between 1 and 100. For the profiles labelled by PREM, $L = 200$. The earth parameters in (a) and (b) are obtained by volume-averaging the PREM model of Dziewonski & Anderson (1981). The viscosity profile of the PREM model (frame c, solid line), has the analytical expression given by eq. (20). The curve with $L = 28$ corresponds to the same viscosity profile shown in Fig. 4. The thickness and viscosity of the lithosphere are 120 km and 10^{50} Pa s, respectively.

where $\eta_0 = 10^{20}$ Pa s is the viscosity just above the CMB ($r = r_c$ = 3480 km) and at radius $r = r_m$ (r_m = 6000 km in this specific case), and η_1 is the viscosity below the asthenosphere-lithosphere boundary ($r_l = 6251$ km). This viscosity profile has the same general convex trend of the one used previously (curve labelled $L = 28$). However, the low-viscosity zone is deeper, and the mantle average viscosity is clearly smaller. As in our previous computations with the 28-layer model, the lithospheric viscosity is fixed at 10^{50} Pa s. The dashed lines of Fig. 9(c) depict a set of approximations to the viscosity profile, with L ranging from 1 to 100.

Examples of Love numbers obtained using the PREM profiles of Fig. 9 are displayed in Figs 10 (low degrees) and 11 (high degrees), respectively. The Love numbers have clearly the same general trend as in the $L = 28$ case (Figs 5 and 6), but the overall relaxation times of the PREM model are clearly smaller than those obtained previously, at both low and large degrees. This is simply a consequence of the smaller average mantle viscosity of PREM with respect to $L = 28$ that can be clearly noticed from Fig. 9(c). However, for the l Love number at short wavelengths (see Fig. 11b), we do not observe a simple reduction of the relaxation timescale. Beside the details in the elastic and geometrical structure of the two models, differences with respect to $L = 28$ can be attributed to the different shape and location of the low-viscosity regions in the two viscosity profiles. This is an expression of the known sensitivity of horizontal deformations to the upper mantle structure, as reported in a number of previous investigations (e.g. Spada *et al.* 1992).

We illustrate in Fig. 12 how the Love numbers of degree 2 obtained by a suite of coarsely layered models ($L = 1, 2, 5, 25$, dashed lines) approximate those computed with the fine-graded PREM model

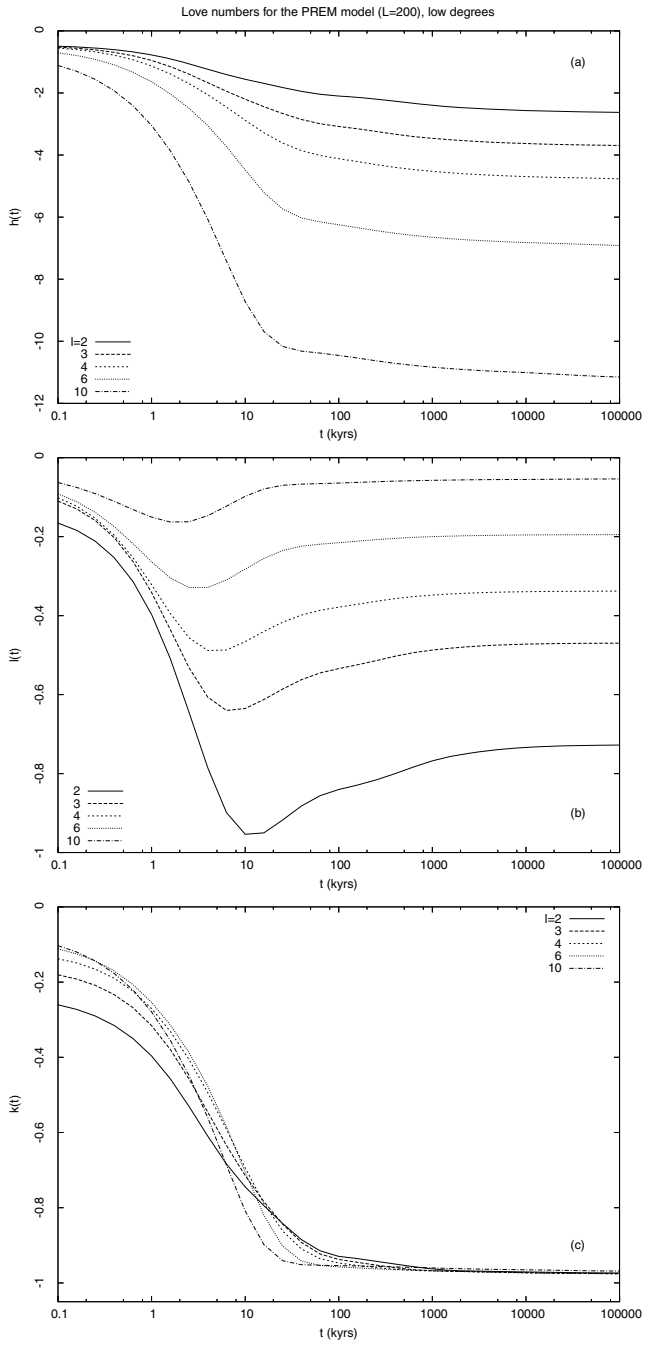

Figure 10. Low-degree ($2 \leq l \leq 10$) time-dependent Love numbers pertaining to the PREM ($L = 200$) profiles of Fig. 9.

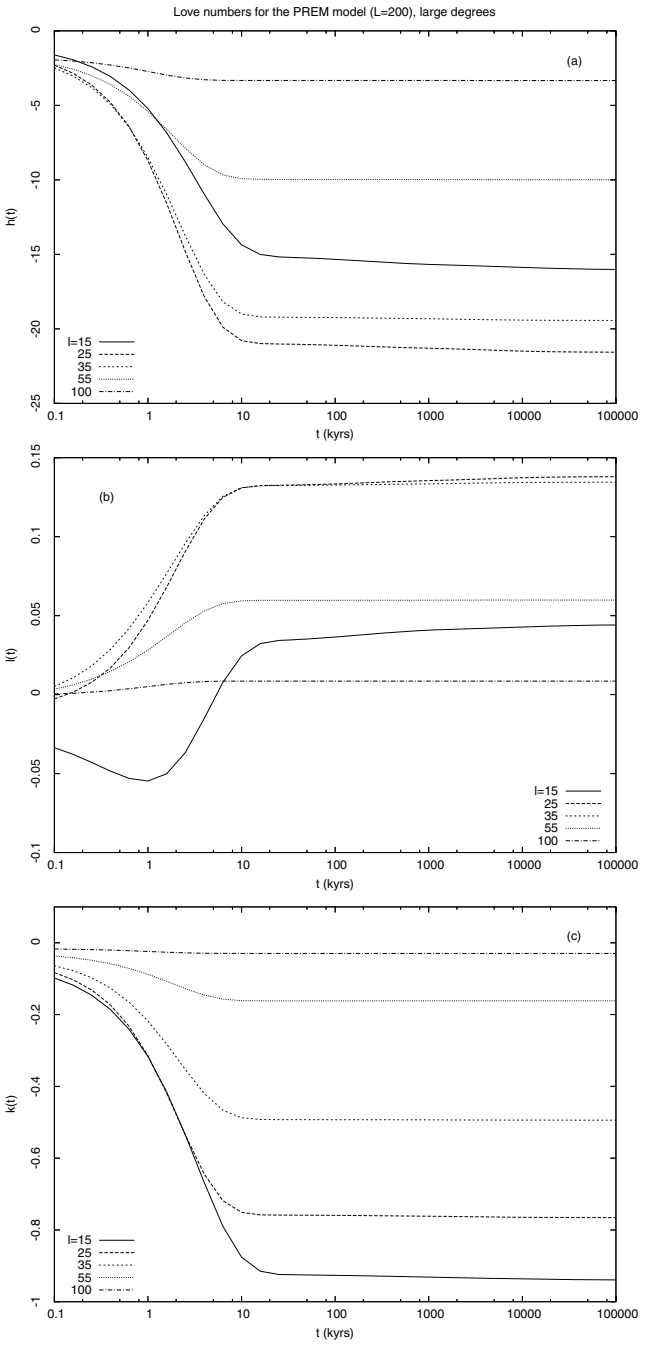

Figure 11. For the PREM ($L = 200$) rheological profiles of Fig. 9, frames (a), (b), and (c) show the h, l, and k Love numbers as a function of time. The harmonic degree ranges between $l = 15$ and $l = 100$.

($L = 200$, solid). Convergence towards the PREM solution is clearly not uniform. In the case of the h Love number (frame a), the PREM model is reasonably well reproduced for times $\leq$10 kyr, but at larger times it is clear that models with small L diverge significantly from PREM, and are well matched at all times if $L \geq 25$. While for the k Love number (frame c), the PREM is satisfactorily fitted with L as small as 3, for the l Love number we notice a divergence from PREM at all timescales, in contrast with what we have found for h in frame (a). However, as for h, a model with $L = 25$ well approximates the PREM result.

To establish general criteria of convergence, valid for any harmonic degree, we have computed the time-averaged misfit between the PREM Love numbers and those based on a L-layer model. The results are shown in Fig. 13, where the minimum number of layers $L_{\min}$ needed to fit PREM to a given accuracy is shown as a function of harmonic degree. The two misfit levels of 5 and 1 per cent are shown. For small degrees, a limited number of layers is sufficient to describe well the PREM model, since long-wavelength deformations are mostly sensitive to the Earth's average structure. This number increases with increasing degree l, until a maximum is reached for $l \sim 20$ in all of the curves shown. When l is further increased, $L_{\min}$ decreases, since at short wavelength the sensitivity to mantle properties is shifted towards the uppermost parts of the layering, so

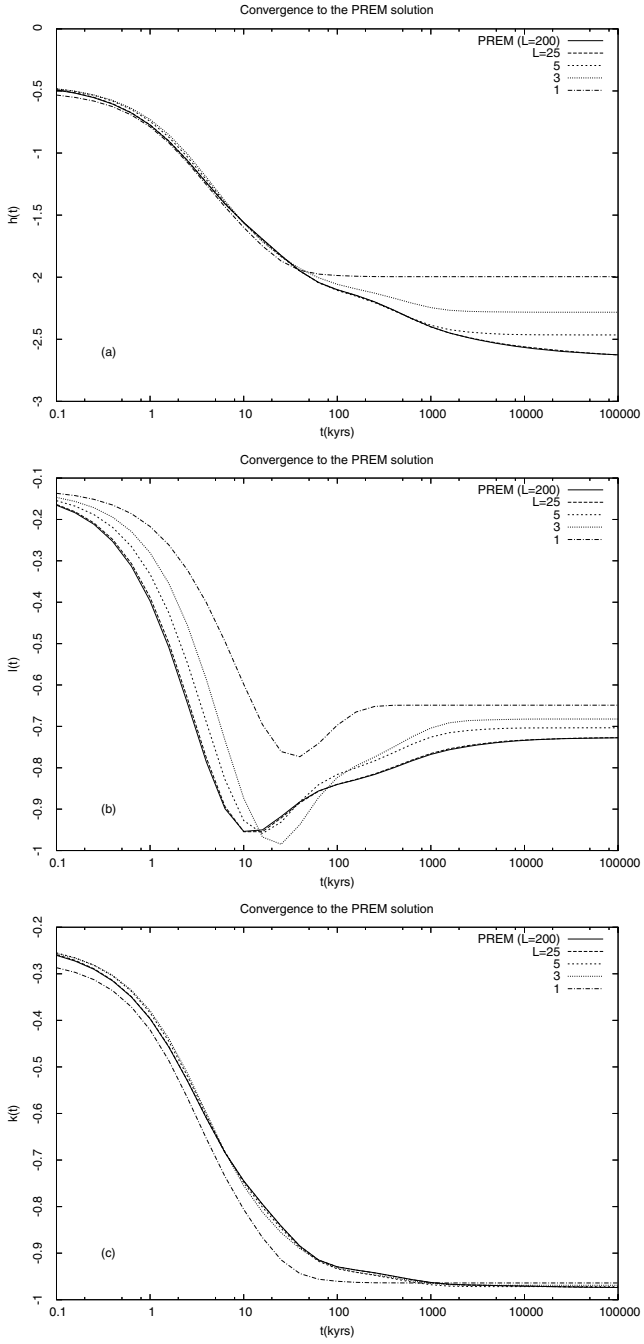

Figure 12. Degree 2 Love numbers corresponding to mantle stratifications with increasing number of layers. The reference solutions, shown by solid lines, pertain to the PREM ($L = 200$) profiles of Fig. 9.

that even a relatively rough stratification below the lithosphere may well match the response of the PREM model. While the general behaviour of the misfit functions for h and k is quite similar, it is clear from Fig. 13 that in order to match the PREM predictions for horizontal deformations $\sim$3 times more mantle layers are needed. This is a clear manifestation of the aforementioned larger sensitivity of horizontal deformations to the fine structure of the upper mantle.

4 CONCLUSIONS

We have shown that the Post–Widder formula can be used successfully to compute the Earth's Love numbers for a set of increasingly

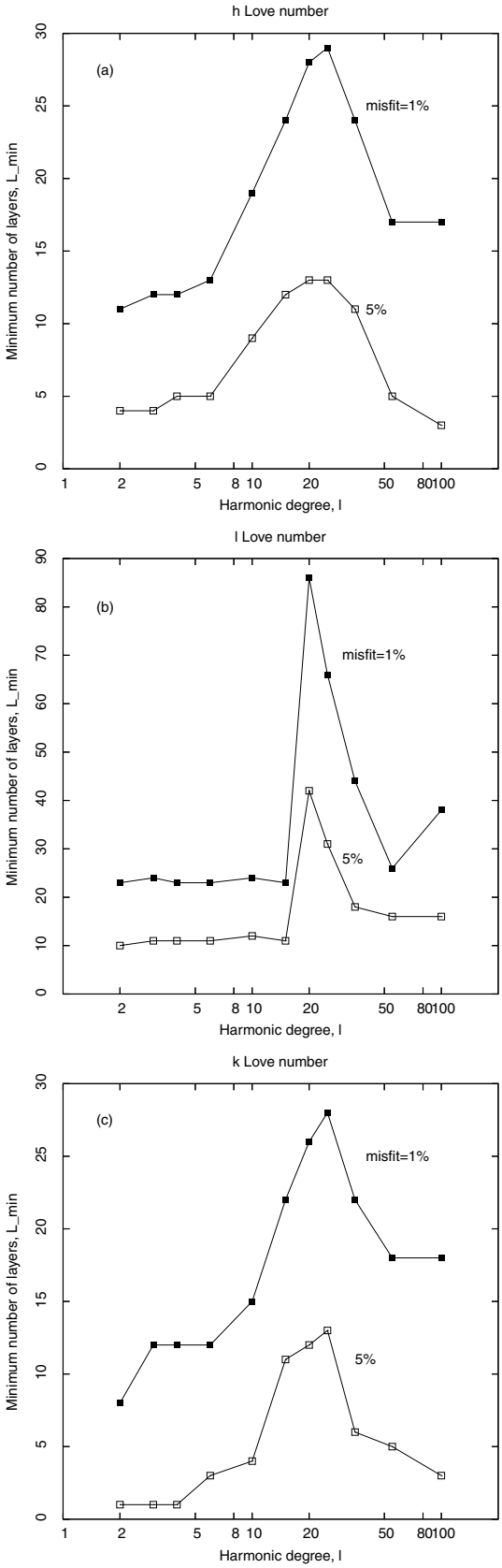

Figure 13. Minimum number of layers $L_{\min}$ required in order to fit the PREM predictions to within 5 and 1 per cent, as a function of the harmonic degree. Notice the different scale used for the l Love number (frame b).

complex earth models. Since the Laplace inverse only requires the nth derivative of the image function along the positive real axis, the discretized Post–Widder formula can be very easily implemented with the aid of Gaver functionals (Abate & Valkó 2004). We did not explore the potential of other real Laplace inversion formulae (Tuan & Duc 2002) that do not involve derivatives and thus may be less prone to the problem of catastrophic cancellation.

With respect to the normal-mode approach, the PWG method allows us to bypass some critical steps, such as the solution of a high-degree polynomial equation for multilayered models, and the search for the residues of the response function. Although we did not consider compressible earth models, Post–Widder is particularly promising in this area, where the secular equation is transcendental and characterized by an infinite number of roots (Vermeersen *et al.* 1996b). It is possible that the presence of unstable 'D modes' in compressible models and of unstable layering in PREM-stratified models be a hindrance for the PWG method, which is based on the sampling of the Laplace-transform function along the real positive axis, where roots associated with unstable modes are located (Vermeersen & Mitrovica 2000); but these instabilities can be successfully dealt with using the normal-mode method, both in spherical (Vermeersen & Mitrovica 2000) and plane geometry (see Klemann *et al.* 2003, and references therein). The results presented here show that when a moderately unstable layering is present in incompressible models, such as in the 28-layer model and in the PREM model we have considered, the PWG method performs well since the probability that a singularity of the transform be sampled is negligible. We speculate that, as far as singularities are isolated points, the PWG method can be applied safely. Abate & Valkó (2004) have shown that in the presence of singularities along the real positive axis the PWG inversion can be performed directly as done here; it might be in order to increase the accuracy, taking advantage of the shift property of the Laplace transform. We will explore this issue in future work.

Other drawbacks of the PWG approach are the slow (logarithmic) convergence and the need of a multiprecision environment (Abate & Valkó 2004). However, we have seen that if the simple but effective Salzer acceleration scheme of is employed, the number of iterations needed to determine the Love numbers can be drastically reduced. This also allows us to reduce the number of significant digits in the computations, with a non-negligible performance improvement and no loss of accuracy. It is possible that the implementation of theoretically more efficient accelerators (Valkó & Abate 2004) further increase the efficiency of the PWG method. The determination of Love numbers for models that include a viscoelastic lithosphere is particularly challenging, as discussed by Vermeersen & Sabadini (1997). Since the PWG approach do not require the solution of the secular equation, it constitutes a particularly valid tool in this context, where the viscoelastic modes are difficult to resolve.

Using the Salzer-accelerated PWG method, we have addressed the problem of identifying the minimum number of layers $L_{\min}$ needed to fit the Love numbers of a realistic earth model including an elastic lithosphere (i.e. the PREM-stratified model of Fig. 9). As pointed out by Vermeersen *et al.* (1996a), the general solution to this problem may help to determine some kind of standard mantle layering for future models and benchmarks. According to our results above, the answer depends on which geodynamic problem is to be solved. If we are concerned with the rotational response of the Earth that only involves the degree 2 k Love number (see e.g. Lambeck 1980), the response of PREM can be matched to within 1 per cent with $L_{\min} \simeq 8$. At the 5 per cent level, $L_{\min} = 1$ is enough (see Fig. 13). However, if we want to solve for the 3-D response of the Earth to a surface load, a relatively large number of layers ($L_{\min} \simeq 90$) must be used to match the PREM response within the same accuracy level 1 per cent. As clearly shown by Fig. 9(b), such a relatively large number of layers is demanded by the horizontal displacement Love number l, whose sensitivity to the mantle layering was first recognized by James & Morgan (1990) and later discussed by Spada *et al.* (1992). If a 5 per cent average misfit is enough for our purposes, $L_{\min}$ reduces to $\sim$40, in agreement with the rule-of-thumb proposed by Vermeersen *et al.* (1996a). In this work, we have dealt with a uniformly stratified mantle. A refinement in the layering strategy, such as the introduction of layers with a depth-dependent thickness, may well help to reproduce the PREM solution with even smaller L. These conclusions were obtained from the specific viscosity profile of Fig. 9(c), but we believe that they will not be substantially modified when other mantle viscosity profiles are employed.

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