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Dark matter density profiles from the Jeans equation

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ABSTRACT

We make a simple analytical study of radial profiles of dark matter structures, with special attention given to the question of the central radial density profile. We let our theoretical assumptions be guided by results from numerical simulations, and show that at any radius where both the radial density profile, ρ, and the phase-space-like density profile, ρ/σ^{ϵ}, are exact power laws, the only allowed density slopes in agreement with the spherically symmetric and isotropic Jeans equation are in the range $1 \leqslant \beta \leqslant 3$, where $\beta \equiv -\mathrm{d}\ln\rho/\mathrm{d}\ln r$. We also allow for a radial variation of these power laws, as well as anisotropy, and show how this allows for more shallow central slopes.

Key words: methods: analytical – galaxies: structure – cosmology: theory – dark matter.

1 INTRODUCTION

The formation and evolution of dark matter (DM) structures are in principle very simple because they only involve gravity. Despite this fact, density profiles of DM haloes have become one of the most challenging issues for our understanding of cold DM structure formation. Numerical simulations provide predictions of steep central density cusps with power-law slopes, $\rho \sim r^{-\beta}$, with β from 1 to 1.5 within a few per cent of the virial radius of the halo (Navarro, Frenk & White 1996; Moore et al. 1998). Recent careful studies (Reed et al. 2003; Diemand, Moore & Stadel 2004b; Navarro et al. 2004) indicate that the resolved region has still not converged on a central density slope, so in principle the central power slope may be even shallower.

The steep inner numerically resolved slopes are, however, not supported by observations. By measuring the rotation curve of a galaxy, one can in principle determine the density profile of its DM halo. Low-surface-brightness galaxies and spirals, where the observed dynamics should be DM-dominated, seem to show slowly rising rotation curves (Rubin et al. 1985; Courteau 1997; Palunas & Williams 2000; de Blok et al. 2001; Salucci 2001; Corbelli 2003; de Blok, Bosma & McGaugh 2003; Swaters et al. 2003), indicating that these DM haloes have constant density cores. Galaxy clusters, where baryons can play even less of a role, may show a similar discrepancy. Arcs (Sand, Treu & Ellis 2002) and strong lensing fits of multiple image configurations and brightnesses (Tyson, Kochanski & dell'Antonio 1998) also indicate shallow cores in clusters. All these observations could be in agreement with *N*-body simulations only if either the very central region is really not cuspy, or if cusps could somehow be erased during galaxy formation.

It is therefore very important to understand if the pure DM central density slopes can really be as steep as indicated by numerical simulations, in order to understand if one needs to invoke baryonic physics to reach agreement with observations. Baryonic structures are often observed to have central cores, which is possibly even understood theoretically (Hansen & Stadel 2003).

Several attempts have been made for an analytical derivation of the density profile (Bertschinger 1985; Syer & White 1998; Subramanian, Cen & Ostriker 2000; Hiotelis 2002; Manrique et al. 2003; Dekel et al. 2003), and none seems to present a clear and simple explanation for the findings of *N*-body codes. We will not attempt to answer the very difficult question of the actual formation of DM structures here, but will instead simply consider the Jeans equation and ask *which* equilibrium DM structures are in agreement with the spherically symmetric Jeans equation. We will be guided by the findings of numerical simulations and only consider the special cases where the phase-space-like density, ρ/σ^{ϵ}, is a power law in radius for some positive ϵ.

The normal use of the Jeans equation for collisionless systems (Hernquist 1990; Tremaine et al. 1994) is to assume a given radial density profile, $\rho(r)$, and then solve the Jeans equation to get the corresponding velocity dispersion, $\sigma^2(r)$. This can be done analytically for sufficiently nice density profiles, and can always be done numerically. The basic result is that the Jeans equation can allow for almost *any* shape of the density profile. An alternative approach is instead to assume the form of the phase-space density, $\rho/\sigma^3(r)$, and then solve the Jeans equation to get the corresponding density profile (Taylor & Navarro 2001). Also, this can always be done numerically, and even analytically for sufficiently nicely behaving velocity dispersions.

We will show below that for sufficiently simple phase-space(-like) densities, this approach can provide analytical insight into the allowed range of density profiles. One example hereof is that if both the central density profile and the phase-space-like density are *exact* power laws, then the central density profile of an isotropic DM structure cannot be more shallow than a Navarro–Frenk–White (NFW) profile with $\beta = 1$.

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2 EXACT POWER LAWS

Let us first consider the case where the coarse-grained radial density profile is an exact power law

$$\rho \sim r^{-\beta}, \tag{1}$$

at a given radius. Now, Taylor & Navarro (2001) observed that the phase-space density, ρ/σ^3, from N-body simulations approximately follows a power law. Recent high-resolution N-body results confirm that this is approximately correct in the equilibrated inner region, where substructures are unimportant (Diemand, private communication).[1] We will here make a slightly weaker assumption, namely that a phase-space-*like* density profile is an exact power law in the very central region

$$\frac{\rho}{\sigma^\epsilon} \sim r^{-\alpha}, \tag{2}$$

with unknown real numbers $\epsilon > 0$ and α. Taylor & Navarro (2001) found $\epsilon = 3$ and $\alpha = 1.875$. Recent high-resolution N-body simulations do in fact support this assumption for the very central numerically resolved region with ϵ of the order of 2–3. We will here not attempt to understand why the phase-space-like density is a power law in the centrally resolved region. A reason therefore must be sought at a deeper level, maybe through a solution to the collisionless Boltzmann equation.

For spherically symmetric and isotropic systems, the Jeans equation can be written (Binney & Tremaine 1987; Taylor & Navarro 2001) through the use of Poisson equation (for a self-gravitating system)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{-r^2}{G\rho}\frac{\mathrm{d}}{\mathrm{d}r}(\sigma^2\rho)\right] = 4\pi\rho r^2. \tag{3}$$

This assumption of a spherical, isotropic system is guided by the numerical N-body results in the central part of the DM structure. The inclusion of anisotropy, i.e. $A_\beta = 1 - \overline{v_\theta^2}/\overline{v_r^2}$, gives another term in the Jeans equation, $2\,A_\beta\,\mathrm{d}(r\sigma^2)/\mathrm{d}r$, and will therefore increase the space of solutions,[2] and we will later show how. We will leave non-spherical structures for a later analysis.

Under the assumption of power laws, equation (3) can now be written as

$$-C_1\,C_2\,r^{2(\alpha-\beta)/\epsilon} = C_3\,r^{2-\beta}, \tag{4}$$

where the two coefficients C_1 and C_2 come from the radial differentiations, e.g. $C_1 = \mathrm{d}\ln(\sigma^2\rho)/\,\mathrm{d}\ln r$, and the last coefficient C_3 is a positive constant. Clearly, the radial power-laws in equation (4) have to agree, giving

$$\beta = \frac{2\,(\epsilon-\alpha)}{(\epsilon-2)}. \tag{5}$$

Moreover, for the Jeans equation, equation (3), to make sense, the product $C_1\,C_2$ must be negative. If the product is positive, then there will be something negative on the left-hand side of equation (4) and something positive on the right-hand side. We can thus find the points where the Jeans equation breaks down by solving $C_1\,C_2 = 0$. This is a simple quadratic equation in α, with solutions

$$\alpha = 2 \pm (\epsilon/2 - 1). \tag{6}$$

That is, when α has the value in equation (6), the left-hand side of the Jeans equation is zero.

Combining the two results in equations (5) and (6) tell us that the *only* allowed values for the density slope are in the range

$$1 \leqslant \beta \leqslant 3, \tag{7}$$

and one thus concludes that for this most simple case of pure power laws, the central density profile of pure DM structures cannot be more shallow than $\rho \sim r^{-1}$. Please note that this result is obtained for rather general power laws like equation (2) with *any* value of ϵ and α. The results of recent N-body simulations tell us that locally the density profile can be approximated by a power law, and furthermore one can always find a value for ϵ such that equation (2) holds true locally. Therefore, for any radius in the resolved region of N-body simulations where substructure is not important, the density profile must be in the range given in equation (7).

When one includes non-isotropic systems, where $A_\beta \neq 0$, then one finds the lower limit to be

$$\beta_{\min} = 1 + A_\beta, \tag{8}$$

which implies that for a negative A_β one can have more shallow profiles, for example, a core in density for sufficiently circular orbits. For purely radial motion, the most shallow profile is $\beta_{\min} = 2$, which is in agreement with numerical findings for the spherical infall model (Lokas & Hoffman 2000). In general A_β can take any value in the range $-\infty < A_\beta < 1$ (Mamon & Lokas 2004), naturally constrained by β being always non-negative (Tremaine et al. 1994; Hansen et al. 2004); however, it should be kept in mind that numerical simulations find very little anisotropy in the very central region, $A_\beta \approx 0$ (Moore et al. 2001). The central isotropy can also be understood from a fundamental statistical mechanics point of view (Hansen et al. 2004).

3 ALMOST POWER LAWS

If the density profile and the phase-space-like density profile are not exact power laws, then our findings in equation (7) may potentially be invalid. In order to investigate this question, we can make expansions around power laws. As we will see, this will also indicate how far beyond the resolved region it makes sense to extend our findings. We therefore write

$$\rho \;\sim\; r^{-\beta(r)}, \tag{9}$$

$$\frac{\rho}{\sigma^\epsilon} \;\sim\; r^{-\alpha(r)}, \tag{10}$$

where $\beta(r)$ and $\alpha(r)$ are now slowly varying functions of radius. We choose a fixed, radially independent ϵ. Now, let us make a Taylor expansion around the radius r_{-1}, where $\beta = 1$, using $\beta' \equiv \mathrm{d}\beta/\mathrm{d}\ln r$ and $\alpha' \equiv \mathrm{d}\alpha/\mathrm{d}\ln r$. It should therefore be kept in mind that this Taylor expansion only holds sufficiently nearby the point of expansion, such that the higher derivative can be ignored, $\beta' \gg \beta''\ln r$. The Jeans equation again looks like equation (4), and the radial powers again lead to the expression for β in equation (5). However, the coefficients are now different:

$$C_1 = -\beta + \frac{2(\alpha-\beta)}{\epsilon} + \ln(r/r_{-1})\,\frac{2}{\epsilon}[\alpha' - \beta'(1+\epsilon/2)],$$

$$C_2 = 1 + \frac{2(\alpha-\beta)}{\epsilon} + \ln(r/r_{-1})\,\frac{2}{\epsilon}(\alpha'-\beta') + \frac{\mathrm{d}\ln C_1}{\mathrm{d}\ln r},$$

[1] Taylor & Navarro (2001) considered spherical bin averages of $\rho(r)$ and $\sigma^3(r)$, and then took the ratio, ρ/σ^3. The actual phase-space density, which consists of spherical averages of ρ/σ^3, differs due to substructures (Arad, Dekel & Klypin 2004; Diemand, Moore & Stadel 2004a). We consider only the equilibrated inner region of the DM structure where there is no difference.

[2] We use an unusual notation for the anisotropy parameter, A_β, to avoid confusion with the β in the density profile.

where $\ln(r/r_{-1})$ appears because we make the expansion around r_{-1}. The first coefficient C_1 is the one determining the most shallow slope, and we find that one has $C_1 = 0$ when

$$\alpha = \frac{\epsilon}{2} + 1 - \frac{\epsilon - 2}{2\epsilon} \ln(r/r_{-1}) [\alpha' - \beta'(1 + \epsilon/2)], \quad (11)$$

which, through equation (5), implies that

$$\beta_{\min} = 1 + \frac{1}{\epsilon} \ln(r/r_{-1}) [\alpha' - \beta'(1 + \epsilon/2)]. \quad (12)$$

One sees that the density slope within radius r_{-1} can be slightly more shallow than $\beta = 1$. This is in agreement with the numerical findings of Taylor & Navarro (2001).

The most recent simulations indeed seem to indicate that the phase-space-like density profile, equation (10), can indeed be well fitted with a power law in the central resolved region, where ϵ is found to be in the range $\epsilon \approx 2$–3 (Diemand, private communication). One can therefore always choose ϵ in such a way that $\alpha' = 0$ locally. Furthermore, it is interesting to make a comparison with a recent beautiful fitting formula valid for the entire resolved region, as presented in Navarro et al. (2004):

$$\beta_{\mathrm{N}}(r) \equiv -\frac{\mathrm{d} \ln \rho}{\mathrm{d} \ln r} = \left(\frac{r}{r_{-1}} \right)^{0.17}, \quad (13)$$

where r_{-1} is the radius where $\beta = 1$. Because this formula gives β smaller than 1, it may lead to a constraint on ϵ from the phase-space-like density. If equation (13) is consistent with the spherical and isotropic Jeans equation, then this β_{N} must be larger than the smallest allowed β as determined from equation (12), in the range where the Taylor expansion leading to equation (12) is valid, i.e. in the vicinity inside r_{-1}. This is solved by

$$\epsilon \geqslant 2. \quad (14)$$

When the numerical *N*-body simulations reach the level of resolution where they can resolve inside r_{-1}, it will be straightforward to test the validity of the Jeans equation through the phase-space-like density profile. Thus, if one finds numerically that the phase-space-like density is indeed a power law, and *only* so with $\epsilon < 2$, while simultaneously the density profile is sufficiently close to a power law (to assure the validity of the Taylor expansion, as quantified after equation 10), then either that region is not resolved numerically or equation (13) breaks down. Clearly, it is possible that the simulations will find no density slope more shallow than $\beta = 1$, in which case there is no constraint on ϵ.

4 CONCLUSIONS

Recent numerical DM simulations show that the phase-space-*like* density profile, ρ/σ^{ϵ}, is well fitted locally with a simple power law with ϵ of the order of 2–3. We show that when the radial density profile is an exact power law, $\rho \sim r^{-\beta}$, the spherically symmetric and isotropic Jeans equations only allow solutions where the density power slope is in the range $1 \leqslant \beta \leqslant 3$. This result is independent of the value of ϵ, and shows that if the central density indeed is a power law, then the density profile cannot be more shallow than $\beta = 1$.

This constraint weakens slightly for a more general density profile where the density power slope, $\beta(r)$, is a function of radius. The inner density profile is then allowed to be as shallow as described in equation (12). Also, for anisotropic systems more shallow profiles are allowed, according to equation (8).

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