PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Effect of a Footprint on Perceived Surface Roughness

D. E. Newland

Proc. R. Soc. Lond. A 1986 **405**, 303-327
doi: 10.1098/rspa.1986.0055

Proc. R. Soc. Lond. A **405**, 303–327 (1986)
Printed in Great Britain

The effect of a footprint on perceived surface roughness

By D. E. Newland
University Engineering Department, Trumpington Street, Cambridge, CB2 1PZ, U.K.

(*Communicated by K. L. Johnson, F.R.S. – Received* 14 *October* 1985)

A two-dimensional homogeneous random surface $\{y(\boldsymbol{X})\}$ is generated from another such surface $\{z(\boldsymbol{X})\}$ by a process of smoothing represented by

$$y(\boldsymbol{X}) = \int_{\infty} \mathrm{d}\boldsymbol{u}\, w(\boldsymbol{u}-\boldsymbol{X})\, z(\boldsymbol{u}),$$

where $w(\boldsymbol{X})$ is a deterministic weighting function satisfying certain conditions. The two-dimensional autocorrelation and spectral density functions of the smoothed surface $\{y(\boldsymbol{X})\}$ are calculated in terms of the corresponding functions of the reference surface $\{z(\boldsymbol{X})\}$ and the properties of the 'footprint' of the contact $w(\boldsymbol{X})$.

When the surfaces are Gaussian, the statistical properties of their peaks and summits are given by the continuous theory of surface roughness. If only sampled values of the surface height are available, there is a corresponding discrete theory. Provided that the discrete sampling interval is small enough, profile statistics calculated by the discrete theory should approach asymptotically those calculated by the continuous theory, but it is known that such asymptotic convergence may not occur in practice. For a smoothed surface $\{y(\boldsymbol{X})\}$ which is generated from a reference surface $\{z(\boldsymbol{X})\}$ by a 'good' footprint of finite area, it is shown in this paper that the expected asymptotic convergence does occur always, even if the reference surface is ideally white. For a footprint to be a good footprint, $w(\boldsymbol{X})$ must be continuous and smooth enough that it can be differentiated twice everywhere, including at its edges. Sample calculations for three footprints, two of which are good footprints, illustrate the theory.

1. Introduction

This paper has arisen from work undertaken by the author and a colleague into the response of wheeled vehicles to road surface irregularities (Cebon & Newland 1984; Cebon 1985; Newland 1986). In assessing the vertical displacements of a vehicle's wheels, it is necessary to allow for the action of the wheels' rubber tyres in enveloping road surface height irregularities. This depends on the scale of the irregularities and the size of the contact patch, which is the region where there is tyre-to-road contact. When the wavelength of the irregularities is small in comparison with the dimensions of a wheel's contact patch, deflections within the tyre absorb the irregularities and the wheel is not lifted significantly.

In practice, all mechanical surfaces whose height is measured by a method involving contact, whether by a wheel or by a probe, will be smoothed by the contacting device. The degree of smoothing depends on the size of the area of contact and the flexibility properties of the measuring device and of the surface.

In the field of vehicle dynamics various models have been devised to represent the interaction that occurs between a rubber-tyred wheel and the ground (Captain *et al.* 1979). In one class of model the enveloping action of the tyre is allowed for by assuming that the vertical displacement of the tyre is a weighted average of the road surface height within its contact patch. In the following analysis, we shall assume that a general homogeneous random surface is smoothed by such an averaging process. A weighted average of the surface height within a contact patch of fixed size is used to define a derived or smoothed surface whose statistical properties will be calculated. We shall assume only that the original or reference surface is homogeneous (stationary in space), and we calculate the two-dimensional autocorrelation and spectral density functions for the smoothed surface in terms of the corresponding functions for the reference surface and the properties of the 'footprint' which defines the averaging process. It has been pointed out before that the averaging of surface height irregularities within a contact patch of fixed size is equivalent to filtering the different wavenumber components of surface roughness (Captain *et al.* 1979; Gillespie *et al.* 1980), but there appears to be no previous general theoretical treatment of the two-dimensional problem.

Recently a paper by another of the author's colleagues has considered a related subject (Greenwood 1984). In his paper, Greenwood has calculated the statistical description of surface roughness for the case when the surface in question is known only by the result of sampling its height on a rectangular grid of constant sampling interval. His results agree with and extend those obtained by Whitehouse & Archard (1970) and Whitehouse & Phillips (1978, 1982). Peaks and summits are defined in terms of the relative heights of adjacent ordinates. The distributions of peak heights and curvatures are obtained for any profile across a homogeneous, Gaussian surface. The distributions of summit heights and mean curvatures are obtained for any homogeneous, Gaussian surface which is also isotropic. When the sampling interval becomes vanishingly small, the discrete results for profile statistics should become asymptotically the same as those worked out earlier by Rice (1944, 1945), Cartwright & Longuet-Higgins (1956), Longuet-Higgins (1957*a*, *b*, 1962) and Nayak (1971) for a continuous profile. Greenwood (1984) shows that, under suitable conditions, this asymptotic convergence of the two theories does occur, but quotes experimental results which indicate that the required conditions may not be satisfied in practice. Then the continuous and discrete theories give different results.

The cause of this lack of agreement is due to the presence of small-wavelength (large-wavenumber) components in the surface height. When the surface is smoothed by a suitable footprint, these large-wavenumber components are eliminated and the statistical properties of profiles across the smoothed surface then become asymptotically the same, whether calculated by the continuous theory or by the discrete theory when the sampling interval becomes vanishingly small.

It will be assumed that the shape and size of the footprint are constant (the so-called fixed footprint model of vehicle dynamics) and that the averaging properties of the footprint are independent of the characteristics of the road surface. Variation of contact patch size due to the vertical dynamics of the wheel and suspension system is not considered.

2. General theory of smoothing

Consider a rigid surface whose height measured from a reference plane is a sample function of a two-dimensional random process. The height of a point on the surface whose position is defined by vector $\boldsymbol{X}$ in the datum plane is given by $z(\boldsymbol{X})$. If the random process $\{z(\boldsymbol{X})\}$ is homogeneous and has a correlation function $R_{zz}(\boldsymbol{x})$ defined by

$$R_{zz}(\boldsymbol{x}) = \mathrm{E}[z(\boldsymbol{X})\, z(\boldsymbol{X}+\boldsymbol{x})] \tag{1}$$

then its two-dimensional spectral density $S_{zz}(\boldsymbol{\gamma})$ is given by

$$S_{zz}(\boldsymbol{\gamma}) = \frac{1}{(2\pi)^2}\int_{\infty} \mathrm{d}\boldsymbol{x}\, R_{zz}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{x}}. \tag{2}$$

The integration extends over the infinite two-dimensional domain of $\boldsymbol{x}$. We shall assume that $R_{zz}(\boldsymbol{x})$ is known for all $\boldsymbol{x}$ and hence that the spectral density function $S_{zz}(\boldsymbol{\gamma})$ is also known. The use of delta functions in the generalized theory of Fourier analysis permits $S_{zz}(\boldsymbol{\gamma})$ to be found subject to very general conditions which appear to include all practicable correlation functions. Since $R_{zz}(\boldsymbol{x})$ is an even function of $\boldsymbol{x}$, it follows from (2) that $S_{zz}(\boldsymbol{\gamma})$ is always real (see, for example, Newland 1984).

Now suppose that a new random surface is defined in such a way that the height $y(\boldsymbol{X})$ of the new surface is a weighted integral of $z(\boldsymbol{X})$ in the vicinity of $\boldsymbol{X}$. For each sample surface making up the ensemble $\{y(\boldsymbol{X})\}$ we define

$$y(\boldsymbol{X}) = \int_{\infty} \mathrm{d}\boldsymbol{u}\, w(\boldsymbol{u}-\boldsymbol{X})\, z(\boldsymbol{u}), \tag{3}$$

where $w(\boldsymbol{X})$ is the same arbitrary deterministic weighting function and $z(\boldsymbol{X})$ is a corresponding sample from the ensemble $\{z(\boldsymbol{X})\}$. We can think of the height of the $z(\boldsymbol{X})$ surface being measured by a probe which records, not the absolute height $z(\boldsymbol{X})$, but a weighted average of this height in the region of $\boldsymbol{X}$. We shall refer to $z(\boldsymbol{X})$ as the original (or reference) surface and $y(\boldsymbol{X})$ as the smoothed (or generated) surface. The (real) weighting function $w(\boldsymbol{X})$ is taken to be always finite and to decay to zero fast enough when $|\boldsymbol{X}| \to \infty$ that the integral in (3) exists whenever $\mathrm{E}[z^2]$ exists. For convenience, we shall use only weighting functions which are normalized so that

$$\int_{\infty} \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X}) = 1. \tag{4}$$

Data and lag footprints

We shall now calculate the autocorrelation function for the smoothed surface, $R_{yy}(\boldsymbol{x})$, in terms of the weighting function $w(\boldsymbol{X})$ and the autocorrelation function for the original surface, $R_{zz}(\boldsymbol{x})$.

Since

$$R_{yy}(\boldsymbol{x}) = \mathrm{E}[y(\boldsymbol{X})\, y(\boldsymbol{X}+\boldsymbol{x})] \tag{5}$$

we have, on substituting from (3),

$$R_{yy}(\boldsymbol{x}) = \mathrm{E}\left[\int_{\infty} \mathrm{d}\boldsymbol{u}_1 \int_{\infty} \mathrm{d}\boldsymbol{u}_2\, w(\boldsymbol{u}_1-\boldsymbol{X})\, w(\boldsymbol{u}_2-\boldsymbol{X}-\boldsymbol{x})\, z(\boldsymbol{u}_1)\, z(\boldsymbol{u}_2)\right]. \tag{6}$$

The expectation operator E may be brought inside the integrations and then, using (1), we obtain

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{u}_1 \int_{\infty} \mathrm{d}\boldsymbol{u}_2\, w(\boldsymbol{u}_1-\boldsymbol{X})\, w(\boldsymbol{u}_2-\boldsymbol{X}-\boldsymbol{x})\, R_{zz}(\boldsymbol{u}_2-\boldsymbol{u}_1). \tag{7}$$

By defining two new variables

$$\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{X} \tag{8}$$

and

$$\boldsymbol{v} = \boldsymbol{u}_2 - \boldsymbol{u} - \boldsymbol{X} - \boldsymbol{x} \tag{9}$$

and noting that $\mathrm{d}\boldsymbol{v} = \mathrm{d}\boldsymbol{u}_2$ when $\boldsymbol{u}$ is constant, (7) can be rewritten as

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{u} \int_{\infty} \mathrm{d}\boldsymbol{v}\, w(\boldsymbol{u})\, w(\boldsymbol{u}+\boldsymbol{v})\, R_{zz}(\boldsymbol{v}+\boldsymbol{x}). \tag{10}$$

The integration over $\boldsymbol{u}$ does not involve $R_{zz}(\boldsymbol{v}+\boldsymbol{x})$ and if we define a new weighting function

$$w'(\boldsymbol{v}) = \int_{\infty} \mathrm{d}\boldsymbol{u}\, w(\boldsymbol{u})\, w(\boldsymbol{u}+\boldsymbol{v}), \tag{11}$$

then an alternative form for (10) is

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{v}\, w'(\boldsymbol{v})\, R_{zz}(\boldsymbol{v}+\boldsymbol{x}) \tag{12}$$

or, on changing the variable again by putting

$$\boldsymbol{s} = \boldsymbol{v} + \boldsymbol{x}, \tag{13}$$

another version of (12) is

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{s}\, w'(\boldsymbol{s}-\boldsymbol{x})\, R_{zz}(\boldsymbol{s}). \tag{14}$$

The use of a prime after a quantity does not denote differentiation; instead it indicates a quantity which has been derived from the corresponding unprimed quantity by an appropriate transformation. In this case $w'(\boldsymbol{v})$ has been derived from $w(\boldsymbol{u})$ by (11). In the terminology of spectral analysis (see, for example, Newland 1984) $w(\boldsymbol{u})$ is called the data window and $w'(\boldsymbol{v})$ is the corresponding lag window. However in the subjects of road roughness in vehicle dynamics and surface roughness in mechanical engineering, the term 'footprint' seems more appropriate than 'window' so that we shall refer to $w(\boldsymbol{u})$ as the *data footprint* and $w'(\boldsymbol{v})$ as the *lag footprint*.

By changing $\boldsymbol{v}$ to $-\boldsymbol{v}$ in (11) and then changing the variable of integration to $\boldsymbol{r}$ where

$$\boldsymbol{r} = \boldsymbol{u} - \boldsymbol{v} \tag{15}$$

it is easy to see that

$$w'(\boldsymbol{v}) = w'(-\boldsymbol{v}) \tag{16}$$

so that the lag footprint $w'(\boldsymbol{v})$ defined by (11) is an even function of $\boldsymbol{v}$ whatever the form of the data footprint $w(\boldsymbol{u})$. Also we note, from (3), that $\boldsymbol{u}$ is a dummy variable for the position vector $\boldsymbol{X}$ and, from (12), that $\boldsymbol{v}$ is a dummy variable for the displacement (lag) vector $\boldsymbol{x}$. The data footprint is therefore often written as $w(\boldsymbol{X})$ and the lag footprint as $w'(\boldsymbol{x})$ and, from (11), the two are related by

$$w'(\boldsymbol{x}) = \int_\infty \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X})\, w(\boldsymbol{X}+\boldsymbol{x}). \tag{17}$$

Frequency and spectral footprints

Following the same terminology, we shall define two footprints in the frequency domain which correspond to the data and lag footprints in the space domain. The Fourier transform of the data footprint, $w(\boldsymbol{X})$, will be called the *frequency footprint*, $W(\gamma)$, and is defined by

$$W(\gamma) = \frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{X}}. \tag{18}$$

The Fourier transform of the lag footprint, $w'(\boldsymbol{x})$, will be called the *spectral footprint*, $W'(\gamma)$, and is defined by

$$W'(\gamma) = \frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{x}\, w'(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{x}}. \tag{19}$$

By changing the variable of integration from $\boldsymbol{x}$ to $-\boldsymbol{x}$ in (19) and using (16), we see that

$$W'(\gamma) = W'^{*}(\gamma) \tag{20}$$

so that the spectral footprint is always a real function. The relationship between $W(\gamma)$ and $W'(\gamma)$ can be obtained by taking Fourier transforms of both sides of (17). After multiplying the right-hand side by $\mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{X}}\,\mathrm{e}^{\mathrm{i}\gamma\cdot\boldsymbol{X}}$ and rearranging terms, we obtain

$$\frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{x}\, w'(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{x}} = \frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X})\, \mathrm{e}^{\mathrm{i}\gamma\cdot\boldsymbol{X}}\int_\infty \mathrm{d}\boldsymbol{x}\, w(\boldsymbol{x}+\boldsymbol{X})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot(\boldsymbol{x}+\boldsymbol{X})} \tag{21}$$

which, with the definitions (18) and (19), gives

$$W'(\gamma) = (2\pi)^2\, W^{*}(\gamma)\, W(\gamma). \tag{22}$$

We can now relate the spectral density of the smoothed surface, $S_{yy}(\gamma)$, to the spectral density of the original surface, $S_{zz}(\gamma)$, by taking the Fourier transform of both sides of (12). After multiplying the right-hand side by $\mathrm{e}^{\mathrm{i}\gamma\cdot\boldsymbol{v}}\,\mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{v}}$ and rearranging terms, we obtain

$$\frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{x}\, R_{yy}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot\boldsymbol{x}} = \frac{1}{(2\pi)^2}\int_\infty \mathrm{d}\boldsymbol{v}\, w'(\boldsymbol{v})\, \mathrm{e}^{\mathrm{i}\gamma\cdot\boldsymbol{v}}\int_\infty \mathrm{d}\boldsymbol{x}\, R_{zz}(\boldsymbol{v}+\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\gamma\cdot(\boldsymbol{v}+\boldsymbol{x})} \tag{23}$$

which with (2), (19) and (20), gives

$$S_{yy}(\gamma) = (2\pi)^2\, W'(\gamma)\, S_{zz}(\gamma) \tag{24}$$

or, on substituting from (22),

$$S_{yy}(\gamma) = (2\pi)^4\, W^*(\gamma)\, W(\gamma)\, S_{zz}(\gamma). \tag{25}$$

These results (24) and (25) allow the spectral density of the smoothed surface to be obtained from the spectral density of the original surface by simply multiplying by the spectral footprint function in (24) and by the square of the frequency footprint function in (25). The theory of the derivation of (24) and (25) is similar to that for the spectral analysis of finite-length records of random functions, which is the basis of the theory of digital spectral analysis (see, for example, Newland (1984)). However the application of this theory to the two-dimensional problem of surface roughness is thought not to have been published before. It will be seen that the fixed one-dimensional data window of digital spectral analysis has been replaced by a moving two-dimensional data footprint, and the moving one-dimensional spectral window of digital spectral analysis has become a fixed two-dimensional spectral footprint.

Three sample footprints

For the purpose of illustration in the following calculations, we consider three sample footprints. We take rectangular axes in the datum plane of the surface with reference directions $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and express the vectors $\boldsymbol{X}$, $\boldsymbol{x}$ and γ in terms of their components along these directions.

For the *rectangular, flat data footprint*, the weighting function $w(\boldsymbol{X}) = w(X_1, X_2)$ is constant over a rectangular area $-a_1 \leqslant X_1 \leqslant a_1$, $-a_2 \leqslant X_2 \leqslant a_2$ and is zero elsewhere. The height of the smoothed surface at $\boldsymbol{X}$ is then an unweighted average of the original surface over a rectangular area $2a_1 \times 2a_2$ centred at $\boldsymbol{X}$. For the *rectangular, cosine data footprint*, the average extends over the same rectangle but now there is a cosine weighting. For any section through $w(X_1, X_2)$ parallel to the X_1 axis, the weighting function has the form $(1+\cos \pi X_1/a_1)$, and similarly for any section parallel to the X_2 axis. For the *normal data footprint*, $w(X_1, X_2)$ is an elliptical Gaussian bell with its major and minor axes aligned with the coordinate axes. The volume under each $w(X_1, X_2)$ surface is normalized to unity in order to satisfy (4).

In table 1, these three data footprint functions are listed, together with their corresponding lag footprints calculated from (17), frequency footprints calculated from (18), and spectral footprints calculated from (19) or from (22). All the functions are shown drawn to scale in figures 1, 2 and 3, and, in order to illustrate more clearly the different shapes of the three spectral footprints, their cross-sections along the γ_1-axes in figures 1 (*d*), 2 (*d*) and 3 (*d*) are plotted again to a logarithmic scale in figure 4.

In the case of the normal data footprint, the area covered by the footprint is infinite, which is obviously a theoretical case only. In any practical application, there must be some truncation to limit the size of the footprint. The flat data footprint (T 1.1) (see table 1), and the cosine data footprint (T 1.5), are examples

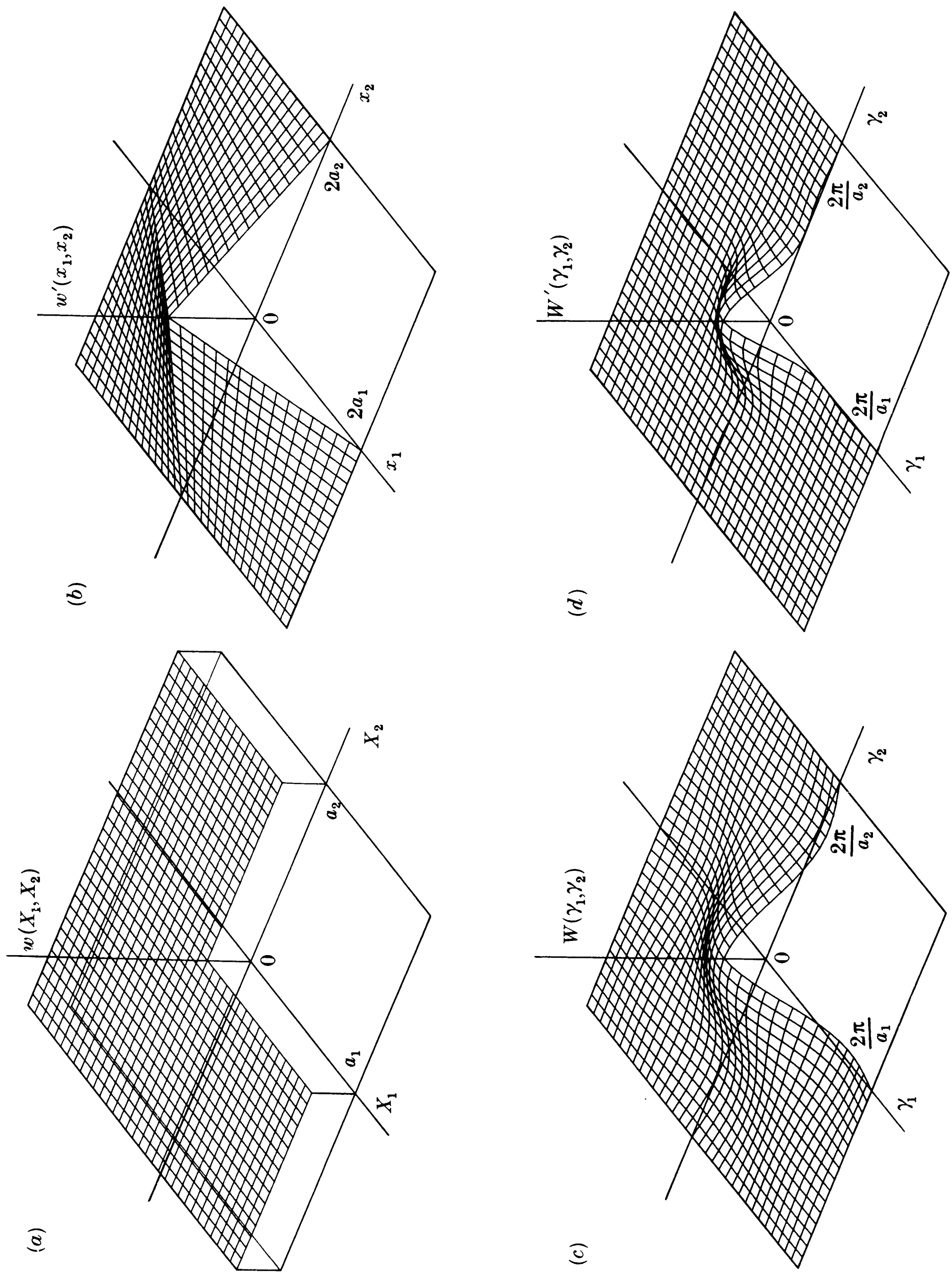

FIGURE 1. (*a*) Rectangular flat data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. The frequency and spectral footprints are drawn for a limited field of wavenumbers only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

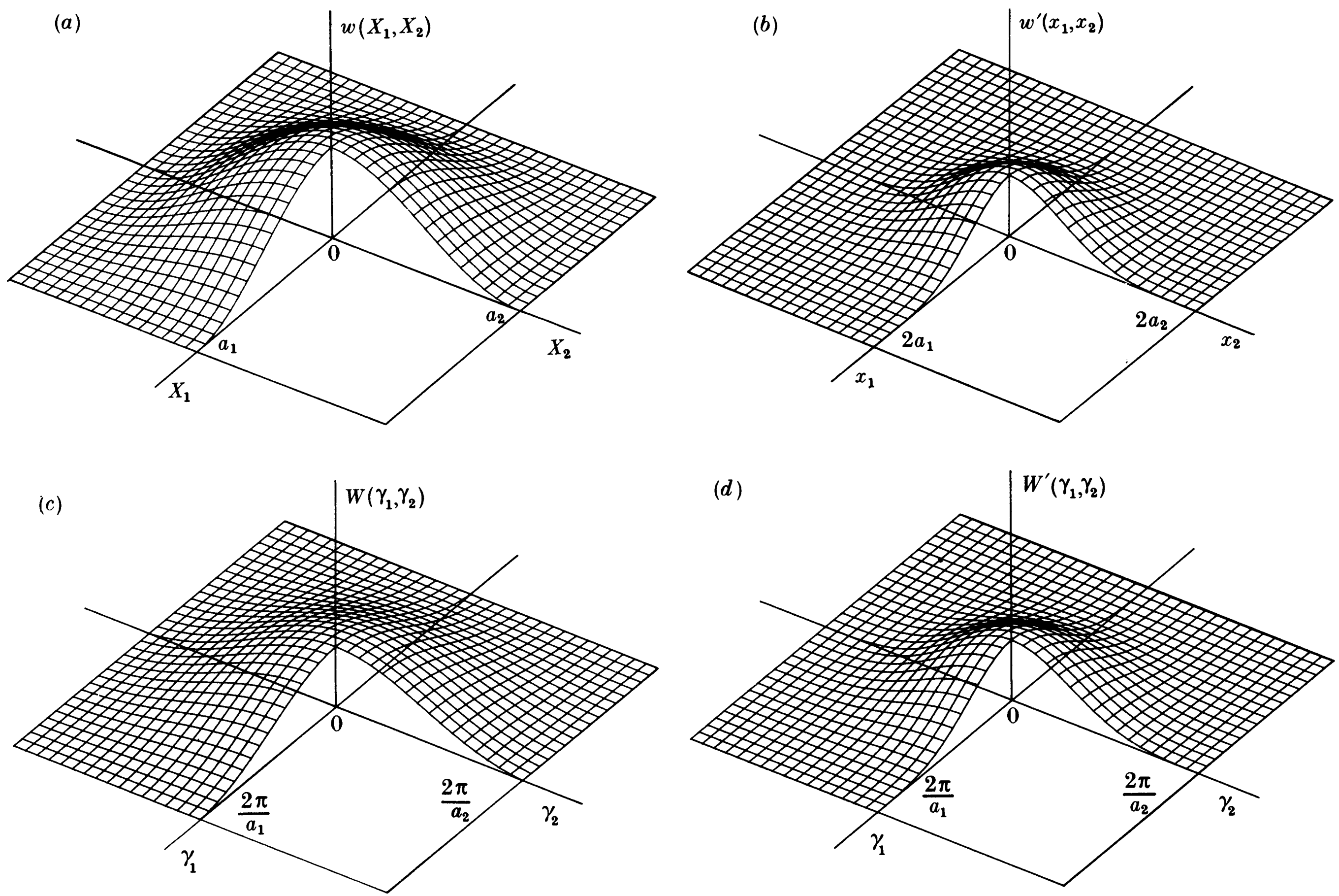

FIGURE 2. (*a*) Rectangular cosine data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. The frequency and spectral footprints are drawn for a limited field of wavenumbers only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

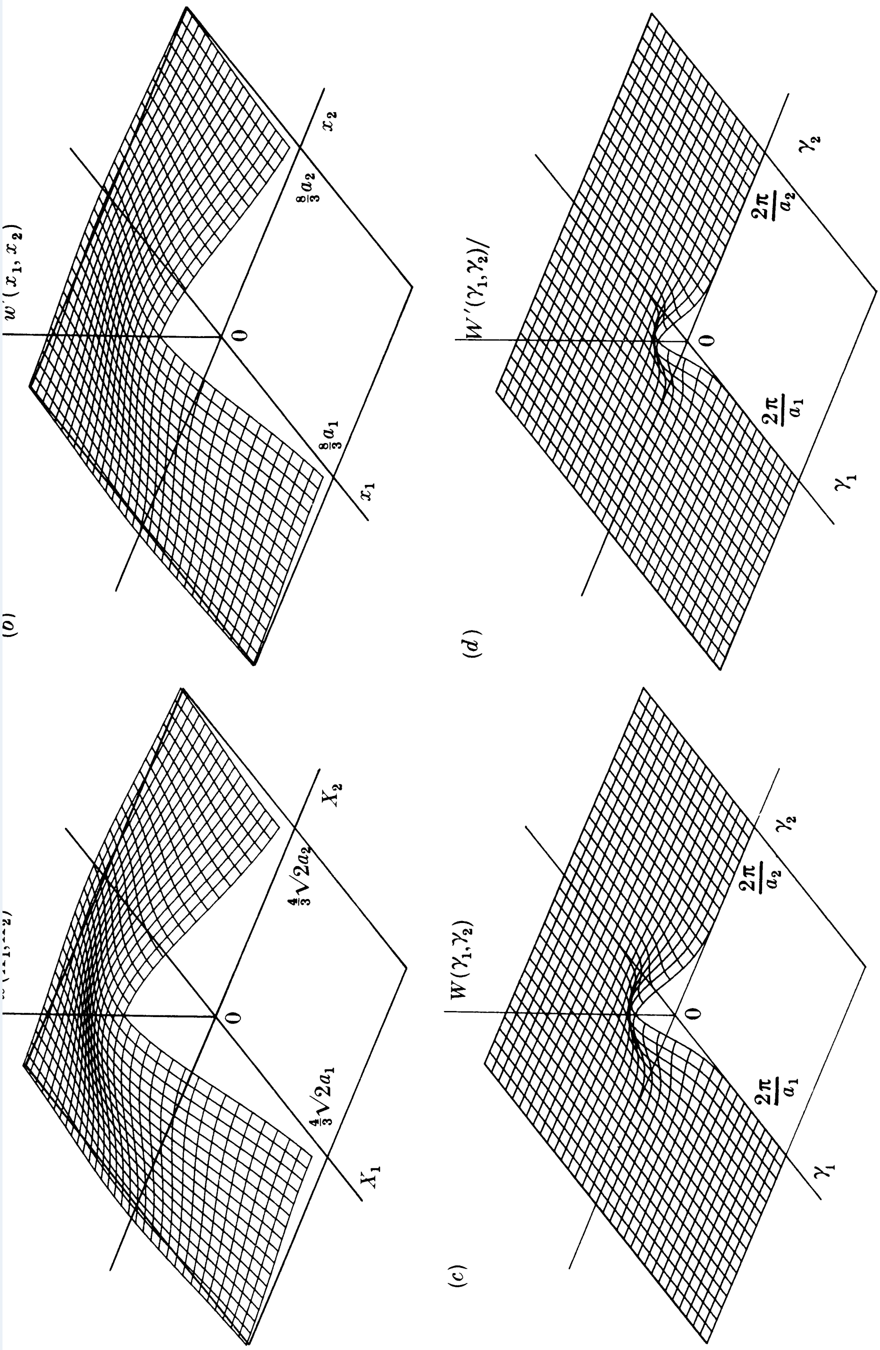

FIGURE 3. (*a*) Normal data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. All the footprints are shown for a limited field of values only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

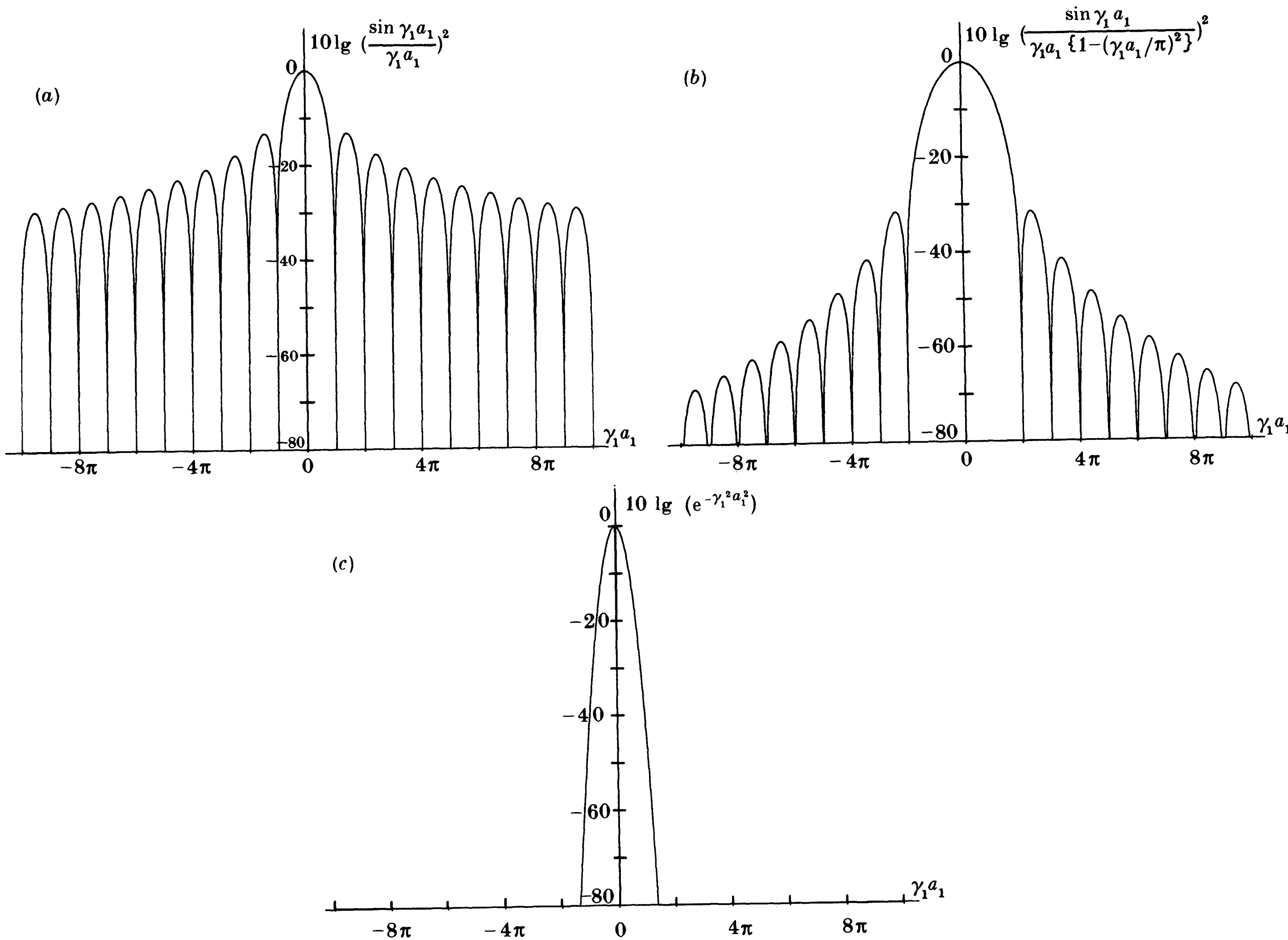

FIGURE 4. Cross-sections through the spectral footprint functions corresponding to (*a*) a rectangular flat data footprint, (*b*) a rectangular cosine data footprint and (*c*) a normal data footprint, when plotted on a logarithmic scale. Values less than -80 dB, which fall below the horizontal axis, have not been plotted.

of footprints of finite area. Later we shall refer to a general footprint of finite area, by which is meant any data footprint satisfying (4) for which $w(\boldsymbol{X})$ is zero for all $|\boldsymbol{X}| > r$ where r is some arbitrary but finite dimension.

The flat data footprint is discontinuous at its edges and its derivatives $\partial w/\partial X_1$ and $\partial w/\partial X_2$ do not exist at $X_1 = \pm a_1$ and $X_2 = \pm a_2$ respectively. The cosine data footprint is not discontinuous and both its first and second derivatives exist everywhere (including at its edges). We see also that, for the corresponding spectral footprints, for large wavenumbers the decay of $W'(\gamma_1, \gamma_2)$ is proportional to $1/\gamma_1^2 \gamma_2^2$ in the case of the flat data footprint and proportional to $1/\gamma_1^6 \gamma_2^6$ in the case of the cosine data footprint. This behaviour is illustrated in figure 4, as is the greater rate of decay of the exponential function in the case of the normal footprint.

3. Roughness properties of a smoothed surface

We shall now consider the application of the previous results to calculations of the statistics of peaks and summits of a smoothed surface. We shall assume that the original surface is a Gaussian surface with zero mean height so that the smoothed surface is also Gaussian with zero mean. The classical theory of Cartwright & Longuet-Higgins (1956), Longuet-Higgins (1957*a*, *b*, 1962) and Nayak (1971) can then be applied. It is known from this theory that the profile statistics for the curve of intersection of the surface with a vertical plane parallel to the reference vector $\boldsymbol{e}$ can be defined in terms of the following three moments of the surface's spectral density:

$$m_0 = \int_\infty \mathrm{d}\boldsymbol{\gamma}\, S_{yy}(\boldsymbol{\gamma}), \tag{26}$$

$$m_2 = \int_\infty \mathrm{d}\boldsymbol{\gamma}\, (\boldsymbol{\gamma}\cdot\boldsymbol{e})^2\, S_{yy}(\boldsymbol{\gamma}), \tag{27}$$

$$m_4 = \int_\infty \mathrm{d}\boldsymbol{\gamma}\, (\boldsymbol{\gamma}\cdot\boldsymbol{e})^4\, S_{yy}(\boldsymbol{\gamma}). \tag{28}$$

Since the directions of the reference axes X_1, X_2 may be chosen arbitrarily, we may assume without loss of generality that the profile direction $\boldsymbol{e}$ is parallel to the X_1-axis, when (27) and (28) simplify to

$$m_2 = \int_\infty \mathrm{d}\boldsymbol{\gamma}\, \gamma_1^2\, S_{yy}(\boldsymbol{\gamma}) \tag{29}$$

and

$$m_4 = \int_\infty \mathrm{d}\boldsymbol{\gamma}\, \gamma_1^4\, S_{yy}(\boldsymbol{\gamma}). \tag{30}$$

If the surface is isotropic, the statistics of its summits can also be defined in terms of these three moments m_0, m_2 and m_4, which are now independent of the direction of the profile across the X_1, X_2 plane.

Since the spectral density of the smoothed surface, $S_{yy}(\boldsymbol{\gamma})$, is defined as the

Fourier transform of its autocorrelation function $R_{yy}(\boldsymbol{x})$, it follows by the inverse transform that

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{\gamma}\, S_{yy}(\boldsymbol{\gamma})\, \mathrm{e}^{\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{x}}. \tag{31}$$

Also, since $\boldsymbol{\gamma}\cdot\boldsymbol{x} = \gamma_1 x_1 + \gamma_2 x_2$ the result of differentiating $\mathrm{e}^{\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{x}}$ with respect to x_1 is to multiply by the factor $\mathrm{i}\gamma_1$. Hence, on differentiating $R_{yy}(\boldsymbol{x})$ with respect to x_1 we have, from (31),

$$\frac{\partial^2}{\partial x_1^2} R_{yy}(\boldsymbol{x}) = -\int_{\infty} \mathrm{d}\boldsymbol{\gamma}\, \gamma_1^2 S_{yy}(\boldsymbol{\gamma})\, \mathrm{e}^{\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{x}} \tag{32}$$

and

$$\frac{\partial^4}{\partial x_1^4} R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{\gamma}\, \gamma_1^4 S_{yy}(\boldsymbol{\gamma})\, \mathrm{e}^{\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{x}}. \tag{33}$$

By putting $\boldsymbol{x} = \boldsymbol{0}$ in (31), (32) and (33) and comparing the results with (26), (29) and (30), we obtain the well-known results that

$$m_0 = R_{yy}(\boldsymbol{0}), \tag{34}$$

$$m_2 = -(\partial^2/\partial x_1^2)\, R_{yy}(\boldsymbol{0}), \tag{35}$$

$$m_4 = (\partial^4/\partial x_1^4)\, R_{yy}(\boldsymbol{0}), \tag{36}$$

where we understand that $(\partial^2/\partial x_1^2)\, R_{yy}(\boldsymbol{0})$ means the value of the partial derivative $(\partial^2/\partial x_1^2)\, R_{yy}(\boldsymbol{x})$ evaluated at $\boldsymbol{x} = \boldsymbol{0}$, and similarly for $(\partial^4/\partial x_1^4)\, R_{yy}(\boldsymbol{x})$.

If any of m_0, m_2, m_4 do not exist, then the classical theory of peak and summit statistics fails. The reason for this is that, when $m_0 \to \infty$ the mean-square height approaches infinity; when $m_2 \to \infty$ but m_0 is finite, the average density of zero-crossings on a profile approaches infinity; when $m_4 \to \infty$ but m_2 is finite, the average density of peaks and summits approaches infinity. If the high wavenumber content of $S_{yy}(\boldsymbol{\gamma})$ is such that the integral in (30) does not converge to a limit, so that m_4 does not exist, then the surface of which $y(\boldsymbol{X})$ is a sample has infinitely many peaks per unit length of profile and infinitely many summits per unit area, and we cannot talk meaningfully about the distributions of these peaks and summits and their curvatures.

The action of smoothing by a data footprint $w(\boldsymbol{X})$ according to (3) blurs the fine details of the topography of the original surface $\{z(\boldsymbol{X})\}$ and small summits present in $z(\boldsymbol{X})$ do not appear as summits in $y(\boldsymbol{X})$. We shall now investigate this consequence of smoothing by calculating the spectral moments m_0, m_2, m_4 for the case of a general surface $\{y(\boldsymbol{X})\}$ derived from a reference surface $\{z(\boldsymbol{X})\}$ by (3).

Spectral moments for a smoothed surface

Starting from (14), we have, on substituting into (34),

$$m_0 = \int_{\infty} \mathrm{d}\boldsymbol{s}\, w'(\boldsymbol{s})\, R_{zz}(\boldsymbol{s}). \tag{37}$$

On differentiating (14) the appropriate number of times and putting $\boldsymbol{x} = \boldsymbol{0}$, and then substituting the results into (35) and (36), we obtain

$$m_2 = -\int_{\infty} \mathrm{d}\boldsymbol{s}\, \frac{\partial^2}{\partial s_1^2} w'(\boldsymbol{s})\, R_{zz}(\boldsymbol{s}) \tag{38}$$

and
$$m_4 = \int_\infty \mathrm{d}\boldsymbol{s} \frac{\partial^4}{\partial s_1^4} w'(\boldsymbol{s}) R_{zz}(\boldsymbol{s}). \tag{39}$$

The lag footprint $w'(\boldsymbol{s})$ is related to the corresponding data footprint $w(\boldsymbol{X})$ by, from (17),
$$w'(\boldsymbol{s}) = \int_\infty \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X})\, w(\boldsymbol{X}+\boldsymbol{s}). \tag{40}$$

By differentiating (40) with respect to s_1, changing the variable of integration from $\boldsymbol{X}$ to $\boldsymbol{X}' = \boldsymbol{X}+\boldsymbol{s}$ and differentiating with respect to s_1 again, and then reverting to the original variable $\boldsymbol{X}$, we find that
$$\frac{\partial^2}{\partial s_1^2} w'(\boldsymbol{s}) = -\int_\infty \mathrm{d}\boldsymbol{X} \frac{\partial}{\partial X_1} w(\boldsymbol{X}) \frac{\partial}{\partial X_1} w(\boldsymbol{X}+\boldsymbol{s}) \tag{41}$$
and, similarly,
$$\frac{\partial^4}{\partial s_1^4} w'(\boldsymbol{s}) = \int_\infty \mathrm{d}\boldsymbol{X} \frac{\partial^2}{\partial X_1^2} w(\boldsymbol{X}) \frac{\partial^2}{\partial X_1^2} w(\boldsymbol{X}+\boldsymbol{s}). \tag{42}$$

These results (40), (41) and (42) may now be used to substitute for $w'(\boldsymbol{s})$ and its derivatives in (37), (38) and (39) to give
$$m_0 = \int_\infty \mathrm{d}\boldsymbol{s} \int_\infty \mathrm{d}\boldsymbol{X}\, w(\boldsymbol{X})\, w(\boldsymbol{X}+\boldsymbol{s})\, R_{zz}(\boldsymbol{s}), \tag{43}$$
$$m_2 = \int_\infty \mathrm{d}\boldsymbol{s} \int_\infty \mathrm{d}\boldsymbol{X} \frac{\partial}{\partial X_1} w(\boldsymbol{X}) \frac{\partial}{\partial X_1} w(\boldsymbol{X}+\boldsymbol{s})\, R_{zz}(\boldsymbol{s}), \tag{44}$$
$$m_4 = \int_\infty \mathrm{d}\boldsymbol{s} \int_\infty \mathrm{d}\boldsymbol{X} \frac{\partial^2}{\partial X_1^2} w(\boldsymbol{X}) \frac{\partial^2}{\partial X_1^2} w(\boldsymbol{X}+\boldsymbol{s})\, R_{zz}(\boldsymbol{s}). \tag{45}$$

We are interested in the conditions that the footprint function $w(\boldsymbol{X})$ must satisfy in order to ensure that the spectral moments m_0, m_2 and m_4 all exist. This depends on the form of the autocorrelation function for the original surface, $R_{zz}(\boldsymbol{s})$.

Good footprints

The most extreme autocorrelation function that we shall consider is that for a theoretically white reference surface of constant spectral density S_0, for which
$$R_{zz}(\boldsymbol{s}) = (2\pi)^2 S_0\, \delta(\boldsymbol{s}). \tag{46}$$

After substituting from (46) into (43), (44) and (45), the integration over $\boldsymbol{s}$ may then be completed to give
$$m_0 = (2\pi)^2 S_0 \int_\infty \mathrm{d}\boldsymbol{X}\, w^2(\boldsymbol{X}), \tag{47}$$
$$m_2 = (2\pi)^2 S_0 \int_\infty \mathrm{d}\boldsymbol{X} \left\{\frac{\partial}{\partial X_1} w(\boldsymbol{X})\right\}^2, \tag{48}$$
$$m_4 = (2\pi)^2 S_0 \int_\infty \mathrm{d}\boldsymbol{X} \left\{\frac{\partial^2}{\partial X_1^2} w(\boldsymbol{X})\right\}^2. \tag{49}$$

These are general expressions for the three spectral moments m_0, m_2, m_4 defined by (26), (29) and (30), for a surface generated by smoothing a white reference surface with a data footprint $w(\boldsymbol{X}) = w(X_1, X_2)$.

For a footprint of finite area, the integrals in (47), (48) and (49) only extend over the area of the footprint, since $w(\boldsymbol{X})$ is identically zero outside the footprint. By definition, $w(\boldsymbol{X})$ is always finite so that, for a footprint of finite area, the integral in (47) always exists and so m_0 always exists. Also for a footprint of finite area, from (48), a sufficient condition for m_2 to exist is for $\partial/\partial X_1 w(\boldsymbol{X})$ to be finite everywhere and, from (49), a sufficient condition for m_4 to exist is for $\partial^2/\partial X_1^2 w(\boldsymbol{X})$ to be finite everywhere. We shall define as a *good footprint* any footprint for which $w(\boldsymbol{X})$ is a continuous, finite function which is smooth enough that its first two derivatives exist everywhere including at the edges of the footprint. Since a profile may be taken in any direction across the surface, the first and second derivatives in any direction across the data footprint must be finite everywhere if the footprint is to be a good footprint.

For a homogeneous reference surface $\{z(\boldsymbol{X})\}$ with a finite mean-square height, the autocorrelation function $R_{zz}(\boldsymbol{s})$ will be finite everywhere, in which case we can see from (43), (44) and (45) that, for a good footprint of finite area, the three moments m_0, m_2, m_4 of the smoothed surface will also always exist. Therefore we conclude the following. For a surface generated by smoothing with a footprint of finite area a homogeneous reference surface which has finite mean-square height, a sufficient condition for the spectral moments m_0, m_2 and m_4 to all exist is for the footprint to be a good footprint as defined above.

For the rectangular flat data footprint defined by (T 1.1), see table 1 and figure 1, $w(\boldsymbol{X})$ is not continuous at the edges of the footprint and so cannot be differentiated here; therefore the flat footprint is not a good footprint. The rectangular cosine data footprint (T 1.5), figure 2, has a first derivative (in any direction) which is continuous and differentiable everywhere and so its second derivative exists everywhere also, and this footprint is a good footprint. Similarly the normal footprint (T 1.9) figure 3, can be differentiated twice; although this footprint has infinite area, $w(\boldsymbol{X})$ and its derivatives decay to zero fast enough as $|\boldsymbol{X}| \to \infty$ to ensure that m_0, m_2 and m_4 defined by (47), (48) and (49) all exist when a white reference surface is smoothed by a normal footprint. Table 2 shows the results of calculating m_0, m_2 and m_4 from each of (47), (48) and (49) for these three different footprints.

4. Discrete calculations of surface roughness

We turn now to the effect of smoothing on calculations of surface roughness by the discrete theory developed by Whitehouse & Archard (1970), Whitehouse & Phillips (1978, 1982) and Greenwood (1984). The following analysis follows the nomenclature used by Greenwood (1984). Beginning with profile properties, if Δ is the sampling interval (units of distance), Greenwood defines an approximate slope

$$m(X) = \{y(X+\Delta)-y(X)\}/\Delta \tag{50}$$

Table 1. Three data footprints with, below each, their corresponding lag, frequency and spectral footprints

1. rectangular flat	2. rectangular cosine	3. normal
	data footprint $w(X_1, X_2)$	
$\frac{1}{4a_1 a_2}$ for $\begin{cases} 0 \leqslant \lvert X_1 \rvert \leqslant a_1, \\ 0 \leqslant \lvert X_2 \rvert \leqslant a_2, \end{cases}$ and 0 elsewhere (T 1.1)	$\frac{1}{4a_1 a_2}\left(1+\cos\frac{\pi X_1}{a_1}\right)\left(1+\cos\frac{\pi X_2}{a_2}\right)$ for $\begin{cases} 0 \leqslant \lvert X_1 \rvert \leqslant a_1 \\ 0 \leqslant \lvert X_2 \rvert \leqslant a_2 \end{cases}$, and 0 elsewhere (T 1.5)	$\frac{1}{2\pi a_1 a_2}\exp\left\{-\frac{1}{2}\left(\frac{X_1^2}{a_1^2}+\frac{X_2^2}{a_2^2}\right)\right\}$ (T 1.9)
	lag footprint $w'(x_1, x_2)$ (from (17))	
$\frac{1}{4a_1 a_2}\left(1-\frac{\lvert x_1 \rvert}{2a_1}\right)\left(1-\frac{\lvert x_2 \rvert}{2a_2}\right)$ for $\begin{cases} 0 \leqslant \lvert x_1 \rvert \leqslant 2a_1 \\ 0 \leqslant \lvert x_2 \rvert \leqslant 2a_2 \end{cases}$, and 0 elsewhere (T 1.2)	$\frac{1}{4a_1 a_2}\left\{1-\frac{\lvert x_1 \rvert}{2a_1}+\frac{3}{4\pi}\sin\frac{\pi \lvert x_1 \rvert}{a_1} + \left(\frac{1}{2}-\frac{\lvert x_1 \rvert}{4a_1}\right)\cos\frac{\pi x_1}{a_1}\right\}\left\{1-\frac{\lvert x_2 \rvert}{2a_2} + \frac{3}{4\pi}\sin\frac{\pi \lvert x_2 \rvert}{a_2}+\left(\frac{1}{2}-\frac{\lvert x_2 \rvert}{4a_2}\right)\cos\frac{\pi x_2}{a_2}\right\}$ for $\begin{cases} 0 \leqslant \lvert x_1 \rvert \leqslant 2a_1 \\ 0 \leqslant \lvert x_2 \rvert \leqslant 2a_2 \end{cases}$, and 0 elsewhere (T 1.6)	$\frac{1}{4\pi a_1 a_2}\exp\left\{-\frac{1}{4}\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}\right)\right\}$ (T 1.10)
	frequency footprint $W(\gamma_1, \gamma_2)$ (from (18))	
$\frac{1}{(2\pi)^2}\left(\frac{\sin \gamma_1 a_1}{\gamma_1 a_1}\right)\left(\frac{\sin \gamma_2 a_2}{\gamma_2 a_2}\right)$ (T 1.3)	$\frac{1}{(2\pi)^2}\left\{\frac{\sin \gamma_1 a_1}{\gamma_1 a_1(1-\gamma_1^2 a_1^2/\pi^2)}\right\}\left\{\frac{\sin \gamma_2 a_2}{\gamma_2 a_2(1-\gamma_2^2 a_2^2/\pi^2)}\right\}$ (T 1.7)	$\frac{1}{(2\pi)^2}\exp\{-\tfrac{1}{2}(\gamma_1^2 a_1^2+\gamma_2^2 a_2^2)\}$ (T 1.11)
	spectral footprint $W'(\gamma_1, \gamma_2)$ (from (19) or (22))	
$\frac{1}{(2\pi)^2}\left(\frac{\sin \gamma_1 a_1}{\gamma_1 a_1}\right)^2\left(\frac{\sin \gamma_2 a_2}{\gamma_2 a_2}\right)^2$ (T 1.4)	$\frac{1}{(2\pi)^2}\left\{\frac{\sin \gamma_1 a_1}{\gamma_1 a_1(1-\gamma_1^2 a_1^2/\pi^2)}\right\}^2\left\{\frac{\sin \gamma_2 a_2}{\gamma_2 a_2(1-\gamma_2^2 a_2^2/\pi^2)}\right\}^2$ (T 1.8)	$\frac{1}{(2\pi)^2}\exp\{-(\gamma_1^2 a_1^2+\gamma_2^2 a_2^2)\}$ (T 1.12)

TABLE 2. SPECTRAL MOMENTS FOR A PROFILE IN THE X_1-DIRECTION ACROSS A SURFACE DERIVED FROM A WHITE REFERENCE SURFACE BY SMOOTHING WITH EACH OF THE THREE SAMPLE FOOTPRINTS IN TABLE 1

	1. rectangular flat	2. rectangular cosine	3. normal
m_0	$(\pi^2/a_1 a_2) S_0$	$(9\pi^2/4a_1 a_2) S_0$	$(\pi/a_1 a_2) S_0$
m_2	does not exist	$(3\pi^4/4a_1^3 a_2) S_0$	$(\pi/2a_1^3 a_2) S_0$
m_4	does not exist	$(3\pi^6/4a_1^5 a_2) S_0$	$(3\pi/4a_1^5 a_2) S_0$
$\alpha = m_0 m_4/m_2^2$	does not exist	3	3

and an approximate curvature

$$\kappa(X) = [\{y(X+\Delta)-y(X)\}-\{y(X)-y(X-\Delta)\}]/\Delta^2 \tag{51}$$

and replaces the three spectral moments m_0, m_2, m_4 of the continuous theory by

$$\sigma^2 = \mathrm{E}[y^2], \tag{52}$$

$$\sigma_m^2 = \mathrm{E}[m^2], \tag{53}$$

and

$$\sigma_\kappa^2 = \mathrm{E}[\kappa^2]. \tag{54}$$

Rather than using σ_m^2 and σ_κ^2 directly, Greenwood finds that it is convenient instead to use the derived parameters

$$r = \sigma_m^2/\sigma\sigma_\kappa \tag{55}$$

and θ, where

$$\sin\theta = \Delta\sigma_\kappa/2\sigma_m. \tag{56}$$

For his 'five-point' summits, Greenwood makes calculations on the ordinates measured at five sampling points distances Δ, $\surd 2\Delta$ and 2Δ apart. If κ_1 and κ_2 are the curvatures in the coordinate directions 1 and 2, defined so that

$$\kappa_1(X_1, X_2) = [\{y(X_1+\Delta, X_2)-y(X_1, X_2)\}-\{y(X_1, X_2)-y(X_1-\Delta, X_2)\}]/\Delta^2 \tag{57}$$

and

$$\kappa_2(X_1, X_2) = [\{y(X_1, X_2+\Delta)-y(X_1, X_2)\}-\{y(X_1, X_2)-y(X_1, X_2-\Delta)\}]/\Delta^2, \tag{58}$$

he finds it necessary to introduce an additional parameter

$$\tau = \mathrm{E}[\kappa_1\kappa_2]/(\mathrm{E}[\kappa_1^2]\,\mathrm{E}[\kappa_2^2])^{\frac{1}{2}}, \tag{59}$$

to define the summit properties.

For an isotropic surface whose autocorrelation function is $R_{yy}(X)$ where $X = (X_1^2+X_2^2)^{\frac{1}{2}}$, from (52),

$$\sigma^2 = R_{yy}(0) \tag{60}$$

from (50) and (53),

$$\sigma_m^2 = 2\{R_{yy}(0)-R_{yy}(\Delta)\}/\Delta^2 \tag{61}$$

and, from (51) and (54),

$$\sigma_\kappa^2 = 2\{3R_{yy}(0)-4R_{yy}(\Delta)+R_{yy}(2\Delta)\}/\Delta^4. \tag{62}$$

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Similarly, on substituting from (57) and (58) into (59) and calculating the ensemble averages, leads to the result that

$$\tau = \frac{2R_{yy}(0) - 4R_{yy}(\Delta) + 2R_{yy}(\sqrt{2}\Delta)}{3R_{yy}(0) - 4R_{yy}(\Delta) + R_{yy}(2\Delta)}. \tag{63}$$

Discrete parameters for a surface smoothed by a good footprint

By writing a Taylor expansion of $R_{yy}(x_1, x_2)$ in the vicinity of $x_1 = x_2 = 0$, we have

$$R_{yy}(\Delta, 0) = R_{yy}(0,0) + \Delta \frac{\partial}{\partial x_1} R_{yy}(0,0) + \frac{\Delta^2}{2}\frac{\partial^2}{\partial x_1^2} R_{yy}(0,0) + \frac{\Delta^3}{6}\frac{\partial^3}{\partial x_1^3} R_{yy}(0,0) + \frac{\Delta^4}{24}\frac{\partial^4}{\partial x_1^4} R_{yy}(0,0) + \int_0^{\Delta} \mathrm{d}x_1 \frac{(\Delta - x_1)^4}{24}\frac{\partial^5}{\partial x_1^5} R_{yy}(x_1, 0) \tag{64}$$

provided that the derivatives involved exist. The fifth order derivative must exist for all values of x_1 in the interval $0 \leqslant x_1 \leqslant \Delta$ but it does not have to be continuous (Jeffreys & Jeffreys 1956). For the case of a surface generated by smoothing another by a good footprint of finite area, we have seen that m_0, m_2 and m_4 exist and so, from (34), (35) and (36), $R_{yy}(0, 0)$ and its second and fourth derivatives all exist. We shall show now that the first and third derivatives are always zero and that the fifth order derivative $(\partial^5/\partial x_1^5)\, R_{yy}(x_1, 0)$ always exists. Although $(\partial^5/\partial x_1^5)\, R_{yy}(x_1, 0)$ will be zero at $x_1 = 0$, there may be a discontinuity here but only so that its value for $x_1 > 0$ is always finite.

From (31), by differentiating with respect to x_1,

$$\frac{\partial}{\partial x_1} R_{yy}(0,0) = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}\gamma_1 \int_{-\infty}^{\infty} \mathrm{d}\gamma_2\, \gamma_1 S_{yy}(\gamma_1, \gamma_2) = 0 \tag{65}$$

by virtue of the property that (see, for example, Newland (1984))

$$S_{yy}(\gamma_1, \gamma_2) = S_{yy}(-\gamma_1, -\gamma_2). \tag{66}$$

Similarly, on account of (66),

$$\frac{\partial^3}{\partial x_1^3} R_{yy}(0,0) = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}\gamma_1 \int_{-\infty}^{\infty} \mathrm{d}\gamma_2\, \gamma_1^3 S_{yy}(\gamma_1, \gamma_2) = 0. \tag{67}$$

For the fifth-order derivative $(\partial^5/\partial x_1^5)\, R_{yy}(x, 0)$ we begin from (7). After changing the variables of integration from $\boldsymbol{u}_1$ to $\boldsymbol{s}_1 = \boldsymbol{u}_1 - \boldsymbol{X}$ and from $\boldsymbol{u}_2$ to $\boldsymbol{s}_2 = \boldsymbol{u}_2 - \boldsymbol{X}$, this becomes

$$R_{yy}(\boldsymbol{x}) = \int_{\infty} \mathrm{d}\boldsymbol{s}_1 \int_{\infty} \mathrm{d}\boldsymbol{s}_2\, w(\boldsymbol{s}_1)\, w(\boldsymbol{s}_2 - \boldsymbol{x})\, R_{zz}(\boldsymbol{s}_2 - \boldsymbol{s}_1) \tag{68}$$

and, on differentiating three times with respect to x_1, and then changing the variables of integration to $\boldsymbol{s}_1' = \boldsymbol{s}_1 - \boldsymbol{x}$ and $\boldsymbol{s}_2' = \boldsymbol{s}_2 - \boldsymbol{x}$ and differentiating twice more, before reverting to the original variables, gives

$$\frac{\partial^5}{\partial x_1^5} R_{yy}(\boldsymbol{x}) = -\int_{\infty} \mathrm{d}\boldsymbol{s}_1 \int_{\infty} \mathrm{d}\boldsymbol{s}_2 \frac{\partial^2}{\partial X_1^2} w(\boldsymbol{s}_1) \frac{\partial^3}{\partial X_1^3} w(\boldsymbol{s}_2 - \boldsymbol{x})\, R_{zz}(\boldsymbol{s}_2 - \boldsymbol{s}_1). \tag{69}$$

The partial derivatives are written with respect to X_1 rather than x_1 because we regard the data footprint as a general function of $\boldsymbol{X}$, $w(\boldsymbol{X})$. For a good footprint, the second derivative $(\partial^2/\partial X_1^2)\,w(\boldsymbol{X})$ is finite; the third derivative $(\partial^3/\partial X_1^3)\,w(\boldsymbol{X})$ may be infinite, but since the second derivative exists, the integral of the third derivative over any finite region must be finite. For the case of a good footprint of finite area and for a reference surface whose mean-square height is finite so that $R_{zz}(\boldsymbol{s}_1-\boldsymbol{s}_1)$ is finite, we conclude from (69) that $(\partial^5/\partial x_1^5)\,R_{yy}(\boldsymbol{x})$ will always exist. Even for a reference surface with a white spectrum, for which (46) applies, by substituting for $R_{zz}(\boldsymbol{s}_2-\boldsymbol{s}_1)$ in (69) and integrating first over $\boldsymbol{s}_1$, we reach the same conclusion that $(\partial^5/\partial x_1)^5 R_{yy}(\boldsymbol{x})$ exists. Provided that this is the case, the last term in the expansion (64) will be of order Δ^5 or smaller. We shall use the notation $O(\Delta^5)$ to mean 'of order Δ^5 or smaller' and, with these results, (64) simplifies to

$$R_{yy}(\Delta, 0) = R_{yy}(0, 0) + \frac{\Delta^2}{2}\frac{\partial^2}{\partial x_1^2}R_{yy}(0, 0) + \frac{\Delta^4}{24}\frac{\partial^4}{\partial x_1^4}R_{yy}(0, 0) + O(\Delta^5). \tag{70}$$

This expansion (70) applies for any surface which has been generated by smoothing a reference surface with a good footprint of finite area, provided that the reference surface has a finite mean-square height, or that it is ideally white. Also (70) is true when a white reference surface is smoothed by the normal footprint defined in table 1 because then $w(\boldsymbol{X})$ approaches zero fast enough when $|\boldsymbol{X}|$ is large to ensure that the required integrals exist even though the domain of the integration is infinite.

For any surface for which (70) is true, we can substitute for $R_{yy}(\Delta) = R_{yy}(\Delta, 0)$ from (70) into (60), (61) and (62) to obtain, from (60) and (34),

$$\sigma^2 = R_{yy}(0) = m_0 \tag{71}$$

from (61) and (35),

$$\sigma_m^2 = -(\partial^2/\partial x_1^2)\,R_{yy}(0) + O(\Delta^2) = m_2 + O(\Delta^2) \tag{72}$$

and, from (62) and (36),

$$\sigma_\kappa^2 = (\partial^4/\partial x_1^4)\,R_{yy}(0) + O(\Delta) = m_4 + O(\Delta). \tag{73}$$

Hence Greenwood's derived parameters r and θ are given by, from (55),

$$r^2 = \frac{m_2^2}{m_0\,m_4} + O(\Delta) \tag{74}$$

and, from (56),

$$\sin\theta = \tfrac{1}{2}\Delta(m_4/m_2)^{\frac{1}{2}} + O(\Delta^2). \tag{75}$$

The additional derived parameter τ for an isotropic surface is, on substituting from (70) into (63),

$$\tau = \tfrac{1}{3} + O(\Delta). \tag{76}$$

In the continuous theory, Nayak (1971) has defined the parameter α by

$$\alpha = m_0\,m_4/m_2^2 \tag{77}$$

and so, from (74),

$$r^2 = 1/\alpha + O(\Delta). \tag{78}$$

When the sampling interval $\Delta \to 0$, then Greenwood's $r \to 1/\sqrt{\alpha}$. Also, from (75), Greenwood's $\theta \to 0$ as $\Delta \to 0$ and, from (76), his parameter $\tau \to \frac{1}{3}$ as $\Delta \to 0$.

Choice of sampling interval

Greenwood (1984) has shown that, when his parameter $\theta \to 0$, his results for profile statistics become indistinguishable from those calculated by the continuous theory. We can see how small the sampling interval needs to be by calculating θ for two examples. We consider a surface $\{y(\boldsymbol{X})\}$ generated by smoothing a white reference surface by (i) the rectangular cosine data footprint (T 1.5) and (ii) the normal data footprint (T 1.9), table 1. Both of these are good footprints for which the expansion (70) is true, so that θ is given by (75). Using the results given in table 2 for m_2 and m_4 for these footprints, we obtain, for the rectangular cosine data footprint,

$$\sin\theta = \pi\Delta/2a_1 + O(\Delta^2) \tag{79}$$

and, for the normal data footprint,

$$\sin\theta = \surd\tfrac{3}{2}\Delta/2a_1 + O(\Delta^2). \tag{80}$$

From these examples, we see that $\theta \to 0$ when $\Delta/a_1 \to 0$, i.e. when the sampling interval Δ is small compared with the footprint dimension in the direction of the profile. In case (i), $2a_1$ is the finite length of the footprint; in case (ii), a_1 is the decay length of the footprint which, if the Gaussian bell represented a probability density, would be the corresponding standard deviation. The width of the footprint (perpendicular to the direction of the profile) is not involved, but it must be the same for the discrete analysis and for the continuous analysis.

For summit properties, Greenwood (1984) has shown that, for the isotropic case, the results of the discrete and continuous theories become very close when $\theta \to 0$ (except for the density of five-point summits differing from the density of geometric summits). To preserve isotropy, the footprint must now have circular symmetry about its centre, and the condition for $\theta \to 0$ becomes $\Delta/a \to 0$ where a is the radius of the footprint.

At present there is neither a discrete analysis nor a complete continuous analysis for the summit properties of a non-isotropic surface. But we are still interested in whether experimental results produced by discrete sampling will represent properly the summit properties of the underlying continuous surface. The above results suggest that there should be close agreement between the discrete and continuous analysis of summits in the non-isotropic case (except for summit densities) when the sampling interval in each coordinate direction is small compared with the length of a good footprint in that direction.

Dependence of discrete statistical properties on the sampling interval

Finally, in this paper, we consider the consequences of measuring the properties of a random surface by discrete sampling at a sampling interval size Δ which is not small compared with the footprint dimension. We shall calculate the dependence on Δ of Greenwood's parameters r, θ and τ for a surface derived by smoothing a theoretically white reference surface by each of the three different footprints listed in table 1.

On substituting for $R_{zz}(\boldsymbol{s})$ for a white reference surface from (46) into (14), evaluating the integral, and using (16), we find that

$$R_{yy}(\boldsymbol{x}) = (2\pi)^2 S_0 w'(\boldsymbol{x}). \tag{81}$$

In order to calculate r and θ, we need the one-dimensional autocorrelation function values $R_{yy}(0)$, $R_{yy}(\Delta)$ and $R_{yy}(2\Delta)$ to substitute into (60), (61) and (62). For a profile parallel to the X_1-axis, these are the same as the two-dimensional autocorrelation function values $R_{yy}(0, 0)$, $R_{yy}(\Delta, 0)$ and $R_{yy}(2\Delta, 0)$ which are given by (81). Hence we have

$$R_{yy}(0) = R_{yy}(0, 0) = (2\pi)^2 S_0 w'(0, 0), \tag{82}$$

$$R_{yy}(\Delta) = R_{yy}(\Delta, 0) = (2\pi)^2 S_0 w'(\Delta, 0) \tag{83}$$

and

$$R_{yy}(2\Delta) = R_{yy}(2\Delta, 0) = (2\pi)^2 S_0 w'(2\Delta, 0). \tag{84}$$

For the discrete summit parameter τ, which applies for an isotropic surface only, we need also the one-dimensional autocorrelation function's value $R_{yy}(\sqrt{2}\Delta)$ which is the same as the two-dimensional function's values $R_{yy}(\Delta, \Delta) = R_{yy}(\sqrt{2}\Delta, 0)$, so that

$$R_{yy}(\sqrt{2}\Delta) = R_{yy}(\sqrt{2}\Delta, 0) = (2\pi)^2 S_0 w'(\sqrt{2}\Delta, 0). \tag{85}$$

Since τ is used for defining the summit properties of an isotropic surface, this parameter is applicable only when the normal footprint is used with $a_1 = a_2$ because only then will the smoothed surface be isotropic. However, for interest, we shall calculate τ also for the two rectangular footprint cases, although for these non-isotropic cases we cannot have $R_{yy}(\Delta, \Delta) = R_{yy}(\sqrt{2}\Delta, 0)$ and so the results do not have relevance to a practical case.

On substituting from (82)–(85) in (60)–(63) and then using (55) and (56), we obtain the results that, for a surface obtained by smoothing with a lag footprint $w'(x_1, x_2)$ a surface with a white spectrum,

$$r = \frac{\sqrt{2}(w'(0, 0) - w'(\Delta, 0))}{\{w'(0, 0)\}^{\frac{1}{2}}\{3w'(0, 0) - 4w'(\Delta, 0) + w'(2\Delta, 0)\}^{\frac{1}{2}}} \tag{86}$$

$$\theta = \sin^{-1}\left[\frac{1}{2}\left\{\frac{3w'(0, 0) - 4w'(\Delta, 0) + w'(2\Delta, 0)}{w'(0, 0) - w'(\Delta, 0)}\right\}^{\frac{1}{2}}\right] \tag{87}$$

and

$$\tau = \frac{2w'(0, 0) - 4w'(\Delta, 0) + 2w'(\sqrt{2}\Delta, 0)}{3w'(0, 0) - 4w'(\Delta, 0) + w'(2\Delta, 0)}. \tag{88}$$

The lag footprints $w'(x_1, x_2)$ which correspond respectively to (1) the rectangular flat data footprint, (2) the rectangular cosine data footprint, and (3) the normal data footprint are given by (T 1.2), (T 1.6) and (T 1.10) in table 1. Using each of these three definitions of $w'(x_1, x_2)$ in turn, the parameters r, θ and τ have been calculated from (86), (87) and (88) for sampling intervals in the range 0 to $4a_1$.

The results are plotted in figures 5, 6 and 7 on the base of non-dimensional sampling interval Δ/a_1.

In figure 5, we see that, for the 'good' footprints, cases 2 and 3, r becomes $1/\surd 3$ when $\Delta = 0$ in agreement with (78) since $\alpha = 3$ in both these cases (see table 2). For the flat footprint, case 1, r becomes zero when $\Delta = 0$ on account of the high wavenumber components which are not smoothed by the sharp edges of this data footprint. In figure 6, we see that $\theta \rightarrow 0$ when $\Delta \rightarrow 0$ for both good footprints, in agreement with (75), but does not do so for the flat footprint, case 1. Even though the sampling interval approaches zero, in case 1 the statistical properties of peaks of the sampled profile will not become asymptotically the same as the corresponding properties of the peaks of a continuous profile This is because high wavenumber components remaining in the smoothed profile (which would have been eliminated by a good footprint) are sufficient to ensure that the moments m_2 and m_4 do not exist for the smoothed surface. Therefore the number of crossings and the number of peaks per unit length of the continuous profile cannot be defined.

In figure 7, the parameter τ is plotted as a function of the same non-dimensional sampling interval Δ/a_1. For the good footprints, τ becomes asymptotically $\frac{1}{3}$ as $\Delta \rightarrow 0$ in agreement with (76). For the flat data footprint, the value of τ fluctuates, the discontinuities occurring at $\Delta = a_1$, $\surd 2a_1$ and $2a_1$ on account of the discontinuous function (T 1.2), table 1, which defines this lag footprint. As mentioned above, cases 1 and 2 involve non-isotropic surfaces, so that only case 3 can be applied to Greenwood's isotropic summit analysis.

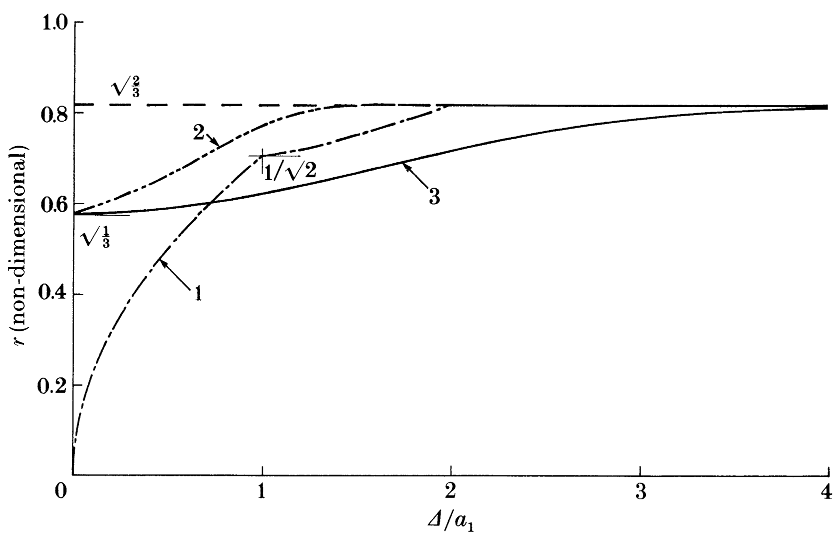

FIGURE 5. Graph of the discrete surface roughness parameter, r, defined by (55) plotted against non-dimensional sampling interval Δ/a_1 for a profile in the X_1-direction across a surface generated from a white reference surface by smoothing with each of the three different data footprints in table 1. 1, flat footprint of length $2a_1$; 2, cosine footprint of length $2a_1$; 3, normal footprint of decay length a_1.

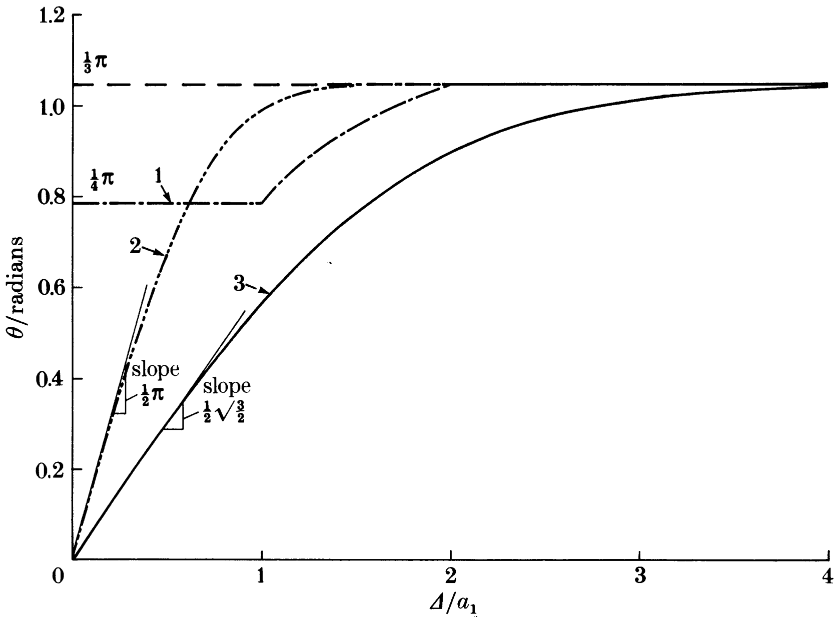

FIGURE 6. Graph of the discrete sampling-interval parameter θ defined by (56) plotted against non-dimensional sampling interval Δ/a_1 for a profile in the X_1-direction across a surface generated from a white reference surface by smoothing with each of the three different data footprints in table 1. 1, flat footprint of length $2a_1$; 2, cosine footprint of length $2a_1$; 3, normal footprint of decay length a_1.

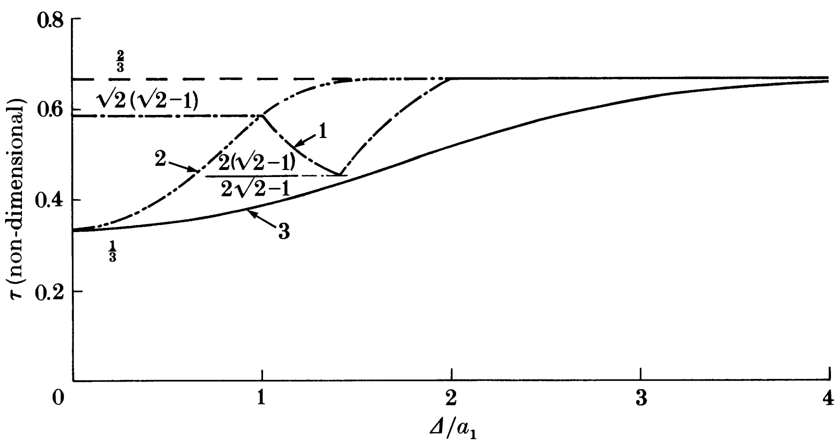

FIGURE 7. Graph of the discrete curvature-correlation parameter τ defined by (59) plotted against non-dimensional sampling interval Δ/a_1 when calculated for a profile in the X_1-direction across a surface generated from a white reference surface by smoothing with each of the three different data footprints in table 1. 1, flat footprint of length $2a_1$; 2, cosine footprint of length $2a_1$; 3, normal footprint of decay length a_1.

Discussion

When the ordinates of a sampled surface have been measured, estimates for the ensemble-averaged correlation functions $R_{yy}(0, 0)$, $R_{yy}(\Delta, 0)$ and $R_{yy}(2\Delta, 0)$ can be calculated for that population of ordinates and, if the surface is isotropic or approximately so, an estimate for $R_{yy}(\sqrt{2}\,\Delta, 0) = R_{yy}(\Delta, \Delta)$ can also be obtained. Then corresponding estimates for Greenwood's parameters r, θ and τ can be calculated by using (55) and (56) with (60)–(62), and (63). Once r, θ and τ are known, the results in Greenwood's paper allow the statistical properties of the peaks and summits as defined in terms of the sampled ordinates to be calculated. Whether such results represent the corresponding statistical properties of the underlying continuous surface depends on the value of the parameter θ. Although the normalized peak and summit height and curvature distributions depend only weakly on θ, the mean values of the heights and curvatures of peaks and summits and the peak and summit densities depend strongly on θ. When $\theta \rightarrow 0$, the profile properties in the discrete and continuous theories become asymptotic and the summit properties become close (although Greenwood shows that the density of his five-point summits is still some 30% greater than the density of geometric summits calculated by the continuous theory). Therefore if the computed properties in the discrete case are to represent properly the continuous surface which has been sampled, the discrete height ordinates must be measured in a way which ensures that $\theta \approx 0$. We have seen that, if the surface is smoothed with a good footprint of finite area before its height is sampled, it is always possible to make θ smaller by reducing the size of the sampling interval. To achieve $\theta \approx 0$ we need a sampling interval which is very small compared with the size of the footprint.

In his paper, Greenwood (1984) shows a graph of the experimental values of the parameters r, θ and τ calculated from the results of measurements by Sayles & Thomas (1979) of the surface roughness of a grit-blasted mild-steel specimen. Measurements have been made for a range of different sampling intervals and the results plotted as a function of the magnitude of the sampling interval. Although these results show a reduction in θ as Δ is decreased, Greenwood says that 'In no case studied has there been any tendency for θ to approach zero'. If the experimental surface had been smoothed by a good footprint before its statistics had been calculated, then we would expect θ to approach zero as Δ is progressively reduced. The practical method of doing this requires investigation, but it appears that the theoretical process of smoothing represented by (3) could be approximated by using a soft-tipped measuring probe or contacting device. In order to have a good footprint, the data weighting function must be continuous and smooth with no sharp edges. This might be achieved by making the tip of the probe softer at its edges than at its centre, with the stiffness per unit area changing smoothly and approaching zero stiffness asymptotically at the edges of the probe. When a constant force is applied to such a probe, its height will then be an approximation for the height of the smoothed surface $y(\boldsymbol{X})$ defined by (3). To make $\theta \rightarrow 0$, the sampling interval has to be chosen to be small compared with the size of the tip of the probe. There is no limit on how small the probe may be, provided that the sampling interval remains small compared with the probe's size. The smoothed surface $y(\boldsymbol{X})$ is a different surface if a smaller probe (and therefore a smaller data footprint) is used to generate it, but the discrete statistics will be asymptotically the same as the statistics calculated by the continuous theory for $y(\boldsymbol{X})$ if Δ is small compared with the footprint's size.

In practice it may be possible to make $\theta \rightarrow 0$ without using a good footprint when the spectral density of the original surface decays fast enough as $\gamma_1, \gamma_2 \rightarrow \infty$. The definition of a good footprint of finite area identifies a footprint which smooths a theoretically white reference surface sufficiently to ensure that the number of summits per unit area of the smoothed surface can be defined. When the original surface is not white, a footprint which does not smooth so effectively may nevertheless be adequate to smooth the already partly-smoothed original surface. The test of whether a footprint is adequate and whether the sampling interval is small enough relative to the size of the footprint, is that Greenwood's parameter θ for the sampled height ordinates must be approximately zero.

5. Conclusions

When one homogeneous random surface is generated from another by a process of smoothing defined by (3), the autocorrelation and spectral density functions of the second surface are related to those of the first by (14) and (24). These equations involve the lag and spectral footprint functions which are defined in terms of the data footprint function by (17) and by (22) with (18). Although the analysis parallels closely that for the spectral analysis of finite-length records of random functions in time-series analysis, its application to the two-dimensional surface roughness problem is thought not to have been published before. The fixed data window used in spectral analysis is replaced by a moving data footprint, and the moving spectral window in spectral analysis is replaced by a fixed spectral footprint.

If the reference surface is Gaussian, then the statistical properties of the smoothed surface can be obtained from the well-known theory of Rice, Cartwright & Longuet-Higgins, Longuet-Higgins, and Nayak, provided that the necessary higher order derivatives of the autocorrelation function for the smoothed surface exist. It has been shown in this paper that, if the smoothing is carried out by a data footprint of finite area which is a good footprint, then these higher order derivatives will exist, even if the reference surface is ideally white. A footprint will be a good footprint if its data footprint function is continuous and smooth enough that it can be differentiated twice in any direction everywhere, including at its edges.

For the case of a surface whose roughness is measured by discrete sampling, the heights of the ordinates depend on the footprint of the sampling device. Provided that this is a good footprint (of finite area) and that the sampling interval is small compared with the size of the footprint, it has been shown that the statistical properties of surface profiles calculated by the discrete theory of Whitehouse & Archard, Whitehouse & Phillips, and Greenwood approach asymptotically the results which the continuous theory would give for the analysis of a continuous smoothed surface generated from the reference surface by the same footprint.

I am most grateful to Dr J. A. Greenwood for his helpful comments after reading earlier drafts of this paper.

References

Captain, K. E., Boghani, A. B. & Wormley, D. N. 1979 *Veh. syst. Dynam.* **8**, 1–32.

Cartwright, D. E. & Longuet-Higgins, M. S. 1956 *Proc. R. Soc. Lond.* A **237**, 212–232.

Cebon, D. & Newland, D. E. 1984 *Proceedings 8th IAVSD-IUTAM symposium on the dynamics of vehicles on roads and tracks*, Cambridge, MA, 1983, 29–42. Lisse: Swets and Zeitlinger.

Cebon, D. 1985 An investigation of the dynamic interaction between wheeled vehicles and road surfaces. Ph.D. thesis, Department of Engineering, University of Cambridge.

Gillespie, T. D., Sayers, M. W. & Segel, L. 1980 *National Cooperative Highway Research Program Report* 228. Washington, D.C.: Transportation Research Board, National Research Council.

Greenwood, J. A. 1984 *Proc. R. Soc. Lond.* A **393**, 133–157.

Jeffreys, H. & Jeffreys, B. 1956 *Methods of Mathematical Physics*, 3rd edn, p. 51. Cambridge: Cambridge University Press.

Longuet-Higgins, M. S. 1957*a* *Phil. Trans. R. Soc. Lond.* A **249**, 321–387.

Longuet-Higgins, M. S. 1957*b* *Phil. Trans. R. Soc. Lond.* A **250**, 157–174.

Longuet-Higgins, M. S. 1962 In *Hydrodynamic Stability*, Proceedings 13th symposium on applied mathematics, Am. Math. Soc. pp. 105–143.

Nayak, P. R. 1971 *Trans. Am. Soc. mech. Engrs J. Lubr. Technol.* **93**, 398–407.

Newland, D. E. 1984 *An Introduction to Random Vibrations and Spectral Analysis*, 2nd edn, London: Longman.

Newland, D. E. 1986 In *Random vibration – status and recent developments* (ed. I. Elishakoff & R. H. Lyon), chap. 16. Amsterdam: Elsevier.

Rice, S. O. 1944 *Bell Syst. tech. J.* **23**, 282–332.

Rice, S. O. 1945 *Bell Syst. tech. J.* **24**, 46–156. (Both the papers by Rice are reprinted in N. Wax (ed.) *Selected Papers on Noise and Stochastic Processes*, 1954. New York: Dover)

Sayles, R. S. & Thomas, T. R. 1979 *Trans. Am. Soc. mech. Engrs J. Lubr. Technol.* **101**, 409–418.

Whitehouse, D. J. & Archard, J. F. 1970 *Proc. R. Soc. Lond.* A **316**, 97–121.

Whitehouse, D. J. & Phillips, M. J. 1978 *Phil. Trans. R. Soc. Lond.* A **290**, 267–298.

Whitehouse, D. J. & Phillips, M. J. 1982 *Phil. Trans. R. Soc. Lond.* A **305**, 441–468.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY
PROCEEDINGS OF

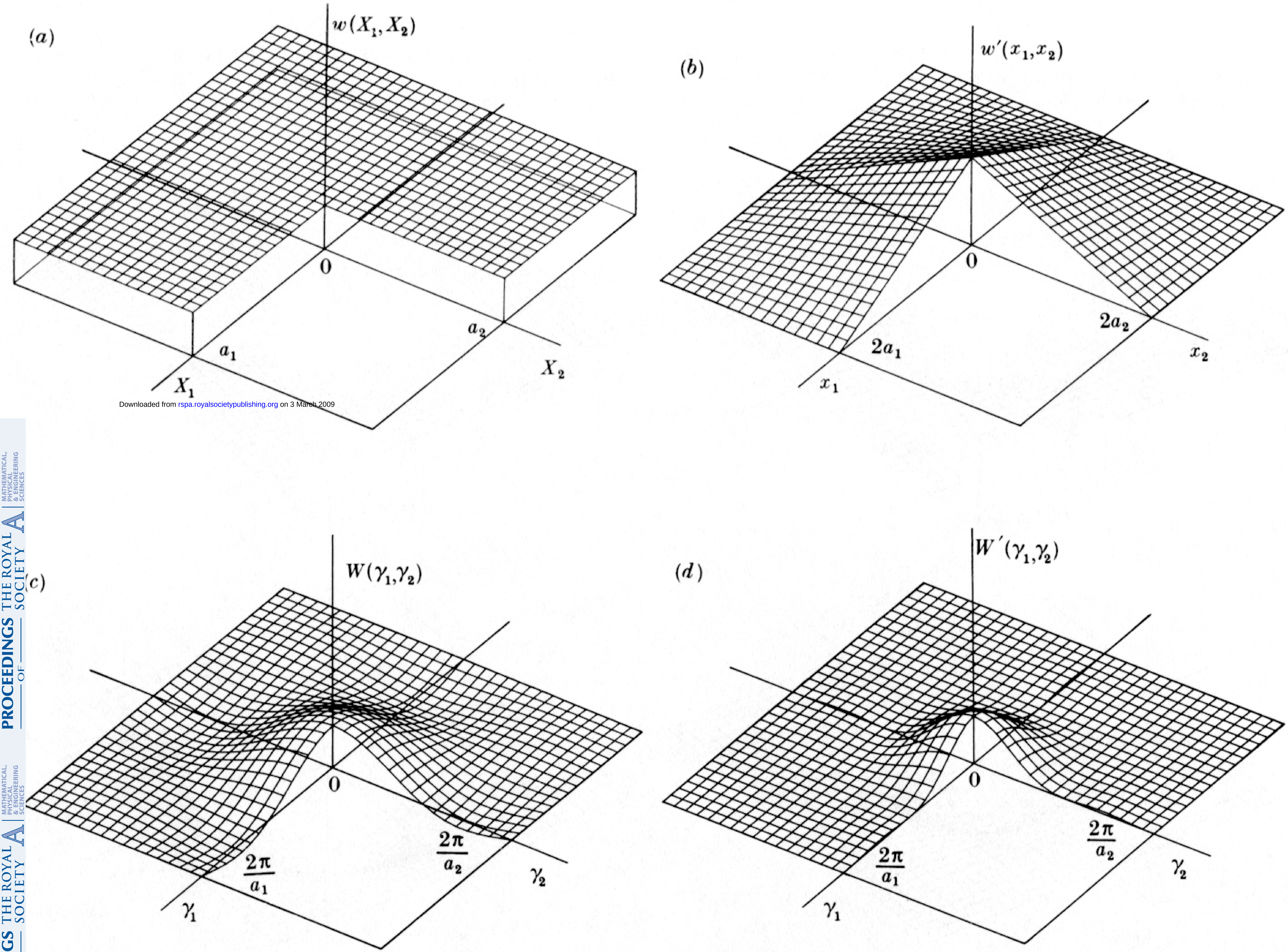

FIGURE 1. (*a*) Rectangular flat data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. The frequency and spectral footprints are drawn for a limited field of wavenumbers only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

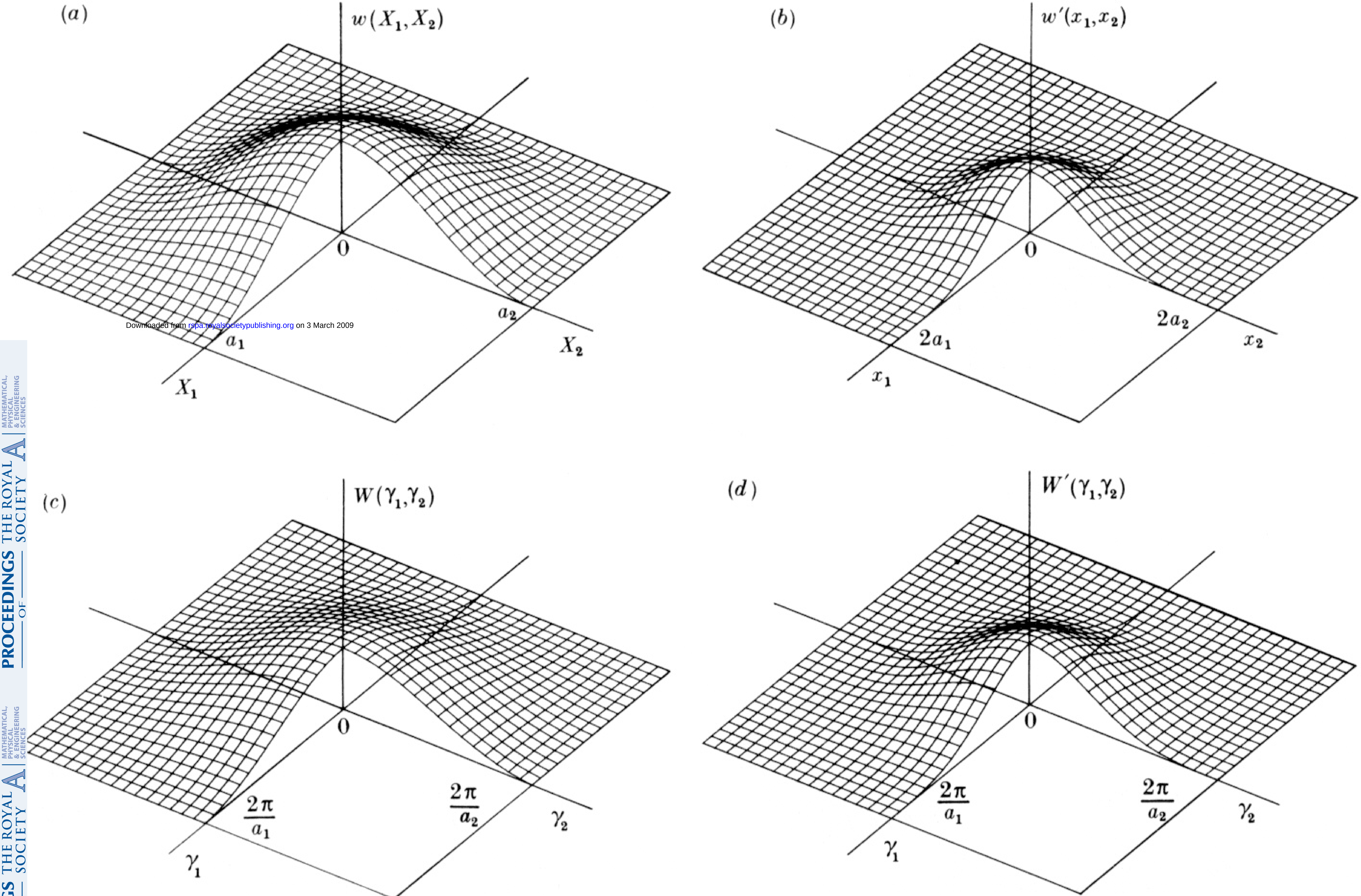

FIGURE 2. (*a*) Rectangular cosine data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. The frequency and spectral footprints are drawn for a limited field of wavenumbers only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

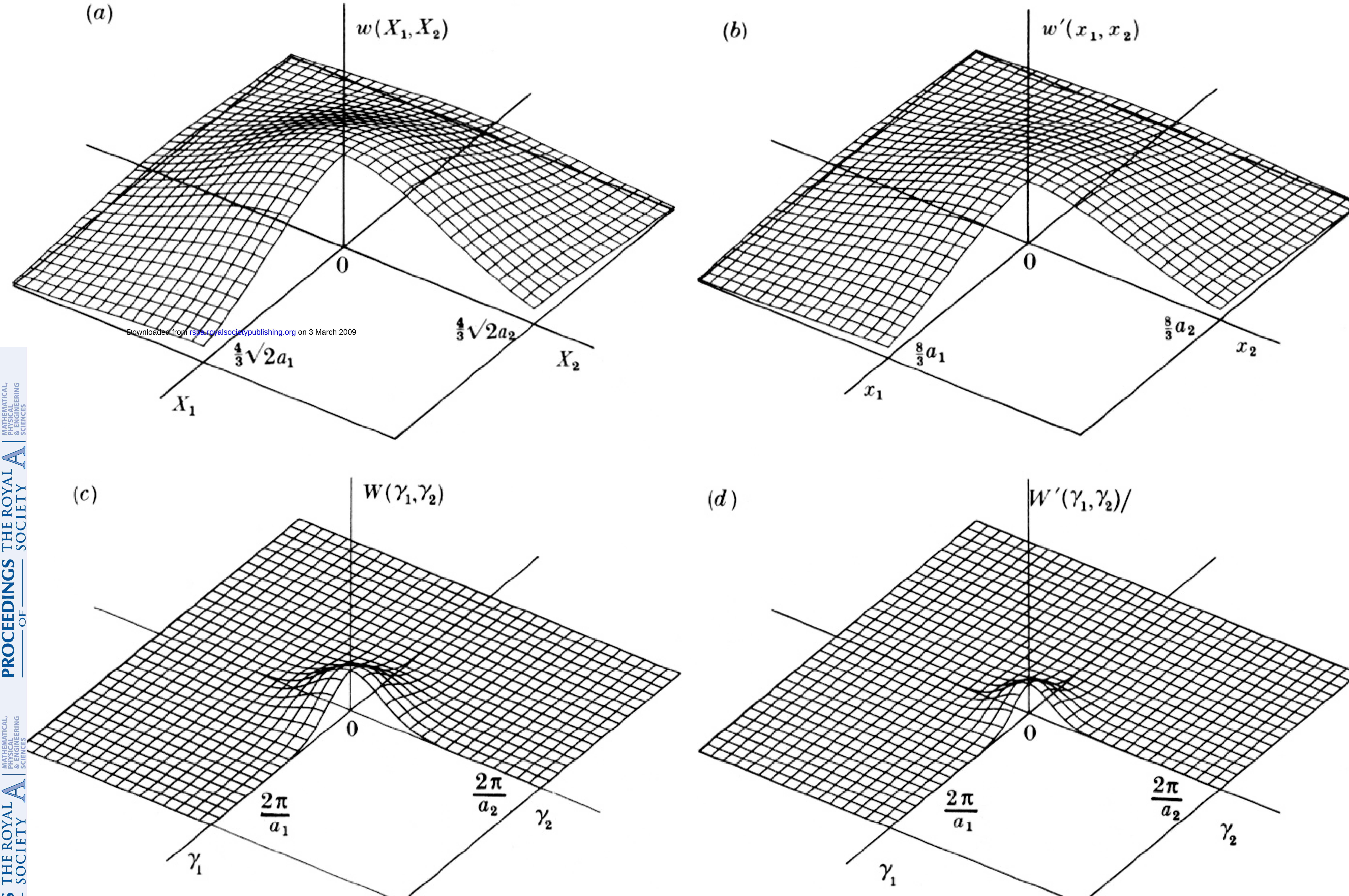

FIGURE 3. (*a*) Normal data footprint and its corresponding (*b*) lag, (*c*) frequency and (*d*) spectral footprints as defined in table 1. All the footprints are shown for a limited field of values only. The front quadrant of each graph has been cut away to show the underlying shape more clearly.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES