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Information filtering via Iterative Refinement

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Abstract. – With the explosive growth of accessible information, expecially on the Internet, evaluation-based filtering has become a crucial task. Various systems have been devised aiming to sort through large volumes of information and select what is likely to be more relevant. In this letter we analyse a new ranking method, where the reputation of information providers is determined self-consistently.

Introduction. – The study of complex networks and of some dynamical processes taking place on these structures has recently attracted a great deal of attention in the physics community [1–4]. The importance of technological networks, such as the Internet, lies mostly in the increased communication capabilities [5, 6], which make information progressively easier to produce and distribute. As storage and transmission costs continue to drop, an overabundance of information threatens to overwhelm its recipients. It is, therefore, crucial to process information in order to present a user only the one that answers best her requests [7].

An important aspect of information filtering regards *scoring systems* in the World Wide Web [8, 9]. They collect evaluations and aggregate them into published scores that are meaningful to the final user. This embraces many different instances, ranging from commercial websites, where buyers evaluate sellers (Ebay, Amazon, etc.) to new generation search engines (Google, Yahoo, etc.), and opinion websites, where people evaluate objects (Epinions, Tailrank, etc.) Since the evaluators carry different expertise, it is important to estimate how accurate a given vote may be and to weight it accordingly. This can be done through the use of raters' reputations [10]. *Reputation* summarises one's past behaviour and has always been used to bear the risk of interacting with strangers. The Internet, while enhancing such a risk, brings in the possibility to find its antidotes [11]. Since nobody knows *a priori* who are the honest and competent evaluators, in fact, online scoring systems often include some measure of their past performance. This gives users an indication on how trustworthy a given piece of information is supposed to be. An expert of the field would probably obtain a high reputation; experts' votes should then count more when aggregating the scores. While reputation is usually obtained by asking users supplementary evaluations about other users, the procedure of *Iterative Refinement* (IR), which can be shown to outperform naive methods [12], does not require to explicitly rate the raters.

The aim of this letter is to study, in a generalised model, the IR method's dependence on the relevant parameters, illustrate the subtle issues in its mathematical underpinning and elaborate on distortions generated by different kinds of cheating. Prior to describing the major focus of this work, we will briefly state the model and define some notations.

Model and algorithm. – To describe our approach in the simplest manner, let us consider N raters evaluating M objects, which can be books, movies or even other raters. Each object l has an intrinsic quality Q_l and each rater i has an intrinsic judging power $1/\sigma_i^2$. Let x_{il} be a random variable representing the rating given by rater i to object l. Intrinsic qualities and judging powers are defined by the first two moments of its distribution:

$$\langle x_{il} \rangle = \mu_{il} = Q_l + \Delta_{il}, \tag{1}$$

$$\left\langle (x_{il} - \mu_{il})^2 \right\rangle = \sigma_i^2 \ , \tag{2}$$

where Δ_{il} is the systematic error of agent i towards object l. Expectation values are taken over the distribution of x_{il}. They can be regarded as ensemble averages, obtained if the evaluations were to be performed infinitely many times. Our aim is then to extract the quality of each object from a single set $\{x_{il}\}$ of evaluations. We thus estimate the intrinsic quality Q_l of object l by a weighted average of the received votes

$$q_l = \sum_{i=1}^{N} f_i x_{il} \ ; \tag{3}$$

the inverse judging power σ_i^2 of rater i is estimated by the sample variance V_i

$$V_i = \frac{1}{M} \sum_{l=1}^{M} (x_{il} - q_l)^2 \ . \tag{4}$$

The unnormalised weights ω_i take the general form

$$\omega_i = V_i^{-\beta} \ , \tag{5}$$

with $\beta \geq 0$ and $f_i = \omega_i / \sum_j \omega_j$. As such, ω_i decreases when V_i increases because rater i has a lower judging power and should be given less credit. We will consider scenarios where β equals 1 or 1/2. The case $\beta = 1/2$, in fact, exhibits scale-changing and translational invariance because q_l becomes a sum of dimensionless random variables; the case $\beta = 1$ corresponds to optimal weights, as explained later in the section *No systematic errors.*

The IR algorithm allows to solve eqs. (3)-(5), thus estimating Q_l and σ_i, via the following recursive procedure: I) Without additional information, set $\omega_i = 1/N \ \forall i = 1, 2, ..., N$. II) Estimate q_l with eq. (3). III) Estimate V_i with eq. (4) and plug it in eq. (5) to find the weights. IV) Repeat from step II). Numerical simulations show that this process converges to the minimum of the cost function $E(\{q_l\}) = \sum_i \left[\sum_l (x_{il} - q_l) V_i^{-\beta} \right]^2$ much faster than other conventional methods.

Analytical approach. – Equation (2) implies that the random variable $(x_{il} - \mu_{il})^2$ has mean σ_i^2 and variance $m_i^2 \sigma_i^4$, which is determined by the distribution of x_{il}; in particular, $m_i^2 = 3$ if the distribution of votes is itself Gaussian. Let us define the variable $\gamma_{ij} =$

$\frac{1}{M}\sum_l (x_{il}-Q_l)(x_{jl}-Q_l)$; provided that x_{il} has finite moments of, at least, order 4, in the large-M limit one obtains

$$\gamma_{ij} \to \sigma_i^2\delta_{ij} + \overline{\Delta_i\Delta_j} + \frac{1}{\sqrt{M}}(e_{ij} + \overline{\Delta_i}h_j + \overline{\Delta_j}h_i) \ . \tag{6}$$

Here the overlined quantities represent averages over the M items, $\overline{x_i} = \frac{1}{M}\sum_l x_{il}$. The Gaussian random variables e_{ij} and h_i have mean zero and variances $\mathrm{var}(e_{ij}) = m_{ij}^2\sigma_i^2\sigma_j^2$ and $\mathrm{var}(h_i) = \sigma_i^2$, where $m_{ij}^2 = 1+\delta_{ij}(m_i^2-1)$. In the following we shall use the notation $g_i = e_{ii}$.

Equation (6) has to be interpreted in probability, as prescribed by the Central Limit Theorem. In its derivation we have further assumed that raters are independent; in fact, the correlation among the variables e_{ij} of different indices diminishes as M increases. If $M \gg 1$, the random variables $\{e_{ij}\}$ are effectively independent, with the first visible triangular correlation of order $1/M^2$ or smaller. From counting the degrees of freedom associated with random numbers, it is desirable to have $M \geq (N+1)/2$. Equation (6) forms the basis of our analytical pursuit in the later development.

The performance of the IR method can be stated by measuring the following mean-squared errors:

$$d_{q_l} = \left\langle (q_l - Q_l)^2 \right\rangle = \left\langle (s_l + \tilde{\Delta}_l)^2 \right\rangle \tag{7}$$

$$d_{\sigma_i} = \left\langle (V_i - \sigma_i^2)^2 \right\rangle = \mathsf{Var}(V_i) + \mathrm{Bias}^2(V_i) \ , \tag{8}$$

with $\mathrm{Bias}(V_i) \equiv \langle V_i\rangle - \sigma_i^2$. In eq. (7) we have separated the systematic error part, making use of the variables $\tilde{\Delta}_l = \sum_i f_i\Delta_{il}$ and $s_l = \sum_j (y_{jl}-Q_l)f_j$, with $y_{il} = x_{il} - \Delta_{il}$. Equations (1), (2) guarantee that the first two moments of $(y_{il}-Q_l)$ are independent of index l, therefore $\left\langle s_l^2\right\rangle = \frac{1}{M}\sum_{l=1}^M \left\langle s_l^2 \right\rangle$. This permits us to employ eq. (6) to obtain

$$\left\langle s_l^2 \right\rangle = \sum_i \sigma_i^2 \left\langle f_i^2\right\rangle + \sum_{i,j}\left\langle f_i f_j \frac{e_{ij}}{\sqrt{M}}\right\rangle . \tag{9}$$

The variable s_l becomes Gaussian in the large-N limit, as long as the weights f_j are fixed and satisfy the Lindeberg condition [13]. However, such inference cannot be drawn easily because the weights and the estimated q_l are tangled up in eqs. (3)-(5). The standard deviation of s_l can, nevertheless, be calculated. The general problem of finding intrinsic values from completely distorted votes is not solvable. In fact, even if one disposed of an infinite number of raters and evaluations, the estimator (3) of Q_l would always be biased of the amount $\langle\tilde{\Delta}_l\rangle$. We shall, in the following, focus our attention on three particular cases of special interest.

No systematic errors. – When $\Delta_{il} = 0\ \forall i, l$, raters are impartial but possess different judging powers. In order to obtain the best-quality estimator one can minimise the mean-squared error $d_q(\{\omega_k\})$ of (3) with respect to the ω_i's. This gives the optimal weights [14], $\beta = 1$ in (5), with minimal $d_q(\{1/\sigma_k^2\}) = 1/\sum_i \sigma_i^{-2}$. Since the law of large numbers guarantees the convergence of $d_q(\{1/\sigma_k\})$ to zero for large N, the same must obviously be true for optimal weights. Unfortunately, it is not possible to state that the choice $\beta = 1$ is optimal if the σ_i^2's are not known in advance. Although the convergence of $q_l \to Q_l$ for $N \to \infty$ is guaranteed, the small deviation $|q_l - Q_l|$ due to finite N will propagate to the estimate of σ_i^2 and render $V_i \neq \sigma_i^2$, even when $M \to \infty$. A recursive procedure allows to calculate the expectation values for $\langle f_i\rangle$; using eq. (6), it is straightforward to show that

$$V_i = \left[\sigma_i^2 + \frac{g_i}{\sqrt{M}}\right](1-2f_i) + \sum_j f_j^2\left(\sigma_j^2 + \frac{g_j}{\sqrt{M}}\right) + 2\sum_{j<k} f_j f_k \frac{e_{jk}}{\sqrt{M}} - 2\sum_{j;j\neq i} f_j\frac{e_{ij}}{\sqrt{M}}. \tag{10}$$

Now we use $\omega_i = V_i^{-\beta}$ and, after iterative substitutions, we may express ω_i in terms of σ_i's and random variables $\{e_{ij}\}$. One may then compute f_i and plug it in eqs. (7)-(9). Let us define $G(b) \equiv \frac{1}{N}\sum_i m_i^2 \sigma_i^{-b}$ and denote by angular brackets a simple average over the raters $\langle y \rangle = \frac{1}{N}\sum_i y_i$. Equipped with this formalism, we perform tedious but straightforward calculations to obtain the following asymptotic expansions, for $M, N \to \infty$, to the first two dominating orders:

$$\left\langle (q-Q)^2 \right\rangle \simeq \frac{1}{N\langle \sigma^{-2\beta}\rangle^2}\left[\left\langle \frac{1}{\sigma^{4\beta-2}} \right\rangle + \frac{\beta \mathcal{C}_1}{N} + \frac{\beta \mathcal{C}_2}{M}\right], \tag{11}$$

$$\text{Bias}(V_i) \simeq \left\langle (q-Q)^2 \right\rangle - \frac{2\sigma_i^{2-2\beta}}{N\langle \sigma^{-2\beta}\rangle}\left[1 + \frac{\beta \mathcal{D}_1}{N} + \frac{\beta \mathcal{D}_2}{M}\right], \tag{12}$$

$$\mathsf{Var}(V_i) \simeq m_i^2 \frac{\sigma_i^4}{M}\left[1 + \frac{(\beta+1)\sigma_i^{-2\beta}}{N\langle \sigma^{-2\beta}\rangle}\right], \tag{13}$$

with complicated constant coefficients([1]). These expressions simplify considerably when taking the limit $\beta = 1/2$ and $\beta = 1$. For instance, eq. (12) takes the forms $\text{Bias}_{\beta=1}(V_i) \simeq -1/\left(N\left\langle \sigma^{-2}\right\rangle\right)$ and $\text{Bias}_{\beta=1/2}(V_i) \simeq 1/(N\langle 1/\sigma\rangle^2) - 2\sigma_i/\left(N\langle \sigma^{-1}\rangle\right)$. The analytical solution allows one to find an *unbiased estimator* for σ_i^2 —up to $\mathcal{O}(1/N^2, 1/NM)$. In applications we may use eq. (4) as an estimator of σ^2 to evaluate $\text{Bias}(V_i)$ and redefine the weights as $\omega_i = 1/(V_i - \text{Bias}(V_i))$. Since we have here $d_{q_l} = s_l$, it suffices to plug eqs. (11)-(13) in eq. (9) to find theoretical expressions for the mean-squared errors. They are shown to match numerical simulations in figs. 1 and 2.

In fig. 1, the mean-squared error of the variance $d_\sigma = \frac{1}{N}\sum_i d_{\sigma_i}$ is plotted against M in log-log scale. Our theoretical prediction becomes very good as soon as $M > 10$. Diamonds and filled circles show simulation results of the IR method where the biased estimator of the variance has been corrected by recursive use of eq. (12): the plateau reached by d_σ for large M disappears because the accuracy of the prediction can be thus improved by two orders of magnitude. The mean-squared error of the quality $d_q = \frac{1}{M}\sum_l d_{q_l}$, on the other hand, can never vanish for large M when N is finite. This is shown in the inset of fig. 1, while the dependence of d_q on N is reported in fig. 2. We have also plotted therein, as a dotted line, the behaviour of the same quantity when the estimator of q_l is just the average unweighted vote received by item l. This illustrates how IR is able to reduce the error. A comparison between the two weighting schemes shows that $\omega_i = 1/V_i$ performs almost always better than $\omega_i = 1/\sqrt{V_i}$. The inset of fig. 2 shows d_σ *vs.* N; the plateau, which is the same for $\beta = 1/2$ and 1, vanishes for $M \to \infty$ when corrected for the bias as before.

Camouflage. – Let us now restart from the general problem of eqs. (1), (2). The case we want to analyse here is that of ratings affected by systematic errors that depend on the rater but not on the ratee, $\Delta_{il} = \Delta_i\ \forall l$. Such a fictitious distortion is instructive to study analytically and can be easily generalised to more interesting cases. In fact, as it alters a rater's scale of evaluation but not the ranking of her preferences, it can serve as a basis to study systems where agents are only asked to sort a set of items in order of increasing quality.

If one knew the values of Δ_i for all i, one could find the optimal weights $\{\omega_k^*\}$ proceeding as described in the absence of systematic errors. Upon minimisation of $d_q(\{\omega_k\})$ with respect

([1])They are given by $\mathcal{C}_1 = \frac{4\langle \sigma^{2-6\beta}\rangle}{\langle \sigma^{-2\beta}\rangle} + 2\frac{\langle \sigma^{2-4\beta}\rangle^2 \langle \sigma^{-2(\beta+1)}\rangle}{\langle \sigma^{-2\beta}\rangle^3} - 6\frac{\langle \sigma^{-4\beta}\rangle\langle \sigma^{2-4\beta}\rangle}{\langle \sigma^{-2\beta}\rangle^2}$, $\mathcal{C}_2 = 4\langle \sigma^{2-4\beta}\rangle + (2\beta - 1)G(4\beta-2) - \frac{(\beta+1)G(2\beta)\langle \sigma^{2-4\beta}\rangle}{\langle \sigma^{-2\beta}\rangle}$, $\mathcal{D}_1 = \frac{2\sigma_i^{-2\beta}}{\langle \sigma^{-2\beta}\rangle} - \frac{\langle \sigma^{2-4\beta}\rangle \sigma_i^{-2}}{\langle \sigma^{-2\beta}\rangle^2} - \frac{2\langle \sigma^{-4\beta}\rangle}{\langle \sigma^{-2\beta}\rangle^2} + \frac{\langle \sigma^{2-4\beta}\rangle\langle \sigma^{-2(\beta+1)}\rangle}{\langle \sigma^{-2\beta}\rangle^3}$ and $\mathcal{D}_2 = \frac{(\beta-1)m_i^2}{2} - \frac{(\beta+1)G(2\beta)}{\langle \sigma^{-2\beta}\rangle} + 2\frac{\langle \sigma^{-2(\beta+2)}\rangle \sigma_i^{-2\beta}}{\langle \sigma^{-2\beta}\rangle}$.

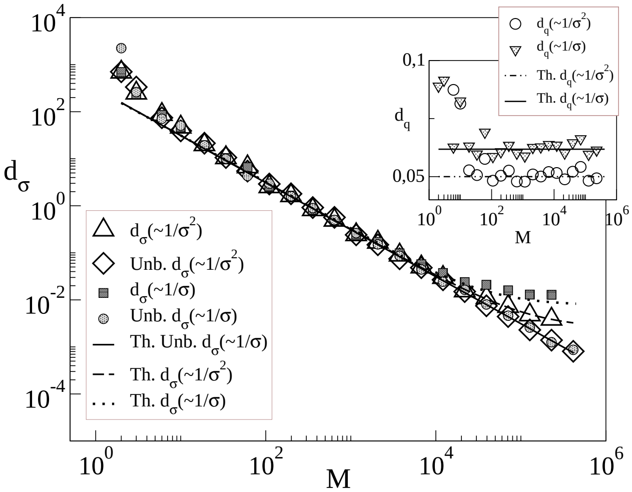

Fig. 1 – Average-squared difference d_σ between given and predicted variance, as a function of M in log-log scale. Symbols represent simulations of the IR method with $\beta = 1$ (triangles) and $\beta = 1/2$ (filled squares) in eq. (5); diamonds and filled circles show simulations of d_σ, where the estimator of the variance has been corrected for the bias. The corresponding theoretical predictions, calculated as explained in the text, fit the data very well. In the inset a similar plot shows the coincidence between the predicted and simulated plateau reached by d_q for large M. Parameters of the simulations: $N = 100$, intrinsic values Q_l uniformly distributed between 10 and 20 and standard deviations σ_i uniformly distributed between 1 and 5; averaged over 10^3 realizations.

to the ω_i's one obtains $\underline{\omega}^* = A^{-1}\underline{1}$, with $A_{ij} = \sigma_i^2\delta_{ij} + \Delta_i\Delta_j$. Here we have used a more compact matrix notation, where $\underline{1}$ is a vector of ones.

Whenever the deviations Δ_i are small, limited to a minority of the population or randomly distributed around zero, they can be somehow detected. In the general case one can only

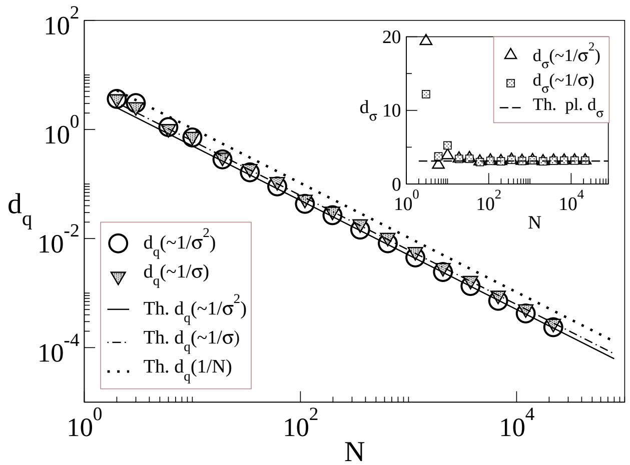

Fig. 2 – Average-squared difference between estimators and intrinsic values, for quality (main) and variance (inset), plotted in log-log scale as a function of N, with $M = 100$, for $\beta = 1/2, 1$. Symbols represent simulation results of the IR method, lines are the corresponding theoretical predictions. The dotted line represents d_q when the quality estimator is just the straight average.

detect, at best, the *relative* systematic errors. In fact $\tilde{\Delta} = \sum_j f_j \Delta_j$ does not depend on l in the presence of camouflage and the relevant quantities only depend on Δ_i under the form $\delta_i = \Delta_i - \tilde{\Delta}$. For instance, the variance can be written as $V_i = \frac{1}{M} \sum_l \left[\sum_j f_j (y_{jl} - y_{il}) + \delta_i \right]^2$. This means that, if we change the Δ_i's while keeping the δ_i's unchanged, we end up with the same result for d_q, only translated by the amount $\tilde{\Delta}$.

In order to estimate analytically the performance of the IR method, we can posit $\tilde{\Delta} = 0$ and solve eqs. (3)-(5) as before. Thus we find $f_i(\{\delta_i\})$, whose term of order zero is $(\sigma_i^2 + \delta_i^2)^{-\beta} / \sum_j (\sigma_j^2 + \delta_j^2)^{-\beta}$. This way we have a formal solution as a function of δ_i, which must comply with the constraint $\sum_i f_i \delta_i = 0$ and can eventually be recovered numerically.

Cheating. – It is interesting to consider the case of one intentional cheater I wanting to boost the value of object L of an amount Δ, all other raters being honest: $\Delta_{il} = \delta_{iI} \delta_{lL} \Delta$. Agent I commits no systematic error in evaluating all objects but L. Still, she would loose credibility and weight as Δ becomes larger; this would eventually diminish her relative influence over object L. It is important to evaluate the difference $\delta q_l = q_l(\Delta) - q_l$ between the estimated value of the object with and without the friendly uprating. In fact a small δq, compared to the lost in credibility of the rater, discourages cheating, and vice versa.

The variance, as defined in eq. (4), can be written as a function of Δ and of the normalised weights. Hence $V_i(\Delta, \{f_i(\Delta)\}) = V_i(0, \{f_i(\Delta)\}) + \delta_{iI} \Delta^2 / M$, where the formal expression of $V_i(0, \{f_i(\Delta)\})$ is equal to that of eq. (10). Iterative asymptotic expansions can be performed the same way we did in the absence of systematic errors. In this case the variables y_{il} are equal to the x_{il}, except for $y_{IL} = x_{IL} - \Delta$. Therefore eq. (3) becomes $q_l(\Delta) = \sum_i f_i(\Delta) y_{il} + \Delta \delta_{lL} f_I(\Delta)$, which implies $q_l(\Delta) - q_l \simeq \Delta \delta_{lL} f_I$. For $\Delta \ll \sqrt{M}$ the average deviation reads $\langle \delta q_L \rangle \simeq \Delta \left(\langle f_I \rangle - \beta \frac{\Delta^2}{\sigma_I^2 M} \right)$. If the value of Δ is comparable to $\sqrt{M}$, on the other hand, the zeroth order of the correction at the thermodynamic limit amounts to

$$\langle \delta q_L \rangle \to \Delta \cdot \left[\left(\sigma_I^2 + \frac{\Delta^2}{M} \right)^{\beta} \sum_{j \neq I} \sigma_j^{-2\beta} + 1 \right]^{-1} . \tag{14}$$

In fig. 3 eq. (14) is shown to fit the simulations fairly well in the rank space. We have compared thereby the scheme $\beta = 1$ (circles) with $\beta = 1/2$ (stars) in the worst case: the best agent is

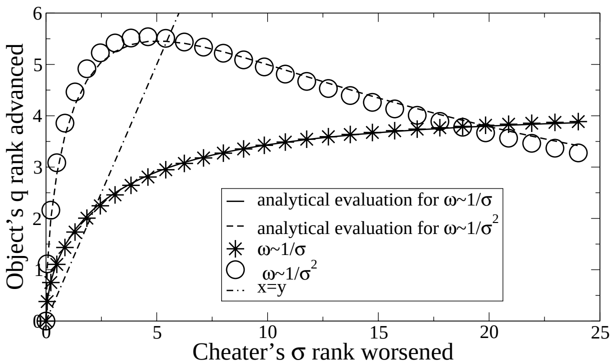

Fig. 3 – Increase of object's quality as a function of the cheater's rank loss, as the value of Δ grows from 0 to 30. Simulations have been carried on with $N = 100$, $M = 100$ and intrinsic values distributed as in fig. 1, except for $\sigma_I = 1$ and $Q_L = 20$. The theoretical estimations are parametric plots of eq. (14) for $\beta = 1$ and $1/2$.

trying to raise the worst object. In the region of moderate cheating the $\omega_i = 1/\sqrt{V_i}$ weighting scheme is less sensitive to cheating. This is particularly important left to the $x = y$ line, where the cheater pays less then what she offers to the object and cheating can be advantageous. However, the relative influence of the cheater is a growing, although saturating, function of Δ. Under the $\omega_i = 1/V_i$ weighting scheme, on the other hand, such an influence starts decreasing once passed a crossover value. There the cheater's reputation is so much damaged by her misbehaviour that, if she attributed a higher value to object L, its estimated rank would diminish. Optimal weights are, therefore, much more resilient to severe cheating.

We just remark that, taking averages without refinement, a cheater would indefinitely increase an object's rank without undergoing any punishment. The transition to the cheater's unfavorable region is the solution of $d_{r(q)} = d_{r_{(\sigma)}}$ in the Δ space.

Conclusion. – In this letter we have analyzed a novel scoring system that aggregates the evaluations of N agents over M objects by use of reputation and weighted averages. Agents, as a result, are ranked according to their judging capability and objects according to their quality. The method can be implemented via an iterative algorithm, where the intrinsic bias of the estimators of the weights can be corrected. We show, with simulations and analytical results, that the method is effective and robust against abuses. The larger the system, the better is the filtering precision. This method can be applied in web-related reputation and scoring systems.

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