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The eigenvalues of the Laplacian with Dirichlet boundary condition in $\mathbb{R}^2$ are almost never minimized by disks

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Abstract Minimization of the Dirichlet eigenvalues of the Laplacian among sets of prescribed measure is a standard problem in shape optimization. The main result of this paper is that in the Euclidean plane, apart from the first four, no Dirichlet eigenvalue can be minimized by disks or disjoint unions of disks.

Keywords Eigenvalues · Laplacian · Dirichlet · Minimization · Disks

Mathematics Subject Classification 35P10 · 49R05

1 Introduction

Shape optimization problems associated to the eigenvalues of the Laplacian are numerous and received a lot of attention since the end of the nineteenth century. The most classical one is the minimization of the eigenvalues of the Laplacian with Dirichlet boundary condition among sets of prescribed measure. In dimension 2, this problem was introduced in the late nineteenth century by Lord Rayleigh. He conjectured that the first eigenvalue is minimized by a disk. Faber and Krahn proved Rayleigh's conjecture in all dimensions in the 1920s. It is a straightforward consequence that the second eigenvalue is minimized by two identical balls. This result is usually attributed to Pólya and Szegő, but it seems to be contained in one of Krahn's older papers. The problem of the minimization of the kth eigenvalue of the Dirichlet–Laplacian for $k > 2$ is still open whatever the dimension.

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Yet, despite many research articles on the subject, still very little is known on optimal shapes (see [7] or [9] for instance). Progresses regarding existence have recently been made [4,11] but very few geometrical information are available on optimal shapes (connectivity, regularity...).

Identifying a minimizer is a very difficult task. Actually, even the computation of the eigenvalues of a given shape is a non-trivial problem. Except for some very particular shapes such as disks and rectangles, analytic formulae do not exist. Nevertheless, the approximation of the eigenvalues of a shape is possible with a relatively good precision by standard Finite Elements Methods tools. Then suspected optimal shapes can be found using some optimization methods. But these shapes usually cannot be described theoretically, for instance with usual functions.

Thus following the works of Pólya and Szegő and the works of Wolf and Keller [15], we restrict in this article our study to the class of unions of disks, eventually the disk. This class always contains an optimal shape. However, based on numerical results obtained in [14] and [2], we cannot expect these optimal shapes to be optimal for the general problem.

More precisely, this article addresses the question: "Is the disk or a disjoint union of disks a minimizer of λ_k?"

We start part 2 by recalling some definitions and classical results in spectral optimization.

Theorem 4 states that the eigenvalues are not locally minimized by a disk apart from the first and perhaps the third. The proof of this Theorem, in part 3, is broken down in three parts. First, in part 3.1 we compute the asymptotic development of order two of the eigenmodes with respect to a radial deformation of the disk. This development has already been established by Wolf and Keller [15] for λ_2 and λ_3 while Rayleigh established the one for λ_1. We provide here a detailed computation leading to the more general result for any λ_k. We then distinguish two different cases. First the case of simple eigenvalues in part 3.2 then in part 3.3 the case of double eigenvalues.

Theorem 5 states that λ_k is not globally minimized by a disk or a disjoint union of disks except perhaps if $1 \leq k \leq 4$ or $k \in \{6, 7, 9\}$. In part 4 we first show Theorem 5 using a theorem due to Wolf and Keller. In fact we use this theorem twice. First, a restricted version allows us to find the minimizer of a given eigenvalue when considering only the class of disjoint unions of disks and disks. Then, we determine which eigenvalues could be minimized by a union of a given number of disks. Finally we rule out possibilities by comparing with the best unions of disks and with rectangles. Notice that Theorem 5 can be improved by comparing the best unions of disks with reliable numerical results. In fact, the numerical schemes introduced in [14] and [2] give a numerical estimate which is greater than the exact one.

Theorem 7 is a local version of Theorem 5 which identifies which eigenvalues are perhaps locally minimized by a disjoint union of a given number of disks. In part 5 we give a local version of the theorem due to Wolf and Keller with a sketch of proof. We then prove Theorem 7 with an induction based on arguments and ideas of the proof of Theorem 5.

2 Definitions and main results

Let Ω be a bounded open set of $\mathbb{R}^2$ and let us denote by $\lambda_k(\Omega)$ the kth eigenvalue of the Laplacian with Dirichlet boundary condition. That is to say there exists a function u such that

$$\begin{cases} -\Delta u = \lambda_k u & \text{on} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

For a given bounded open set Ω in the plane we know that

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \to +\infty.$$

We are looking for open sets $\Omega_k \subseteq \mathbb{R}^2$ with $|\Omega_k| = 1$ which are optimal in the following sense:

$$\lambda_k(\Omega_k) = \min\{\lambda_k(\Omega);\ \Omega \subseteq \mathbb{R}^2 \quad \text{bounded open}, \quad |\Omega| = 1\}. \tag{2}$$

The question of the existence of such optima is a difficult one. For years there have only been partial answers, see [5,8] for instance. But recently there have been improvements. Indeed we can cite the following result due to Bucur [4].

Theorem 1 *For every $k \in \mathbb{N}$, the problem*

$$\min\{\lambda_k(A),\ A \subseteq \mathbb{R}^N,\ |A| = c\}$$

has at least one solution in the family of quasi-open sets. Moreover, every solution is bounded and has finite perimeter.

Notice that the first part of this theorem (existence) has also been proved in [11] by Mazzoleni and Pratelli contemporarily by other means. Notice that there is no more general existence result too.

It is easy to show the following property.

Property 1 (Homogeneity) *Let $c > 0$ be a real. Then*

$$\lambda_j(c\Omega) = c^{-2}\lambda_j(\Omega). \tag{3}$$

Using this property, it is straightforward that an equivalent problem to (2) is to find $\Omega_k \subseteq \mathbb{R}^2$ such that

$$|\Omega_k|\lambda_k(\Omega_k) = \min\{|\Omega|\lambda_k(\Omega);\ \Omega \subseteq \mathbb{R}^2 \text{open}\}. \tag{4}$$

Definition 1
1. We say that Ω_k is a minimizer of λ_k if it is a solution of (2)⇔(4).
2. We say that a regular set Ω_k is a local minimizer of λ_k if for all analytical deformations F we have, $|F(\Omega_k)|\lambda_k(F(\Omega_k)) \geq |\Omega_k|\lambda_k(\Omega_k)$.

We refer to [7] for a proof of the following two theorems.

Theorem 2 (Faber–Krahn)

$$\lambda_1(B) = \min\{\lambda_1(\Omega),\ \Omega \subset \mathbb{R}^2 \ \textit{open},\ |\Omega| = 1\}$$

where B is the disk of area 1.

Theorem 3 (Krahn–Szegő) $\min\{\lambda_2(\Omega),\ \Omega \subset \mathbb{R}^2 \textit{open},\ |\Omega| = 1\}$ *is realized by the union of two identical disks.*

The only other useful result for us is a result of Wolf and Keller who proved in [15] that λ_3 is locally minimized by a disk, considering particular radial deformations. We use the same ideas to show our first result:

Theorem 4 *The eigenvalues of the Laplacian with Dirichlet boundary condition λ_k with $k > 3$ are not locally minimized by the disk in dimension 2 among sets of constant measure.*

Notice that this result agrees with the available numerical results (see Fig. 2). To complete Theorems 2 and 3 we now have the following global result:

Theorem 5 – *λ_3 is perhaps minimized by the disk but is not by any disjoint unions of disks,*
- *λ_4 and λ_6 are not minimized by a disjoint union of disks nor by the disk except perhaps by two particular unions of two disks (known radii),*
- *λ_7 and λ_9 are not minimized by a disjoint union of disks nor by the disk except perhaps by two particular unions of three disks (known radii),*
- *λ_5, λ_8 and the eigenvalues λ_k with $k \geq 10$ are not minimized by the disk nor by a disjoint union of disks.*

Remark 1 Theorem 5 can be improved using two numerical upper estimates for λ_6 and λ_7. As noticed previously, the tools used in [14] and [2] give reliable numerical upper bounds which makes it possible to conclude the proof of the improved theorem:

Theorem 6 – *λ_3 is perhaps minimized by the disk but is not by any disjoint unions of disks,*
- *λ_4 is not minimized by a disjoint union of disks nor by the disk except perhaps by a particular union of two disks (whose radii are in the ratio $\sqrt{j_{0,1}/j_{1,1}}$),*
- *the eigenvalues λ_k with $k \geq 5$ are not minimized by the disk nor by any disjoint unions of disks.*

Also, notice that Theorem 4 is a local result whereas Theorem 5 is global. Nevertheless we can obtain a local version of Theorem 5:

Theorem 7 *Let $n \in \mathbb{N}^*$. If a disjoint union of n disks minimizes locally the eigenvalue λ then λ must be some λ_{n+2k} with $k \in \{0, \ldots, n\}$.*

Remark 2 We will not show that λ_{n+2k} with $k \in \{0, \ldots, n\}$ is minimized locally by a disjoint union of n disks. We will not give arguments indicating that they are not. But we will show that any other cases are not possible.

3 Proof of Theorem 4

3.1 Eigenvalues on a set obtained by a small deformation of a disk

To show that the disk is not a local minimizer, we consider shapes obtained by small deformations of the disk. We choose a relevant perturbation such that the eigenvalue of the corresponding shape is smaller than the one of the disk. The aim of this section is to estimate the eigenvalues of such profiles.

Since we only work on radial deformations, we use polar coordinates in $\mathbb{R}^2$. Let us introduce parameters (r, θ) satisfying

$$\begin{cases} x = r\cos(\theta), \\ y = r\sin(\theta) \end{cases}$$

with $r \in]0, R[$, $R > 0$, $\theta \in [0, 2\pi[$.

We recall the analytic expression of the eigenvalues and eigenfunctions of the Dirichlet–Laplacian of the disk.

Theorem 8 *The eigenvalues and eigenfunctions of the disk of radius R (normalized for the L^2-norm) for Dirichlet–Laplacian are given by*

$$
\begin{aligned}
\lambda_{0,p} &= \frac{j_{0,p}^2}{R^2}, \quad p \geq 1,\\
u_{0,p}(r,\theta) &= \sqrt{\frac{1}{\pi}} \frac{1}{R\left|J_0'(j_{0,p})\right|} J_0\left(\frac{j_{0,p}r}{R}\right), \quad p \geq 1,\\
\lambda_{m,p} &= \frac{j_{m,p}^2}{R^2}, \quad m,p \geq 1, \quad \text{multiplicity } 2\\
u_{m,p}(r,\theta) &= \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{R|J_m'(j_{m,p})|} J_m\left(\frac{j_{m,p}r}{R}\right)\cos(m\theta) \\ \sqrt{\frac{2}{\pi}} \frac{1}{R|J_m'(j_{m,p})|} J_m\left(\frac{j_{m,p}r}{R}\right)\sin(m\theta) \end{cases}, \quad m,p \geq 1,
\end{aligned}
\tag{5}
$$

where $j_{m,p}$ is the pth zero of the Bessel function J_m.

This result is easy to show and elements can be found in [7].

We will consider here a disk of radius 1. Thus, we consider the case of the constraint measure π.

We consider deformations of the unit disk in the following sense: for $\varepsilon \geq 0$ a small parameter, the boundary points of the new domain Ω_ε are described by the parameterization $(R(\theta,\varepsilon),\theta)$ where

$$
R(\theta,\varepsilon) = 1 + \varepsilon \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{in\theta} + \varepsilon^2 \sum_{n=-\infty}^{\infty} b_n \mathrm{e}^{in\theta} + O(\varepsilon^3) \tag{6}
$$

with $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$ for all n.

We will see in Sects. 3.2 and 3.3 that we need a second order development.

Using $\left(\sum\limits_{n=-\infty}^{\infty} a_n \mathrm{e}^{in\theta}\right)^2 = \sum\limits_{n,l=-\infty}^{\infty} a_l a_n \mathrm{e}^{i(l+n)\theta}$, $a_n a_{-n} = |a_n|^2$ and $\int_0^{2\pi} \mathrm{e}^{in\theta}\mathrm{d}\theta = 0$ for $n \neq 0$ we show that the area of Ω_ε is

$$
A(\varepsilon) = \int_0^{2\pi}\int_0^{R(\theta,\varepsilon)} r\,\mathrm{d}r\mathrm{d}\theta = \pi\left[1 + 2\varepsilon a_0 + \varepsilon^2\left(2b_0 + \sum_{n=-\infty}^{\infty}|a_n|^2\right) + O(\varepsilon^3)\right]. \tag{7}
$$

We will deal with the measure constraint in two times. First, let us make $A(\varepsilon) = \pi + O(\varepsilon^3)$. We thus obtain the conditions

$$
a_0 = 0 \quad \text{and} \quad b_0 = -\frac{1}{2}\sum_{n=-\infty}^{\infty}|a_n|^2. \tag{8}
$$

Then, from Property 1, we will have to compare $A(\varepsilon)\lambda(\Omega_\varepsilon)$ with $\pi\lambda(\Omega_0)$.

Let us consider an eigenvalue λ of the disk. From Theorem 8 we know that there exist $m > 0$ and $p \geq 0$ such that $\lambda = j_{m,p}^2$. So let us fix them.

Now, Part VII.6.5 of [10, pp. 423–426], gives us an expression of the eigenvalues and eigenfunctions of the Laplacian on the new domains. For some more details we can also refer to the pp. 155–160 of [12] and to [13] or [6] for details on the theorem used in [12].

For the following and for simplicity, let us denote $\lambda(\Omega_\varepsilon) = \omega^2$ and $u(r,\theta,\varepsilon)$ an associated eigenfunction. Note that even if it is not explicit, they depend on m and p. Since the eigenfunctions given in (5) define a basis and since $J_{-n} = (-1)^n J_n, \forall n$, let us write

$$u(r,\theta,\varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon) J_n(\omega r) \mathrm{e}^{in\theta}, \quad \text{with} \quad A_{-n} = (-1)^n \overline{A_n} \tag{9}$$

and with

$$A_n(\varepsilon) = \delta_{|n|m}\alpha_n + \varepsilon\beta_n + \varepsilon^2\gamma_n + O(\varepsilon^3) \tag{10}$$

where $\delta_{|n|m} = 0$ if $|n| \neq m$ and else $\delta_{|n|m} = 1$. Since $A_{-n} = (-1)^n \overline{A_n}$ we deduce that $\alpha_{-n} = (-1)^n \overline{\alpha_n}$, $\beta_{-n} = (-1)^n \overline{\beta_n}$, $\gamma_{-n} = (-1)^n \overline{\gamma_n}$.

Remark 3 For $m \neq 0$ $\alpha_m \neq 0$ and must satisfy

$$u(r,\theta,0) = \left(\alpha_m \mathrm{e}^{im\theta} + \overline{\alpha_m \mathrm{e}^{im\theta}}\right) J_m(\omega r), \tag{11}$$

$u(r,\theta,0)$ being an eigenfunction on the disk associated with $\lambda(\Omega_0) = j_{m,p}^2$. Thus, if we choose $\alpha_m = 1$, the eigenfunction associated with $j_{m,p}^2$ is $u_{m,p} = 2J_m(j_{m,p}^2 r)\cos(m\theta)$ and if we choose $\alpha_m = i$, $u_{m,p} = -2J_m(j_{m,p}^2 r)\sin(m\theta)$.

From [10] we know that we can write

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3). \tag{12}$$

Thus, remembering $\lambda(\Omega_\varepsilon) = \omega^2$

$$\lambda(\Omega_\varepsilon) = \omega_0^2 + 2\varepsilon\omega_0\omega_1 + \varepsilon^2\left(2\omega_0\omega_2 + \omega_1^2\right) + O(\varepsilon^3). \tag{13}$$

We now want to express $\omega_0, \omega_1, \omega_2$ with $(a_n), (b_n), (\alpha_n), (\beta_n) \ldots$

The Dirichlet boundary condition becomes

$$u\left(R(\theta,\varepsilon),\theta,\varepsilon\right) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon) J_n\left(\omega R(\theta,\varepsilon)\right) \mathrm{e}^{in\theta} = 0. \tag{14}$$

Remark that $\omega R = \omega_0 + (\omega - \omega_0) + \omega(R-1)$. Using this in (14) and expanding J_n in a Taylor series

$$\sum_{n=-\infty}^{\infty} A_n(\varepsilon)\Big[J_n(\omega_0) + J_n'(\omega_0)\left(\omega - \omega_0 + \omega(R-1)\right) + \frac{1}{2} J_n''(\omega_0)\left(\omega - \omega_0 + \omega(R-1)\right)^2 + O(\varepsilon^3)\Big] \mathrm{e}^{in\theta} = 0. \tag{15}$$

Using (6) for R, (10) for A_n and (12) for ω in (15) we obtain the equation

$$\begin{aligned}0 =& \sum_n \delta_{|n|m}\alpha_n J_n(\omega_0)\mathrm{e}^{in\theta}\\&+\varepsilon\sum_n\left(\beta_n J_n(\omega_0)+\delta_{|n|m}\alpha_n J_n'(\omega_0)\left[\omega_1+\omega_0\sum_l a_l\mathrm{e}^{il\theta}\right]\right)\mathrm{e}^{in\theta}\\&+\varepsilon^2\sum_n\left(\gamma_n J_n(\omega_0)+\beta_n J_n'(\omega_0)\left[\omega_1+\omega_0\sum_l a_l\mathrm{e}^{il\theta}\right]\right.\\&+\delta_{|n|m}\alpha_n\left[J_n'(\omega_0)\left(\omega_2+\omega_1\sum_l a_l\mathrm{e}^{il\theta}+\omega_0\sum_l b_l\mathrm{e}^{il\theta}\right)\right.\\&\left.\left.+\frac{1}{2}J_n''(\omega_0)\left(\omega_1^2+2\omega_0\omega_1\sum_l a_l\mathrm{e}^{il\theta}+\omega_0^2\left(\sum_l a_l\mathrm{e}^{il\theta}\right)^2\right)\right]\right)\mathrm{e}^{in\theta}+O(\varepsilon^3). \qquad (16)\end{aligned}$$

This equation is true if and only if the coefficients ahead of ε^j, $j = 0, 1, 2\ldots$, are all equal to zero.

We now have to separate the cases $m = 0$, that is to say simple eigenvalues of the disks, and $m > 0$, double eigenvalues.

Case $m = 0$: simple eigenvalues

Lemma 1 *With previous notations, if* $\lambda(\Omega_0) = j_{0,p}^2$ *then*

$$A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{0,p}^2\left(1+4\varepsilon^2\sum_{l>0}\left(1+\frac{j_{0,p}J_l'(j_{0,p})}{J_l(j_{0,p})}\right)|a_l|^2\right)+O(\varepsilon^3). \qquad (17)$$

Proof ***Term in*** ε^0
$\alpha_0 J_0(\omega_0) = 0$. But $\alpha_0 \neq 0$ else $u(r, \theta, 0) = 0$. So $J_0(\omega_0) = 0$ that is to say $\omega_0 = j_{0,p}$.

Term in ε^1
$J_0(\omega_0) = 0$ and $a_0 = 0$ so

$$0 = \alpha_0\omega_1 J_0'(\omega_0) + \sum_{n\neq 0}\left[\alpha_0 J_0'(\omega_0)\omega_0 a_n + \beta_n J_n(\omega_0)\right]\mathrm{e}^{in\theta}.$$

For $n \neq 0$, $\beta_n = -\alpha_0\omega_0 a_n \frac{J_0'(\omega_0)}{J_n(\omega_0)}$ and for $n = 0$, $\alpha_0\omega_1 J_0'(\omega_0) = 0$ so $\omega_1 = 0$.

Term in ε^2
Using $\omega_1 = 0$ and $J_0(\omega_0) = 0$

$$\begin{aligned}0 =& \sum_n\left(\gamma_n J_n(\omega_0)+\beta_n J_n'(\omega_0)\omega_0\sum_l a_l\mathrm{e}^{il\theta}\right)\mathrm{e}^{in\theta}\\&+\alpha_0\left[J_0'(\omega_0)\left(\omega_2+\omega_0\sum_l b_l\mathrm{e}^{il\theta}\right)+\frac{1}{2}J_0''(\omega_0)\omega_0^2\sum_{n,l}a_l a_n\mathrm{e}^{i(l+n)\theta}\right].\end{aligned}$$

Then, for $n = 0$ we obtain

$$\omega_2 = -\omega_0 b_0 - \sum_{n\neq 0}\omega_0\frac{\beta_n}{\alpha_0}a_{-n}\frac{J_n'(\omega_0)}{J_0'(\omega_0)} - \frac{1}{2}\omega_0^2\frac{J_0''(\omega_0)}{J_0'(\omega_0)}\sum_{n\neq 0}|a_l|^2.$$

Using $\beta_n = -\alpha_0\omega_0 a_n \frac{J_0'(\omega_0)}{J_n(\omega_0)}$ for $n \neq 0$, $J_0''(\omega_0) = -\frac{1}{\omega_0} J_0'(\omega_0)$, $J_{-n}' = (-1)^n J_n'$ and $|a_{-n}| = |\overline{a_n}| = |a_n|$

$$\omega_2 = \omega_0 \sum_{n\neq 0} \left(1 + \omega_0 \frac{J_n'(\omega_0)}{J_n(\omega_0)}\right) |a_n|^2 = 2\omega_0 \sum_{n>0} \left(1 + \omega_0 \frac{J_n'(\omega_0)}{J_n(\omega_0)}\right) |a_n|^2.$$

In conclusion, replacing $\omega_0, \omega_1, \omega_2$ by these values in (13) and considering $A(\varepsilon) = \pi + O(\varepsilon^3)$ we deduce (17). □

Case $m > 0$: double eigenvalues

Lemma 2 *With previous notations if* $\lambda(\Omega_0) = j_{m,p}^2$, $m > 0$ *then*

$$\begin{aligned} A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{m,p}^2 \Bigg(1 - 2\varepsilon a_{2m} \frac{\overline{\alpha_m}}{\alpha_m} \\ + 2\varepsilon^2 \Bigg[2|a_{2m}|^2 + \sum_{|l|\neq m} \left(1 + j_{m,p} \frac{J_l'(j_{m,p})}{J_l(j_{m,p})}\right) |a_{m-l}|^2 + \left(\frac{\beta_m}{\overline{\alpha_m}} \overline{a_{2m}} - \frac{\overline{\beta_m}}{\alpha_m} a_{2m}\right) \\ + \frac{\overline{\alpha_m}}{\alpha_m} \left(-b_{2m} + \sum_{|l|\neq m} \left(\frac{1}{2} + j_{m,p} \frac{J_l'(j_{m,p})}{J_l(j_{m,p})}\right) a_{m+l} a_{m-l}\right)\Bigg]\Bigg) + O(\varepsilon^3). \qquad (18) \end{aligned}$$

Remark 4 From Remark 3 we know that for a double eigenvalue we can choose different α_m such that one of the eigenvalues becomes of the form $j_{m,p}^2\,(1 - 2\varepsilon a_{2m}) + O(\varepsilon^2)$, for instance for $\alpha_m = 1$, and the other one $j_{m,p}^2\,(1 + 2\varepsilon a_{2m}) + O(\varepsilon^2)$, for instance for $\alpha_m = i$.

Proof ***Term in*** ε^0
With $J_{-m} = (-1)^m J_m$ and $\alpha_{-m} = (-1)^m \overline{\alpha_m}$

$$\alpha_m J_m(\omega_0) e^{im\theta} + \alpha_{-m} J_{-m}(\omega_0) e^{-im\theta} = 2\mathrm{Re}\left(\alpha_m e^{im\theta}\right) J_m(\omega_0) = 0 \quad \forall\theta$$

so $J_m(\omega_0) = 0$ that is to say $\omega_0 = j_{m,p}$.

Term in ε^1

$$\begin{aligned} 0 = \sum_{|n|\neq m} \left[\beta_n J_n(\omega_0) + (\alpha_m a_{n-m} + \overline{\alpha_m} a_{n+m})\, \omega_0 J_m'(\omega_0)\right] e^{in\theta} \\ + (\alpha_m \omega_1 + \overline{\alpha_m}\omega_0 a_{2m}) J_m'(\omega_0) e^{im\theta} + (\overline{\alpha_m}\omega_1 + \alpha_m \omega_0 a_{-2m}) J_m'(\omega_0) e^{-im\theta}. \end{aligned}$$

For $n \neq |m|$ $\beta_n = -\omega_0\, (a_{n-m}\alpha_m + a_{n+m}\overline{\alpha_m})\, \frac{J_m'(\omega_0)}{J_n(\omega_0)}$, for $n = m$ we obtain $\frac{\omega_1}{\omega_0} = -a_{2m} \frac{\overline{\alpha_m}}{\alpha_m}$ and for $n = -m$, $\frac{\omega_1}{\omega_0} = -\overline{a_{2m} \frac{\overline{\alpha_m}}{\alpha_m}}$ so $\frac{\omega_1}{\omega_0} \in \mathbb{R}$ and $\omega_1 \in \mathbb{R}$. Therefore, if $a_{2m} \neq 0$, $\frac{a_{2m}}{\overline{a_{2m}}} = \left(\frac{\alpha_m}{\overline{\alpha_m}}\right)^2$ so $\omega_1 = -\omega_0 a_{2m} e^{-2i\arg(\alpha_m)}$, which is also true if $a_2 m = 0$.
In conclusion $\omega_1 = -a_{2m}\omega_0 \frac{\overline{\alpha_m}}{\alpha_m}$ and $\omega_1^2 = \omega_0^2 |a_{2m}|^2$.

Term in ε^2

With $J_{-m} = (-1)^m J_m$, $\alpha_{-m} = (-1)^m \overline{\alpha_m}$, $J''_m(\omega_0) = -\frac{1}{\omega_0} J'_m(\omega_0)$ by definition of Bessel functions and $\left(\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{in\theta}\right)^2 = \sum_{n,l=-\infty}^{\infty} a_l a_n \mathrm{e}^{i(l+n)\theta}$

$$0 = \sum_n \left(\gamma_n J_n(\omega_0) + \beta_n J'_n(\omega_0) \left[\omega_1 + \omega_0 \sum_l a_l \mathrm{e}^{il\theta} \right] \right) \mathrm{e}^{in\theta}$$
$$+ \alpha_m J'_m(\omega_0) \left(\omega_2 - \frac{\omega_1^2}{2\omega_0} + \omega_0 \sum_l b_l \mathrm{e}^{il\theta} - \frac{\omega_0}{2} \sum_{n,l} a_l a_n \mathrm{e}^{i(l+n)\theta} \right) \mathrm{e}^{im\theta}$$
$$+ \overline{\alpha_m} J'_m(\omega_0) \left(\omega_2 - \frac{\omega_1^2}{2\omega_0} + \omega_0 \sum_l b_l \mathrm{e}^{il\theta} - \frac{\omega_0}{2} \sum_{n,l} a_l a_n \mathrm{e}^{i(l+n)\theta} \right) \mathrm{e}^{-im\theta}.$$

For $n = m$, with (8), we have that

$$\omega_2 = \frac{\omega_1^2}{2\omega_0} + \omega_0 \sum_l |a_l|^2 - \omega_0 \frac{\overline{\alpha_m}}{\alpha_m} \left(b_{2m} - \frac{1}{2} \sum_l a_{m+l} a_{m-l} \right) - \frac{\beta_m}{\alpha_m} \omega_1 - \frac{\omega_0}{\alpha_m} \sum_l \beta_l a_{m-l} \frac{J'_l(\omega_0)}{J'_m(\omega_0)}.$$

Using the previous expressions for β_n, $|n| \neq m$, ω_1, ω_1^2 and with $\beta_{-m} = (-1)^m \beta_m$ and $J'_{-m} = (-1)^m J'_m$ we deduce

$$\frac{\omega_2}{\omega_0} = \underbrace{\frac{|a_{2m}|^2}{2} + \sum_l |a_l|^2 + \omega_0 \sum_{|l| \neq m} |a_{m-l}|^2 \frac{J'_l(\omega_0)}{J_l(\omega_0)}}_{\varGamma} + \underbrace{\left(\frac{\beta_m}{\overline{\alpha_m}} \overline{a_{2m}} - \frac{\overline{\beta_m}}{\alpha_m} a_{2m} \right)}_{\varUpsilon_1}$$
$$+ \underbrace{\frac{\overline{\alpha_m}}{\alpha_m} \left(-b_{2m} + \sum_{|l| \neq m} \left(\frac{1}{2} + \omega_0 \frac{J'_l(\omega_0)}{J_l(\omega_0)} \right) a_{m+l} a_{m-l} \right)}_{\varUpsilon_2}.$$

Similarly, for $n = -m$, $\frac{\omega_2}{\omega_0} = \varGamma + \overline{\varUpsilon_1 + \varUpsilon_2}$.

Thus, since $\varGamma \in \mathbb{R}$, $\frac{\omega_2}{\omega_0} = \varGamma + \varUpsilon_1 + \varUpsilon_2 = \varGamma + \overline{\varUpsilon_1 + \varUpsilon_2} = \overline{\varGamma + \varUpsilon_1 + \varUpsilon_2}$ so $\frac{\omega_2}{\omega_0} \in \mathbb{R}$ so $\omega_2 \in \mathbb{R}$. Furthermore, $\varGamma \in \mathbb{R}$ so $\varUpsilon_1 + \varUpsilon_2 \in \mathbb{R}$ so, in particular, $\varUpsilon_1 + \varUpsilon_2 = \pm|\varUpsilon_1 + \varUpsilon_2|$. Moreover $\varUpsilon_1 \in i\mathbb{R}$ so $\varUpsilon_1 = -Im(\varUpsilon_2)$.

With $\sum_l |a_l|^2 = |a_{2m}|^2 + \sum_{|l| \neq m} |a_{m-l}|^2$ we obtain

$$\frac{\omega_2}{\omega_0} = \frac{3}{2} |a_{2m}|^2 + \sum_{|l| \neq m} \left(1 + \omega_0 \frac{J'_l(\omega_0)}{J_l(\omega_0)} \right) |a_{m-l}|^2 + \left(\frac{\beta_m}{\overline{\alpha_m}} \overline{a_{2m}} - \frac{\overline{\beta_m}}{\alpha_m} a_{2m} \right)$$
$$+ \frac{\overline{\alpha_m}}{\alpha_m} \left(-b_{2m} + \sum_{|l| \neq m} \left(\frac{1}{2} + \omega_0 \frac{J'_l(\omega_0)}{J_l(\omega_0)} \right) a_{m+l} a_{m-l} \right).$$

In conclusion, replacing $\omega_0, \omega_1, \omega_2$ by these values in (13) and considering $A(\varepsilon) = \pi + O(\varepsilon^3)$ we deduce (18). □

We obtained asymptotic developments of the eigenvalues on domains with respect to small deformations of the disk. Now, if we find some families (a_n) and (b_n) such that $A(\varepsilon)\lambda(\varOmega_\varepsilon) <$

$\pi\lambda(\Omega_0) = \pi j_{m,p}^2$ for corresponding m and p, then the disk is not a local minimizer for the corresponding eigenvalue.

On the contrary, Wolf and Keller have shown that for these deformations, λ_3 is always greater than $j_{1,1}^2$.

3.2 Simple eigenvalues of the unit disk

Lemma 3 *Let λ_k be an eigenvalue of the Dirichlet–Laplacian which is simple for the disk. Then, for $k \neq 1$, λ_k is not locally minimized by the disk among sets of constant measure.*

Remark 5 It is easy to show that for λ_k simple for the disk, the disk is a critical point for the function $t \mapsto |\Omega_t|\lambda_k(\Omega_t)$. We can use the expressions for the derivatives with respect to the domain given in pp. 38–39 of [7] prove it or remark that in (17) we do not have terms in ε. So we have an example of a lot of critical points which are not (local) minimizers.

The case of the simple eigenvalues corresponds to the case $m = 0$, so $A(\varepsilon)\lambda_k(\Omega_\varepsilon)$ is given by (17). If we can find a l such that $1 + \frac{j_{0,p} J_l'(j_{0,p})}{J_l(j_{0,p})} < 0$ we can show that the disk is not a local minimizer. In fact we have the following estimates:

Lemma 4

$$1 + \frac{j_{0,1} J_3'(j_{0,1})}{J_3(j_{0,1})} > 0 \quad \textit{and} \quad 1 + \frac{j_{0,p} J_3'(j_{0,p})}{J_3(j_{0,p})} < 0 \quad \forall p \geq 2.$$

Proof of Lemma 3 From Lemma 4, if we choose (a_n) given by $a_i = 0, \forall |i| \neq 3$, $a_3 \neq 0$ small enough and $a_{-3} = \overline{a_3}$, and (b_n) such that $b_0 = -|a_3|^2$ and $b_n = 0$ for $n \neq 0$,

$$A(\varepsilon)\lambda_k(\Omega_\varepsilon) = \pi j_{0,p}^2 + 4\pi\varepsilon^2 j_{0,p}^2 \underbrace{\left(1 + \frac{j_{0,p} J_3'(j_{0,p})}{J_3(j_{0,p})}\right)}_{<0} |a_3|^2 + O(\varepsilon^3) \quad \forall p \geq 2.$$

Remark that it corresponds to the case

$$R(\theta, \varepsilon) = 1 + 2\varepsilon\,[\mathrm{Re}(a_3)\cos(3\theta) - \mathrm{Im}(a_3)\sin(3\theta)] - \varepsilon^2 |a_3|^2 + O(\varepsilon^3).$$

So there exist shape obtained from the unit disk by small variations for which simple eigenvalues apart from the first are less than the ones of the unit disk. This concludes the proof. □

Proof of Lemma 4 To prove the lemma we use the following classical results on Bessel functions:

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}_+^*, \quad x J_n' = n J_n - x J_{n+1} \tag{19}$$

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}_+^*, \quad x J_n' = -n J_n + x J_{n-1} \tag{20}$$

$$\text{and} \qquad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}_+^*, \quad \frac{2n}{x} J_n = J_{n-1} + J_{n+1}. \tag{21}$$

These results can be found for instance in [1, pp. 358–361].

From (21) and $J_0(j_{0,p}) = 0$ we deduce that

$$\frac{2}{j_{0,p}} J_1(j_{0,p}) = J_0(j_{0,p}) + J_2(j_{0,p}) = J_2(j_{0,p}) \quad \Rightarrow \quad \frac{J_2(j_{0,p})}{J_1(j_{0,p})} = \frac{2}{j_{0,p}}. \tag{22}$$

From (19) and (22)

$$\frac{J_1'(j_{0,p})}{J_1(j_{0,p})} = \frac{1}{j_{0,p}} - \frac{J_2(j_{0,p})}{J_1(j_{0,p})} = -\frac{1}{j_{0,p}}.$$

Therefore,

$$1 + \frac{j_{0,p} J_1'(j_{0,p})}{J_1(j_{0,p})} = 1 + j_{0,p} \times \left(-\frac{1}{j_{0,p}}\right) = 0, \quad \forall p \in \mathbb{N}^*.$$

Moreover, (22) is

$$\frac{J_2(j_{0,p})}{J_1(j_{0,p})} = \frac{2}{j_{0,p}} \quad \Rightarrow \quad \frac{J_1(j_{0,p})}{J_2(j_{0,p})} = \frac{j_{0,p}}{2}.$$

With this equality and (20)

$$\frac{J_2'(j_{0,p})}{J_2(j_{0,p})} = -\frac{2}{j_{0,p}} + \frac{J_1(j_{0,p})}{J_2(j_{0,p})} = -\frac{2}{j_{0,p}} + \frac{j_{0,p}}{2} = \frac{j_{0,p}^2 - 4}{2 j_{0,p}}.$$

Since $j_{0,p} \geq j_{0,1} > 2, \forall p \in \mathbb{N}^*$, $\frac{J_2'(j_{0,p})}{J_2(j_{0,p})} > 0$ and

$$1 + \frac{j_{0,p} J_2'(j_{0,p})}{J_2(j_{0,p})} > 0, \quad \forall \quad p \in \mathbb{N}^*.$$

Now, (19) gives

$$\frac{J_2'(j_{0,p})}{J_2(j_{0,p})} = \frac{2}{j_{0,p}} - \frac{J_3(j_{0,p})}{J_2(j_{0,p})} = \frac{j_{0,p}^2 - 4}{2 j_{0,p}} \quad \Rightarrow \quad \frac{J_2(j_{0,p})}{J_3(j_{0,p})} = \frac{2 j_{0,p}}{8 - j_{0,p}^2}.$$

Therefore with this equality and (19)

$$\frac{J_3'(j_{0,p})}{J_3(j_{0,p})} = \frac{-3}{j_{0,p}} + \frac{J_2(j_{0,p})}{J_3(j_{0,p})} = \frac{-3}{j_{0,p}} + \frac{2 j_{0,p}}{8 - j_{0,p}^2} = \frac{5 j_{0,p}^2 - 24}{j_{0,p}\left(8 - j_{0,p}^2\right)}.$$

Let us define $f(x) = 1 + x\frac{5x^2-24}{x(8-x^2)} = 4\frac{x^2-4}{8-x^2}$.

Then $f(x) > 0\, \forall x \in \left]2, 2\sqrt{2}\right[$ and $f(x) < 0\, \forall x \in]0, 2[\cup \left]2\sqrt{2}, +\infty\right[$.

But $j_{0,1} \in \left]2, 2\sqrt{2}\right[$ so $f(j_{0,1}) > 0$ whereas $j_{0,k} \geq j_{0,2} > 2\sqrt{2}$ so $f(j_{0,k}) < 0\, \forall k \geq 2$ so we proved lemma 4. □

3.3 Double eigenvalues of the unit disk

Lemma 5 *Let λ_s be an eigenvalue of the Dirichlet–Laplacian which is double for the disk. Then, for $s \neq 3$, λ_s is not locally minimized by the disk among sets of constant measure.*

We are in the case $m \neq 0$ and $A(\varepsilon)\lambda(\Omega_\varepsilon)$ is given by (18).

For simplicity let us choose $a_{2m} = 0$. Therefore, (18) becomes

$$A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{m,p}^2 \Bigg(1 + 2\varepsilon^2 \Bigg[\sum_{|l|\neq m} \left(1 + j_{m,p}\frac{J_l'(j_{m,p})}{J_l(j_{m,p})}\right)|a_{m-l}|^2 + \frac{\overline{\alpha_m}}{\alpha_m} \underbrace{\left(-b_{2m} + \sum_{|l|\neq m}\left(\frac{1}{2} + j_{m,p}\frac{J_l'(j_{m,p})}{J_l(j_{m,p})}\right) a_{m+l}a_{m-l}\right)}_{\Psi}\Bigg]\Bigg) + O(\varepsilon^3).$$

It is then easy to nullify Ψ (choice of b_{2m}). So we just have to study

$$A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{m,p}^2 \left(1 + 2\varepsilon^2 \sum_{|l|\neq m} \left(1 + j_{m,p}\frac{J_l'(j_{m,p})}{J_l(j_{m,p})}\right)|a_{m-l}|^2\right) + O(\varepsilon^3). \tag{23}$$

Thus, if we find a l such that

$$\left(1 + \frac{j_{m,p}J_l'(j_{m,p})}{J_l(j_{m,p})}\right) + \left(1 + \frac{j_{m,p}J_{2m-l}'(j_{m,p})}{J_{2m-l}(j_{m,p})}\right) < 0$$

($|a_{m-l}| = |a_{l-m}|$) we will be able to find some (a_n) and (b_n) satisfying the necessary conditions and such that $A(\varepsilon)\lambda(\Omega_\varepsilon) < \pi\lambda(\Omega_0)$, which proves Lemma 5.

Lemma 6

$$\forall m > 1,\ \forall p \in \mathbb{N}^* \quad \left(1 + \frac{j_{m,p}J_{m+2}'(j_{m,p})}{J_{m+2}(j_{m,p})}\right) + \left(1 + \frac{j_{m,p}J_{m-2}'(j_{m,p})}{J_{m-2}(j_{m,p})}\right) < 0.$$

Lemma 7

$$\left(1 + \frac{j_{1,1}J_2'(j_{1,1})}{J_2(j_{1,1})}\right) + \left(1 + j_{1,1}\frac{J_4'(j_{1,1})}{J_4(j_{1,1})}\right) > 0$$
$$\textit{and} \quad \left(1 + \frac{j_{1,p}J_2'(j_{1,p})}{J_2(j_{1,p})}\right) + \left(1 + j_{1,p}\frac{J_4'(j_{1,p})}{J_4(j_{1,p})}\right) < 0 \quad \forall p \geq 2.$$

Proof of Lemma 5 The idea of this proof is similar to the previous one.

Let us begin with the case $m > 1$, $p \in \mathbb{N}^*$.

From Lemma 6, if we choose (a_n) given by $a_i = 0, \forall |i| \neq 2$, $a_2 \neq 0$ small enough and $a_{-2} = \overline{a_2}$, and for (b_n), such that $b_0 = -|a_2|^2$, $b_k = 0$ for $k \neq 0$, then

$$A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{m,p}^2 + 4\pi\varepsilon^2 j_{m,p}^2 |a_2|^2 \times \underbrace{\left(\left(1 + \frac{j_{m,p}J_{m+2}'(j_{m,p})}{J_{m+2}(j_{m,p})}\right) + \left(1 + \frac{j_{m,p}J_{m-2}'(j_{m,p})}{J_{m-2}(j_{m,p})}\right)\right)}_{<0} + O(\varepsilon^3).$$

Remark that it corresponds to the case

$$R(\theta,\varepsilon) = 1 + 2\varepsilon\left[\mathrm{Re}(a_2)\cos(2\theta) - \mathrm{Im}(a_2)\sin(2\theta)\right] - \varepsilon^2|a_2|^2 + O(\varepsilon^3).$$

So there exist shapes obtained from the unit disk by small variations for which $A(\varepsilon)\lambda(\Omega_\varepsilon) < \pi\lambda(\Omega_0)$ when $\lambda(\Omega_0) = j_{m,p}^2$, $m > 1$ and $p > 0$.

Now for $m = 1$, $p > 1$, from Lemma 7, if we choose (a_n) given by $a_i = 0, \forall |i| \neq 3$, $a_3 \neq 0$ small enough and $a_{-3} = \overline{a_3}$, and for (b_n) such that $b_0 = -|a_3|^2$, $b_k = 0$ then

$$A(\varepsilon)\lambda(\Omega_\varepsilon) = \pi j_{1,p}^2 + 4\pi\varepsilon^2 j_{1,p}^2 |a_3|^2 \times \underbrace{\left(\left(1 + \frac{j_{1,p} J_2'(j_{1,p})}{J_2(j_{1,p})}\right) + \left(1 + \frac{j_{1,p} J_4'(j_{1,p})}{J_4(j_{1,p})}\right)\right)}_{<0} + O(\varepsilon^3) \quad \forall p \geq 2.$$

Remark that it corresponds to the case

$$R(\theta, \varepsilon) = 1 + 2\varepsilon\,[\mathrm{Re}(a_3)\cos(3\theta) - \mathrm{Im}(a_3)\sin(3\theta)] - \varepsilon^2|a_3|^2 + O(\varepsilon^3).$$

So there exist shapes obtained from the unit disk by small variations for which $A(\varepsilon)\lambda(\Omega_\varepsilon) < \pi\lambda(\Omega_0)$ when $\lambda(\Omega_0) = j_{1,p}^2$ and $p > 1$.

Notice that $j_{1,1}^2 = \lambda_2(\Omega_0) = \lambda_3(\Omega_0)$. □

Proof of Lemma 6 One more time, we use (21) and $J_m(j_{m,p}) = 0$ so

$$\frac{2(m+1)}{j_{m,p}} J_{m+1}(j_{m,p}) = J_m(j_{m,p}) + J_{m+2}(j_{m,p}) = J_{m+2}(j_{m,p}) \Rightarrow \frac{J_{m+2}(j_{m,p})}{J_{m+1}(j_{m,p})} = \frac{2(m+1)}{j_{m,p}}. \tag{24}$$

(19) and (24) gives

$$\frac{J_{m+1}'(j_{m,p})}{J_{m+1}(j_{m,p})} = \frac{m+1}{j_{m,p}} - \frac{J_{m+2}(j_{m,p})}{J_{m+1}(j_{m,p})} = -\frac{m+1}{j_{m,p}}. \tag{25}$$

Therefore,

$$1 + \frac{j_{m,p} J_{m+1}'(j_{m,p})}{J_{m+1}(j_{m,p})} = 1 + j_{m,p} \times \left(-\frac{m+1}{j_{m,p}}\right) = -m < 0, \quad \forall p \in \mathbb{N}^*, \quad \forall m \geq 1.$$

Then we rewrite (25)

$$\frac{J_{m+2}(j_{m,p})}{J_{m+1}(j_{m,p})} = \frac{2(m+1)}{j_{m,p}} \Rightarrow \frac{J_{m+1}(j_{m,p})}{J_{m+2}(j_{m,p})} = \frac{j_{m,p}}{2(m+1)}.$$

With this equation and (20)

$$\frac{J_{m+2}'(j_{m,p})}{J_{m+2}(j_{m,p})} = -\frac{m+2}{j_{m,p}} + \frac{J_{m+1}(j_{m,p})}{J_{m+2}(j_{m,p})} = -\frac{m+2}{j_{m,p}} + \frac{j_{m,p}}{2(m+1)} = \frac{j_{m,p}^2 - 2(m+2)(m+1)}{2(m+1)\,j_{m,p}}. \tag{26}$$

So

$$1 + \frac{j_{m,p} J_{m+2}'(j_{m,p})}{J_{m+2}(j_{m,p})} = \frac{j_{m,p}^2 - 2(m+1)^2}{2(m+1)}, \quad \forall p \in \mathbb{N}^*. \tag{27}$$

On the other hand, for $m > 1$, (21) gives

$$\frac{2(m-1)}{j_{m,p}} J_{m-1}(j_{m,p}) = J_m(j_{m,p}) + J_{m-2}(j_{m,p}) = J_{m-2}(j_{m,p}) \Rightarrow \frac{J_{m-1}(j_{m,p})}{J_{m-2}(j_{m,p})} = \frac{j_{m,p}}{2(m-1)}.$$

With this equation and (19)

$$\frac{J'_{m-2}(j_{m,p})}{J_{m-2}(j_{m,p})} = \frac{m-2}{j_{m,p}} - \frac{J_{m-1}(j_{m,p})}{J_{m-2}(j_{m,p})} = \frac{m-2}{j_{m,p}} - \frac{j_{m,p}}{2(m-1)}$$
$$= \frac{2(m-1)(m-2) - j_{m,p}^2}{2(m-1)\,j_{m,p}}. \tag{28}$$

So

$$1 + \frac{j_{m,p}J'_{m-2}(j_{m,p})}{J_{m-2}(j_{m,p})} = \frac{2(m-1)^2 - j_{m,p}^2}{2(m-1)}, \quad \forall p \in \mathbb{N}^*.$$

Still for $m > 1$, $\forall p \in \mathbb{N}^*$, from (27) and (28)

$$\left(1 + \frac{j_{m,p}J'_{m+2}(j_{m,p})}{J_{m+2}(j_{m,p})}\right) + \left(1 + \frac{j_{m,p}J'_{m-2}(j_{m,p})}{J_{m-2}(j_{m,p})}\right)$$
$$= \frac{j_{m,p}^2 - 2(m+1)^2}{2(m+1)} + \frac{2(m-1)^2 - j_{m,p}^2}{2(m-1)} = -\frac{j_{m,p}^2 + 2(m+1)(m-1)}{(m+1)(m-1)}$$
$$= -\left(\frac{j_{m,p}^2}{(m+1)(m-1)} + 2\right) < 0.$$

□

Proof of Lemma 7 Now, with (26) and (19) we have

$$\frac{J'_{m+2}(j_{m,p})}{J_{m+2}(j_{m,p})} = \frac{m+2}{j_{m,p}} - \frac{J_{m+3}(j_{m,p})}{J_{m+2}(j_{m,p})} = \frac{j_{m,p}^2 - 2(m+1)(m+2)}{2(m+1)\,j_{m,p}}$$
$$\Rightarrow \frac{J_{m+2}(j_{m,p})}{J_{m+3}(j_{m,p})} = \frac{2(m+1)\,j_{m,p}}{4(m+1)(m+2) - j_{m,p}^2}.$$

Therefore, with this equation and (20)

$$\frac{J'_{m+3}(j_{m,p})}{J_{m+3}(j_{m,p})} = \frac{-(m+3)}{j_{m,p}} + \frac{J_{m+2}(j_{m,p})}{J_{m+3}(j_{m,p})}$$
$$= \frac{-(m+3)}{j_{m,p}} + \frac{2(m+1)\,j_{m,p}}{4(m+1)(m+2) - j_{m,p}^2}$$
$$= \frac{(3m+5)\,j_{m,p}^2 - 4(m+1)(m+2)(m+3)}{j_{m,p}\left[4(m+1)(m+2) - j_{m,p}^2\right]}. \tag{29}$$

In the particular case $m = 1$, we obtain, respectively, from (25) and (29)

$$1 + \frac{j_{1,p}J'_2(j_{1,p})}{J_2(j_{1,p})} = -1 \quad \text{and} \quad 1 + j_{1,p}\frac{J'_4(j_{1,p})}{J_4(j_{1,p})} = 1 + \frac{8j_{1,p}^2 - 96}{24 - j_{1,p}^2}$$

so

$$\left(1 + \frac{j_{1,p}J'_2(j_{1,p})}{J_2(j_{1,p})}\right) + \left(1 + j_{1,p}\frac{J'_4(j_{1,p})}{J_4(j_{1,p})}\right) = \frac{8j_{1,p}^2 - 96}{24 - j_{1,p}^2}.$$

Let us define $f(x) = \frac{8x^2-96}{24-x^2}$. Then $f(x) > 0$ for $x \in]2\sqrt{3}, 2\sqrt{6}[$ and $f(x) < 0$ for $x \in [0, 2\sqrt{3}[\cup]2\sqrt{6}, +\infty[$. But $j_{1,1} \in]2\sqrt{3}, 2\sqrt{6}[$ so $f(j_{1,1}) > 0$ whereas $j_{1,p} \geq j_{1,2} > 2\sqrt{6}$ so $f(j_{1,p}) < 0 \, \forall p \geq 2$. So we proved Lemma 7. □

Conclusion With Lemmas 3 and 5 we have proved exactly Theorem 4.

Remark 6 It is clear that if λ_k is not a local minimum, then it is not a global minimum.

4 Proof of Theorem 5

In this section, we use the previous results on disks to obtain information on the optimality of disjoint unions of disks. For that, we need a result linking eigenvalues of disjoint unions of disks and of disks. Wolf and Keller obtained such a result in [15].

Let us define the open set Ω_n^* by $|\Omega_n^*| = 1$ and satisfying

$$\lambda_n\left(\Omega_n^*\right) = \min\{\lambda_n(\Omega);\ \Omega \text{ open set, } |\Omega| = 1\}$$

and

$$\lambda_n^* = \lambda_n\left(\Omega_n^*\right) = \min\{\lambda_n(\Omega);\ \Omega \text{ open set, } |\Omega| = 1\}.$$

Theorem 9 (Wolf–Keller, dimension N) *Suppose that $\Omega_n^* \in \mathbb{R}^N$ is the union of at least two disjoint open sets of positive measure. Then*

$$\left(\lambda_n^*\right)^{N/2} = \left(\lambda_i^*\right)^{N/2} + \left(\lambda_{n-i}^*\right)^{N/2} = \min_{1\leq j\leq \frac{n-1}{2}} \left[\left(\lambda_j^*\right)^{N/2} + \left(\lambda_{n-j}^*\right)^{N/2}\right] \tag{30}$$

where i is a value of $1 \leq j \leq \frac{n-1}{2}$ minimizing the sum $(\lambda_j^*)^{N/2} + (\lambda_{n-j}^*)^{N/2}$. Moreover,

$$\Omega_n^* = \left[\left(\frac{\lambda_i^*}{\lambda_n^*}\right)^{1/2} \Omega_i^*\right] \bigcup \left[\left(\frac{\lambda_{n-i}^*}{\lambda_n^*}\right)^{1/2} \Omega_{n-i}^*\right] \quad \textit{(disjoint union).} \tag{31}$$

This theorem gives an iterative way of finding candidates for unions of sets minimizing eigenvalues of the Dirichlet–Laplacian. It can easily be restricted to disks and disjoint unions of disks of same measure.

Therefore, we can obtain the disjoint union of disks, eventually the disk, of measure one which minimizes each eigenvalue. We can do exact computations. But for simplicity I used a computer to obtain the results given in Fig. 1. Notice that the results given here are approximations but I did them at the very end of the process. Moreover these results are essential for the proof of Theorem 5.

Now we want to identify if there exists such a union of disks which can be a minimizer of a given eigenvalue. For that, we use one more time iteratively Theorem 9 to find candidates. We can then compare with results of Fig. 1 to determine if the union candidate is better than any other union of disks. If the candidate is not in Fig. 1 then it cannot be a minimizer by construction. We also have the possibility to compare with the eigenvalues of the rectangles since the formulae are known. In fact, recall that if $\Omega = [0, L] \times \left[0, \frac{1}{L}\right]$ then $\lambda_{m,n} = \pi^2\left(\frac{m^2}{L^2} + L^2 n^2\right), \forall m, n \geq 1$.

With these observations and results of previous sections we can establish Theorem 5.

Proof of Theorem 5 – Suppose that Ω_k^* is a union of two disjoint disks. According to Theorem 9, there exists an $1 \leq i \leq k$ such that Ω_i^* and Ω_{n-i}^* are both disks. We deduce

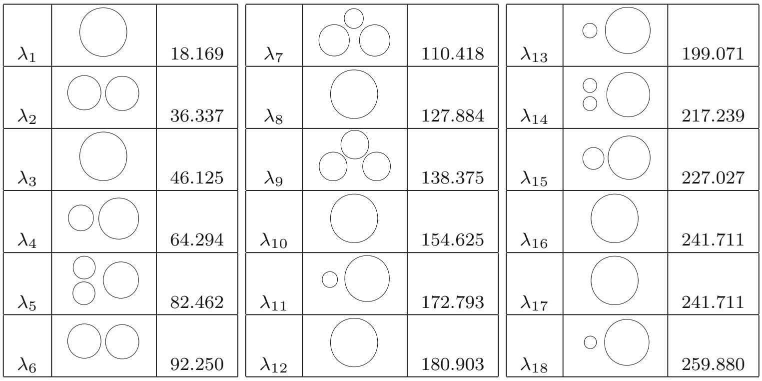


Fig. 1 Disjoint unions of disks, eventually the disk, of measure one which minimizes eigenvalues of the Laplacian with Dirichlet boundary condition and their approximated values

that at best $i \in \{1, 3\}$ and $n - i \in \{1, 3\}$. Therefore if $k \notin \{2, 4, 6\}$ λ_k is not minimized by a union of two disjoint disks.

– Suppose that Ω_k^* is a union of three disjoint disks. According to Theorem 9, there exists an $1 \le i \le k$ such that Ω_i^* is a disk and Ω_{n-i}^* is a union of two disjoint disks. It follows that at best $i \in \{1, 3\}$ and $n - i \in \{2, 4, 6\}$. Therefore, the only possibilities are $k = 3$, $k = 5$, $k = 7$ and $k = 9$.
  But we have to exclude $k = 3$ since the eigenvalue of one disk is smaller than for any union of disks (cf. Fig. 1).
  Then, Antunes and Freitas in [3] found the minimizing rectangle for λ_5 (whose lengths are $(5/3)^{1/4}$ and its inverse). The eigenvalue of this rectangle ($\simeq$81.5463) is smaller than the one for any union of three disks ($\simeq$82.462 for the best union of three disks from Fig. 1).
  So if $k \notin \{7, 9\}$ λ_k is not minimized by a union of three disjoint disks.
– Suppose that Ω_k^* is a union of four disjoint disks. We have here to consider two cases:
  - there exists an $1 \le i \le k$ such that Ω_i^* is a disk and Ω_{n-i}^* is a union of three disjoint disks. It follows that at best $i \in \{1, 3\}$ and $n - i \in \{7, 9\}$. Therefore, the only possibilities are $k = 8$, $k = 10$ and $k = 12$,
  - there exists an $1 \le i \le k$ such that Ω_i^* and Ω_{n-i}^* are both unions of two disjoint disks. It follows that at best $i \in \{2, 4, 6\}$ and $n - i \in \{2, 4, 6\}$. Therefore, the only possibilities are $k = 4$, $k = 6$, $k = 8$, $k = 10$ and $k = 12$.

  But for all these last three eigenvalues, the eigenvalue of one disk is smaller than the ones of any union of disks (cf. Fig. 1).
  On the other hand λ_4 and λ_6 are smaller in the case of the union of two disks than in the case of a union of four disks (cf. Fig. 1).
  So $\forall k$, λ_k is not minimized by a union of four disjoint disks.
– Suppose that Ω_k^* is a union of five disjoint disks. We have here to consider two cases:
  - there exists an $1 \le i \le k$ such that Ω_i^* is a disk and Ω_{n-i}^* is a union of four disjoint disks. From previous considerations this is not possible,
  - there exists an $1 \le i \le k$ such that Ω_i^* is a union of two disjoint disks and Ω_{n-i}^* is a union of three disjoint disks. It follows that at best $i \in \{2, 4, 6\}$ and $n - i \in \{7, 9\}$. Therefore the only possibilities are $k = 9$, $k = 11$, $k = 13$ and $k = 15$.

But λ_9 is smaller in the case of a union of three disks than in the case of a union of five disks (cf. Fig. 1). Likewise, λ_{11}, λ_{13} and λ_{15} are smaller in the case of a union of two disks than in the case of a union of five disks (cf. Fig. 1).
So $\forall k$, λ_k is not minimized by a union of five disjoint disks.

– Suppose that Ω_k^* is a union of six disjoint disks. We have to consider three distinct cases:
  - there exists an $1 \leq i \leq k$ such that Ω_i^* is a disk and Ω_{n-i}^* is a union of five disjoint disks. This is not possible from previous items,
  - there exists an $1 \leq i \leq k$ such that Ω_i^* is a union of two disks and Ω_{n-i}^* is a union of four disjoint disks. This is not possible from previous items,
  - there exists an $1 \leq i \leq k$ such that Ω_i^* and Ω_{n-i}^* are both unions of three disjoint disks. It follows that at best $i \in \{7, 9\}$ and $n - i \in \{7, 9\}$. Therefore, the only possibilities are $k = 14$, $k = 16$ and $k = 18$.

But λ_{14} is smaller in the case of a union of three disks than in the case of a union of six disks (cf. Fig. 1). Likewise, λ_{16} (resp. λ_{18}) is smaller in the case of one disk (resp. the union of two disks) than in the case of a union of six disks (cf. Fig. 1).
So $\forall k$, λ_k is not minimized by a union of six disjoint disks.

– Suppose that Ω_k^* is a union of m disjoint disks with $m > 6$. The different possibilities are:
  - there exists an $1 \leq i \leq k$ such that Ω_i^* is a disk and Ω_{n-i}^* is a union of $m - 1 > 5$ disjoint disks, but we see by induction that it is not possible,
  - there exists an $1 \leq i \leq k$ such that Ω_i^* is a union of two disjoint disks and Ω_{n-i}^* is a union of $m - 2 > 4$ disjoint disks, but we see by induction that it is not possible,
  - there exists an $1 \leq i \leq k$ such that Ω_i^* is a union of three disjoint disks and Ω_{n-i}^* is a union of $m - 3 > 3$ disjoint disks, but we see by induction that it is not possible,
  - the other possibilities of unions of disks for Ω_i^* are not possible.

So $\forall k$, λ_k is not minimized by a union of more than 6 disjoint disks. □

Theorem 5 can be improved using numerical results. In fact if we find shapes, not necessarily explicit, for which the eigenvalues are less than the ones of the best union of disks, then these unions cannot be minimizers. For that we can use the numerical results of Oudet [14], the improved ones of Freitas and Antunes [2] or the more recent ones given in Fig. 2. These last results were obtained by Oudet and me using finite element methods and are consistent with the results of Freitas and Antunes.

We then can see that shapes exist for which λ_6 and λ_7 are smaller than for any union of disks. Taking this into account in the previous proof simplify it and the possibility for λ_9 disappears.

Remark 7 Notice that Antunes and Freitas used a method different from the one used by Oudet and me to obtain these shapes. Moreover both these methods ensure that the numerical values are greater than the real ones.

5 Proof of Theorem 7

The main idea of the proof of Theorem 5 can be used to show Theorem 7. For that we first need to state a corollary of Theorem 9 in the case of local minima.

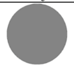
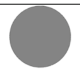
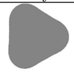
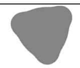
i	Ω_i^*	λ_i^*
1		18.169
2		36.337
3		46.126
4		64.306
5		78.166
6		88.502
7		106.211
8		118.970
9		132.493
10		142.746
11		159.821
12		173.035
13		186.977
14		199.286
15		209.954

Fig. 2 Shapes numerically obtained suggested as minimizers of the eigenvalues of the Laplacian with Dirichlet boundary condition. These results obtained by Oudet and me are consistent with the ones obtained by Freitas and Antunes in [2] (cf. Remark 7)

Let us denote λ_k^* a local minimum of λ_k achieved by Ω_k^*, open set of measure 1.

Corollary 1 (Local version of Theorem 9) *Let λ_n^* be a local minimum of λ_n achieved by Ω_n^* with $|\Omega_n^*| = 1$. Suppose that $\Omega_n^* = \Omega_1 \cup \Omega_2$ (disjoint) with $|\Omega_1| > 0$ and $|\Omega_2| > 0$. Then there exists $1 \leq i < n$ such that*

1. *Ω_1 is a local minimizer of λ_i and Ω_2 is a local minimizer of λ_{n-i},*
2. *if $\lambda_i^* = |\Omega_1|^{2/N}\lambda_i(\Omega_1)$ and $\lambda_{n-i}^* = |\Omega_2|^{2/N}\lambda_{n-i}(\Omega_2)$ then $(\lambda_n^*)^{N/2} = (\lambda_i^*)^{N/2} + (\lambda_{n-i}^*)^{N/2}$,*
3. *if $\Omega_i^* = \frac{\Omega_1}{|\Omega_1|^{1/N}}$ and $\Omega_{n-i}^* = \frac{\Omega_2}{|\Omega_2|^{1/N}}$ then*

$$\Omega_n^* = \left[\left(\frac{\lambda_i^*}{\lambda_n^*}\right)^{1/2}\Omega_i^*\right] \bigcup \left[\left(\frac{\lambda_{n-i}^*}{\lambda_n^*}\right)^{1/2}\Omega_{n-i}^*\right] \quad \textit{(disjoint union).}$$

Proof of Corollary 1 The proof of this corollary is based on the proof of Theorem 5 as we can find in [7]. Thus we will not do details here.

We first show that there exists an $1 \leq i < n$ such that $\lambda_i(\Omega_1) = \lambda_{n-i}(\Omega_2) = \lambda_n^*$ with the same arguments as in pp. 74–75 of [7].

Suppose that $\lambda_i(\Omega_1)$ is not a local minimum of λ_i. So there exists a vector field V such that $\forall \varepsilon > 0$ $\lambda_i(\Omega_1) > \lambda_i(\Omega_1 + \varepsilon V)$ with $|\Omega_1| = |\Omega_1 + \varepsilon V|$.

Let now be $\eta > 0$ and $\Omega_1'' = (1-\eta)\Omega_1' = (1-\eta)(\Omega_1 + \varepsilon V)$. For η small enough, $\lambda_i(\Omega_1') < \lambda_i(\Omega_1'') < \lambda_{n-i}(\Omega_2)$. Let be $\Omega' = \Omega_1'' \cup \Omega_2'$ with $\Omega_2' = \kappa\Omega_2$ such that $|\Omega_1''| + |\Omega_2'| = 1$. Thus $\lambda_{n-i}(\Omega_2) > \lambda_{n-i}(\Omega_2')$ and so $\lambda_n(\Omega') = \max\{\lambda_{n-i}(\Omega_2'), \lambda_i(\Omega_1'')\} < \lambda_{n-i}(\Omega_2) = \lambda_n^*$, which contradicts the hypothesis of local minimality of Ω_n^*.

Thus, $\lambda_i(\Omega_1)$ is a local minimum of λ_i with the constraint measure $|\Omega_1|$.

We similarly show that $\lambda_{n-i}(\Omega_2)$ is a local minimum of λ_{n-i} with the constraint measure $|\Omega_2|$.

Now let us define $\Omega_i^* = \frac{\Omega_1}{|\Omega_1|^{1/N}}$. It is easy to show that $|\Omega_i^*| = 1$ and $\lambda_i(\Omega_i^*) = \lambda_i^* = |\Omega_1|^{2/N}\lambda_n^*$.

Similarly, for $\Omega_{n-i}^* = \frac{\Omega_2}{|\Omega_2|^{1/N}}$ we have $|\Omega_{n-i}^*| = 1$ and $\lambda_{n-i}(\Omega_{n-i}^*) = \lambda_{n-i}^* = |\Omega_2|^{2/N}\lambda_n^*$.

Since $|\Omega_1| + |\Omega_2| = 1$, $(\lambda_i^*)^{N/2} + (\lambda_{n-i}^*)^{N/2} = (\lambda_n^*)^{N/2}$.

Finally let us define

$$\widetilde{\Omega} = \left[\left(\frac{(\lambda_i^*)^{N/2}}{(\lambda_i^*)^{N/2} + (\lambda_{n-i}^*)^{N/2}} \right)^{1/N} \Omega_i^* \right] \bigcup \left[\left(\frac{(\lambda_{n-i}^*)^{N/2}}{(\lambda_i^*)^{N/2} + (\lambda_{n-i}^*)^{N/2}} \right)^{1/N} \Omega_{n-i}^* \right].$$

With arguments similar to [7] we show that $|\widetilde{\Omega}| = 1$ and $\lambda_n(\widetilde{\Omega}) = \lambda_n^*$. □

Before giving the proof of Theorem 7, let us exhibit the result for the first values of n.

A union of	1	disjoint disks can locally minimize at most	λ_1 and λ_3.
	2		λ_2, λ_4 and λ_6.
	⋮		⋮

Proof of Theorem 7 Let us do an induction. The result is clearly true for $n = 1$ from what precedes. Suppose now that the theorem is true from 1 to $n - 1$ and let us prove that it is true for n. We are looking for ks for which λ_k can possibly be locally minimized by a disjoint union of n disks. So there exists an $1 \leq m \leq n$ such that $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 a disjoint union of m disks and Ω_2 a disjoint union of $n - m$ disks. According to Corollary 1, Ω can be a local minimizer of λ_i only if $i = i_1 + i_2$ where Ω_1 is a local minimizer of λ_{i_1} and Ω_2 is a local minimizer of λ_{i_2}. From the hypothesis of the induction we have that

$$i_1 \in \{m + 2l_1;\ l_1 = 0, \ldots, m\} \quad \text{and} \quad i_2 \in \{n - m + 2l_2;\ l_2 = 0, \ldots, (n - m)\}.$$

Therefore,

$$i = i_1 + i_2 \in \{n + 2l;\ l = 0, \ldots, n\} \quad (\text{independent of } m)$$

which concludes this proof. □

Remark 8 This theorem does not say that these unions of disks are local minimizers of the corresponding eigenvalue but only that it is possible.

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References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables. Dover, New York (2012)
2. Antunes, P.R., Freitas, P.: Numerical optimization of low eigenvalues of the Dirichlet and Neumann Laplacians. J. Optim. Theory Appl. **154**(1), 235–257 (2012)
3. Antunes, P.R., Freitas, P.: Optimal spectral rectangles and lattice ellipses. In: Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, vol. 469, no. 2150 (2013)
4. Bucur, D.: Minimization of the k-th eigenvalue of the Dirichlet Laplacian. Arch. Ration. Mech. Anal. **206**(3), 1073–1083 (2012)
5. Buttazzo, G., Dal Maso, G.: An existence result for a class of shape optimization problems. Arch. Ration. Mech. Anal. **122**(2), 183–195 (1993)
6. Gustafson, K.: The RKNG (Rellich, Kato, Sz-Nagy, Gustafson) perturbation theorem for linear-operators in Hilbert and Banach-space. Acta Sci. Math. **45**(1–4), 201–211 (1983)
7. Henrot, A.: Extremum Problems for Eigenvalues of Elliptic Operators. Springer, Berlin (2006)
8. Henrot, A., Bucur, D.: Minimization of the third eigenvalue of the Dirichlet Laplacian. In: Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, vol. 456, no. 1996, pp. 985–996 (2000)

9. Henrot, A., Pierre, M.: Variation et optimisation de formes: une analyse géométrique, vol. 48. Springer, New York (2006)
10. Kato, T.: Perturbation Theory for Linear Operators, vol. 132. Springer, Berlin (1995)
11. Mazzoleni, D., Pratelli, A.: Existence of minimizers for spectral problems. J. Math. Pures Appl. **100**(3), 433–453 (2013)
12. Micheletti, A.M.: Perturbazione dello spettro dell'operatore di Laplace, in relazione ad una variazione del campo. Ann. Sc. Norm. Super. Pisa. **26**(1), 151–169 (1972)
13. Nagy, BdS: Perturbations des transformations autoadjointes dans l'espace de Hilbert. Comment. Math. Helv. **19**(1), 347–366 (1946)
14. Oudet, É.: Numerical minimization of eigenmodes of a membrane with respect to the domain. ESAIM Control Optim. Calc. Var. **10**(03), 315–330 (2004)
15. Wolf, S.A., Keller, J.B.: Range of the first two eigenvalues of the Laplacian. Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **447**(1930), 397–412 (1994)