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DOMINATING AND UNBOUNDED FREE SETS

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Abstract. We prove that every analytic set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with σ-bounded sections has a not σ-bounded closed free set. We show that this result is sharp. There exists a closed set with bounded sections which has no dominating analytic free set, and there exists a closed set with non-dominating sections which does not have a not σ-bounded analytic free set. Under projective determinacy analytic can be replaced in the above results by projective.

§1. Introduction. Let $E \subseteq X \times X$. A set $F \subseteq X$ is called free for E if, for any $x, y \in F$ with $x \neq y$, we have $(x, y) \notin E$. It is of considerable interest to find smallness conditions on E which guarantee the existence of large free sets. In some results of this type one assumes that X is a Polish space and imposes various topological or measure theoretic smallness restrictions on sections of the set E. Examples of such theorems can be found in [My1], [My2], and [NPS]. In the paper, we will assume most of the time that $X = {}^{\omega}\omega$. We consider a notion of smallness called σ-boundedness and prove that if $E \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ has all sections E_x, $x \in {}^{\omega}\omega$, σ-bounded and is analytic, then we can find a closed superperfect, so not σ-bounded, set free for E. We also show that this result is sharp. That is, on the one hand, one cannot get the free set in this theorem any bigger—there is a closed set E with all sections bounded for which there does not exist an analytic dominating free set—and, on the other hand, one cannot make the sections any bigger—there exists a closed set E with all sections non-dominating for which there does not exist a not σ-bounded analytic free set. The latter example clearly is nowhere dense. Hence, we obtain that Mycielski's result [My1] is sharp for the Baire space in a strong sense: there exists a closed nowhere dense set in $({}^{\omega}\omega)^2$ with no superperfect free set. Under projective determinacy, the above results are established for projective, rather than merely analytic, sets.

Below we present some notation and notions used in the sequel. For $x, y \in {}^{\omega}\omega$, we write $x \leq y$ ($x \leq^* y$) if for all (for all except finitely many) n, $x(n) \leq y(n)$ holds. A set $F \subseteq {}^{\omega}\omega$ is called *bounded* (*σ-bounded*) if it has an upper bound with respect to the ordering $\leq$ ($\leq^*$). F is called *dominating* if it is cofinal with respect to $\leq^*$, i.e., $(\forall x \in {}^{\omega}\omega)(\exists y \in F) x \leq^* y$. The cardinal coefficient $\mathfrak{b}$ ($\mathfrak{d}$) is defined as the smallest cardinality of a not σ-bounded (dominating) set.

Given a tree $p \subseteq {}^{<\omega}\omega$, the set of its branches is denoted with $[p]$. We let $|\sigma|$ denote the length of $\sigma \in {}^{<\omega}\omega$. For $\sigma \in {}^{<\omega}\omega$ and $n \in \omega$, we let $\sigma\hat{\ }n$ be the sequence

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of length $|\sigma| + 1$ with initial segment σ and last coordinate n. A tree $p \subseteq {}^{<\omega}\omega$ is called *superperfect* if $p \neq \emptyset$ and for every $\sigma \in p$ there exists $\tau \in p$ such that $\sigma \subseteq \tau$ and $\{n \in \omega : \tau\hat{\ }n \in p\}$ is infinite. Such τ are called *infinite splitnodes*. The set of all infinite splitnodes is denoted by $\mathrm{Split}(p)$. For $\sigma \in \mathrm{Split}(p)$, by $\mathrm{Succ}_p(\sigma)$ we denote the set of infinite successor splitnodes of σ, i.e. those $\tau \in \mathrm{Split}(p)$ for which $\sigma \subseteq \tau, \sigma \neq \tau$ and there is no ρ with $\sigma \subseteq \rho \subseteq \tau$, $\rho \neq \sigma, \tau$ and $\rho \in \mathrm{Split}(p)$. A tree $p \subseteq {}^{<\omega}\omega$ is called *uniform* if p is superperfect and for every $\sigma \in \mathrm{Split}(p)$ there exists $n \in \omega$ such that every member of $\mathrm{Succ}_p(\sigma)$ has length n.

§2. Large free sets.

THEOREM 2.1. *Let $E_n \subseteq ({}^{\omega}\omega)^2$, $n \in \omega$, be such that for any open $U \subseteq ({}^{\omega}\omega)^2$ the projection of $U \cap E_n$ on the first coordinate has the Baire property. If, for all $x \in {}^{\omega}\omega$ and all $n \in \omega$, $(E_n)_x$ is bounded, then there exists a superperfect tree $p \subseteq {}^{<\omega}\omega$ such that $[p]$ is free for $\bigcup_n E_n$.*

PROOF. Without loss of generality we can assume that $E_n \subseteq E_{n+1}$ for $n \in \omega$. Our goal is to construct a mapping $s \rightarrow \sigma_s$ from ${}^{<\omega}\omega$ to ${}^{<\omega}\omega$ so that $s \subseteq t \Rightarrow \sigma_s \subseteq \sigma_t$ and $s \perp t \Rightarrow \sigma_s \perp \sigma_t$ with certain additional properties. To this end fix a sequence of finite trees $T_n \subseteq {}^{<\omega}\omega$, $n \in \omega$, such that $T_0 = \{\emptyset\}$, $T_{n+1} = T_n \cup \{s\}$ for some $s \notin T_n$ with $s|l \in T_n$ for any $l < |s|$, and $\bigcup_n T_n = {}^{<\omega}\omega$. After stage n in addition to σ_s for $s \in T_n$, we will have produced a number $k_n \in \omega$ and sets $B_s^p \subseteq N_{\sigma_s}$ for $s \in T_n$ and $p \in \omega$. (Here and below $N_\sigma = \{x \in {}^{\omega}\omega : \ x||\sigma| = \sigma\}$.) We require that the following conditions hold:

(1) $k_{n+1} > k_n$;
(2) if $s\hat{\ }k \in T_{n+1} \setminus T_n$, then $\sigma_{s\hat{\ }k}(|\sigma_s|) \geq k_n$;
(3) B_s^p is open and dense in N_{σ_s};
(4) $N_{\sigma_s} \subseteq B_\emptyset^{p-1} \cap \cdots \cap B_{s|p-1}^0$ where $p = |s|$;
(5) if $s \in T_{n+1} \setminus T_n$, then $\bigcup\{(E_n)_x : \ x \in \bigcap_p B_s^p\} \cap N_{\sigma_t\hat{\ }k} = \emptyset$ for any $k \geq k_{n+1}$ and any $t \in T_n$.

For $n = 0$, let $\sigma_\emptyset = \emptyset$, $k_0 = 0$, and $B_\emptyset^p = {}^{\omega}\omega$, $p \in \omega$. Assuming that the construction has been carried out up to stage n, we show how to proceed at stage $n + 1$. Let $s \in {}^{<\omega}\omega$ be such that $T_{n+1} = T_n \cup \{s\}$. Put $s' = s|(|s| - 1)$. Note that $s' \in T_n$ so $\sigma_{s'}$ is already defined. Let $T_n = \{s_i : \ i \leq r\}$ for some $r \in \omega$. For $i \leq r$ define inductively σ_i, with $\sigma_{s'} \subseteq \sigma_i \subseteq \sigma_j$, $\sigma_{s'} \neq \sigma_i$, if $i \leq j$, and $\sigma_i(|\sigma_{s'}|) = k_n$. We will additionally produce $m_i \in \omega$ and $B_i \subseteq N_{\sigma_i}$ comeager in N_{σ_i}. The sequence σ_s will be an extension of σ_r. If σ_{i-1} has been defined (if $i = 0$, put $\sigma_{-1} = \sigma_{s'}\hat{\ }k_n$), let, for $m \in \omega$,

$$A_m = \{x \in N_{\sigma_{i-1}} : \ \forall y \in {}^{\omega}\omega\big((x, y) \in E_n \Rightarrow \forall j \leq \max_{t \in T_n} |\sigma_t| \ y(j) < m\big)\}.$$

Note that by our assumptions on E_n, $\bigcup_m A_m = N_{\sigma_{i-1}}$ and each set A_m has the Baire property. (Each A_m is the complement of the projection on the first coordinate of a set of the form $U \cap E_n$ for some open $U \subseteq ({}^{\omega}\omega)^2$.) Thus, there exist $\sigma_i \supseteq \sigma_{i-1}$ and $m_i \in \omega$ such that $A_{m_i} \cap N_{\sigma_i}$ is comeager in N_{σ_i}. This produces σ_i and m_i for $i \leq r$. Let $B_i = A_{m_i} \cap N_{\sigma_i}$. Now (3) enables us to find $\sigma \supseteq \sigma_r$ such that $N_\sigma \subseteq B_\emptyset^{|s'|} \cap \cdots \cap B_{s'}^0$; let $\sigma_s = \sigma$. Put $k_{n+1} = \max\{k_n, m_0, m_1, \ldots, m_r\} + 1$.

Let B_s^p, $p \in \omega$, be open dense subsets of N_{σ_s} such that $\bigcap_p B_s^p \subseteq \bigcap_{i \leq r} B_i$. It is straightforward to check that (1)–(5) hold.

Define $p \subseteq {}^{<\omega}\omega$ by letting

$$\sigma \in p \Leftrightarrow (\exists s \in {}^{<\omega}\omega)\, \sigma \subseteq \sigma_s.$$

Clearly p is a tree. We will prove that it is superperfect and that $[p]$ is free for E. To see that it is superperfect, it is enough to show that, for any $s \in {}^{<\omega}\omega$, $\sigma_s{}^\frown m \in p$ for infinitely many m. But this is guaranteed by the fact that $\sigma_s \subseteq \sigma_{s^\frown k}$ for any $k \in \omega$ and by (1) and (2).

To prove that $[p]$ is free for E, let $x, y \in [p]$, $x \neq y$. We need to show that $(x, y) \notin E_n$ for all n. Fix n. Let $s, u \in {}^{<\omega}\omega$ be such that $\sigma_s \subseteq x$, $\sigma_u \subseteq y$. By making s and u long enough, we can guarantee that $\sigma_s \perp \sigma_u$ and, by making s perhaps even longer, that $s \in T_{l+1} \setminus T_l$ and $u \in T_l$ for some $l \geq n$. Now elongate u, if needed, so that u is the longest member of T_l with $\sigma_u \subseteq y$. Find $k \in \omega$ and $l' \in \omega$ such that $\sigma_{u^\frown k} \subseteq y$ and $u^\frown k \in T_{l'+1} \setminus T_{l'}$. Then $n \leq l < l'$. Note that by (2) and (1)

$$y(|\sigma_u|) = \sigma_{u^\frown k}(|\sigma_u|) \geq k_{l'} \geq k_{l+1}.$$

On the other hand, by (4), $x \in \bigcap_p B_s^p$ whence, by (5), $y \notin (E_l)_x$. It follows that $(x, y) \notin E_l$, so $(x, y) \notin E_n$. ⊣

COROLLARY 2.2. *Let $E \subseteq ({}^\omega\omega)^2$ be analytic. If, for all $x \in {}^\omega\omega$, E_x is σ-bounded, then there exists a superperfect tree $p \subseteq {}^{<\omega}\omega$ such that $[p]$ is free for E.*

If all projective games are determined, the same conclusion holds if E is assumed to be projective.

PROOF. Using the theorem of Burgess and Hillard (see [Ke2, Theorem 35.43]), we can write E as $E = \bigcup_n E_n$ where, for each n, E_n is analytic, and $(E_n)_x$ is bounded for any $x \in {}^\omega\omega$.

Under projective determinacy, if E is projective, $E = \bigcup_n E_n$ with each E_n projective and $(E_n)_x$ bounded for $x \in {}^\omega\omega$ by a result of Kechris [Ke2, Exercise 39.24]. The projection of $U \cap E_n$ has the Baire property for open $U \subseteq ({}^\omega\omega)^2$ since if projective determinacy holds, all projective sets have the Baire property. ⊣

REMARK 2.3. Corollary 2.2 admits the following generalization: Let X be a Polish space which is not the union of a countable family of compact sets, and let $E \subseteq X^2$ be analytic. If each section E_x, $x \in X$, can be covered by countably many compact sets, then there is a closed set $F \subseteq X$ which is free for E and cannot be covered by countably many compact sets.

This fact follows immediately from its particular case, Theorem 2.1, after noticing that by Hurewicz's theorem (see [Ke2, Theorem 7.10]) X contains a closed subset homeomorphic to ${}^\omega\omega$.

REMARK 2.4. For abstract sets we can strengthen Corollary 2.2 in two directions, as follows.

(a) Suppose $E \subseteq ({}^\omega\omega)^2$ is symmetric such that E_x is σ-bounded for all $x \in {}^\omega\omega$. There exists a dominating set of size $\mathfrak{d}$ which is free for E.

(b) Suppose $E \subseteq ({}^\omega\omega)^2$ is symmetric such that E_x is not dominating for all $x \in {}^\omega\omega$. There exists a set of size $\mathfrak{b}$ which is not σ-bounded and is free for E.

Proofs of these facts are by transfinite induction arguments and are left to the reader. Note that here we cannot drop the requirement that E be symmetric. Indeed let $E = \{(x, y) \in ({}^\omega\omega)^2 : y \leq^* x\}$. Let $A \subseteq {}^\omega\omega$ be dominating. There must exist $x, y \in A$ with $y <^* x$, and hence $(x, y) \in E$.

§3. Larger free sets, larger sections—counterexamples. In this section, we prove two theorems which establish sharpness of the conclusion and the assumption in Corollary 2.2.

THEOREM 3.1. *There exists a symmetric closed $E \subseteq ({}^\omega\omega)^2$ such that for all $x \in {}^\omega\omega$, E_x is bounded and no dominating analytic set is free for E.*

Moreover, if all projective games are determined, no dominating projective set is free for E.

PROOF. For $x \in {}^\omega\omega$ and $l \in \omega$, define

$$k_x^l(0) = x(l) \qquad \text{and} \qquad k_x^l(n+1) = \max_{i<k_x^l(n)} x(i).$$

Now define $E \subseteq ({}^\omega\omega)^2$ by letting $(x, y) \in E$ if and only if $x = y$ or if $x \neq y$, then

$$(\forall n \in \omega)\ x(n) \leq k_y^l(n+1) \ \text{ and } \ y(n) \leq k_x^l(n+1) \ \text{ for } \ l = |x \cap y|.$$

Here and below $x \cap y$ stands for the longest $\sigma \in {}^{<\omega}\omega$ with $\sigma \subseteq x$ and $\sigma \subseteq y$. Checking that E is closed and symmetric is straightforward. Note that if $(x, y) \in E$, then for any $n \in \omega$ we have $y(n) = x(n)$ or $y(n) \leq \max\{k_x^l(n+1) : l \leq n\}$. Thus, E_x is bounded for any $x \in {}^\omega\omega$. It remains to see that no dominating analytic set is free for E.

Let p be a uniform tree. We will construct $x, y \in [p]$ so that $x \neq y$ and $(x, y) \in E$. Let ρ be an infinite splitnode of p with $|\rho| > 0$. We can find $\sigma_0, \tau_0 \in \mathrm{Succ}_p(\rho)$ with $\sigma_0(|\rho|) \neq \tau_0(|\rho|)$, $\sigma_0(|\rho|) > \max\{\rho(0), |\sigma_0|\}$, and $\tau_0(|\rho|) > \max\{\rho(0), |\tau_0|\}$. Suppose we have constructed $\sigma_n, \tau_n \in \mathrm{Split}(p)$ with $|\sigma_n| > n$, and $|\tau_n| > n$. We may choose $\sigma_{n+1} \in \mathrm{Succ}_p(\sigma_n)$ and $\tau_{n+1} \in \mathrm{Succ}_p(\tau_n)$ so that

$$\sigma_{n+1}(|\sigma_n|) > \max\{\tau_n(n), |\sigma_{n+1}|\} \ \text{ and } \ \tau_{n+1}(|\tau_n|) > \max\{\sigma_n(n), |\tau_{n+1}|\}.$$

If we let $x = \bigcup\{\sigma_n : n < \omega\}$ and $y = \bigcup\{\tau_n : n < \omega\}$, then x, y are as desired. To see this, note that $\rho = x \cap y$. Let $l = |\rho|$. A simple induction shows that $|\sigma_n| < k_x^l(n)$. (Indeed, it clearly holds for $n = 0$, and if we assume $|\sigma_n| < k_x^l(n)$, then $|\sigma_{n+1}| < \sigma_{n+1}(|\sigma_n|) \leq \max_{i<k_x^l(n)} x(i) = k_x^l(n+1)$.) Using this we get

$$y(n) = \tau_n(n) < \sigma_{n+1}(|\sigma_n|) = x(|\sigma_n|) \leq \max_{i<k_x^l(n)} x(i) = k_x^l(n+1).$$

A similar argument shows that the same condition holds with the roles of x and y interchanged. So, $(x, y) \in E$.

By [Sp], every dominating analytic set and, more generally, if all projective games are determined, every dominating projective set, contains $[p]$ for a uniform tree p. Thus, the theorem follows. ⊣

REMARK 3.2. We actually proved a bit more than is claimed in Theorem 3.1: we showed that for no uniform tree the set of all its branches is free for E and such sets are not necessarily dominating.

THEOREM 3.3. *There exists a symmetric closed $E \subseteq ({}^{\omega}\omega)^2$ whose all sections E_x, $x \in {}^{\omega}\omega$, are non-dominating and which does not have a not σ-bounded analytic free set.*
Moreover, if all projective games are determined, then E does not have a not σ-bounded projective free set.

PROOF. For $x, y \in {}^{\omega}\omega$ and $l \in \omega$, define

$$k^l_{x,y}(0) = \max\{x(l), y(l)\} \qquad \text{and} \qquad k^l_{x,y}(n+1) = \max_{i<k^l_{x,y}(n)}\{x(i), y(i)\}.$$

Define $E \subseteq ({}^{\omega}\omega)^2$ by letting $(x, y) \in E$ if and only if $x = y$ or if $x \neq y$, then for $l = |x \cap y|$,

$$(\forall n \in \omega)(\exists i, j < k^l_{x,y}(2n))\ n < i,\ n < j,\ y(i) < x(i), \text{ and } x(j) < y(j).$$

Checking that E is symmetric and closed is routine. If $(x, y) \in E$, then the definition of E guarantees that for any $n \in \omega$ there exists $i_n > n$ with $x(i_n) > y(i_n)$, that is, $\exists^\infty i\ x(i) > y(i)$. Thus, E_x is not dominating. To finish the proof, it suffices to show that no analytic set which is not σ-bounded is free for E.

Let p be a superperfect tree. We will find $x, y \in [p]$ such that $x \neq y$ and $(x, y) \in E$. Let ρ be an infinite splitnode of p. We recursively construct a sequence $\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots$ (in this order) of elements of $\mathrm{Split}(p)$ so that $\sigma_n \subseteq \sigma_{n+1}$, $\tau_n \subseteq \tau_{n+1}$, and $|\sigma_0| < |\tau_0| < |\sigma_1| < |\tau_1| < \cdots$. We put $\sigma_0 = \rho$ and let τ_0 be any member of $\mathrm{Succ}_p(\rho)$. When choosing σ_{n+1} and τ_{n+1}, we also maintain the following two conditions:

(1) $\sigma_{n+1}(|\sigma_n|) > \max\{|\tau_n|, \tau_n(|\sigma_n|)\}$;
(2) $\tau_{n+1}(|\tau_n|) > \max\{|\sigma_{n+1}|, \sigma_{n+1}(|\tau_n|)\}$.

It is not difficult to see that such a construction is possible.

Let $x = \bigcup_n \sigma_n$ and $y = \bigcup_n \tau_n$. From the fact that $\rho = \sigma_0$ and $\rho \subseteq \tau_0, \rho \neq \tau_0$ and from (1) for $n = 0$, we have $x \cap y = \rho$. Let $l = |\rho|$. By induction we show that for all $n \in \omega$

$$|\tau_n| < k^l_{x,y}(2n) \text{ and } |\sigma_{n+1}| < k^l_{x,y}(2n+1).$$

By (1), $|\tau_0| < \sigma_1(|\sigma_0|) = x(l) \leq k^l_{x,y}(0)$. Using this and (2), we get $|\sigma_1| < \tau_1(|\tau_0|) = y(|\tau_0|) \leq \max_{i<k^l_{x,y}(0)} y(i) \leq k^l_{x,y}(1)$. So, the assertion is verified for $n = 0$. Assume it is true for n. Then by (1) and the inductive assumption,

$$|\tau_{n+1}| < \sigma_{n+2}(|\sigma_{n+1}|) \leq \max_{i<k^l_{x,y}(2n+1)} x(i) \leq k^l_{x,y}(2n+2).$$

Using this and (2), we obtain

$$|\sigma_{n+2}| < \tau_{n+2}(|\tau_{n+1}|) \leq \max_{i<k^l_{x,y}(2n+2)} y(i) \leq k^l_{x,y}(2n+3),$$

and the inductive proof of our assertion is complete. Combining it with the obvious fact that $n < |\sigma_n|$, we get

$$n < |\sigma_n| < |\tau_n| < k^l_{x,y}(2n).$$

Note now that by (1) and (2), $x(|\sigma_n|) > y(|\sigma_n|)$ and $y(|\tau_n|) > x(|\tau_n|)$. Since this holds for all $n \in \omega$, we see that (x, y) fulfills the condition defining E, so $(x, y) \in E$.

By the theorem of Kechris [Ke1] and Saint Raymond [SR] (see [Ke2, Corollary 21.23 and Exercise 21.24]), each analytic set which is not σ-bounded contains $[p]$

for a superperfect tree p. Also, as proved by Kechris in [Ke1], under the assumption of projective determinacy, each not σ-bounded projective set contains $[p]$ for a superperfect tree p. Thus, the theorem is proved. ⊣

The set E from Theorem 3.3 clearly is nowhere dense. We conclude that Mycielski's result [My1] quoted in the introduction is sharp for the Baire space in the following sense.

COROLLARY 3.4. *There exists a symmetric, closed nowhere dense set $E \subseteq ({}^{\omega}\omega)^2$ which does not have a not σ-bounded free analytic set.*

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