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Abstract

Finitely additive martingales are the counterpart of finitely additive measures over filtered probability space. We study the structure of the Yosida Hewitt decomposition in such setting and obtaing a full characterisation. Based on this result we introduce a "conditional expectation" operator for finitely additive measures which has some properties in common with ordinary conditional expectation. We address then the problem of computing the expectation of random elements generated by a given class of stochastic processes. On the basis of a notion of coherence for processes, akin to the no arbitrage principle in mathematical finance, we give conditions under which such expectation may be computed explicitely.

Keywords: coherence, finitely additive measures, finitely additive martingales, semimartingale, Doob-Meyer decomposition, equivalent martingale measure, fundamental theorem of asset pricing

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Short title: F.a. martingales.

1. Introduction.

In the foundations of subjective probability, finitely additive (henceforth f.a.) measures arise from an attitude of coherence in the face of uncertainty (see, among others, [10], [11], [12], [14] and [17]). This is defined with reference to the class $\mathcal{K}$ of bets that an individual, viewed as a bookie, stands ready to accept: coherence prescribes that no such bet may produce a strictly positive net outcome. In the space of bounded functions on a set Ω, a coherent probability is associated with the functional separating $\mathcal{K}$ and $\mathcal{B}(\Omega)_+$. The mathematical properties of probability follow then from the structure of the class of bets considered.

In this paper we consider individuals betting on the relative increase of a process K over some interval – typically over time. At a point of their choice bettors enter the game placing a stake and at any successive point they may choose to walk off, receiving the amount of the stake times the increment of K over that interval. Stakes are simple and bettors may only place a finite though arbitrary number of them (a more precise definition of coherence will be given in section 7, see (7.1)). An example of this framework is offered by financial markets, in which case K is the price of an asset.

In any setting in which information evolves according to some parameter, the structure of the separating measure (or rather its restrictions), viewed itself as a process, becomes important and a study of its properties leads us straight to the theory of f.a. stochastic processes. Despite its interest and importance, this theory did not receive but little attention since the first attempts made by Bochner (see e.g. [4]) – especially so if compared to the well established theory of stochastic processes. It is clear that, interpreting stochastic processes as ordered collections of random elements, and viewing in turn (if possible) random elements as densities of countably additive (henceforth c.a.) measures, most results on stochastic processes can be read as statements on families of c.a. measures. It is then challenging to investigate whether these claims extend to the f.a. setting. One of the purposes of this paper is to show that indeed this is not a pure *curiosum*.

In the next section we shall introduce a few definitions together with a key result on this subject, due to Armstrong: this essentially provides a f.a. analogous of the well known Doob-Meyer decomposition. In the following section 3 we shall further characterize such decomposition. In section 4, based on the preceding results, we introduce an operator acting on f.a. measures but akin in some suitable sense to ordinary conditional expectation for c.a. measures. This tool will play a key role in the analysis of the following section 5 in which we shall apply our findings to the computation of the f.a. expectation of random elements, generated by a stochastic process. We show that such expectation can be approximated fairly well by a functional for which there exists a fully analytical formulation. In section 6 we treat a special case in which such formulation can be further refined. Eventually in section 7 we shall return to the initial problem of a coherent assessment of probability i.e. of separation and we shall prove that the expected value of the elements of our set of bets can be computed explicitly at least in one special case of interest, and can be approximated conveniently well in more general cases.

Although the focus of the paper is on subjective probability, there is a strong, and perhaps little known, analogy with a much debated issue in mathematical finance (the so called *Fundamental Theorem of Asset Pricing*) to which we shall briefly refer in the last section.

2. Definitions and preliminaries

The setting for the following sections shall be as follows. Ω and Δ are arbitrary sets, with Δ linearly ordered. $\mathcal{A}$ and $\mathcal{A}_\delta$ will be algebras of subsets of Ω; by $\mathcal{F}_\delta$, $\mathcal{F}$ we shall denote their induced σ algebras respectively. A *filtration is* a collection $(\mathcal{A}_\delta : \delta \in \Delta)$ of sub algebras of $\mathcal{A}$ which increases with δ. The collection $\tilde{\lambda} = (\lambda_\delta : \delta \in \Delta)$, with $\lambda_\delta \in ba(\Omega, \mathcal{A}_\delta)$ for each $\delta \in \Delta$, is a *f.a. stochastic process*; a c.a. stochastic process $\tilde{\lambda}$ is a f.a. process such λ_δ is c.a. for each $\delta \in \Delta$. $\tilde{\lambda}$ is *bounded* whenever $\left\|\tilde{\lambda}\right\| \triangleq \bigvee_{\delta\in\Delta} \|\lambda_\delta\| < \infty$. $\tilde{\alpha} \geq \tilde{\beta}$ stands for $\alpha_\delta \geq \beta_\delta$ for each $\delta \in \Delta$; $\tilde{\alpha} > \tilde{\beta}$ for $\tilde{\alpha} \geq \tilde{\beta}$ but $\tilde{\beta} \nleq \tilde{\alpha}$. By $\mu|\mathcal{C}$ we mean the restriction of the measure μ to the algebra $\mathcal{C} \subset \mathcal{A}$; by μ_b we mean the measure $\mu_b(A) = \mu(b1_A)$ defined for each $b \in L^1(\Omega, \mathcal{A}, \mu)$ and $A \in \mathcal{A}$.

At times it shall be convenient to "complete" Δ by replacing it with $\Delta^* = \Delta \cup \{\delta_0, \delta_\infty\}$ and defining the order $\geq^*$ on Δ^* by letting $\delta_\infty \geq^* \varepsilon \geq^* \delta \geq^* \delta_0$ for any $\delta,\ \varepsilon \in \Delta$ such that $\varepsilon \geq \delta$, where $\geq$ is the order on Δ. To δ_∞ and δ_0 we shall associate the algebras $\mathcal{A}_{\delta_\infty} \triangleq \bigvee_{\delta\in\Delta} \mathcal{A}_\delta$ and $\mathcal{A}_{\delta_0} \triangleq \bigwedge_{\delta\in\Delta} \mathcal{A}_\delta$ respectively. Such completion – bearing no loss of generality – will often be implicit.

The most important result on f.a. processes was obtained by T. Armstrong in a noteworthy paper ([1], but see [2] as well) in which it was shown that some form of the core theorem of stochastic processes, the Doob Meyer decomposition, indeed holds in the f.a. model. More precisely:

Lemma 1 (Armstrong [1], Proposition 6.1 and Corollary 6.1.1, p. 288). *Let the positive, bounded f.a. stochastic process $\tilde{\lambda}$ satisfy*

$$\lambda_\delta \leq \lambda_\varepsilon | \mathcal{A}_\delta \tag{2.1}$$

$$(\textit{resp. } \lambda_\delta \geq \lambda_\varepsilon | \mathcal{A}_\delta) \tag{2.2}$$

for each $\delta \leq \varepsilon$. Then

1. *there exists a collection $\hat{\lambda} = \left(\hat{\lambda}_\delta : \delta \in \Delta\right)$ of bounded, positive, f.a. measures on $(\Omega, \mathcal{A})$ that increase (resp. decrease) with δ and such that $\lambda_\delta = \hat{\lambda}_\delta \Big| \mathcal{A}_\delta$ for each $\delta \in \Delta$;*

2. *$\tilde{\lambda}$ admits the decomposition*

$$\lambda_\delta = \mu_\delta + \alpha_\delta \tag{2.3}$$

$$(\textit{resp. } \lambda_\delta = \mu_\delta - \alpha_\delta) \tag{2.4}$$

where μ_δ and α_δ are positive, bounded f.a. measures on $(\Omega, \mathcal{A}_\delta)$ satisfying

(i). for each $\delta \leq \varepsilon$

$$\mu_\delta = \mu_\varepsilon | \mathcal{A}_\delta \tag{2.5}$$

(ii). $\alpha_\delta = \hat{\alpha}_\delta | \mathcal{A}_\delta$ where $\hat{\alpha} \triangleq (\hat{\alpha}_\delta : \delta \in \Delta)$ is a collection of positive f.a. measures on $(\Omega, \mathcal{A})$ increasing with δ and such that $\inf_{\delta\in\Delta} \|\hat{\alpha}_\delta\| = 0$.

This lemma is the backbone of what follows and we shall therefore adopt its notation and terminology. If a f.a. process satisfies (2.1) (resp. (2.2), resp. (2.5)) it will be referred to as a f.a. *submartingale* (resp. *supermartingale* resp. *martingale*) and in this case $\hat{\lambda}_\delta$ will always denote the extension of λ_δ to $(\Omega, \mathcal{A})$ with the properties described in the first claim: we shall refer to $\hat{\lambda}$ as the *extension* of $\tilde{\lambda}$. A f.a. process with the properties of $\tilde{\alpha}$ in lemma 1 claim 2 is an *increasing f.a. processes*. By analogy with the theory of stochastic processes, the f.a. processes $\tilde{\mu}$ and $\tilde{\alpha}$ appearing in the decompositions (2.3) and (2.4) will be indicated as the *characteristics* of $\tilde{\lambda}$.

When $\lambda \in ba(\Omega, \mathcal{A})$, $\lambda = \lambda^c + \lambda^\perp$ will be its Yosida and Hewitt decomposition into a c.a. (λ^c) and a purely finitely additive (henceforth p.f.a., $\lambda^\perp$) part. By $\mathcal{I}_\mathcal{A}$ we shall indicate the ideal of all $\mathcal{A}$ measurable $\lambda^\perp$ null sets. It is well known (see [3], theorem 10.3.3, p. 244) that when $\mathcal{A}$ is a σ algebra and $P \in ca(\Omega, \mathcal{A})$ then for each $\varepsilon > 0$ there exists a set $F \in \mathcal{I}_\mathcal{A}$ such that $P(F^c) \leq \varepsilon$.

Throughout the paper we adopt the convention $\frac{0}{0} = 0$.

3. The Structure of F.A. Martingales

F.a. martingales are the counterpart of f.a. measures on filtered f.a. measurable spaces and this is what makes them so important. If $\lambda \in ba(\Omega, \mathcal{A})$ we denote $\tilde{\lambda} = (\lambda|\mathcal{A}_\delta : \delta \in \Delta)$ which is clearly a f.a. martingale. The converse is also true: the equation

$$\lambda(F) = \lambda_\delta(F)$$

where $F \in \mathcal{A}_\delta$ defines in fact a positive, f.a. measure on $\left(\Omega, \bigcup_{\delta\in\Delta} \mathcal{A}_\delta\right)$ which may in turn be extended to the whole of $(\Omega, \mathcal{A})$ (a multiplicity of such extensions is known to exist, see [3], chapter 3). In the setting of ordinary stochastic processes, a full correspondence can only be established between c.a. measures and uniformly integrable martingales: this will be a bijection (see, for example, [13], proposition III.3.5, p. 154). L^1 bounded martingales, which are *a fortiori* f.a. martingales, do generate a measure, but only a f.a. one, unless uniform integrability is satisfied. The extreme difficulty of establishing this property is a good reason why f.a. should be considered with interest in these models. For the rest of the paper, let λ be a f.a. probability on $(\Omega, \mathcal{A})$, $\lambda_\delta = \lambda|\mathcal{A}_\delta$ and $\tilde{\lambda} = (\lambda_\delta : \delta \in \Delta)$. If $\mathcal{G}$ is a sub algebra of $\mathcal{A}$ we write $\lambda^c_\mathcal{G} \triangleq (\lambda|\mathcal{G})^c$ (resp. $\lambda^\perp_\mathcal{G} = (\lambda|\mathcal{G})^\perp$) and $\lambda^c_\delta \triangleq \lambda^c_{\mathcal{A}_\delta}$ (resp. $\lambda^\perp_\delta = \lambda^\perp_{\mathcal{A}_\delta}$).

Projecting a f.a. measure on its c.a. part and restricting it to some sub algebra are linear operations that do not commute, i.e. in general $\lambda^c_\delta \neq \lambda^c|\mathcal{A}_\delta$. To see this point more clearly, take the opposite case: then, λ^c is the c.a. extension of λ^c_δ to $(\Omega, \mathcal{A})$. This situation is quite exceptional, especially if $\mathcal{A}$ is considerably "larger" than $\mathcal{A}_\delta$. In general one would expect that c.a. is not preserved by taking arbitrary extensions, or, in other terms, that a p.f.a. component comes in and the probability mass assigned in accordance to the c.a. criterion shrinks the larger the algebra of events considered. More precisely, when $\varepsilon \geq \delta$ it is clear that $\lambda^c_\varepsilon | \mathcal{A}_\delta$ is a c.a., positive measure on $(\Omega, \mathcal{A}_\delta)$; furthermore, given that $0 \leq \lambda^\perp_\delta$,

$$\lambda^c_\varepsilon | \mathcal{A}_\delta \leq \lambda^c_\varepsilon | \mathcal{A}_\delta + \lambda^\perp_\varepsilon \Big| \mathcal{A}_\delta = \lambda_\delta$$

i.e.

$$\lambda^c_\varepsilon | \mathcal{A}_\delta \leq \sup \left\{ \mu \in ca\,(\Omega, \mathcal{A}_\delta)_+ : \mu \leq \lambda_\delta \right\} \triangleq \lambda^c_\delta$$

$\tilde{\lambda}^c = (\lambda^c_\delta : \delta \in \Delta)$ is then a f.a. supermartingale and, according to lemma 1, it admits the decomposition $\lambda^c_\delta = \mu^c_\delta - \alpha^c_\delta$. Analogously $\tilde{\lambda}^\perp = \left(\lambda^\perp_\delta : \delta \in \Delta\right)$ is a f.a. submartingale and decomposes as $\lambda^\perp_\delta = \mu^\perp_\delta + \alpha^\perp_\delta$. The inequalities $\lambda^\perp_\delta \geq \mu^\perp_\delta \vee \alpha^\perp_\delta \geq 0$ imply that both $\mu^\perp_\delta$ and $\alpha^\perp_\delta$ are p.f.a. measures, absolutely continuous with respect to $\lambda^\perp_\delta$. The analogous conclusion for $\tilde{\lambda}^c$ is more indirect and to establish we shall use the following lemma

Lemma 2. *Let $\tilde{\xi}$ be a f.a. positive supermartingale and $m \in ca\,(\Omega, \mathcal{A})_+$. Then, $\tilde{\xi}^m \triangleq \tilde{\xi} \wedge \tilde{m}$ is a positive, c.a. supermartingale and admits an extension $\hat{\xi}^m$ (as of lemma 1) such that $0 \leq \hat{\xi}^m_\delta \leq m$.*

Proof. Let $m \in ca\,(\Omega, \mathcal{A})_+$. Clearly, if $\varepsilon \geq \delta$

$$\begin{aligned} \xi^m_\varepsilon | \mathcal{A}_\delta &= (\xi_\varepsilon \wedge m_\varepsilon) | \mathcal{A}_\delta \\ &\leq \xi_\varepsilon | \mathcal{A}_\delta \wedge m_\varepsilon | \mathcal{A}_\delta \\ &\leq \xi_\delta \wedge m_\delta \\ &= \xi^m_\delta \end{aligned}$$

Let $\mathcal{D}$ be the collection of finite subsets of Δ (directed by inclusion), $d = \{\delta_1 \leq \delta_2 \leq \ldots \leq \delta_N\} \in \mathcal{D}$, and define $\phi_N \triangleq \xi^m_{\delta_N}$ and $\phi_n \triangleq \left(\xi^m_{\delta_n} - \xi^m_{\delta_{n+1}}\right) \Big| \mathcal{A}_{\delta_n}$ for $n = 1, \ldots, N-1$. Clearly $0 \leq \phi_n \leq m_{\delta_n} - \sum_{j=n+1}^N \phi_j \big| \mathcal{A}_{\delta_n}$. By Hahn Banach applied recursively, ϕ_n admits a positive extension $\hat{\phi}_n$ to $(\Omega, \mathcal{A})$ dominated by $m - \sum_{j=n+1}^N \hat{\phi}_j$ and therefore c.a.. For $n = 1, \ldots, N$, define $\hat{\xi}^m_{\delta_n} \triangleq \sum_{j=n}^N \hat{\phi}_j$ so that $0 \leq \hat{\xi}^m_{\delta_{n+1}} \leq \hat{\xi}^m_{\delta_n} \leq m$. Let now $\hat{\xi}^{m,d} = \left(\hat{\xi}^{m,d}_\delta : \delta \in \Delta\right)$ be defined by

$$\hat{\xi}^{m,d}_\delta \triangleq \hat{\xi}^m_{\delta_1} 1_{\{\delta \leq \delta_1\}}(\delta) + \sum_{n=1}^{N-1} \hat{\xi}^m_{\delta_{n+1}} 1_{]\delta_n, \delta_{n+1}]}(\delta) \tag{3.1}$$

$\xi^{m,d}_\delta \triangleq \hat{\xi}^{m,d}_\delta \Big| \mathcal{A}_\delta$, and $\tilde{\xi}^{m,d} \triangleq \left(\xi^{m,d}_\delta : \delta \in \Delta\right)$. From (3.1) we deduce the following properties of $\hat{\xi}^{m,d}$ and $\tilde{\xi}^{m,d}$:

(*i*) $\hat{\xi}^{m,d}$ is a decreasing collection of c.a. measures dominated by m (i.e. $\tilde{\xi}^{m,d}$ is a c.a. supermartingale and $\tilde{\xi}^{m,d} \leq \tilde{m}$);

(*ii*) $\tilde{\xi}^{m,d} \leq \tilde{\xi}^{m}$ and $\xi_\delta^{m,d} = \xi_\delta$ when $\delta \in d$, since

$$\xi_\delta^{m,d} = \xi_{\delta_1}^m \Big| \mathcal{A}_\delta 1_{\{\delta \leq \delta_1\}} + \sum_n \xi_{\delta_{n+1}}^m \Big| \mathcal{A}_\delta 1_{]\delta_n, \delta_{n+1}]} \leq \xi_\delta^m$$

Endow the space $\prod_{\delta \in \Delta} ba(\Omega, \mathcal{A})$ with the product topology obtained after assigning to each coordinate space the vague topology. For each $d \in \mathcal{D}$, the set $\mathcal{U}^d$ of elements satisfying (*i*) and (*ii*) above is closed and its intersection with the sphere of radius $\left\| \tilde{\xi}^m \right\|$ is non empty. It follows that $\bigcap_{d \in \mathcal{D}} \mathcal{U}^d$ is itself non empty: if $\hat{\xi}^m \in \bigcap_{d \in \mathcal{D}} \mathcal{U}^d$ then necessarily $\tilde{\xi}^m = \left(\hat{\xi}_\delta^m \Big| \mathcal{A}_\delta : \delta \in \Delta \right)$. ■

Theorem 1. *Let $\tilde{\xi}$ be a f.a. positive supermartingale with decomposition $\tilde{\mu} - \tilde{\alpha}$.*

1. *If ξ_δ is c.a., then there exists a decomposition where μ_δ and α_δ are c.a.;*
2. *If $\xi_\delta \ll P_\delta$ for some $P \in ca(\Omega, \mathcal{F})_+$, then there exists a decomposition where $\mu_\delta \ll P_\delta$ and $\alpha_\delta \ll P_\delta$.*

Proof. Consider the set $\hat{Z}$ of decreasing families $\hat{\zeta} \in \prod_{\delta \in \Delta} ca(\Omega, \mathcal{A})_+$ such that $\hat{\zeta}_\delta \Big| \mathcal{A}_\delta \leq \xi_\delta$ for each $\delta \in \Delta$ and let $\hat{Z}_0$ be a linearly ordered subset. If $\mathbf{A}$ is the collection of all finite subsets of $\hat{Z}_0$ – directed by inclusion, as usual – let $\hat{\zeta}^a = \sup_{\hat{\zeta} \in a} \hat{\zeta}$: $\hat{\zeta}^a \in \hat{Z}_0$. Then $\hat{\zeta}^* \triangleq \lim_a \hat{\zeta}^a \geq \hat{\zeta}$ for each $\hat{\zeta} \in \hat{Z}_0$, furthermore

$$\begin{aligned} \hat{\zeta}_\delta^* &= \lim_a \hat{\zeta}_\delta^a \\ &\geq \lim_a \hat{\zeta}_\varepsilon^a \\ &= \hat{\zeta}_\varepsilon^* \end{aligned}$$

for $\varepsilon \geq \delta$ and

$$\begin{aligned} \hat{\zeta}_\delta^* \Big| \mathcal{A}_\delta &= \left(\lim_a \hat{\zeta}_\delta^a \right) \Big| \mathcal{A}_\delta \\ &= \lim_a \left(\hat{\zeta}_\delta^a \Big| \mathcal{A}_\delta \right) \\ &\leq \xi_\delta \end{aligned}$$

so that $\hat{\zeta}^* \in \hat{Z}$. By Zorn's lemma, we can therefore find a maximal element $\hat{\zeta}$ for $\hat{Z}$: let $\tilde{\zeta} = \left(\hat{\zeta}_\delta \Big| \mathcal{A}_\delta : \delta \in \Delta \right)$. With the notation of the previous lemma, from $\hat{\xi}^m \in \hat{Z}$ we get $\tilde{\xi} \geq \tilde{\zeta} \geq \tilde{\xi}^m$. But then, since $\tilde{\xi} = \bigvee_{m \in ca(\Omega, \mathcal{A})_+} \tilde{\xi}^m$, we conclude that $\hat{\zeta}$ is a c.a. extension of $\tilde{\xi}$ and we denote it consequently by $\hat{\xi}$.

As in [1], $\mu \triangleq \bigvee_{\delta \in \Delta} \hat{\xi}_\delta \in ca(\Omega, \mathcal{A})_+$, $\mu_\delta \geq \xi_\delta$ and $\tilde{\mu}$ is obviously a c.a. martingale. $\hat{\alpha} \triangleq \mu - \hat{\xi}$ is a collection of positive, c.a. measures on $(\Omega, \mathcal{F})$ increasing with δ and

$$\begin{aligned} \bigwedge_{\delta \in \Delta} \|\hat{\alpha}_\delta\| &= \bigwedge_{\delta \in \Delta} \hat{\alpha}_\delta(\Omega) \\ &= \mu(\Omega) - \bigvee_{\delta \in \Delta} \xi_\delta(\Omega) \\ &= 0 \end{aligned}$$

By definition, $\tilde{\alpha} \triangleq \tilde{\mu} - \tilde{\xi}$ is a c.a. increasing process and the decomposition $\tilde{\mu} - \tilde{\alpha}$ satisfies the claim. ■

The flow of information – as parametrized by δ – causes a shift of the probability mass between the c.a. and p.f.a. components of λ_δ and it is important to understand wether this phenomenon is smooth or not, i.e. if the interplay between the characteristics of λ is continuous with respect to the filtration. To this end we introduce the following definition

Definition 1. *The characteristics of λ are left (resp. right) continuous with respect to the filtration if for any monotone increasing (resp. decreasing) sequence $\langle \delta_n \rangle_{n\in\mathbb{N}}$ in Δ,*

$$\Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} \triangleq \lim_n \left(\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} - \hat{\lambda}^{\perp}_{\delta_n} \right) \Big| \bigvee_n \mathcal{A}_{\delta_n} = 0$$

(resp. $\Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}} \triangleq \lim_n \left(\lambda^{\perp}_{\delta_n} - \lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}} \right) \Big| \bigwedge_n \mathcal{A}_{\delta_n} = 0$).

Since it is not difficult to establish (as will clearly emerge from the proof of the following lemma) that $\Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} = -\Delta\lambda^{c}_{\bigvee_n \mathcal{A}_{\delta_n}}$ and $\Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}} = -\Delta\lambda^{c}_{\bigwedge_n \mathcal{A}_{\delta_n}}$, the above definition is indeed a condition on λ characteristics.

Lemma 3. *The characteristics of λ are left continuous with respect to the filtration. If $\mathcal{A}$ and $\mathcal{A}_\delta$ are σ algebras, then the characteristics of λ are also right continuous with respect to the filtration.*

Proof. Clearly, $0 \leq \Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} \leq \lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}}$. If $F \in \mathcal{A}_{\delta_{n_0}}$ and $n > n_0$

$$\begin{aligned} \Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}}(F) &\leq \lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}}(F) - \lambda^{\perp}_{\delta_n}(F) \\ &= -\lambda^{c}_{\bigvee_n \mathcal{A}_{\delta_n}}(F) + \lambda^{c}_{\delta_n}(F) \\ &\leq \lambda^{c}_{\delta_n}(F) \\ &= \mu^{c}_{\delta_n}(F) - \alpha^{c}_{\delta_n}(F) \\ &\leq \mu^{c}(F) \end{aligned}$$

i.e. $\Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} \leq \lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} \wedge \mu^{c}_{\bigvee_n \mathcal{A}_{\delta_n}}$ proving the first claim.

Let now $\langle \delta_n \rangle_{n\in\mathbb{N}}$ be a monotone decreasing sequence and $F \in \bigwedge_n \mathcal{A}_{\delta_n}$

$$\begin{aligned} \Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) &\leq \lambda^{\perp}_{\delta_n}(F) - \lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) \\ &= \lambda^{c}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) - \lambda^{c}_{\delta_n}(F) \\ &\leq \lambda^{c}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) \\ &\leq \mu^{c}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) \\ &= \mu^{c}(F) \end{aligned} \tag{3.2}$$

By (3.2), $\Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}$ is then c.a. and, by the Hahn Banach theorem, it admits a c.a. extension, ϕ, to the whole of $(\Omega, \mathcal{A})$. Let $\mathcal{N}$ be the collection of subsets of Ω which are null with respect to the inner measure generated by $\Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}$, i.e.

$$\mathcal{N} = \left\{F \subset \Omega : \Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}(E) = 0,\ E \subset F,\ E \in \mathcal{A}_{\delta}\right\}$$

Clearly, $\bigcup_n \mathcal{I}_{\mathcal{A}_{\delta_n}} \subset \mathcal{N}$ and $G \cap \{\phi(G|\,\mathcal{A}_{\varepsilon}) = 0\} \in \mathcal{N}$ for each $G \in \mathcal{A}$ and $\varepsilon \geq \delta$. It is well known that the collection

$$\mathcal{B} = \left\{F\Delta N : F \in \bigwedge_n \mathcal{A}_{\delta_n}, N \in \mathcal{N}\right\}$$

is an algebra with the property that if $F_1\Delta N_1 = F_2 \Delta N_2 \in \mathcal{B}$ then $F_1 \Delta F_2 \in \mathcal{N}$. An extension β of $\Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}$ to $(\Omega, \mathcal{B})$ can then be defined as usual by letting $\beta(F\Delta N) = \Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F)$. For any $F \in \bigwedge_n \mathcal{A}_{\delta_n}$ and $F_n \in \mathcal{I}_{\mathcal{A}_{\delta_n}}$

$$\begin{aligned}
\phi(F) &= \Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}(F) \\
&= \beta\left(F \cap F_n^c \cap \left\{\phi\left(F_n^c \middle| \bigwedge_n \mathcal{A}_{\delta_n}\right) > 0\right\}\right) \\
&= \Delta\lambda^{\perp}_{\bigwedge_n \mathcal{A}_{\delta_n}}\left(F \cap \left\{\phi\left(F_n^c \middle| \bigwedge_n \mathcal{A}_{\delta_n}\right) > 0\right\}\right) \\
&= \phi\left(F \cap \left\{\phi\left(F_n^c \middle| \bigwedge_n \mathcal{A}_{\delta_n}\right) > 0\right\}\right)
\end{aligned}$$

i.e. $\phi(F_n^c|\bigwedge_n \mathcal{A}_{\delta_n})$ is positive ϕ a.s.: let $G = \left\{\phi\left(\bigcup_{n>m} F_n^c \middle| \bigwedge_n \mathcal{A}_{\delta_n}\right) > 0\right\}$. Let

$$f_m \triangleq \phi\left(\bigcup_{n>m} F_n^c \middle| \bigwedge_n \mathcal{A}_{\delta_n}\right)^{-1} 1_{G \cap \bigcup_{n>m} F_n^c}$$

The sequence $\langle f_m \rangle_{m\in\mathbb{N}}$ converges ϕ a.s. to the indicator of $G \cap \left(\bigcap_m \bigcup_{n>m} F_n^c\right) \in \bigwedge_n \mathcal{A}_{\delta_n}$: indeed $f_m = 1_G$ on $\bigcap_m \bigcup_{n>m} F_n^c$ while if $\omega \in \bigcap_{n>n_\omega} F_n$ for some n_ω, then $f_m = 0$ when $m > n_\omega$. Then $f^* \triangleq \sup_n f_n < \infty$, ϕ a.s.. But then for $F \in \bigwedge_n \mathcal{A}_{\delta_n}$

$$\begin{aligned}
\phi(F) &= \lim_m \phi\left(\frac{\phi\left(\bigcup_{n>m} F_n^c|\bigwedge_n \mathcal{A}_{\delta_n}\right)}{\phi\left(\bigcup_{n>m} F_n^c|\bigwedge_n \mathcal{A}_{\delta_n}\right)} 1_{G\cap F}\right) \\
&= \lim_m \phi(f_m 1_F) \\
&= \lim_r \lim_m \phi(1_F(f_m \wedge r)) + \lim_r \lim_m \phi\left(1_F (f_m - r)^+\right) \\
&= \phi\left(F \cap \bigcap_m \bigcup_{n>m} F_n^c\right) + \lim_r \lim_m \phi\left(1_F(f_m - r)^+\right)
\end{aligned}$$

In other words, ϕ vanishes outside the set $\left(\bigcap_m \bigcup_{n>m} F_n^c\right) \cup \{r < f^* < \infty\}$ for any r. However, for any c.a. probability measure Q and any $\eta > 0$, the sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ and the integer r can be chosen such that $Q\left(\bigcap_m \bigcup_{n>m} F_n^c\right) + Q\left(r < f^* < \infty\right) < \eta$, or, equivalently, ϕ is p.f.a.. It follows that $\phi = 0$. ■

Remark 1. *Since* $\lim_n \left(\hat{\lambda}^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} - \hat{\lambda}^{\perp}_{\delta_n}\right) > 0$ *if an only if* $0 < \lim_n \left(\hat{\lambda}^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}} - \hat{\lambda}^{\perp}_{\delta_n}\right)(\Omega) = \Delta\lambda^{\perp}_{\bigvee_n \mathcal{A}_{\delta_n}}(\Omega)$*, it follows that continuity with respect to the filtration carries over to the extension* $\hat{\lambda}^{\perp}$ *as well.*

Remark 2. *Consider the case when the filtration is discontinuous and, in particular, let* $\delta_n \uparrow \delta$ *but* $\mathcal{A}_\delta \supsetneq \bigvee_n \mathcal{A}_{\delta_n}$*. As illustrated in the proof of lemma 3,* $\lim_n \left(\lambda^{\perp}_{\delta} - \hat{\lambda}^{\perp}_{\delta_n}\right)\Big| \mathcal{A}_\delta \leq \lambda^{\perp}_{\delta}$*, while* $\lim_n \left(\lambda^{\perp}_{\delta} - \hat{\lambda}^{\perp}_{\delta_n}\right)\Big| \bigvee_n \mathcal{A}_{\delta_n} \leq \mu^c_{\bigwedge_n \mathcal{A}_{\delta_n}}$*. Although in principle the "distance" between* $\bigvee_n \mathcal{A}_{\delta_n}$ *and* $\mathcal{A}_\delta$ *may be small – even* $\mathcal{A}_\delta = \sigma\left(\bigvee_n \mathcal{A}_{\delta_n}\right)$ *– nevertheless we have here a general situation in which a c.a. measure* $\lim_n \left(\lambda^{\perp}_{\delta} - \hat{\lambda}^{\perp}_{\delta_n}\right)\Big| \bigvee_n \mathcal{A}_{\delta_n}$ *admits a p.f.a. extension (see [5] for explicit examples of c.a. measures on an algebra admitting p.f.a. extensions to the generated* σ *algebra).*

In the case of supermartingales or submartingales driven by a given, c.a. probability measure P the property of *continuity with respect to the filtration* simply translates into that of continuity in mean. Clearly, martingales are filtration continuous and it is well known that when $\Delta = \mathbb{R}_+$, the filtration is right continuous and X is either a supermartingale or a submartingale, this property is equivalent to the existence of a right continuous modification of the process X (see [16]).

We summarize the results of this section in the following

Proposition 1. *Let* $(\Omega, \mathcal{A})$ *be a probability space,* $(\mathcal{A}_\delta : \delta \in \Delta)$ *a filtration of algebras (resp.* σ *algebras) with* $\bigwedge_\delta \mathcal{A}_\delta$ *finite and* λ *a positive, bounded, f.a. set function on* $(\Omega, \mathcal{A})$*. Let* λ_δ*,* λ^c_δ *and* $\lambda^{\perp}_\delta$ *be defined as above; let* $\overline{\lambda^c_\delta}$ *be the c.a. extension of* λ^c_δ *to* $(\Omega, \mathcal{F}_\delta)$*. Then there exist:*

1. *a c.a. probability measure* P *such that*

$$P|\,\mathcal{A}_\delta \gg \lambda^c_\delta \tag{3.3}$$

2. *a positive, filtration left continuous (resp. continuous)* P *supermartingale*

$$X = M - A \tag{3.4}$$

on $(\Omega, \mathcal{F}, P; (\mathcal{F}_\delta : \delta \in \Delta))$*, such that*

$$\overline{\lambda^c_\delta} = P_{X_\delta}|\,\mathcal{F}_\delta$$

where

(i). M is a positive, uniformly integrable martingale and

(ii). A is an increasing, integrable process such that $A_0 = 0$.

3. *a positive, filtration left continuous (resp. continuous), bounded f.a. submartingale*

$$\lambda_\delta^\perp = \mu_\delta^\perp + \alpha_\delta^\perp$$

where

(i). $\mu^\perp$ is a p.f.a. martingale and

(ii). $\alpha^\perp$ is a p.f.a., increasing process;

Furthermore, for $F \in \mathcal{A}_\delta$ and each $\delta, \varepsilon \in \Delta$, $\varepsilon > \delta$,

$$\left(\lambda_\varepsilon^\perp - \lambda_\delta^\perp\right)(F) = \left(\lambda_\delta^c - \lambda_\varepsilon^c\right)(F) = P\left(1_F\left(A_\varepsilon - A_\delta\right)\right) \tag{3.5}$$

Proof. Replacing Δ by Δ^*, $(\lambda_\delta^c : \delta \in \Delta^*)$ and $\left(\lambda_\delta^\perp : \delta \in \Delta^*\right)$ are easily seen to be a c.a. positive supermartingale and a positive f.a. submartingale respectively. By lemma 1 and theorem 1, we obtain the decompositions $\lambda_\delta^\perp = \mu_\delta^\perp + \alpha_\delta^\perp$ and $\lambda_\delta^c = \mu_\delta^c - \alpha_\delta^c$ with $\tilde{\mu}^\perp$ (resp. $\tilde{\mu}^c$) and $\tilde{\alpha}^\perp$ (resp. $\tilde{\alpha}^c$) a positive, p.f.a. (resp. c.a.) martingale and increasing process respectively. In case $\mu_{\delta_\infty}^c = 0$, $\tilde{\lambda}^c = 0$ so that $\tilde{\lambda}^\perp$ is a p.f.a. martingale; since $\mathcal{A}_{\delta_0}$ is finite $0 = \lambda_{\delta_0}^\perp(\Omega) = \lambda_\delta^\perp(\Omega)$, i.e. $\lambda = 0$. Let $P = \mu_{\delta_\infty}^c(\Omega)^{-1}\mu_{\delta_\infty}^c$; it is obvious that $P_\delta \gg \mu_\delta^c$ and the inequality $\mu_\delta^c \geq \lambda_\delta^c \vee \alpha_\delta^c$ implies (3.3) and $P_\delta \gg \alpha_\delta^c$. Let $\hat{\alpha}_\delta^c$ be the extension of α_δ^c to the whole of $(\Omega, \mathcal{A})$. Since $0 \leq \hat{\alpha}_\delta^c \leq \hat{\alpha}_{\delta_\infty}^c = \alpha_{\delta_\infty}^c$ we conclude that $\hat{\alpha}_\delta^c$ is c.a. and absolutely continuous with respect to P. Let $M_\delta = \frac{d\mu_\delta^c}{dP_\delta}$ and $A_\delta = \frac{d\hat{\alpha}_\delta^c}{dP}$. By construction for each $F \in \mathcal{F}_\delta$, $P(1_F M_\delta) = P(1_F M_{\delta_\infty})$ so that $\{M_\delta : \delta \in \Delta^*\}$ is uniformly integrable. A is clearly a.s. increasing. Continuity with respect to the filtration follows from lemma 3. (3.5) is a direct consequence of the martingale nature of $\tilde{\lambda}$ by which

$$\begin{aligned} 0 &= \left(\lambda_\varepsilon - \lambda_\delta\right)(F) \\ &= \left(\lambda_\varepsilon^c - \lambda_\delta^c\right)(F) + \left(\lambda_\varepsilon^\perp - \lambda_\delta^\perp\right)(F) \\ &= -P\left(1_F\left(A_\varepsilon - A_\delta\right)\right) + \left(\lambda_\varepsilon^\perp - \lambda_\delta^\perp\right)(F) \end{aligned}$$

an the proof is complete. ■

Remark 3. *The c.a. probability measure P mentioned in the first claim of the preceding proposition will be referred to as the c.a. probability generated by λ. By theorem 1, if Q is any arbitrary c.a. probability measure that satisfies (3.3), then the characteristics of $\tilde{\lambda}^c$ may be constructed to be absolutely continuous with respect to Q and so will be the measure P generated by λ. Clearly the supermartingale X defined in (3.4) will have in this case a Q version, namely $X^Q = UX$ where U is the uniformly integrable martingale generated by the Radon Nikodým derivative of P with respect to Q. Under P we have $M_\delta = \mu_{\delta_\infty}^c(\Omega)$, P a.s..*

Proposition 1 can be considered as the extension to the f.a. case of the theorem stating that the density of one c.a. probability with respect to another is described by a uniformly integrable martingale (see e.g. [13], proposition III.3.5, p. 154). One of the implications of this proposition is that lack of c.a. receives a rather convenient characterization in filtered probability spaces, where it is associated with the process A. This makes f.a. measures quite tractable in this context, as the following sections will confirm.

In section 5 we shall restrict our model to the usual setting for which proposition 1 specializes as follows.

Corollary 1. *Let $(\Omega, \mathcal{F}; (\mathcal{F}_t : t \in \mathbb{R}_+))$ be a filtered probability space with $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for each t and $\mathcal{F}_0$ finite. Let λ be a f.a. probability on $(\Omega, \mathcal{F})$ generating the c.a. probability P. Then under P:*

1. *there exists a uniformly integrable martingale M and a predictable, increasing process of integrable variation A such that the supermartingale $X = M - A$ satisfies*
$$\lambda^c_\tau = P_{X_\tau} | \mathcal{F}_\tau$$
for any stopping time τ with $P(\tau < \infty) = 1$.

2. *If σ and τ are stopping times and $F \in \mathcal{F}_\sigma$ then*
$$\left(\lambda^\perp_\tau - \lambda^\perp_\sigma\right)(F \cap \{\tau \geq \sigma\}) = P\left(1_{F\cap\{\tau\geq\sigma\}}(A_\tau - A_\sigma)\right) \tag{3.6}$$
and, for $f \in \mathcal{B}(\Omega, \mathcal{F})$
$$\hat{\lambda}^\perp_{\tau\wedge\sigma}\left(f1_{\{\tau\geq\sigma\}}\right) = \hat{\lambda}^\perp_\sigma\left(f1_{\{\tau\geq\sigma\}}\right) \tag{3.7}$$
for any pair of extensions $\hat{\lambda}^\perp_{\tau\wedge\sigma}$ and $\hat{\lambda}^\perp_\sigma$ to $(\Omega, \mathcal{F})$ of $\lambda^\perp_{\tau\wedge\sigma}$ and $\lambda^\perp_\sigma$ satisfying $\hat{\lambda}^\perp_{\tau\wedge\sigma} \leq \hat{\lambda}^\perp_\sigma$.

Proof. If the filtration is right continuous the supermartingale X in (3.4) will be right continuous in mean since it is filtration continuous: i.e. it admits a *càdlàg* modification. Since $0 \leq X \leq \mu_{\delta_\infty}(\Omega)^{-1}$, this is clearly a process of class D so that the properties of M and A follow from the theorem of Doob Meyer.

Let $F^n_k \triangleq \{(k-1)\,2^{-n} < \tau \leq k2^{-n}\} \in \mathcal{F}_{k2^{-n}}$ and $\tau^n = \sum_k k2^{-n}1_{F^n_k}$. This definition implies (i) $\tau^n \downarrow \tau$, P a.s., (ii) $F^n_k = F^{n+1}_{2k-1} \cup F^{n+1}_{2k}$ and (iii) $F \in \mathcal{F}_{\tau^n}$ if and only if $F \cap F^n_k \in \mathcal{F}_{k2^{-n}}$ for any k. Let $F \subset F^n_k$, $F \in \mathcal{F}_{\tau_n}$. Then

$$\begin{aligned}
\lambda^c_{\tau^n}(F) &\triangleq \inf\left\{\sum_i \lambda(G_i) : G_i \in \mathcal{F}_{\tau^n_k};\ \bigcup_i G_i = F; G_i \cap G_j = \varnothing\right\} \\
&= \inf\left\{\sum_i \lambda(G_i) : G_i \in \mathcal{F}_{k2^{-n}};\ \bigcup_i G_i = F; G_i \cap G_j = \varnothing\right\} \qquad \text{(by } (iii))\\
&\triangleq \lambda^c_{k2^{-n}}(F)
\end{aligned}$$

For arbitrary $F \in \mathcal{F}_\tau$

$$\begin{aligned}
\lambda^c_{\tau^n}(F) &= \sum_k \lambda^c_{\tau^n}(F \cap F^n_k) + \lambda^c_{\tau^n}(F \cap \{\tau = \infty\}) \\
&= \sum_k \lambda^c_{k2^{-n}}(F \cap F^n_k) \\
&= \sum_k P\left(X_{k2^{-n}} 1_{F \cap F^n_k}\right) \\
&= P\left(X_{\tau^n} 1_F\right)
\end{aligned}$$

(i) implies $\mathcal{F}_{\tau^{n+1}} \subset \mathcal{F}_{\tau^n}$. From (ii) we deduce that on F^{n+1}_k, $X_{\tau^{n+1}} = X_{k2^{-n-1}}$ while

$$X_{\tau^n} = \begin{cases} X_{k2^{-n-1}} & k \text{ even} \\ X_{(k+1)2^{-n-1}} & \tau \text{ odd} \end{cases}$$

It follows that when $F \in \mathcal{F}_{\tau^{n+1}}$

$$\begin{aligned}
P\left(X_{\tau^n} 1_F\right) &= \sum_k P\left(X_{\tau^n} 1_{F \cap F^{n+1}_k}\right) \\
&\leq \sum_k P\left(X_{\tau^{n+1}} 1_{F \cap F^{n+1}_k}\right) \\
&= P\left(X_{\tau^{n+1}} 1_F\right)
\end{aligned}$$

Let $\mathcal{G}_n = \mathcal{F}_{\tau^n}$ and $X_n = X_{\tau^n}$, then $P \circ X_n = \lambda^c_{\mathcal{G}_n}$ so that $(X_n : n \in \mathbb{N})$ is filtration continuous. From lemma 3 and the fact that $\mathcal{F}_\tau = \bigwedge_n \mathcal{G}_n$ we conclude that for $F \in \mathcal{F}_\tau$

$$\begin{aligned}
\lambda^c_\tau(F) &= \lim_n \lambda^c_{\mathcal{G}_n}(F) \\
&= \lim_n P_{X_n}(F) \\
&= P\left(1_F \lim_n X_{\tau^n}\right) \\
&= P\left(1_F X_\tau\right)
\end{aligned}$$

the last line following from X being a.s. right continuous.

Let $F \in \mathcal{F}_\sigma$. Since $F \cap \{\tau \geq \sigma\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$, then

$$\begin{aligned}
\left(\lambda^\perp_\tau - \lambda^\perp_\sigma\right)(F \cap \{\tau \geq \sigma\}) &= -\left(\lambda^c_\tau - \lambda^c_\sigma\right)(F \cap \{\tau \geq \sigma\}) \\
&= P\left(1_{F \cap \{\tau \geq \sigma\}}\left(X_\sigma - X_\tau\right)\right) \\
&= P\left(1_{F \cap \{\tau \geq \sigma\}}\left(A_\tau - A_\sigma\right)\right)
\end{aligned}$$

Let $f \in \mathcal{B}(\Omega, \mathcal{F})$ and $\hat{\lambda}^\perp_\sigma$ and $\hat{\lambda}^\perp_{\tau\wedge\sigma}$ be the above mentioned extensions. Then

$$\begin{aligned}
0 &\leq \left|\left(\hat{\lambda}^\perp_\sigma - \hat{\lambda}^\perp_{\tau\wedge\sigma}\right)\left(f 1_{\{\tau \geq \sigma\}}\right)\right| \\
&\leq \|f\| \left(\lambda^\perp_\sigma - \lambda^\perp_{\tau\wedge\sigma}\right)(\tau \geq \sigma) \\
&= 0
\end{aligned}$$

by (3.6). ■

Property (3.5) suggests that A should be viewed as the predictable "compensator" of the f.a. process $\tilde{\lambda}^{\perp}$. This simple fact shall have deep consequences – at least for the case covered in the above corollary 1. Let in fact $\mathcal{P}_0$ be the algebra of finite unions of sets of the form $F \times]s,t]$, where $F \in F_s$, $s < t$ and $s,t \in \mathbb{R}_+$. For any f.a. submartingale $\tilde{\xi}$ consider the Doléans measure $V\tilde{\xi}$ on $\left(\tilde{\Omega}, \mathcal{P}_0\right)$ associated to $\tilde{\xi}$ via the equation

$$V\tilde{\xi}\left(\bigcup_{n=1}^{N} F_n \times]s_n,t_n]\right) = \sum_{n=1}^{N}\left(\xi_{t_n}^{\perp} - \xi_{s_n}^{\perp}\right)(F_n)$$

By corollary 1, $V\tilde{\lambda}^{\perp}$admits a c.a. extension to $\mathcal{P} \triangleq \sigma(\mathcal{P}_0)$, the predictable σ algebra (as proved in [7] and [15]). In other words, any probability measure m, though f.a. when acting on random elements, is c.a. when viewed as acting on processes (through the map V) and it is obvious that this transposition from random elements to processes is likely to be crucial when computing the expectation of random elements generated by stochastic processes. In the next sections we shall establish this loose relationship in clearer terms.

4. A F.A. Conditional Expectation

As is well known, conditional expectation, in the sense established by Kolmogorov, is not available when c.a. fails. Nevertheless, we shall construct an operator that, while not possessing all properties of c.a. conditional expectation, still is almost as useful from a practical point of view.

Let $\mathcal{I}_\delta = \mathcal{I}_{\mathcal{F}_\delta}$ and $\varepsilon \in \Delta$. $\hat{\lambda}_\varepsilon^{\perp} \in ba(\Omega, \mathcal{F})$ has the following two noteworthy properties:

(a). for each $\delta \leq \varepsilon$ and any $B \in \mathcal{I}_\delta$, $\hat{\lambda}_\varepsilon^{\perp}(B) = \left(\hat{\lambda}_\varepsilon^{\perp} - \hat{\lambda}_\delta^{\perp}\right)(B) = P\left((A_\varepsilon - A_\delta) 1_B\right)$;

(b). for each $\delta < \varepsilon$, $P_{A_\varepsilon - A_\delta}[\mathcal{I}_\delta]$ is dense in $P_{A_\varepsilon - A_\delta}[\mathcal{F}_\delta]$.

We shall fix the above situation in the following definition:

Definition 2. *Let $\xi \in ba(\Omega, \mathcal{F})_+$, $m \in ca(\Omega, \mathcal{F})_+$, $\mathcal{G}$ a sub σ algebra of $\mathcal{F}$ and $\mathcal{I}$ an ideal in $\mathcal{G}$. m is a compensating measure for ξ over $\mathcal{G}$ with respect to $\mathcal{I}$ if (i) $\xi(B) = m(B)$ for each $B \in \mathcal{I}$ and (ii) $m[\mathcal{I}]$ is dense in $m[\mathcal{G}]$.*

For the rest of this section let $\xi \in ba(\Omega, \mathcal{F})_+$ and let $m \in ca(\Omega, \mathcal{F})_+$ be a compensating measure for ξ over $\mathcal{G}$ with respect to $\mathcal{I}$.

Let Π be the family of all finite, disjoint collections of elements of $\mathcal{I}$. For $\pi \in \Pi$ and $f \in \mathcal{B}(\Omega, \mathcal{F})$, define the map $\xi(\cdot|\,\pi) : \mathcal{B}(\Omega, \mathcal{F}) \to \mathcal{B}(\Omega, \mathcal{G})$ by setting

$$\xi(f|\,\pi) = \sum_{F \in \pi} \frac{\xi(f 1_F)}{m(F)} 1_F$$

We then have the following properties

1. $|\xi(f|\pi)| \leq \|f\|$;

2. For any $F \in \pi_0 \leq \pi$ (the order being defined by refinement)

$$\xi(f1_F) = \xi(\xi(f|\pi)1_F) = m(\xi(f|\pi)1_F) \tag{4.1}$$

Let $\mathbb{S}_f$ be the subset of $L^1(\Omega, \mathcal{G}, m)$ consisting of those elements U such that $|U| \leq \|f\|$, m a.s.. Being closed in the weak topology, $\mathbb{S}_f$ is in fact a compact set (see [8], theorem IV.8.9, p. 292). Furthermore, for each $\pi \in \Pi$ the subsets

$$\mathcal{U}_\pi(f) = \{U \in \mathbb{S}_f : m(U1_F) = \xi(f1_F), F \in \pi\} \tag{4.2}$$

is weakly closed and non empty. By the finite intersection property we conclude that the intersection $\mathcal{U}(f) = \bigcap_{\pi\in\Pi} \mathcal{U}_\pi(f)$ is non empty. Let $\xi(f|\mathcal{I}) \in \mathcal{U}(f)$: since for each $F \in \mathcal{I}$ there exists $\pi \in \Pi$ such that $F \in \pi$, $\xi(f|\mathcal{I})$ satisfies (4.1) for any set $F \in \mathcal{I}$; furthermore, since $|\xi(f|\mathcal{I})| \leq \|f\|$, m a.s. we can choose a version satisfying such inequality pointwise. We have therefore proved the existence claim in the following proposition.

Proposition 2. *Let $f \in \mathcal{B}(\Omega, \mathcal{F})$. Then there exists $\xi(f|\mathcal{I}) \in \mathcal{B}(\Omega, \mathcal{G})$ such that $|\xi(f|\mathcal{I})| \leq \|f\|$, m a.s. and*

$$\xi(f1_F) = \xi(\xi(f|\mathcal{I})1_F) = m(\xi(f|\mathcal{I})1_F) \tag{4.3}$$

for each $F \in \mathcal{I}$. Furthermore, the conditional expectation operator

$$\xi(\cdot|\mathcal{I}) : \mathcal{B}(\Omega, \mathcal{F}) \to \mathcal{B}(\Omega, \mathcal{G})$$

satisfies the following properties

1. *$\xi(\cdot|\mathcal{I})$ is positive;*

2. *$\xi(f1_F|\mathcal{I}) = \xi(f|\mathcal{I})1_F$ for any $F \in \mathcal{G}$;*

Proof. (Existence). An alternative proof can also be given, based on the Radon Nikodým approach to conditional expectation. This will help clarifying some other aspects. Let $\mathcal{K}$ be the closed linear subspace of $\mathcal{B}(\Omega, \mathcal{G})$ generated by the indicators of the sets in $\mathcal{I}$: plainly, if $f \in \mathcal{B}(\Omega, \mathcal{G})$ and $F \in \mathcal{I}$, then $f1_F \in \mathcal{K}$. Clearly ξ and m coincide on $\mathcal{K}$. For any $f \in \mathcal{B}(\Omega, \mathcal{F})_+$, define the positive linear functional ξ_f over $\mathcal{K}$ by writing $\xi_f(K) = \xi(fK)$.

$$\begin{aligned} \xi_f(K) &\leq \xi(|fK|) \\ &\leq \|f\|\,\xi(|K|) \\ &= \|f\|\,m(|K|) \end{aligned}$$

Let $\psi(K) = \|f\| m(|K|)$ for $K \in \mathcal{B}(\Omega, \mathcal{G})$. ξ_f is then dominated by the sub additive functional ψ over $\mathcal{K}$ and, by the Hahn Banach theorem, it admits an extension, $\hat{\xi}_f$, to $\mathcal{B}(\Omega, \mathcal{G})$ that still satisfies $0 \leq \hat{\xi}_f \leq \psi$. $\hat{\xi}_f$ is therefore a bounded linear functional on $\mathcal{B}(\Omega, \mathcal{G})$ and the equation $\beta_f(A) = \hat{\xi}_f(1_A)$ defines $\beta_f \in ba(\Omega, \mathcal{G})_+$. Since, by construction, $0 \leq \beta_f(A) \leq \|f\| m(A)$, β_f is c.a. and absolutely continuous with respect to $m|\mathcal{G}$. Let b_f be its Radon Nikodým derivative: clearly $|b_f| \leq \|f\|$, m a.s. so that, replacing, if necessary, b_f by $b_f 1_{\{|b_f| \leq \|f\|\}}$, we have $b_f \in \mathcal{B}(\Omega, \mathcal{G})$. For $F \in \mathcal{I}$, $b_f 1_F \in \mathcal{I}$ and so

$$\begin{aligned} \xi(f 1_F) &= \xi_f(F) \\ &= \hat{\xi}_f(F) \\ &= \beta_f(F) \\ &= m(b_f 1_F) \\ &= \xi(b_f 1_F) \end{aligned}$$

For $f \in \mathcal{B}(\Omega, \mathcal{F})$, let

$$\xi(f|\mathcal{I}_\delta) = b_{f \vee 0} - b_{f \wedge 0}$$

(4.3) is therefore proved. Properties (1) and (2) are indeed obvious. ■

5. F.A. Expectation of Stochastic Processes

In this section we shall consider an ordinary filtered measurable space $(\Omega, \mathcal{F}; (\mathcal{F}_t : t \in \mathbb{R}_+))$ endowed with a f.a. probability λ, the filtration being right continuous and $\mathcal{F}_0$ finite. P shall denote the c.a. probability generated by λ (see remark 3). Let $\tilde{\Omega} \triangleq \Omega \times \mathbb{R}_+$ and $\tilde{\mathcal{F}} \triangleq \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$. Corollary 1 applies.

Let H be a pair $\left(\langle t_i \rangle_{i=0}^I, \langle F_i \rangle_{i=0}^I\right)$ of finite sequences such that (i) t_i is a finite stopping time, (ii) $0 = t_0 \leq t_1 \leq \ldots \leq t_I < \infty$, P a.s. and (iii) $F_i \in \mathcal{I}_{t_i}$. Let $\mathcal{H}$ be the family of all such pairs of sequences H. By a stochastic process we simply mean a function $K : \tilde{\Omega} \to \mathbb{R}$ such that $K_t \triangleq K(\cdot, t)$ is $\mathcal{F}_t$ measurable. A bounded process is an element of $\mathcal{B}\left(\tilde{\Omega}\right)$. If K is a stochastic process and τ a stopping time K^τ is the stopped process defined via $K_t^\tau = K_{t \wedge \tau}$. We assume that K_∞ exists (otherwise, replace K by K^t for $t \in \mathbb{R}_+$). For $H \in \mathcal{H}$, let $D_i^H(K) \triangleq 1_{F_i}\left(K_{t_{i+1}} - K_{t_i}\right)$ and

$$K_t^H = K_0 1_{F_0} + \sum_{i=0}^I D_i^H\left(K^t\right) \tag{5.1}$$

where $t_{I+1} = \infty$. Choose extensions $\hat{\lambda}_{t_{i+1} \wedge t}^\perp$ of $\lambda_{t_{i+1} \wedge t}^\perp$ to $(\Omega, \mathcal{F})$ such that $\hat{\lambda}_{t_i \wedge t}^\perp \leq \hat{\lambda}_{t_{i+1} \wedge t}^\perp \leq \hat{\lambda}_t^\perp$.

Let us begin by defining the following quantities

$$I_{\lambda^\perp}^H(K)_t \triangleq \lambda_t^\perp(K_0 1_{F_0}) + \sum_{i=0}^I \left(\hat{\lambda}_t^\perp - \hat{\lambda}_{t_{i+1} \wedge t}^\perp\right)\left(D_i^H(K)\right) \tag{5.2}$$

and

$$\left[\lambda^{\perp},K\right]_t^H \triangleq \sum_{i=0}^{I} \hat{\lambda}^{\perp}_{t_{i+1}\wedge t}\left(D_i^H\left(K\right)\right) \tag{5.3}$$

Then, clearly, $\hat{\lambda}_t^{\perp}\left(K_{\infty}^H\right) = I_{\lambda^{\perp}}^H\left(K\right)_t + \left[\lambda^{\perp},K\right]_t^H$. We aim at representing $I_{\lambda^{\perp}}^H\left(K\right)$ and $\left[\lambda^{\perp},K\right]^H$ as explicitly as possible. For what concerns (5.2) we have

$$\begin{aligned}
I_{\lambda^{\perp}}^H\left(K\right)_t &= \sum_{i=0}^{I}\left(\hat{\lambda}^{\perp}_{t_{I+1}\wedge t}-\hat{\lambda}^{\perp}_{t_{i+1}\wedge t}\right)\left(D_i^H\left(K\right)\right)+\lambda_t^{\perp}\left(K_0 1_{F_0}\right)\\
&= \sum_{j=1}^{I}\left(\hat{\lambda}^{\perp}_{t_{j+1}\wedge t}-\hat{\lambda}^{\perp}_{t_j\wedge t}\right)\left(\sum_{i=0}^{j-1}D_i^H\left(K\right)\right)+\lambda_t^{\perp}\left(K_0 1_{F_0}\right)\\
&= \sum_{j=0}^{I}\left(\hat{\lambda}^{\perp}_{t_{j+1}\wedge t}-\hat{\lambda}^{\perp}_{t_j\wedge t}\right)\left(K_{t_j}^H\right)+\lambda^{\perp}_{t_1\wedge t}\left(K_0 1_{F_0}\right)\\
&= \sum_{j=0}^{I}\left(\lambda^{\perp}_{t_{j+1}\wedge t}-\lambda^{\perp}_{t_j\wedge t}\right)\left(K_{t_j}^H 1_{\{t_j<t\}}\right)+\lambda^{\perp}_{t_1\wedge t}\left(K_0 1_{F_0}\right)\\
&= P\sum_{j=0}^{I}\left(A_{t_{j+1}}^t-A_{t_j}^t\right)K_{t_j}^H+P\left(A_{t_1}^t K_0 1_{F_0}\right)
\end{aligned}$$

i.e.

$$I_{\lambda^{\perp}}^H\left(K\right)_t = P\int_0^t H\cdot K^H dA + P\left(A_{t_1}^t K_0 1_{F_0}\right) \tag{5.4}$$

where $H\cdot X \triangleq \sum_{j=0}^{I} X_{t_j} 1_{]]t_j,t_{j+1}]]}$. It is quite natural to investigate convergence properties as H increases according to some convenient notion. To this end let us introduce the following:

Definition 3. *A Riemann sequence $\tilde{H}$ in $\mathcal{H}$ is any sequence $\langle H_n\rangle_{n\in\mathbb{N}}$, $H_n = \left(\langle t_i^n\rangle_{i=0}^{I^n}, \langle F_i^n\rangle_{i=0}^{I^n}\right) \in \mathcal{H}$ satisfying (i) for each t there exists a scalar $\delta_t^n \geq \bigvee_{i\leq I_n}\left(t_i^n\wedge t - t_{i-1}^n\wedge t\right)$ such that $\delta_t^n \downarrow 0$ and (ii) $\sum_{i=0}^{I^n} P\left(F_i^{nc}\right)\to 0$.*

Theorem 2. *Let $\mathcal{H}$ be defined as above and $I_{\lambda^{\perp}}^H\left(K\right)_t$ as in (5.2). Then, for any Riemann sequence $\tilde{H} = \langle H_n\rangle_{n\in\mathbb{N}}$ and for any bounded process K, the limit $I_{\lambda^{\perp}}\left(K\right)_t \triangleq \lim_n I_{\lambda^{\perp}}^{H_n}\left(K\right)_t$ exists and is equal to*

$$I_{\lambda^{\perp}}\left(K\right)_t = P\int_0^t K_- dA \tag{5.5}$$

Proof. $\left|K_{t_j}^H - K_{t_j}\right| \leq 2\left\|K\right\| 1_{\bigcup_{i\leq I} F_i^c}$ and, by (5.4),

$$\left|I_{\lambda^{\perp}}^{H_n}\left(K\right)_t - P\int_0^t K_- dA\right| \leq P\int_0^t\left(\left|H_n\cdot\left(K^{H_n}-K\right)\right| + \left|H_n\cdot K - K_-\right|\right)dA + \left|P\left(A_{t_1^n}^t K_0 1_{F_0^n}\right)\right|$$

The two terms on the right hand side are easily seen to converge to 0: the first by the definition of Riemann sequence and bounded convergence for stochastic integrals (see [13], theorem I.4.31, p. 46); the second one by right continuity of A. ■

In order to obtain an analogous result for $\left[\lambda^{\perp}, K\right]_t^H$ we need to be more explicit about H. Most difficulties arise here because we cannot assume at this stage that the sequence $\left\langle\left[\lambda^{\perp}, K^{H_n}\right]_t\right\rangle_{n\in\mathbb{N}}$ converges when $\tilde{H} = \langle H_n\rangle_{n\in\mathbb{N}}$ is a Riemann sequence in $\mathcal{H}$. This issue has an immediate solution if we require coherence, as in the next section. As a piece of notation replace the index H_n simply by n, let m_A be the measure on $\left(\tilde{\Omega}, \mathcal{P}\right)$ defined via the equation $m_A(B) = P\int 1_B dA$ and let LIM denote the Banach limit.

Theorem 3. *Let $\tilde{H} = \langle H_n\rangle_{n\in\mathbb{N}}$ be a Riemann sequence in $\mathcal{H}$ and K a bounded stochastic process. Then*

1. *The quantity*
$$\left[\lambda^{\perp}, K\right]_t^{\tilde{H}} \triangleq \operatorname*{LIM}_n \left[\lambda^{\perp}, K\right]_t^{H_n}$$
is well defined.

2. *There exists a continuous linear map*
$$f_{\tilde{H}}^{\perp} : \mathcal{B}\left(\tilde{\Omega}, \tilde{\mathcal{F}}\right) \to L^{\infty}\left(\tilde{\Omega}, \mathcal{P}, m_A\right)$$
such that
$$\left[\lambda^{\perp}, K\right]_t^{\tilde{H}} = P\int_0^t f_{\tilde{H}}^{\perp}(K)\, dA$$

3. *If K is càdlàg, then there exists a Riemann sequence $\tilde{H}_K$ such that*
$$\left[\lambda^{\perp}, K\right]_t^{\tilde{H}_K} = P\sum_r \Delta K_{\upsilon_r}^{\perp} \Delta A_{\upsilon_r} 1_{\{\upsilon_r\le t\}} + \Phi_{\tilde{H}_K}^{\perp}(\Delta K; t)$$
where $\Delta K_t^{\perp} \triangleq \lambda_t^{\perp}\left(\Delta K_t\middle|\mathcal{F}_{t-}\right)$ and $\langle\upsilon_r\rangle_{r\in\mathbb{N}}$ is the collection of predictable stopping times exhausting the accessible part of the jump times of K.

Proof. (claim 1). If $F_i \in \mathcal{I}_{t_i}$, by (3.7),

$$\begin{aligned}
\hat{\lambda}_{t_{i+1}\wedge t}^{\perp}\left(F_i \cap \{t_i \ge t\}\right) &= \hat{\lambda}_{t_i\wedge t}^{\perp}\left(F_i \cap \{t_i \ge t\}\right) \\
&\le \lambda_{t_i}^{\perp}\left(F_i \cap \{t_i \ge t\}\right) \\
&= 0
\end{aligned}$$

and since $F_i \cap \{t_i < t\} \in \mathcal{I}_{t_i\wedge t}$, we obtain from proposition 2,

$$\begin{aligned}
\hat{\lambda}_{t_{i+1}\wedge t}^{\perp}\left(D_i^H(K)\right) &= \hat{\lambda}_{t_{i+1}\wedge t}^{\perp}\left(1_{\{t_i<t\}} D_i^H(K)\right) \\
&= P\left\{\hat{\lambda}_{t_{i+1}\wedge t}^{\perp}\left(D_i^H(K)\middle|\mathcal{I}_{t_i\wedge t}\right) 1_{\{t_i<t\}}\left(A_{t_{i+1}}^t - A_{t_i}^t\right)\right\} \\
&= P\left\{\hat{\lambda}_{t_{i+1}\wedge t}^{\perp}\left(D_i^H(K)\middle|\mathcal{I}_{t_i\wedge t}\right)\left(A_{t_{i+1}}^t - A_{t_i}^t\right)\right\}
\end{aligned}$$

Let $f_n^{\perp}(K;t) \triangleq \sum_i \hat{\lambda}^{\perp}_{t^n_{i+1}\wedge t}\left(D^n_i(K)\middle|\mathcal{I}_{t^n_i\wedge t}\right)1_{]]t^n_i,t^n_{i+1}]]}$: $\left\|f_n^{\perp}\right\| \le 2\|K\|$ and

$$\left[\lambda^{\perp},K\right]^{H_n}_t = P\int_0^t f_n^{\perp}(K;t)\,dA \tag{5.6}$$

The first claim follows from (5.6) and the fact that $f_n^{\perp}(K;t)$ is bounded uniformly with respect to n.

(claim 2). For every $g \in L^1\left(\tilde{\Omega},\mathcal{P},m_A\right)$, the quantity $\mathrm{LIM}_n\, m_A\left(f_n^{\perp}(K;t)\,g\right)$ is bounded in absolute value by $2\|K\|\,\|g\|$ and, by linearity of the Banach limit, it may be interpreted as the value assigned to g by some linear bounded operator on $L^1\left(\tilde{\Omega},\mathcal{P},m_A\right)$. Then

$$\operatorname*{LIM}_n P\int f_n^{\perp}(K;t)\,g dA = P\int_0^t f_{\tilde{H}}^{\perp}(K;t)\,g dA$$

for some $f_{\tilde{H}}^{\perp}(K;t) \in L^{\infty}\left(\tilde{\Omega},\mathcal{P},m_A\right)$ such that $\left\|f_{\tilde{H}}^{\perp}(K;t)\right\| \le 2\|K\|$. $f_{\tilde{H}}^{\perp}(\cdot;t)$ is linear since $f_n^{\perp}(\cdot;t)$ is.

Take $F \in \mathcal{F}$, $u > t$, n such that $t+\delta^u_n < u$ and let

$$Y^n_i \triangleq P_{A^u_{t^n_{i+1}}-A^u_{t^n_i}}\left(1_F\,\frac{A^t_{t^n_{i+1}}-A^t_{t^n_i}}{A^u_{t^n_{i+1}}-A^u_{t^n_i}}\middle|\mathcal{F}_{t^n_i\wedge u}\right)$$

Based on $\left\{A^t_{t^n_{i+1}} > A^t_{t^n_i}\right\} \subset \{t^n_i < t\} \subset \left\{t^n_{i+1}\wedge u < t+\delta^u_n\right\}$, we can establish the following facts:
(i)

$$\begin{aligned}
P\left\{\hat{\lambda}^{\perp}_{t^n_{i+1}\wedge u}\left(D^n_i(K)\middle|\mathcal{I}_{t^n_i\wedge u}\right)1_F\left(A^t_{t^n_{i+1}}-A^t_{t^n_i}\right)\right\} &= P\left\{\hat{\lambda}^{\perp}_{t^n_{i+1}\wedge u}\left(D^n_i(K)\middle|\mathcal{I}_{t^n_i\wedge u}\right)\right.\\
&\qquad \left. 1_F\frac{A^t_{t^n_{i+1}}-A^t_{t^n_i}}{A^u_{t^n_{i+1}}-A^u_{t^n_i}}\left(A^u_{t^n_{i+1}}-A^u_{t^n_i}\right)\right\}\\
&= P\left\{\hat{\lambda}^{\perp}_{t^n_{i+1}\wedge u}\left(D^n_i(K)\middle|\mathcal{I}_{t^n_i\wedge u}\right)Y^n_i\right.\\
&\qquad \left. 1_{\{t^n_i<t\}}\left(A^u_{t^n_{i+1}}-A^u_{t^n_i}\right)\right\}\\
&= \hat{\lambda}^{\perp}_{t^n_{i+1}\wedge u}\left(D^n_i(K)\,Y^n_i 1_{\{t^n_i<t\}}\right)
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\le \sum_i\left(\lambda^{\perp}_{t^n_{i+1}\wedge u}-\lambda^{\perp}_{t^n_{i+1}\wedge t}\right)(t^n_i<t)\\
&= \sum_i\left(\lambda^{\perp}_{t^n_{i+1}\wedge u}-\lambda^{\perp}_{t^n_{i+1}\wedge t}\right)\left(t^n_i<t\le t^n_{i+1}\right)\\
&\le P\sum_i\left(\left(A_{t+\delta^u_n}-A_t\right)1_{\{t^n_i<t\le t^n_{i+1}\}}\right)\\
&\le P\left(A_{t+\delta^u_n}-A_t\right)
\end{aligned}$$

(iii) for $g_i \in L^\infty\left(\Omega, \mathcal{F}_{t_i^n \wedge u}, P\right)$

$$\begin{aligned}
0 &\leq \left| P\left\{ \left(P_{A^u_{t^n_{i+1}} - A^u_{t^n_i}}\left(F \middle| \mathcal{F}_{t^n_i \wedge u}\right) - Y^n_i \right) g_i \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) \right\} \right| \\
&\leq \|g_i\| P\left\{ \left(P_{A^u_{t^n_{i+1}} - A^u_{t^n_i}}\left(F \middle| \mathcal{F}_{t^n_i \wedge u}\right) - Y^n_i \right) \left(A^u_{t^n_{i+1}} - A^u_{t^n_i} \right) 1_{\{t^n_i < t\}} \right\} \\
&= \|g_i\| P\left\{ \left(A^u_{t^n_{i+1}} - A^t_{t^n_{i+1}} \right) 1_{\{t^n_i < t\} \cap F} \right\} \\
&\leq \|g_i\| P\left(\left(A_{t+\delta^u_n} - A_t \right) 1_{\{t^n_i < t \leq t^n_{i+1}\}} \right)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \left| P\left\{ \left(P_{A^u_{t^n_{i+1}} - A^u_{t^n_i}}\left(F \middle| \mathcal{F}_{t^n_i \wedge u}\right) - 1_F \right) g_i \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) \right\} \right| \\
&= \left| P\left\{ \left(P_{A^u_{t^n_{i+1}} - A^u_{t^n_i}}\left(F \middle| \mathcal{F}_{t^n_i \wedge u}\right) - 1_F \right) g_i \left(A^t_{t^n_{i+1}} - A^u_{t^n_{i+1}} \right) 1_{\{t^n_i < t \leq t^n_{i+1}\}} \right\} \right| \\
&\leq 2\|g_i\| P\left\{ 1_{\{t^n_i < t \leq t^n_{i+1}\}} \left(A_{t+\delta^n_u} - A_t \right) \right\}
\end{aligned}$$

i.e.

$$\begin{aligned}
\sum_i \left| P\left(1_F - Y^n_i\right) g_i \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) \right| &\leq 3P \sum_i 1_{\{t^n_i < t \leq t^n_{i+1}\}} g_i \left(A_{t+\delta^n_u} - A_t \right) \\
&\leq 3 \sup_i \|g_i\| P\left(A_{t+\delta^u_n} - A_t \right)
\end{aligned}$$

From $(i) - (iii)$, the fact that $Y^n_i 1_{\{t^n_i < t\}} \in \mathcal{F}_{t^n_i \wedge t}$ and right continuity of A we conclude:

$$\begin{aligned}
P\int_0^t f^\perp_{\tilde{H}}(K;t) 1_F dA &= \operatorname*{LIM}_n P \sum_i \hat{\lambda}^\perp_{t^n_{i+1} \wedge u}\left(D^n_i(K) \middle| \mathcal{I}_{t^n_i \wedge u} \right) 1_F \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) && \text{(def)} \\
&= \operatorname*{LIM}_n \sum_i \hat{\lambda}^\perp_{t^n_{i+1} \wedge u}\left(D^n_i(K) Y^n_i 1_{\{t^n_i < t\}} \right) && (i) \\
&= \operatorname*{LIM}_n \sum_i \hat{\lambda}^\perp_{t^n_{i+1} \wedge t}\left(D^n_i(K) Y^n_i 1_{\{t^n_i < t\}} \right) && (ii) \\
&= \operatorname*{LIM}_n \sum_i \hat{\lambda}^\perp_{t^n_{i+1} \wedge t}\left(D^n_i(K) \middle| \mathcal{I}_{t^n_i \wedge t} \right) Y^n_i \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) && \text{(def)} \\
&= \operatorname*{LIM}_n P \sum_i \hat{\lambda}^\perp_{t^n_{i+1} \wedge t}\left(D^n_i(K) \middle| \mathcal{I}_{t^n_i \wedge t} \right) 1_F \left(A^t_{t^n_{i+1}} - A^t_{t^n_i} \right) && (iii) \\
&= P\int_0^t f^\perp_{\tilde{H}}(K;u) 1_F dA && \text{(def)}
\end{aligned}$$

and, *a fortiori*, $P\int_s^t f^\perp_{\tilde{H}}(K;t) 1_F dA = P\int_s^t f^\perp_{\tilde{H}}(K;u) 1_F dA$ for each $F \in \mathcal{F}_s$. The two c.a. measures induced by the integrals $P\int f^\perp_{\tilde{H}}(K;t)\, dA$ and $P\int f^\perp_{\tilde{H}}(K;u)\, dA$ over the space $\left(\tilde{\Omega}, \mathcal{P}\right)$ coincide on the π system $\{F \times]s,t] : F \in \mathcal{F}_s,\ s < t\}$ and therefore coincide over the whole of $\left(\tilde{\Omega}, \mathcal{P}\right)$, i.e. $f^\perp_{\tilde{H}}(K;t) 1_{]0,t]} = f^\perp_{\tilde{H}}(K;u) 1_{]0,t]}$, m_A a.s.. We may then define $f^\perp_{\tilde{H}}(K)$ implicitly by setting

$$f^\perp_{\tilde{H}}(K) \triangleq \lim_n f^\perp_{\tilde{H}}(K;n) 1_{]0,n]}$$

Clearly, $f^{\perp}_{\tilde{H}}(K) \in L^{\infty}\left(\tilde{\Omega}, \mathcal{P}, m_A\right)$ and $\left[\lambda^{\perp}, K\right]^{\tilde{H}}_t = P\int_0^t f^{\perp}_{\tilde{H}}(K)\, dA$ which proves the second claim.

(claim 3). To prove the last claim we need to construct $\tilde{H}_K$ explicitly. To this end consider the following sequence $\tau^n = \langle \tau^n_i \rangle_{i=0}^{I_n}$ of stopping times: $\tau^n_0 = 0$ and

$$\tau^n_i = \inf\left\{t > \tau^n_{i-1} : \left|K_t - K_{\tau^n_{i-1}}\right| \vee \left|t - \tau^n_{i-1}\right| \geq 2^{-n}\right\} \tag{5.7}$$

with I_n such that $P\left(\tau^n_{I_n} < 2^n\right) < 2^{-n}$. These are indeed stopping times since K is *càdlàg* (actually, the filtration need not be complete, the stopping time τ^n_i is only P a.s. equal to a stopping time of the given filtration (see [13], lemma I.1.19, p. 5), but such distinction will not be relevant in what follows).

Choose $F^n_i \in \mathcal{I}_{\tau^n_i}$ such that $\sum_i P\left(F^{nc}_i\right) \to 0$: then, by construction, the sequence $\tilde{H}_K = \langle H_n \rangle_{n\in\mathbb{N}}$ with $H_n = \left(\langle \tau^n_i \rangle^{I^n}_{i=0}, \langle F^n_i \rangle^{I^n}_{i=0}\right)$ is a Riemann sequence in $\mathcal{H}$, as defined above. Let $\Delta^r X = \Delta X 1_{\{|\Delta X| > 2^{-r}\}}$ for each *càdlàg* process X. Since $\left|K_{\tau^n_{i+1}} - K_{\tau^n_i} - \Delta^n K_{\tau^n_{i+1}}\right| \leq 2^{-n+1}$, then

$$\begin{aligned}
\sum_i \left|\hat{\lambda}^{\perp}_{\tau^n_{i+1}\wedge t}\left(\left(D^n_i(K) - \Delta^r K_{\tau^n_{i+1}}\right) 1_{F^n_i \cap \{\tau^n_i < t\}}\right)\right| &\leq 2^{-n+1} \sum_i \hat{\lambda}^{\perp}_{\tau^n_{i+1}\wedge t}\left(F^n_i \cap \{\tau^n_i < t\}\right) \\
&\leq 2^{-n+1} \sum_i P\left(A^t_{\tau^n_{i+1}} - A^t_{\tau^n_i}\right) \\
&\leq 2^{-n+1} P(A_t)
\end{aligned}$$

Let $\bigcup_j [[\sigma_j]]$ be the thin set supporting the jumps of K (the existence of which follows from K being *càdlàg*), with the σ_j's stopping times and observe that we have $\{\Delta^n K \neq 0\} \subset \bigcup_i [[\tau^n_i]]$.

Since

$$\sum_i \hat{\lambda}^{\perp}_{\tau^n_{i+1}\wedge t}\left(\left|\Delta K_{\tau^n_{i+1}} - \Delta^n K_{\tau^n_{i+1}}\right| 1_{F^n_i \cap \{\tau^n_i < t\}}\right) \leq 2^{-n} P(A_t)$$

then

$$\begin{aligned}
\left[\lambda^{\perp}, K\right]^{\tilde{H}_K}_t &= \operatorname*{LIM}_n \sum_i \hat{\lambda}^{\perp}_{\tau^n_{i+1}\wedge t}\left(\Delta^n K_{\tau^n_{i+1}} 1_{F^n_i \cap \{\tau^n_i < t\}}\right) \\
&= \operatorname*{LIM}_n \sum_i \hat{\lambda}^{\perp}_{\tau^n_{i+1}\wedge t}\left(\Delta K_{\tau^n_{i+1}} 1_{F^n_i \cap \{\tau^n_i < t\}}\right) \\
&= \operatorname*{LIM}_n \sum_j \hat{\lambda}^{\perp}_{\sigma_j\wedge t}\left(\Delta K_{\sigma_j} 1_{D^n_j}\right)
\end{aligned}$$

where $D^n_j \triangleq \bigcup_i \left(\left\{\tau^n_{i+1} = \sigma_j\right\} \cap F^n_i\right)$. Splitting each jump time into its accessible and totally inaccessible part, we obtain

$$\left[\lambda^{\perp}, K\right]^{\tilde{H}_K}_t = \operatorname*{LIM}_n \sum_r \hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r}\right) + P\int_0^t f^{\perp}_{\tilde{H}_K}\left(\widetilde{\Delta K}\right) dA$$

where the stopping times υ_r are predictable, $D^n_r \triangleq \bigcup_i \left(\left\{\tau^n_{i+1} = \upsilon_r\right\} \cap F^n_i\right)$ and $\widetilde{\Delta K}$ is the totally inaccessible jump part of K. Let $\mathcal{I}_{\upsilon_r-} \triangleq \mathcal{I}_{\mathcal{F}_{\upsilon_r-}}$ and $G^m_r \in \mathcal{I}_{\upsilon_r-}$. Then

$$\begin{aligned}
\hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r\cap G^m_r}\right) &= \hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r\cap G^m_r\cap\{\upsilon_r\le t\}}\right) \\
&= P\left\{\hat{\lambda}^{\perp}_{\upsilon_r}\left(\Delta K_{\upsilon_r}|\mathcal{I}_{\upsilon_r-}\right) 1_{D^n_r\cap G^m_r\cap\{\upsilon_r\le t\}}\Delta A_{\upsilon_r}\right\}
\end{aligned}$$

which converges to $P\left\{\hat{\lambda}^{\perp}_{\upsilon_r}\left(\Delta K_{\upsilon_r}|\mathcal{I}_{\upsilon_r-}\right) 1_{\{\upsilon_r\le t\}}\Delta A_{\upsilon_r}\right\}$ as $P\left(G^m_r \cap D^n_r\right) \to 1$. But then we eventually obtain

$$\begin{aligned}
\left[\lambda^{\perp}, K\right]^{\tilde{H}_K}_t &= \operatorname*{LIM}_n \lim_m \sum_r \hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r\cap G^m_r}\right) + \operatorname*{LIM}_n \lim_m \sum_r \hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r\cap G^{mc}_r}\right) \\
&\quad + P\int_0^t f^{\perp}_{\tilde{H}_K}\left(\widetilde{\Delta K}\right) dA \\
&= P\sum_r \Delta K^{\perp}_{\upsilon_r}\Delta A_{\upsilon_r} 1_{\{\upsilon_r\le t\}} + \Phi^{\perp}_{\tilde{H}_K}\left(\Delta K\right)
\end{aligned}$$

where $\Phi^{\perp}_{\tilde{H}_K}\left(\Delta K\right) \triangleq \mathrm{LIM}_n \lim_{k,m} \sum_r \hat{\lambda}^{\perp}_{\upsilon_r\wedge t}\left(\Delta K_{\upsilon_r} 1_{D^n_r\cap G^{mc}_r}\right) + P\int_0^t f^{\perp}_{\tilde{H}_K}\left(\widetilde{\Delta K}\right) dA$. ■

6. λ Predictable Processes

It would be desirable to characterize explicitly the quantity $\left[\lambda^{\perp}, K\right]^{\tilde{H}}$ further than for the class of continuous processes – for which clearly $\left[\lambda^{\perp}, K\right]^{\tilde{H}_K}_t = 0$. This is a difficult problem for which only partial answers are available. To this end let us introduce the following definition

Definition 4. *A stopping time σ is λ predictable if it admits a P announcing sequence $\langle\sigma^r\rangle_{r\in\mathbb{N}}$ that satisfies $\lim_n \lambda\left(\sigma - \sigma^r \le 2^{-n}\right) = 0$. A process K is λ predictable whenever its jump times are λ predictable and K_τ is $\mathcal{F}_{\tau-}$ measurable for any λ predictable stopping time τ.*

The condition introduced would clearly be true were the measure λ c.a.. It actually means that the full predictive content of the announcing sequence for σ is obtained even without actually going all the way up to σ itself. A special case that meets definition 4 is the one in which the jump times σ_j ake on a finite number of values.

Theorem 4. *Let K be a bounded, càdlàg process. Then:*

1. *If the jump times of K are λ predictable*

$$\left[\lambda^{\perp}, K\right]^{\tilde{H}_K}_t = P\sum_r \Delta K^{\perp}_{\upsilon_r\wedge t}\Delta A_{\upsilon_r\wedge t} \tag{6.1}$$

so that $\mathrm{LIM}_n\, \lambda^{\perp}\left(K^{H_{K,n}}_t\right) = P\int_0^t K^{\perp} dA$;

2. If K is λ predictable

$$\left[\lambda^{\perp}, K\right]_t^{\tilde{H}_K} = P\sum_r \Delta K_{\upsilon_r\wedge t}\Delta A_{\upsilon_r\wedge t} \tag{6.2}$$

so that $\mathrm{LIM}_n\, \lambda^{\perp}\left(K_t^{H_{K,n}}\right) = P\int_0^t KdA$.

Proof. Clearly under the current assumptions, $\widetilde{\Delta K} = 0$. Consider the case in which each υ_r is announced by the sequence $\langle \upsilon_r^k\rangle_{r\in\mathbb{R}}$, where $\upsilon_r^k < \upsilon_r$ on $\{\upsilon_r > 0\}$ and $\upsilon_r^k \uparrow \upsilon_r$, P a.s.. Exploiting the fact that (i) $\mathcal{F}_{\upsilon_r-} = \bigvee_k \mathcal{F}_{\upsilon_r^k}$ (in fact $\{\upsilon_r > s\} = \bigcup_r \{\upsilon_r^k > s\}$), (ii) $\hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(\{\tau_i^n \geq \upsilon_r^k\}\cap F_i^n\right) \leq \lambda^{\perp}_{\tau_i^n}\left(F_i^n\right) = 0$, (iii) $\tau_{i+1}^n \leq \tau_i^n + 2^{-n}$ (by definition (5.7)) and lemma 3, we obtain (putting $\lambda^{\perp}_{\upsilon_r\wedge t-} \triangleq \lambda^{\perp}_{\mathcal{F}_{\upsilon_r\wedge t-}}$)

$$\begin{aligned}
\lambda^{\perp}_{\upsilon_r\wedge t-}\left(D_r^n\right) &= \lim_k \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(D_r^n\right)^1 \\
&= \lim_k \sum_i \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(\tau_{i+1}^n = \upsilon_r; F_i^n\right) \\
&\leq \lim_k \sum_i \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(\tau_{i+1}^n = \upsilon_r; \upsilon_r^k > \tau_i^n\right) \\
&\leq \lim_k \sum_i \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(\tau_{i+1}^n = \upsilon_r; \upsilon_r - \upsilon_r^k \leq 2^{-n}\right)
\end{aligned}$$

Let $G\subset\mathbb{N}$,

$$J_n^k \triangleq \left\{\omega : \left(\upsilon_r - \upsilon_r^k\right)(\omega) \in \left]2^{-n-1}, 2^{-n}\right]\right\}$$

and $m_k(G) \triangleq \left(\hat{\lambda}^{\perp}_{\upsilon_r\wedge t-} - \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\right)\left(\bigcup_{i\in G} J_i^k\right)$: clearly, $m_k \in ba\left(\mathbb{N}, 2^{\mathbb{N}}\right)_+$. Lemma 3 implies

$$0 = \lim_k \left(\lambda^{\perp}_{\upsilon_r\wedge t-} - \lambda^{\perp}_{\upsilon_r^k\wedge t}\right)(\Omega) \geq \lim_k m_k(G)$$

In other words, for each $G\subset\mathbb{N}$ the sequence $\langle m_k(G)\rangle_{k\in\mathbb{N}}$ converges to 0. By virtue of Phillips lemma (see [3], theorem 8.3.3, p. 206), we can conclude that for each $\varepsilon > 0$ there exists an integer k_ε^1 such that $k > k_\varepsilon^1$ implies $m_k([n, m[) < 2^{-1}\varepsilon$ for all $n < m$. Let, for each $\varepsilon > 0$, k_ε^2 be that integer such that $k > k_\varepsilon^2$ implies $m_k([n_0, \infty[) < 2^{-1}\varepsilon$ and put $k_\varepsilon = k_\varepsilon^1 \vee k_\varepsilon^2$. Obviously, when $k > k_\varepsilon$ and for arbitrary n, we have

$$\begin{aligned}
\left(\hat{\lambda}^{\perp}_{\upsilon_r\wedge t-} - \hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\right)\left(\upsilon_r^k \geq \upsilon_r - 2^{-n}\right) &= m_k([n, \infty[) \\
&\leq m_k([n_0, \infty[) + m_k([n_0\wedge n, n_0[) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

i.e., $\hat{\lambda}^{\perp}_{\upsilon_r^k\wedge t}\left(\upsilon_r^k \geq \upsilon_r - 2^{-n}\right)$ converges uniformly with respect to n. Therefore, given that

$$\hat{\lambda}^{\perp}_{\upsilon_r\wedge t-}\left(\upsilon_r^k \geq \upsilon_r - 2^{-n}; \upsilon_r = \tau_{i+1}^n\right) \leq \hat{\lambda}^{\perp}_t\left(\upsilon_r = \tau_{i+1}^n\right)$$

and that $\sum_r \hat{\lambda}^{\perp}_{\upsilon_r \wedge t-}\left(\upsilon_r = \tau^n_{i+1}\right) \leq \hat{\lambda}^{\perp}_t \left(\bigcup_i \{\upsilon_r = \tau^n_{i+1}\}\right) \leq \hat{\lambda}^{\perp}_t (\Omega)$, we can appeal to dominated convergence and obtain

$$\begin{aligned}
\operatorname*{LIM}_n \sum_r \lambda^{\perp}_{\mathcal{F}_{\upsilon_r \wedge t-}}(D^n_r) &\leq \sum_i \lim_n \sum_r \hat{\lambda}^{\perp}_{\upsilon_r \wedge t-}\left(\upsilon^k_r \geq \upsilon_r - 2^{-n}; \upsilon_r = \tau^n_{i+1}\right) \\
&= \sum_i \sum_r \lim_n \hat{\lambda}^{\perp}_{\upsilon_r \wedge t-}\left(\upsilon^k_r \geq \upsilon_r - 2^{-n}; \upsilon_r = \tau^n_{i+1}\right) \\
&= \sum_i \sum_r \lim_n \lim_k \hat{\lambda}^{\perp}_{\upsilon^k_r \wedge t}\left(\upsilon^k_r \geq \upsilon_r - 2^{-n}; \upsilon_r = \tau^n_{i+1}\right) \\
&= \sum_i \sum_r \lim_k \lim_n \hat{\lambda}^{\perp}_{\upsilon^k_r \wedge t}\left(\upsilon^k_r \geq \upsilon_r - 2^{-n}; \upsilon_r = \tau^n_{i+1}\right) \\
&= 0
\end{aligned}$$

The second claim is a consequence of the first one and of proposition 2. ■

Observe that (6.2) coincides with the representation of $P[K, A]$ whenever K is a predictable semimartingale and A a predictable process of finite variation. In case K is not predictable we recall that $P[K, A] = P \sum_r \Delta K^p_{\upsilon_r} \Delta A_{\upsilon_r}$ where $\cdot^p$ is the predictable projection of K. (6.1) is then the exact analogous of this formula in which $\Delta K^{\perp}_{\upsilon_r}$ should be interpreted as the "orthogonal" predictable projection, i.e. the predictable projection according to $\lambda^{\perp}$.

7. Coherence

Let's now return to our original problem as outlined in the introduction, the setting being as in the previous section. Let $\mathcal{S}$ be the set of bounded stopping times of the filtration and $\mathbb{K}$ a convex subset of $\mathcal{B}\left(\tilde{\Omega}\right)$ such that for each $K \in \mathbb{K}$

(i) K_t is $\mathcal{F}_t$ measurable for each $t \in \mathbb{R}_+$,

(ii) for each $\omega \in \Omega$, $K_t(\omega)$ is right continuous and

(iii) for each $K \in \mathbb{K}$ there exists $T \in \mathbb{R}_+$ such that $K = K^T$.

We define

$$\mathcal{K} = \left\{ \sum_{i=1}^{I} c_i \left(K^i_{\tau_{i+1}} - K^i_{\tau_i}\right) : c_i \in \mathcal{L}(\mathcal{F}_{\tau_i}), \tau_{i+1} \geq \tau_i, \tau_i \in \mathcal{S}, K^i \in \mathbb{K} \right\}$$

where $\mathcal{L}(\mathcal{F})$ are the simple, $\mathcal{F}$ measurable functions. Clearly $\mathcal{K} \subset \mathcal{B}(\Omega, \mathcal{F})$ and the elements of $\mathcal{K}$ are easily seen to be simple stochastic integrals (computed at ∞) with respect to processes in $\mathbb{K}$. We define coherence accordingly, i.e.

$$\mathcal{K} \cap \mathcal{B}(\Omega, \mathcal{F})_+ = \{0\} \tag{7.1}$$

Let λ denote the f.a., probability measure separating $\mathcal{K}$ and $\mathcal{B}(\Omega, \mathcal{F})$ and P the c.a. probability it induces: since $\mathcal{K}$ is a vector space, $\lambda[\mathcal{K}] = 0$. The linear structure of $\mathcal{K}$ implies $\lim_n \lambda^{\perp}(k_n) = -\lim_n \lambda^c(k_n) = -\lambda^c(k) = \lambda^{\perp}(k)$. In other words, $\left[\lambda^{\perp}, K\right]_t^{H_n}$ converges and its limit does not depend on the sequence $\tilde{H}$ in $\mathcal{H}$ and, then, neither does $f_{\tilde{H}}^{\perp}(K)$ – which we denote accordingly by $f(K)$.

Proposition 3. *Every stochastic process $K \in \mathbb{K}$ is a P special semimartingale.*

Proof. For each $K \in \mathbb{K}$ let $\phi(K)_t \triangleq K_{t-} + f(K)_t - P(K_\tau | \mathcal{F}_t)$. If τ is any stopping time, from theorems 2 and 3 and proposition I.3.14 in [13] and $K_\tau \in \mathcal{K}$, it follows that

$$
\begin{aligned}
0 &= \lambda(K_\tau) \\
&= P\left\{M_\tau K_\tau - A_\tau K_\tau + \int_0^\tau (K_- + f(K))\, dA\right\} \\
&= P\left\{M_\tau K_\tau + \int_0^\tau \phi(K)\, dA\right\}
\end{aligned}
$$

The process $Y_t \triangleq M_t K_t + \int_0^t \phi(K)\, dA$ is adapted and right continuous, starts at 0 and admits a terminal variable (by definition of $\mathbb{K}$): it is therefore a P uniformly integrable martingale (see [13], lemma I.1.44, p. 11). Because $\int_0^t \phi(K)\, dA$ is a process of integrable variation, MK is a special P semimartingale i.e. K is a P special semimartingale. ∎

As a last implication we have the following result establishing conditions under which the separating measure can be taken to be c.a..

Corollary 2. *Let $Q \gg P$. Assume that the processes in $\mathbb{K}$ are λ predictable and that $\mathcal{F}_0$ is finite. Then:*

1. *$K \in \mathbb{K}$ is a uniformly integrable martingale under P so that $P[\mathcal{K}] = 0$.*
2. *If all processes $K \in \mathbb{K}$ are Q semimartingales, then there exists a positive Q local martingale Z^Q such that $Z_0^Q = 1$ and that $Z^Q K$ is a local martingale under Q.*

Proof. If τ is any stopping time $K_\tau \in \mathcal{K}$. Given that $K \in \mathbb{K}$ is a semimartingale, we obtain by Ito's lemma

$$
\begin{aligned}
\lambda(K_\tau) &= Q\left(X_\tau^Q K_\tau\right) + Q\int_0^\tau K dA^Q \\
&= Q\left\{\int_0^\tau X_-^Q dK + \int_0^\tau K_- dM^Q + \left[M^Q, K\right]_\tau\right\} \\
&= Q\left\{\int_0^\tau X_-^Q dK + \left[M^Q, K\right]_\tau\right\}
\end{aligned}
$$

By the same argument invoked in the proof of proposition 3, $Y^Q \triangleq \int X_-^Q dK + [M^Q, K]$ is then a Q uniformly martingale. Observe that $\left\{X_-^Q = 0\right\} \subset \left\{X^Q = 0\right\}$ Q a.s. and that, therefore, $\int_{\left\{X_-^Q=0\right\}} dM^Q = -\int_{\left\{X_-^Q=0\right\}} dA^Q$: the former is then a local martingale while the latter a predictable process of finite variation and by the Doob Meyer decomposition both must be a.s. equal to 0. Define then

$$Z^Q = \mathcal{E}\left(\int \frac{1}{X_-^Q} dM^Q\right) \tag{7.2}$$

From $X^Q \geq 0$ it follows that $\frac{\Delta M^Q}{X_-^Q} \geq -1$ a.s.: Z^Q is then a positive local martingale (see [13], theorem 4.61, p. 59). Let $\mathcal{L}\left(Z^Q\right) = \int \left(X_-^Q\right)^{-1} dM^Q$, the stochastic logarithm of Z^Q. By Ito's lemma,

$$\begin{aligned} Z^Q K &= \int K_- dZ^Q + \int Z_-^Q dK + \left[Z^Q, K\right] \\ &= \int K_- dZ^Q + \int Z_-^Q dK + \int Z_-^Q d\left[\mathcal{L}\left(Z^Q\right), K\right] \\ &= \int K_- dZ^Q + \int Z_-^Q X_-^Q dY^Q \end{aligned}$$

We conclude that $Z^Q K$ is a Q local martingale. Clearly $Z^P = 1$, which completes the proof. ■

Remark 4. *Let $U_t = \frac{dP_t}{dQ_t}$. Then it should be obvious from the first claim of the preceding corollary that the uniformly integrable martingale U satisfies the second claim. From (7.2) and $M^Q = U$, $X^Q = XU$ we obtain $Z^Q = \mathcal{E}\left(\int \frac{1}{X_- U_-} dU\right) = \mathcal{E}\left(\int \frac{1}{X_-} d\mathcal{L}(U)\right)$. Then Z^Q and U differ unless $X = 1$, P a.s. i.e. unless λ is c.a..*

For the case covered by the preceding corollary, a c.a. separating measure can be obtained, by replacing λ by P. However this is not harmless since we cannot draw the conclusion that λ vanishes on P null sets. The existence of a "martingale" measure such as P in the above corollary (or of a state price density such as Z), is a crucial step in mathematical finance.

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