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The Lagrangian cobordism group of T^2

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Abstract We compute the Lagrangian cobordism group of the standard symplectic 2-torus and show that it is isomorphic to the Grothendieck group of its derived Fukaya category. The proofs use homological mirror symmetry for the 2-torus.

Keywords Symplectic manifolds · Lagrangian submanifolds · Lagrangian cobordisms · Fukaya categories · Homological mirror symmetry

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Contents

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1 Introduction

The Fukaya category $\mathscr{F}(M)$ of a symplectic manifold (M, ω) is an A_∞-category whose objects are the Lagrangian submanifolds of M, and whose morphism spaces are Floer cochain groups. Its derived category $D\mathscr{F}(M)$ is triangulated, and thus, it possesses a mechanism for generating objects by taking cones of morphisms. Recent work of Biran–Cornea [5,6] provides a way of understanding cone decompositions of objects in $D\mathscr{F}(M)$ geometrically via Lagrangian cobordisms. According to their definition, a Lagrangian cobordism $V : (L_1, \dots, L_r) \rightsquigarrow (L'_1, \dots, L'_s)$ between tuples of Lagrangians in M is a Lagrangian submanifold V of $\mathbb{R}^2 \times M$ with cylindrical ends corresponding to the L_i and L'_j. Figure 1 displays the projection of such a cobordism to $\mathbb{R}^2$. The main result of [5,6] is that a Lagrangian cobordism of the form

$$V : L \rightsquigarrow (L_1, \dots, L_s)$$

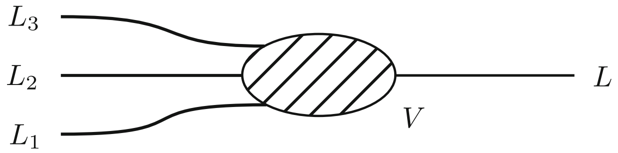

Fig. 1 Projection of a Lagrangian cobordism $V : L \rightsquigarrow (L_1, L_2, L_3)$

leads to an iterated cone decomposition of L in $D\mathscr{F}(M)$ whose "building blocks" are the L_i.

Partial information about the triangulated structure of $D\mathscr{F}(M)$ is captured by its *Grothendieck group* $K_0(D\mathscr{F}(M))$. It is generated by the objects of $\mathscr{F}(M)$, that is, the Lagrangians in M, with relations coming from exact triangles in $D\mathscr{F}(M)$. On the cobordism side, one can naturally define a *Lagrangian cobordism group* $\Omega_{\mathrm{Lag}}(M)$. It is generated by the Lagrangians in M, or a suitable subset thereof, and has relations coming from Lagrangian cobordisms. As an immediate consequence of Biran–Cornea's results, there exists a surjective group homomorphism

$$\Theta : \Omega_{\mathrm{Lag}}(M) \to K_0(D\mathscr{F}(M)),$$

induced by $L \mapsto L$. It is natural to ask whether Θ is an isomorphism, and if not, what its kernel is. This question formalizes the question to what extent the triangulated structure of $D\mathscr{F}(M)$ can be explained geometrically by the existence of Lagrangian cobordisms. In general, nothing is known about the kernel of Θ.

Main results The case we consider is when (M, ω) is the standard symplectic 2-torus $(T^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega_{\mathrm{std}} = dx \wedge dy)$. The version of the Lagrangian cobordism group $\Omega_{\mathrm{Lag}}(T^2)$ we study is generated by non-contractible simple closed curves in T^2.

Our first main result is the following statement.

Theorem 1.1 *The natural group homomorphism*

$$\Theta : \Omega_{\mathrm{Lag}}(T^2) \to K_0(D\mathscr{F}(T^2))$$

is an isomorphism.

Theorem 1.1 has implications of three different kinds. First, as already indicated above, it tells us that the set of relations in $K_0(\mathscr{F}(T^2))$ is generated by ones coming from Lagrangian cobordisms. Second, together with Theorem 1.2, it can be regarded as a computation of $K_0(D\mathscr{F}(T^2))$, which to the author's knowledge has not been carried out before (but cf. [1] for a computation of $K_0(D\mathscr{F}(\Sigma))$ for higher genus surfaces). Third, it gives information about Lagrangian cobordisms in $\mathbb{R}^2 \times T^2$. There are two known constructions of such cobordisms based on Hamiltonian isotopy and on Lagrangian surgery, but we do not know whether the resulting cobordisms are the only ones that exist. Theorem 1.1 does not rule out that there are more, but it (or rather its proof) shows that the known cobordisms generate the set of relations in $\Omega_{\mathrm{Lag}}(T^2)$.

The second main result answers the question what $\Omega_{\mathrm{Lag}}(T^2)$ looks like

Theorem 1.2 *There exists a canonical short exact sequence*

$$0 \to \mathbb{R}/\mathbb{Z} \xrightarrow{\zeta} \Omega_{\mathrm{Lag}}(T^2) \xrightarrow{\eta} H_1(T^2; \mathbb{Z}) \to 0.$$

The map η is the obvious one, given by $[L]_\Omega \mapsto [L]_{H_1}$. The map ζ takes $x \in \mathbb{R}/\mathbb{Z}$ to the class represented by the boundary of a cylinder of area x. In other words, $\zeta(x)$ is represented by the difference $L - L'$, with L any non-contractible curve, and L' any isotopic curve such that the area swept out during any isotopy from L' to L is x (this area is determined by the two curves up to an integer). Since two such curves are Hamiltonian isotopic if and only if they bound a cylinder of area 0, the injectivity of ζ says that $\Omega_{\mathrm{Lag}}(T^2)$ distinguishes different Hamiltonian isotopy classes.

While most of the proof of Theorem 1.2 is quite elementary, it is not clear how to rule out in a direct way possible cobordisms that might obstruct the injectivity of ζ. Our proof relies on the connection to homological mirror symmetry discussed below.

1.1 Proving Theorem 1.1 using HMS

We will first prove an extended version of Theorem 1.1 for which we consider a Fukaya category $\mathscr{F}^\sharp(T^2)$ defined over a Novikov field Λ (consisting of formal power series with $\mathbb{C}$-coefficients), whose objects are Lagrangians in T^2 that are decorated with certain local systems of Λ-vector spaces.[1] The definition of the cobordism group is also modified accordingly: The Lagrangians as well as the cobordisms carry local systems, and there are additional relations coming from short exact sequences of local systems. We denote the resulting group by $\Omega^\sharp_{\mathrm{Lag}}(T^2)$. As in the case without local systems, there is a natural surjective group homomorphism $\Theta^\sharp : \Omega^\sharp_{\mathrm{Lag}}(T^2) \to K_0(D\mathscr{F}^\sharp(T^2))$.

Theorem 1.3 *The natural group homomorphism*

$$\Theta^\sharp : \Omega^\sharp_{\mathrm{Lag}}(T^2) \to K_0(D\mathscr{F}^\sharp(T^2))$$

is an isomorphism.

As the main ingredient in the proof of Theorem 1.3, apart from Biran–Cornea's theory, we use that T^2 is one of the symplectic manifolds for which the homological mirror symmetry conjecture has been proven. The statement of relevance to us is the result of Abouzaid–Smith [3], who on their way to HMS for T^4 construct a triangulated equivalence $D^b(X) \simeq D^\pi \mathscr{F}(T^2)$ between the derived category of coherent sheaves of an elliptic curve X defined over the Novikov field Λ and the split-closed derived Fukaya category of T^2. (A more refined version of HMS for T^2 was recently proven by Lekili–Perutz [14], but this is not needed for our purposes.)

One can adapt Abouzaid–Smith's result to our setting, such as to obtain a triangulated equivalence

$$D^b(X) \simeq D\mathscr{F}^\sharp(T^2).$$

[1] We remark that from certain perspectives, $\mathscr{F}^\sharp(T^2)$ might actually be a more natural version of the Fukaya category to look at than $\mathscr{F}(T^2)$.

(In particular, this shows that taking the split closure of $D\mathscr{F}(T^2)$ or using appropriate local systems is equivalent here.) What needs to be checked is that Abouzaid–Smith's functor, which is given explicitly only on a small collection of split generators, takes every sheaf to a Lagrangian with a local system (as opposed to some summand of some non-trivial complex). We do so by matching up sheaves and Lagrangians with local systems in an inductive manner, using Atiyah's classification of vector bundles on elliptic curves [4], respectively, surgery and cobordisms to understand the structures of D^bX and $D\mathscr{F}^\sharp(T^2)$. The result of this procedure is stated in Proposition 7.3.

The resulting isomorphism between Grothendieck groups

$$K_0(D^b(X)) \cong K_0(D\mathscr{F}^\sharp(T^2))$$

allows us to understand relations in $K_0(D\mathscr{F}^\sharp(T^2))$ via the well-understood group $K_0(D^b(X))$, and to check that the "obvious" inverse to the map $\Theta^\sharp : \Omega^\sharp_{\mathrm{Lag}}(T^2) \to K_0(D\mathscr{F}^\sharp(T^2))$ is well defined. This proves Theorem 1.3, from which Theorem 1.1 follows.

1.2 Organization of the paper

In Sect. 2, we recall the definition of Lagrangian cobordisms and define the groups $\Omega_{\mathrm{Lag}}(M)$ and $\Omega^\sharp_{\mathrm{Lag}}(M)$, and Sect. 3 serves to describe the Fukaya category $\mathscr{F}^\sharp(T^2)$. In Sect. 4, we explain how Lagrangian cobordisms lead to iterated cone decompositions in the derived Fukaya category, focusing on the (small) modifications necessary to make Biran–Cornea's proofs work in our setting. In Sect. 5, we describe in detail the Lagrangian cobordisms resulting from surgering curves in surfaces and discuss some specifics in the case of T^2. Section 6 serves to explain and prove as much as we can about $\Omega_{\mathrm{Lag}}(T^2)$ without any mirror symmetry considerations; in particular, we prove Theorem 1.2 modulo the injectivity of the map $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2)$. In Sect. 7, we examine Abouzaid–Smith's mirror functor (or rather, a version of it that is adapted to our setting); we provide a description of its action on objects that is explicit enough to enable the computations in $K_0(D\mathscr{F}^\sharp(T^2))$ required for the proofs of the main theorems in Sect. 8. Appendix 1 collects some facts on triangulated categories, cone decompositions, and Grothendieck groups; Appendix 2 explains how to get exact triangles from short exact sequences of local systems; and finally, Appendix 3 assembles a couple of statements from Atiyah's classification of vector bundles on elliptic curves [4], which are used in Sect. 7.

2 The Lagrangian cobordism group

2.1 Lagrangian cobordisms

We start by recalling some definitions from [5]. For a symplectic manifold (M, ω), we denote by $(\widetilde{M}, \widetilde{\omega})$ the symplectic manifold obtained by equipping $\widetilde{M} = \mathbb{R}^2 \times M$ with the split symplectic form $\widetilde{\omega} = \omega_{\mathrm{std}} \oplus \omega$, where $\omega_{\mathrm{std}} = dx \wedge dy$ is the standard symplectic form on $\mathbb{R}^2$. We denote by $\pi : \widetilde{M} \to \mathbb{R}^2$ the projection to the first factor, and given any subset $S \subset \mathbb{R}^2$, we write $V|_S = V \cap \pi^{-1}(S)$.

We say that two ordered collections $(L_i)_{i=1}^r$ and $(L'_j)_{j=1}^s$ of Lagrangian submanifolds of M are *Lagrangian cobordant* if there exists a compact cobordism $\left(V; \coprod_i L_i, \coprod_j L'_j\right)$ together with a Lagrangian embedding $V \to [0,1] \times \mathbb{R} \times M \subset \mathbb{R}^2 \times M$ with cylindrical ends, in the sense that there is some $\varepsilon > 0$ such that

$$V|_{[0,\varepsilon)\times\mathbb{R}} = \coprod_{i=1}^{r} [0,\varepsilon) \times \{i\} \times L_i \quad \text{and} \quad V|_{(1-\varepsilon,1]\times\mathbb{R}} = \coprod_{j=1}^{s} (1-\varepsilon, 1] \times \{j\} \times L'_j.$$

The Lagrangian submanifold V of $\widetilde{M}$ is called a *Lagrangian cobordism* with positive ends $(L'_j)_{j=1}^s$ and negative ends $(L_i)_{i=1}^r$. The terminology is that V goes from $(L'_j)_{j=1}^s$ to $(L_i)_{i=1}^r$, and we denote this relationship by

$$V : (L'_1, \ldots, L'_s) \rightsquigarrow (L_1, \ldots, L_r).$$

Example 1 Hamiltonian isotopy: Let $\Phi : M \to M$ be a Hamiltonian diffeomorphism, and let $\phi : [0,1] \times M \to M$ be a Hamiltonian isotopy with $\phi(0,\cdot) = \mathrm{id}$, $\phi(1,\cdot) = \Phi$, which is generated by a Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ such that $H(t,\cdot) \equiv 0$ for t close to 0 and 1 (this condition can be achieved by suitably reparametrizing any given Hamiltonian isotopy). Then, for any Lagrangian submanifold $L \subset M$, the map

$$[0,1] \times L \to \mathbb{R}^2 \times M, \quad (t,x) \mapsto (t, \, -H(t,\phi_t(x)), \, \phi_t(x)),$$

defines a Lagrangian cobordism $V : \phi_1(L) \rightsquigarrow L$.

Example 2 Lagrangian surgery: Let $L_0, L_1 \subset M$ be two transversely intersecting Lagrangian submanifolds. One can resolve the intersection points by cutting out small neighborhoods and gluing in Lagrangian handles diffeomorphic to $[-1,1] \times S^{n-1}$ (where $n = \frac{1}{2}\dim M$). This produces a new Lagrangian submanifold which we denote by $L_0 \# L_1$. Biran–Cornea [5] show that there exists a Lagrangian cobordism

$$V : L_0 \# L_1 \rightsquigarrow (L_0, L_1).$$

This construction will be of prime importance later on; we will describe it in more detail in Sect. 5.

2.2 Extra data

Whenever Lagrangians come equipped with extra data, such as orientations, $Spin$ or Pin structures, gradings, or local systems, it makes sense to consider cobordisms over which these data extend. We say that two collections $(L_i)_{i=1}^r$ and $(L'_j)_{j=1}^s$ of Lagrangians decorated with such extra data are Lagrangian cobordant if there exists a Lagrangian cobordism $V : (L'_j)_{j=1}^s \rightsquigarrow (L_i)_{i=1}^r$ between the underlying Lagrangians together with choices of the same types of extra data for V which restrict to the given data on the ends (provided there exists a suitable notion of restricting to boundary components).

2.3 The Lagrangian cobordism group

Let $\mathscr{L}$ be the set of (suitably qualified) Lagrangian submanifolds of M, and let $\langle \mathscr{L} \rangle$ be the free Abelian group generated by $\mathscr{L}$. Denote by $R \subset \langle \mathscr{L} \rangle$ the subgroup generated by all expressions

$$L_1 + \cdots + L_r - L'_1 - \cdots - L'_s \in \langle \mathscr{L} \rangle$$

such that there is a (suitably qualified) Lagrangian cobordism $V : (L'_1, \ldots, L'_s) \rightsquigarrow (L_1, \ldots, L_r)$. The Lagrangian cobordism group corresponding to $\mathscr{L}$ and R is then defined as

$$\Omega_{\mathrm{Lag}}(M) = \langle \mathscr{L} \rangle / R,$$

where we suppress the dependence of $\mathscr{L}$ and R in the notation. An analogous definition applies when Lagrangians carry extra data, in which case one imposes relations coming from cobordisms with the same types of extra data.

As indicated, it usually makes sense to constrain the Lagrangians and cobordisms one admits in the definition of $\Omega_{\mathrm{Lag}}(M)$, because Lagrangian cobordism without any additional condition is a quite flexible notion. One possibility is to require all Lagrangians and cobordisms to be (uniformly) monotone, as is done in [5,6]. In the case $M = T^2$, this paper deals with, we will require that the Lagrangians are non-contractible curves and that the cobordisms have vanishing Maslov class.

Remark 2.1 Constraining the cobordisms one allows in the definition of $\Omega_{\mathrm{Lag}}(M)$ obviously has an effect on what equality in $\Omega_{\mathrm{Lag}}(M)$ means. For example, the identity $[L] = [L']$ in the monotone version of $\Omega_{\mathrm{Lag}}(M)$ does not necessarily mean that there exists a monotone cobordism $L' \rightsquigarrow L$. The identity might instead come from a monotone cobordism $(L', K) \rightsquigarrow (L, K)$; from this, one *can* create a cobordism $L' \rightsquigarrow L$ by connecting the ends corresponding to K, but in general at the cost of losing monotonicity.

2.4 Additional relations from local systems

Next, we define a variant of the Lagrangian cobordism group that one can define whenever the Lagrangians one studies carry local systems. Let $\mathscr{L}^\sharp$ be the set of all pairs (L, E) where L is a Lagrangian submanifold in M and where E is a local system on L (both with suitable qualifications depending on the context), and let $\langle \mathscr{L}^\sharp \rangle$ be the free Abelian group generated by $\mathscr{L}^\sharp$.

To define the subgroup $R^\sharp \subset \langle \mathscr{L}^\sharp \rangle$ of relations we impose, consider first all expressions of the form

$$(L_1, E_1) + \cdots + (L_r, E_r) - (L'_1, E'_1) - \cdots - (L'_s, E'_s) \in \langle \mathscr{L}^\sharp \rangle \tag{1}$$

such that there is a Lagrangian cobordism $V : (L'_1, \ldots, L'_s) \rightsquigarrow (L_1, \ldots, L_r)$ together with a local system E on V which restricts to the E'_i or E_j on the respective ends.

Second, consider all expressions of the form

$$(L, E) - (L, E') - (L, E'') \in \langle \mathscr{L}^\sharp \rangle \tag{2}$$

such that there exists a short exact sequence $0 \to E' \to E \to E'' \to 0$ of local systems on L. Now, define $R^\sharp$ to be the subgroup of $\langle \mathscr{L}^\sharp \rangle$ generated by all expressions of types (1) and (2) and set

$$\Omega^\sharp_{\mathrm{Lag}}(M) = \langle \mathscr{L}^\sharp \rangle / R^\sharp.$$

Similar to before, we suppress $\mathscr{L}^\sharp$ and $R^\sharp$ from the notation, keeping in mind that the group really depends on them.

3 The Fukaya category of T^2

In this section, we describe the constructions of the Fukaya categories $\mathscr{F}(T^2)$ and $\mathscr{F}^\sharp(T^2)$ appearing in Theorems 1.1 and 1.3, following essentially the general recipe in [19]. The ground field over which these A_∞-categories are defined is the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i q^{a_i} \,\middle|\, c_i \in \mathbb{C},\ a_i \in \mathbb{R},\ a_i < a_{i+1},\ \lim_{i\to\infty} a_i = \infty \right\}.$$

We will first describe the objects of $\mathscr{F}^\sharp(T^2)$ and $\mathscr{F}(T^2)$, and then sketch the construction of morphism spaces, A_∞-compositions and derived categories.

3.1 Objects

The objects of $\mathscr{F}^\sharp(T^2)$ are tuples (L, α, P, E) as follows:

- $L \subset T^2$ is a non-contractible simple closed curve,
- $\alpha : L \to \mathbb{R}$ is a grading of L,
- P is a Pin structure on L,
- E is a local system of Λ-vector spaces on L.

These structures are subject to certain conditions which we will describe below. The pair (α, P) is called a *brane structure* on L, and the triple (L, α, P) is called a *Lagrangian brane*. We usually suppress the brane structure and/or the local system from the notation.

The objects of $\mathscr{F}(T^2)$ are Lagrangian branes $L = (L, \alpha, P)$ with L, α and P as above. We regard them as objects of $\mathscr{F}^\sharp(T^2)$ by equipping them with trivial rank-one local systems, such as to make $\mathscr{F}(T^2)$ a full A_∞-subcategory of $\mathscr{F}^\sharp(T^2)$. In view of that, we would not define the morphism spaces of $\mathscr{F}(T^2)$ separately.

3.1.1 Gradings

The Lagrangian Grassmannian $\mathrm{Gr}(\mathbb{R}^{2n})$ of $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ can be naturally identified with $U(n)/O(n)$, and this identification induces a map $\det^2 : \mathrm{Gr}(\mathbb{R}^{2n}) \to S^1$. A *grading* of a Lagrangian subspace $\Lambda \in \mathrm{Gr}(\mathbb{R}^{2n})$ is a number $\alpha \in \mathbb{R}$ such that $e^{2\pi i \alpha} = \det^2(\Lambda)$. (See [19] for the general definition.)

Since the tangent bundle of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is $T^2 \times \mathbb{R}^2$, the Gauß map associated with a Lagrangian $L \subset T^2$ can be viewed as a map $\Gamma_L : L \to \mathrm{Gr}(\mathbb{R}^2)$. A *grading* of L is a continuous function $\alpha : L \to \mathbb{R}$ such that $\alpha(x)$ is a grading of $T_x L \in \mathrm{Gr}(\mathbb{R}^2)$ in the previous sense, that is, α is a lift of the map $\det^2 \circ \Gamma_L : L \to S^1$ with respect to the covering $\mathbb{R} \to S^1$, $\alpha \mapsto e^{2\pi i \alpha}$. Note that every non-contractible $L \subset T^2$ possesses a grading because its Γ_L is null-homotopic. Objects of $\mathscr{F}^\sharp(T^2)$ are allowed to carry all possible gradings.

A grading α on a curve $L \subset T^2$ induces an orientation of L because we can view $e^{\pi i \alpha} \in S^1$ as a point in $\mathrm{Gr}^{\mathrm{or}}(\mathbb{R}^2)$, the Grassmannian of oriented lines in $\mathbb{R}^2$; changing the grading by ± 1 reverses the induced orientation. In the following, it will sometimes be convenient to have an orientation present, and we will always equip graded curves with this induced orientation. Conversely, given an oriented curve L, we can assign a *standard grading* to it as follows. Suppose that L has slope $(p, q) \in H_1(T^2; \mathbb{Z})$. Then, there is a unique number $\alpha_0 \in [-1, 1)$ such that

$$e^{\pi i \alpha_0} = \frac{p + iq}{\sqrt{p^2 + q^2}}.$$

Viewed as a constant function, α_0 is a grading for any of the linear Lagrangians to which L is isotopic. We define the standard grading of L to be the function $\alpha : L \to \mathbb{R}$ induced by α_0 via any isotopy connecting L to a linear Lagrangian (Fig. 2).

3.1.2 Pin structures

A Pin structure on an rank n vector bundle F is a principal Pin_n-bundle P together with a choice of two-sheeted covering $P \to P_O(F)$, which is equivariant with respect to the action of Pin_n on both sides. Here, $P_O(F)$ is the bundle of orthonormal frames of F with respect to some auxiliary metric, which is acted upon by Pin_n via the homomorphism $Pin_n \to O_n$. A Pin structure on a manifold is a Pin structure on its tangent bundle. Pin structures generalize the more familiar notion of $Spin$ structures to

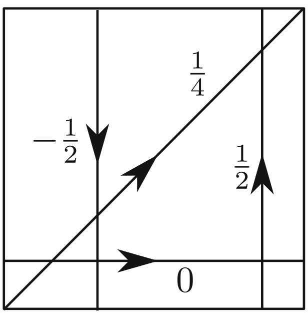

Fig. 2 Oriented curves on T^2 and their standard grading

non-oriented (even non-orientable) bundles. If bundle F is orientable, a Pin structure is the same as an equivalence class of $Spin$ structures, where two $Spin$ structures are equivalent if they are obtained from one another by "reversing the orientation". (See [19, Section (11i)] for some and [12] for more background on Pin_n and Pin structures.)

The orthonormal frame bundle of a circle L is simply $L \sqcup L$, and we can therefore think of a Pin structure on L as a double cover of L. Hence, L admits precisely two different Pin structures, corresponding to the trivial, respectively, the non-trivial 2-sheeted cover. The latter is referred to as the bounding Pin structure, because it is obtained by restricting the unique Pin structure on the disc to the boundary circle. The Lagrangians appearing in the definition of $\mathscr{F}^\sharp(T^2)$ will be equipped with the bounding Pin structure. (Allowing also the other Pin structure would be possible, but would lead to some redundancy.)

3.1.3 Local systems

Let X be a topological space. A local system E on X assigns a vector space E_x to every point $x \in X$ and an isomorphism $\pi_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ to every path $\gamma : [0, 1] \to X$ which depends only on the homotopy class of the path relative to the end points, in a way which is compatible with concatenation of paths. We call E_x the *fiber* of E at x, and π_γ the *parallel transport* in E along γ. A local system E yields a representation $\eta_{x_0} : \pi_1(X, x_0) \to \mathrm{GL}(E_{x_0})$ for every $x_0 \in X$. Assuming that X is path-connected, E is determined up to isomorphism by one single such representation η_{x_0}; in fact, E is isomorphic to a local system whose fibers are all equal to E_{x_0} and with parallel transport maps constructed from η_{x_0}.

If the base space is an oriented circle L, a local system E on L is hence determined by specifying a single vector space F_E and a *monodromy* isomorphism $M_E \in \mathrm{GL}(F)$, which encodes the parallel transport along the preferred generator of $\pi_1(L)$. The local systems E we allow in the definition of $\mathscr{F}^\sharp(T^2)$ are as follows: They have fiber $F_E = \Lambda^{\oplus n}$ for some $n \geq 1$, and all eigenvalues of the monodromy $M_E \in \mathrm{GL}_n(\Lambda)$ have norm 1 with respect to the non-Archimedean norm on Λ defined by

$$\left| \sum_{i=0}^{\infty} c_i q^{a_i} \right| = e^{-a_0}.$$

3.2 Morphisms

Let $L_i \equiv (L_i, \alpha_i, P_i, E_i)$, $i = 0, 1$, be objects of $\mathscr{F}^\sharp(T^2)$ such that L_0 and L_1 intersect transversely. The corresponding space of *Floer cochains* is the graded Λ-vector space with ith graded component

$$CF^i(L_0, L_1) = \bigoplus_{\substack{y \in L_0 \cap L_1 \\ i(y) = i}} \mathrm{Hom}(E_{0,y}, E_{1,y}),$$

where $\mathrm{Hom}(E_{0,y}, E_{1,y})$ is the space of homomorphisms between the fibers of the local systems at $y \in L_0 \cap L_1$, and $i(y)$ is the *index* of y, defined by

$$i(y) \equiv i(y; L_0, L_1) = \lfloor \alpha_1(y) - \alpha_0(y) \rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the next lowest integer.

Let $Z = \mathbb{R} \times [0, 1]$ be the strip with coordinates (s, t) and equipped with the usual complex structure j_Z. Given $y_0, y_1 \in L_0 \cap L_1$, denote by $\mathcal{M}_Z(y_0, y_1)$ the space of all maps $u : Z \to T^2$ satisfying

$$\partial_s u + J(t, u)\partial_t u = 0 \tag{3}$$

for a generic ω-compatible t-dependent almost complex structure J on T^2, and with boundary and asymptotic conditions given by

$$u(s, 0) \in L_0, \;\; u(s, 1) \in L_1, \quad \lim_{s \to -\infty} u(s, \cdot) = y_0, \quad \lim_{s \to +\infty} u(s, \cdot) = y_1.$$

$\mathcal{M}_Z(y_0, y_1)$ carries a natural $\mathbb{R}$-action given by translation in the $\mathbb{R}$-variable. We denote by $\mathcal{M}^{1+1}(y_0, y_1) = \mathcal{M}_Z(y_0, y_1)/\mathbb{R}$ the moduli space obtained by quotienting out this action.

The Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)[1]$ is then defined on generators $\phi_1 \in \mathrm{Hom}(E_{0,y_1}, E_{1,y_1})$ of $CF(L_0, L_1)$ by

$$\partial \phi_1 = (-1)^{i(y_1)} \bigoplus_{y_0 \in L_0 \cap L_1} \sum_u \mathrm{sgn}(u)\, \pi_1^u \circ \phi \circ \pi_0^u \, q^{\omega(u)},$$

where the second sum runs over the zero-dimensional component $\mathcal{M}^{1+1}(y_0, y_1)^0$ of the moduli space (which is a discrete set). Here, π_i^u, for $i = 0, 1$, denotes parallel transport in the local system E_i along the boundary component $u(\mathbb{R} \times \{i\}) \subset L_i$ of the strip $u(Z)$, and $q^{\omega(u)}$ is an element of Λ that encodes the symplectic area of the strip u. Finally, $\mathrm{sgn}(u) \in \{\pm 1\}$ is a sign whose determination we will describe in Sect. 3.4.

The definition of $CF(L_0, L_1)$ for non-transversely intersecting L_0, L_1 requires the use of Hamiltonian perturbations: One fixes a Floer datum (H, J) for every such pair, consisting of a Hamiltonian function H such that $\phi_1^H(L_0) \pitchfork L_1$ and an almost complex structure J, and then considers an analogue of Eq. (3) with an additional perturbation term on the right-hand side.

In either case, the space of morphisms in $\mathscr{F}^\sharp(T^2)$ from L_0 to L_1 is defined as a graded vector space over Λ by

$$\hom(L_0, L_1) = CF(L_0, L_1),$$

and the A_∞-structure map $\mu^1 : \hom(L_0, L_1) \to \hom(L_0, L_1)[1]$ of order one is the Floer differential ∂.

3.3 A_∞-compositions

We give a brief description of the higher A_∞-compositions

$$\mu^d : \hom(L_{d-1}, L_d) \otimes \cdots \otimes \hom(L_0, L_1) \to \hom(L_0, L_d)[2-d], \quad d \geq 2,$$

again limiting ourselves to the case of mutually transverse L_i. We refer to [19] for the proof that these really define an A_∞-structure.

Let $L_0, \ldots, L_d$ be objects of $\mathscr{F}^\sharp(T^2)$, and let $y_0 \in L_0 \cap L_d$ and $y_i \in L_{i-1} \cap L_i$, for $i = 1, \ldots, d$. Moreover, let S be a disc with one incoming and d outgoing boundary punctures. The moduli space of $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$ of *pseudo-holomorphic polygons* associated with this collection Lagrangians and points is the set of all maps $u \in C^\infty(S, T^2)$ solving the equation

$$Du(z) + J(z, u) \circ Du(z) \circ j_S = 0,$$

where J is a generic ω-compatible almost complex structure depending on $z \in S$, with boundary conditions given by the L_i and asymptotic conditions at the punctures given by the y_i as indicated in Fig. 3. The precise general definitions require again choices of Floer data and additional choices of perturbation data that lead to the appearance of an inhomogeneity on the right-hand side of the equation; see [19, Section 8(f)] for details.

Given now $\phi_1, \ldots, \phi_d$ with $\phi_i \in \mathrm{Hom}(E_{i-1,y_i}, E_{i,y_i})$, the corresponding output of μ^d is

$$\mu^d(\phi_d, \ldots, \phi_1) = (-1)^\dagger \bigoplus_{y_0 \in L_0 \cap L_d} \sum_u \mathrm{sgn}(u)\, \pi_d^u \circ \phi_d \circ \pi_{d-1}^u \circ \cdots \circ \phi_1 \circ \pi_0^u\, q^{\omega(u)},$$

which is an element of $\mathrm{Hom}(E_{0,y_0}, E_{d,y_0}) \subset CF(L_0, L_d)$. Here, the second sum runs over all elements $u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d)^0$, the zero-dimensional component of the moduli space. π_i^u is the parallel transport in E_i along the boundary component of $u(S)$ that lies in L_i, and $q^{\omega(u)} \in \Lambda$ encodes the symplectic area of u. The determination of the sign $\mathrm{sgn}(u) \in \{\pm 1\}$ will be explained below; $(-1)^\dagger$ is an additional sign, with $\dagger = \sum_{k=1}^d k i(y_k)$.

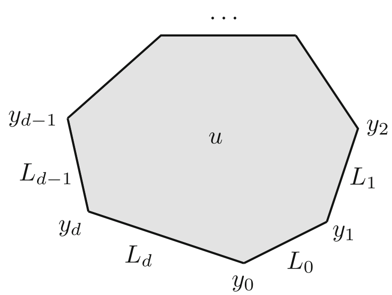

Fig. 3 A pseudo-holomorphic polygon

Remark 3.1 The set $\mathcal{M}^{d+1}(y_0, \dots, y_d)^0$ is *infinite* in general (cf. [18] for the case $d = 2$). Nevertheless, the sums above converge in $\mathrm{Hom}(E_{0,y_0}, E_{d,y_0})$ with respect to the topology induced by the non-Archimedean norm on Λ. This follows from two facts: First, the monodromies of the E_i have norm 1; second, the areas $\omega(u)$ of the polygons entering the count tend to ∞, as one can see by thinking of their lifts to the universal cover of T^2.

3.4 Signs

Let L_0, L_1 be Lagrangian branes intersecting transversely at some point $y \in L_0 \cap L_1$. One can associate to y, considered as a morphism from L_0 to L_1, a one-dimensional real vector space $o(y)$ called the *orientation space* of y. We refer to [19] for the precise definition (also cf. Sect. 4.4 below). At this point, we only remark that $o(y)$ depends on a choice of homotopy class of paths from $T_y L_0$ to $T_y L_1$ in the Lagrangian Grassmannian; this homotopy class in turn is canonically determined by the brane structures.

Given Lagrangian branes $L_0, \dots, L_d$ and intersection points y_i as in Sect. 3.3, there exists a preferred isomorphism

$$\Lambda^{\mathrm{top}}\left(T_u\mathcal{M}^{d+1}(y_0, \dots, y_d)\right) \cong o(y_0) \otimes o(y_1)^\vee \otimes \dots \otimes o(y_d)^\vee$$

for every regular $u \in \mathcal{M}^{d+1}(y_0, \dots, y_d)$, see [19, Section (12b)]. In particular, whenever u is isolated (i.e., an element of the zero-dimensional component of the moduli space), this isomorphism yields a preferred element

$$c_u \in o(y_0) \otimes o(y_1)^\vee \otimes \dots \otimes o(y_d)^\vee,$$

because $\Lambda^{\mathrm{top}}(T_u\mathcal{M}^{d+1}(y_0, \dots, y_d)) = \mathbb{R}$ for isolated u.

The signs $\mathrm{sgn}(u)$ are then defined as follows: We choose, arbitrarily and once and for all, an orientation $\mathfrak{o}_{y_i}$ of $o(y_i)$ for every y_i, which induces an orientation for every $o(y_0) \otimes o(y_1)^\vee \otimes \dots \otimes o(y_d)^\vee$. Then, we set $\mathrm{sgn}(u) = \pm 1$ according to whether c_u is positive or negative with respect to this orientation.

Remark 3.2 An equivalent way of dealing with signs would be to adopt Seidel's basis-free approach from [19] and to define Floer complexes as $CF(L_0, L_1) = \bigoplus_y |o(y)|_\Lambda \otimes \mathrm{Hom}(E_{0,y}, E_{1,y})$, where $|o(y)|_\Lambda$ is the Λ-*normalization* of $o(y)$, that is, $|o(y)|_\Lambda$ is the one-dimensional Λ-vector space obtained by taking the vector space generated by the two orientations of $o(y)$ and imposing the relation that the two generators add up to zero. The A_∞-compositions would then defined by combining the preferred elements $|c_u|_\Lambda \in |o(y_0)|_\Lambda \otimes |o(y_1)|_\Lambda^\vee \otimes \dots \otimes |o(y_d)|_\Lambda^\vee$ induced by the c_u with the parallel transport maps π_i^u. The approach to signs we have chosen is slightly more geometric (though a bit less elegant), since the generators of Floer complexes are actual homomorphisms between fibers of local systems. To translate between the two approaches, one identifies the two versions of the Floer complexes by mapping a generator $\phi \in \mathrm{Hom}(E_{0,y}, E_{1,y})$ of the first version to $[\mathfrak{o}_y] \otimes \phi \in |o(y)|_\Lambda \otimes \mathrm{Hom}(E_{0,y}, E_{1,y})$, which is a generator of the second version; here, $\mathfrak{o}_y$ is the chosen orientation for $o(y)$. This intertwines the respective versions of the A_∞-compositions.

3.5 The derived Fukaya category

The derived Fukaya category $D\mathscr{F}^\sharp(T^2)$ is constructed from $\mathscr{F}^\sharp(T^2)$ by a purely algebraic procedure. One first completes it to a triangulated A_∞-category $\overline{\mathscr{F}^\sharp}(T^2)$ and then sets $D\mathscr{F}^\sharp(T^2) = H^0(\overline{\mathscr{F}^\sharp}(T^2))$. A further completion that formally introduces images for all idempotent morphisms yields the split-closed derived Fukaya category $D^\pi\mathscr{F}^\sharp(T^2)$.

These categories are triangulated categories in the classical sense, meaning that they possess a shift functor and a class of exact triangles (see Appendix 1 for a brief description of what that means). The shift functor is realized geometrically by changing the brane structure in a certain way. In our case, its effect is simply a shift of the grading by 1,

$$(L, \alpha, P, E)[1] = (L, \alpha - 1, P, E).$$

(See [19, (11k)]; the Pin structure remains unchanged since we are dealing with curves that have trivial tangent bundle).

4 Cobordisms and cone decompositions

Let (M, ω) be a symplectic manifold and suppose that $\mathscr{F}^\sharp(M)$ is defined as a graded A_∞-category over the Novikov field Λ, with objects Lagrangian branes carrying local systems.

Theorem 4.1 *Let $L, L_1, \ldots, L_r \in \mathscr{F}^\sharp(M)$ and suppose that there exists a Lagrangian cobordism $V : L \rightsquigarrow (L_1, \ldots, L_k)$. Then, L admits an iterated cone decomposition in $D\mathscr{F}^\sharp(M)$ with linearization $(L_1, \ldots, L_k)$.*

This holds provided that certain technical conditions are satisfied, which will be addressed in Sect. 4.1. We suppress brane structures and local systems from the notation, i.e., we write L instead of (L, α, P, E). By a Lagrangian cobordism between such objects, we mean a Lagrangian cobordism between the underlying Lagrangians, which is equipped with the same type of extra data in a compatible way. The definitions of *iterated cone decomposition* and *linearization* are given in Appendix 1.

Theorem 4.1, which will follow from Proposition 4.7 below, which is an adaptation of an analogous statement in [6] to the present setting, and this section serves to explain the (small) modifications of Biran–Cornea's arguments required for its proof.

4.1 Differences to the setting in [6]

The version of the Fukaya category considered in [6] is linear over $\mathbb{Z}/2\mathbb{Z}$ and ungraded, and has as objects plain Lagrangians L, without brane structures or local systems. Our main point will be to explain the inclusion of gradings and signs in the proofs given in

[6]; including the local systems is standard and does not pose any additional difficulties. Some small differences are due to the fact that we work with Floer cohomology (in contrast to Floer homology).

Biran–Cornea require that (M, ω), as well as all Lagrangians L and cobordisms V involved, are uniformly monotone, and that the maps

$$\pi_1(L) \to \pi_1(M) \quad \text{and} \quad \pi_1(V) \to \pi_1(\widetilde{M})$$

induced by the inclusions vanish for all of them. These conditions are needed to prove compactness of moduli spaces of pseudo-holomorphic curves. More precisely, the conditions on fundamental groups are used to obtain bounds on the areas of such curves; we do not have to impose these conditions, because we use the Novikov field Λ to encode series of curves with areas tending to infinity. The monotonicity assumption is needed to rule out the bubbling off of pseudo-holomorphic discs.

Discs bubbles can, however, also be excluded in some situations that do not fit into the monotone setting. For example, we will need Theorem 4.1 in the case that $L, L_1, \ldots, L_r$ are non-contractible curves in T^2 and V is a Lagrangian cobordism coming from iterated surgery of such curves, as described in Sect. 5. Our curves clearly cannot bound any discs as they are non-contractible. Moreover, our cobordisms will be shown to have vanishing Maslov class (Proposition 5.2); as the expected dimension of the moduli space of Maslov zero pseudo-holomorphic discs with boundary on a Lagrangian in a symplectic 4-manifold is -1, such discs exist only for almost complex structures J belonging to a codimension one stratum $\mathcal{J}_0 \subset \mathcal{J}$ of the space of compatible almost complex structures. Thus, bubbling can be excluded as long as one only works with almost complex structure J in some fixed component of $\mathcal{J} \setminus \mathcal{J}_0$. (This also means that the resulting objects are only invariant with respect to changing J within this component, but that is not a problem for our specific purposes.)

For the arguments of this section, it will not matter which precise mechanism prevents disc bubbling. We will simply assume that we are in one of these favorable settings.

4.2 The categories $\mathscr{B}_{\gamma,h}$

As in [6], we consider A_∞-categories $\mathscr{B}_{\gamma,h}$ associated with pairs (γ, h) as follows: First, $\gamma \subset \mathbb{R}^2$ is a properly embedded curve diffeomorphic to $\mathbb{R}$ with "horizontal ends" whose y-coordinates are in $\frac{1}{2}\mathbb{Z}$; second, $h : \mathbb{R}^2 \to \mathbb{R}$ is a *profile function*, whose associated *extended profile function* we denote by $h' : \mathbb{R}^2 \to \mathbb{R}$.

These terms are defined in Sections 3.2 and 4.1 of [6]. What is really only important for us is that h and h' are assumed to be such that $\Phi_1^{h'}(\gamma)$ looks as in Fig. 4, where $\Phi_1^{h'}$ is the time one map of the Hamiltonian flow of h', and γ is a typical specimen of the type of curves considered, that is, there should be an *odd* number of intersection points $o_1, \ldots, o_\ell \in \gamma \cap \Phi_1^{h'}(\gamma)$, and $\Phi_1^{h'}(\gamma)$ should have horizontal ends such that its negative end lies below the negative end of γ.

Remark 4.2 Note that in [6], the requirement for h' is that $(\Phi_1^{h'})^{-1}(\gamma)$ looks as in Fig. 4. The difference is due to our use of *co*homology.

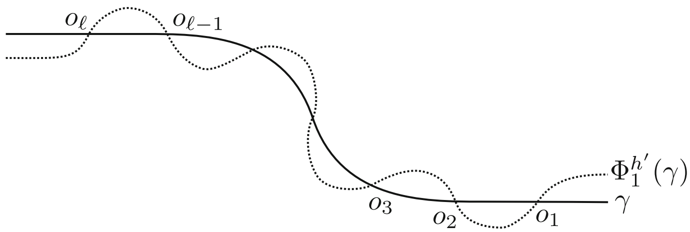

Fig. 4 Curves γ and $\Phi_1^{h'}(\gamma)$ used to define $\mathscr{B}_{\gamma,h}$

The *objects* of $\mathscr{B}_{\gamma,h}$ are pairs $(\widetilde{L}, \widetilde{E})$, where $\widetilde{L}$ is a Lagrangian brane in $\widetilde{M} = \mathbb{R}^2 \times M$ whose underlying Lagrangian is of the form $\widetilde{L} = \gamma \times L$ for some $L \in \mathrm{Ob}\, \mathscr{F}^\sharp(M)$, and $\widetilde{E}$ is a local system on $\widetilde{L}$. We usually abbreviate $\widetilde{L} \equiv (\widetilde{L}, \widetilde{E})$.

The space of *morphisms* in $\mathscr{B}_{\gamma,h}$ between two such $\widetilde{L}_0$ and $\widetilde{L}_1$ is

$$\hom_{\mathscr{B}_{\gamma,h}}(\widetilde{L}_0, \widetilde{L}_1) = CF(\widetilde{L}_0, \widetilde{L}_1),$$

where the Floer complex on the right-hand side is defined with respect to a Floer datum $(\widetilde{H}, \widetilde{J})$ for $(\widetilde{L}_0, \widetilde{L}_1)$ which is chosen compatibly with the Floer datum (H, J) for (L_0, L_1) used to define the Floer complex $CF(L_0, L_1) = \hom_{\mathscr{F}^\sharp(M)}(L_0, L_1)$.

Concretely, the first part of the Floer datum is the time-dependent Hamiltonian function $\widetilde{H} = h' \oplus H$. With this choice, the intersection points of $\Phi_1^{\widetilde{H}}(\widetilde{L}_0)$ and $\widetilde{L}_1$ are of the form (o_j, y), with $o_j \in \Phi_1^{h'}(\gamma) \cap \gamma$ and $y \in \Phi_1^H(L_0) \cap L_1$. Hence, $CF(\widetilde{L}, \widetilde{L}')$ is generated by elements of the form

$$\phi^{o_j} \in \mathrm{Hom}(\widetilde{E}_{0,(o_j,y)}, \widetilde{E}_{1,(o_j,y)}),$$

$j = 1, \dots, \ell$, and there is an obvious splitting of vector spaces

$$CF(\widetilde{L}_0, \widetilde{L}_1) = \bigoplus_{j=1}^{\ell} CF(\widetilde{L}_0, \widetilde{L}_1)^{o_j}, \tag{4}$$

where $CF(L_0, L_1)^{o_j}$ is the summand consisting of homomorphisms between the fibers of the local systems over (o_j, y). The second part of the Floer datum is the time-dependent almost complex structure $\widetilde{J}(t) = i_{h'}(t) \oplus J(t)$, with $i_{h'}(t) = (\phi_t^{h'})_* i$, where i is the standard complex structure on $\mathbb{R}^2$.

With these choices of Floer data and further choices of perturbation data, one constructs the A_∞-compositions $\mu^d_{\mathscr{B}_{\gamma,h}}$ by combining the description in [6] with obvious modifications due to the presence of local systems and the use of cohomology, and building in signs as described in Sect. 3.4.

4.3 The functors $c_j : \mathscr{B}_{\gamma,h} \to \mathscr{F}^\sharp(M)$

For every odd j with $1 \leq j \leq \ell$, there exists a natural A_∞-functor

$$c_j \equiv c_{\gamma,h,j} : \mathscr{B}_{\gamma,h} \to \mathscr{F}^\sharp(M),$$

which will turn out to be a quasi-isomorphism. We start by describing the action of these functors on objects; their full definition will be given in Sect. 4.6.

Let $(\widetilde{L}, \widetilde{E})$ be an object of $\mathscr{B}_{\gamma,h}$, and denote by $\widetilde{\alpha} : \widetilde{L} \to \mathbb{R}$ the grading and by $\widetilde{P}$ the Pin structure which together form the brane structure on $\widetilde{L}$. Recall that $\widetilde{L} = \gamma \times L$ for some Lagrangian $L \subset M$. The image of $(\widetilde{L}, \widetilde{E})$ under c_j, for all j, is

$$c_j(\widetilde{L}, \widetilde{E}) = (L, E),$$

where L and E obtained from $\widetilde{L}$ and $\widetilde{E}$ as follows. Consider the inclusion $L \hookrightarrow \widetilde{L}$ of L as a fiber of $\widetilde{L} \to \gamma$ over the negative horizontal end of γ. Then, the grading, the Pin structure, and the local system on L are the pullbacks of the corresponding data on $\widetilde{L}$ via this inclusion. With the definitions of gradings, Pin structures, and local systems given in Sect. 3.1, it should be fairly obvious what pulling back means. As for pulling back a Pin structure $\widetilde{S}$ from $\widetilde{L}$, we include $P_O(TL) \hookrightarrow P_O(T\widetilde{L})$ by completing frames in TL to frames in $T\widetilde{L}$ using the right-pointing tangent vector to γ; then, we restrict $\widetilde{P}$ accordingly, thinking of it as a double cover of $P_O(T\widetilde{L})$.

4.4 Curves in M versus curves in $\widetilde{M}$

To relate the A_∞-compositions in $\mathscr{B}_{\gamma,h}$ in terms with those of $\mathscr{F}^\sharp(M)$, we need to add to the discussion in [6] a verification that $\widetilde{J}$-holomorphic curves $\widetilde{u}$ in $\widetilde{M}$ living in a fiber of $\widetilde{M} \to \mathbb{R}^2$ carry the same sign as the corresponding J-holomorphic curves u in M.

We need a bit of preparation for that. Take objects $\widetilde{L}_0$ and $\widetilde{L}_1$ of $\mathscr{B}_{\gamma,h}$ and let $L_0 = c_j(\widetilde{L}_0)$ and $L_1 = c_j(\widetilde{L}_1)$ be the corresponding objects of $\mathscr{F}^\sharp(M)$. Let

$$\widetilde{y} = (o_j, y) \in \Phi_1^{\widetilde{H}}(\widetilde{L}_0) \cap L_1,$$

where $o_j \in \Phi_1^{h'}(\gamma) \cap \gamma$ for some $j = 1, \ldots, \ell$, and $y \in \Phi_1^H(L_0) \cap L_1$. Here, $(\widetilde{H}, \widetilde{J})$, and (H, J) are Floer data corresponding to $(\widetilde{L}_0, \widetilde{L}_1)$ and (L_0, L_1), which are related as described in Sect. 4.2.

We now associate an index and an orientation space to the intersection points y and $\widetilde{y}$ as described in [19, Section 11]. To explain this for y, consider the Lagrangian subspaces

$$\Lambda_0 = T_y \Phi_1^H(L_0) \quad \text{and} \quad \Lambda_1 = T_y L_1$$

of $T_y M$; these spaces carry natural brane structures induced by those on L_0 and L_1. Choose a generic path

$$\lambda \in \Omega^-(\mathrm{Gr}(T_y M); \Lambda_0, \Lambda_1),$$

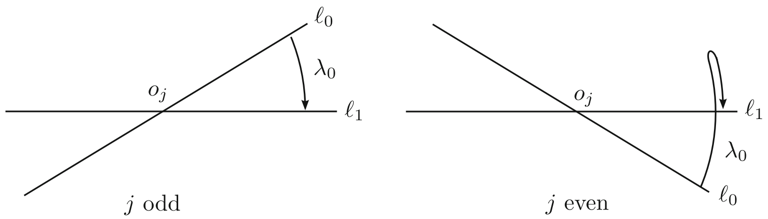

Fig. 5 Paths $\lambda \in \Omega^-(\mathrm{Gr}(\mathbb{R}^2); \ell_0, \ell_1)$

such that the Lagrangian subbundle of $[0, 1] \times T_y M$ obtained from λ admits a grading which restricts to the given gradings of Λ_0 and Λ_1 on the ends; here, $\Omega^-(\mathrm{Gr}(T_y M); \Lambda_0, \Lambda_1)$ is the space of all paths from Λ_0 to Λ_1 in $\mathrm{Gr}(T_y M)$ such that the *crossing form* $q_\lambda(s)$ is negative definite at $s = 1$ (see [19, Section (11f)] for the definition). The index and orientation space of y are then defined as

$$i(y) = \sum_{s<1} \mathrm{sign}(q_\lambda(s)) \quad \text{and} \quad o(y) = \bigotimes_{s<1} (\lambda(s) \cap \lambda(1))^{\mathrm{sign}(q_\lambda(s))}.$$

The genericity assumption on λ implies that the intersection $\lambda(s) \cap \lambda(1)$, $s < 1$, is $\{0\}$ except at finitely many points, where it is a one-dimensional real vector space. Thus, the definitions make sense, and $o(y)$ is a one-dimensional real vector space. The definitions of $i(\widetilde{y})$ and $o(\widetilde{y})$ are analogous.

Lemma 4.3 *Given y and $\widetilde{y} = (o_j, y)$ as above, the indices satisfy $i(\widetilde{y}) = i(y)$ if j is odd, and $i(\widetilde{y}) = i(y) + 1$ if j is even. Moreover, there are canonical isomorphisms $o(\widetilde{y}) \cong o(y)$ if j is odd, and $o(\widetilde{y}) \cong T_{o_j}\gamma \otimes o(y)$ if j is even.*

Proof Set $\ell_0 = T_{o_j}(\Phi_1^{h'}(\gamma))$ and $\ell_1 = T_{o_j}\gamma$. Choose two generic paths $\lambda \in \Omega^-(\mathrm{Gr}(\mathbb{R}^2); \ell_0, \ell_1)$ and $\lambda' \in \Omega^-(\mathrm{Gr}(T_y M); \Lambda_0, \Lambda_1)$, which satisfy the required compatibility with the gradings. For λ, this means that it looks as shown in Fig. 5, depending on the parity of j (the negative definiteness condition on the crossing form means λ must approach ℓ_1 from above at $s = 1$). Then, the path $\widetilde{\lambda} = (\lambda, \lambda') : [0, 1] \to \mathrm{Gr}(T_{\widetilde{y}}\widetilde{M})$ lies in $\Omega^-(\mathrm{Gr}(T_{\widetilde{y}}\widetilde{M}); \widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ and is also compatible with the gradings, as a consequence of how the gradings on the L_i and the $\widetilde{L}_i$ are related. The claimed statements now follows immediately from the definitions of the indices and of the orientation spaces. $\square$

Recall that the moduli spaces $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$ appearing in the definition of the A_∞-compositions of $\mathscr{F}^\sharp(M)$ are spaces of maps $u : S \to M$ satisfying a Cauchy–Riemann type equation, where S is a $(d + 1)$-pointed disc. One commonly views $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$ as the zero set of a section of $\mathcal{E} \to \mathcal{B}$, where $\mathcal{B}$ is a Banach manifold of maps $u : S \to M$ which are locally of class $W^{1,p}$ and satisfy appropriate boundary and asymptotic conditions, and where $\mathcal{E} \to \mathcal{B}$ is the vector bundle whose fiber at u is $L^p(S; \Lambda^{0,1}T^*S \otimes E)$, with $E = u^*TM$. The linearization of the defining section at $u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d)$ is a non-degenerate Cauchy–Riemann operator (in the sense of [19, Section (8h)])

$$D_u : W^{1,p}(S; E, F) \to L^p(S; \Lambda^{0,1}T^*S \otimes E),$$

where $F \subset E|_{\partial S}$ is the Lagrangian subbundle over ∂S induced by the Lagrangians L_i to which u maps the components of ∂S. Analogous statements apply to the moduli spaces $\mathcal{M}^{d+1}(\widetilde{y}_0, \ldots, \widetilde{y}_d)$ used to define $\mathscr{B}_{\gamma,h}$.

Suppose now that choices of Floer data have been made as in Sect. 4.3, and choices of perturbation data as in [6, Section 4.2]. Let $o_j \in \Phi_1^{h'}(\gamma) \cap \gamma$ for some *odd* j. Then, for every curve $u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d)$, the curve $\widetilde{u} = (o_j, u)$ lies in $\mathcal{M}^{d+1}(\widetilde{y}_0, \ldots, \widetilde{y}_d)$ (after applying a small perturbation which only affects the $\mathbb{C}$-component of the curve–in [6] this is called a naturality transformation). This is a consequence of the choice of perturbation datum used in the equation defining $\mathcal{M}^{d+1}(\widetilde{y}_0, \ldots, \widetilde{y}_d)$, which around the odd points splits into a planar part and a vertical part that is identical to the perturbation datum appearing in the equation defining $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$. On the level of linearized operators, we have a splitting

$$D_{\widetilde{u}} = D_{\widetilde{u},\mathbb{C}} \oplus D_u$$

with respect to the canonical identifications

$$\begin{aligned} W^{1,p}(S; \widetilde{E}, \widetilde{F}) \;&\cong\; W^{1,p}(S; \mathbb{C}, T_{o_j}\gamma) \oplus W^{1,p}(S; E, F) \\ L^p(S; \Lambda^{0,1}T^*S \otimes \widetilde{E}) \;&\cong\; L^p(\Lambda^{0,1}T^*S \otimes \mathbb{C}) \oplus L^p(\Lambda^{0,1}T^*S \otimes E) \end{aligned}$$

of the relevant spaces of sections (here, $E = u^*TM$, $\widetilde{E} = \widetilde{u}^*T\widetilde{M}$, and $\widetilde{F}$, F are the Lagrangian subbundles over ∂S corresponding to the boundary conditions for $\widetilde{u}$ and u). Again, as a consequence of the form the Floer and perturbation data around the o_j with j odd, the planar operator $D_{\widetilde{u},\mathbb{C}}$ has index $D_{\widetilde{u},\mathbb{C}} = 0$, $\ker D_{\widetilde{u},\mathbb{C}} = \{0\}$ and $\operatorname{coker} D_{\widetilde{u},\mathbb{C}} = \{0\}$. Therefore, there are canonical isomorphisms

$$\ker D_{\widetilde{u}} \;\cong\; \ker D_u,$$

and moreover, the map $u \mapsto \widetilde{u}$ identifies

$$\mathcal{M}^{d+1}(y_0, \ldots, y_d) \cong \mathcal{M}^{d+1}(\widetilde{y}_0, \ldots, \widetilde{y}_d).$$

Remark 4.4 In fact, if the number of outgoing points is $d = 1$, all these statements are also true in case j is even, because the index of the planar operator $D_{\widetilde{u},\mathbb{C}}$ is zero in this case. In contrast, this index is negative for even j if d is ≥ 2 (cf. the relationship between $i(y)$ and $i(\widetilde{y})$ in Lemma 4.3 and the index formula for operators in [19, Prop. 11.13]).

4.5 Compatible choices of signs

We orient all moduli spaces $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$, $d \geq 1$, by choosing, as in Sect. 3.4, an orientation for every $o(y)$, and requiring that the canonical identification

$$\Lambda^{\mathrm{top}}\left(T_u\mathcal{M}^{d+1}(y_0,\ldots,y_d)\right)\cong o(y_0)\otimes o(y_1)^\vee\otimes\cdots\otimes o(y_d)^\vee$$

be orientation-preserving. In particular, this yields a sign $\mathrm{sgn}(u)\in\{\pm 1\}$ for every regular curve $u\in\mathcal{M}^{d+1}(y_0,\ldots,y_d)$ of index zero.

To express the A_∞-compositions of $\mathscr{B}_{\gamma,h}$ (partially) in terms of those of $\mathscr{F}^\sharp(M)$, we must orient the moduli spaces $\mathcal{M}^{d+1}(\widetilde{y}_0,\ldots,\widetilde{y}_d)$ in a compatible way. The requirement is as follows: Given some $\widetilde{y}=(o_j,y)$ with odd j and an orientation of $o(y)$ as above, we orient $o(\widetilde{y})$ such that the canonical identification

$$o(\widetilde{y})\cong o(y)$$

from Lemma 4.3 matches up the orientations. (For the $\widetilde{y}=(o_j,y)$ with even j, it will not matter how the $o(\widetilde{y})$ are oriented). The orientations of the $\mathcal{M}^{d+1}(\widetilde{y}_0,\ldots,\widetilde{y}_d)$ and in particular the signs $\mathrm{sgn}(\widetilde{u})\in\{\pm 1\}$ of isolated curves $\widetilde{u}$ are then determined as described above.

Lemma 4.5 *Given intersection points $y_0,\ldots,y_d$ and corresponding $\widetilde{y}_0,\ldots,\widetilde{y}_d$ with $\widetilde{y}_i=(o_j,y_i)$ for some* odd *j, the canonical identification $\mathcal{M}^{d+1}(y_0\ldots,y_d)\cong\mathcal{M}^{d+1}(\widetilde{y}_0,\ldots,\widetilde{y}_d)$ is orientation-preserving. In particular,* $\mathrm{sgn}(u)=\mathrm{sgn}(\widetilde{u})$ *for all isolated curves u and $\widetilde{u}=(o_j,u)$.*

Proof The statement follows from the commutativity of the diagram

$$\begin{tikzcd}
\Lambda^{\mathrm{top}}(T_u\mathcal{M}^{d+1}(y_0,\ldots,y_d)) \arrow[r, "\cong"] \arrow[d, "\cong"] & \Lambda^{\mathrm{top}}(T_{\widetilde{u}}\mathcal{M}^{d+1}(\widetilde{y}_0,\ldots,\widetilde{y}_d)) \arrow[d, "\cong"] \\
\Lambda^{\mathrm{top}}(\ker D_u) \arrow[r, "\cong"] \arrow[d, "\cong"] & \Lambda^{\mathrm{top}}(\ker D_{\widetilde{u}}) \arrow[d, "\cong"] \\
o(y_0)\otimes o(y_1)^\vee\otimes\cdots\otimes o(y_d)^\vee \arrow[r, "\cong"] & o(\widetilde{y}_0)\otimes o(\widetilde{y}_1)^\vee\otimes\cdots\otimes o(\widetilde{y}_d)^\vee
\end{tikzcd}$$

in which the vertical isomorphisms are the canonical ones, and where the first row is induced by $\mathcal{M}^{d+1}(y_0\ldots,y_d)\cong\mathcal{M}^{d+1}(\widetilde{y}_0,\ldots,\widetilde{y}_d)$, the second by the splitting $D_{\widetilde{u}}=D_{\widetilde{u},\mathbb{C}}\oplus D_u$ and the fact that $\ker D_{\widetilde{u},\mathbb{C}}=0$, and the third by the canonical identifications from Lemma 4.3. □

4.6 Definition of the functors $c_j:\mathscr{B}_{\gamma,h}\to\mathscr{F}^\sharp(M)$

We now complete the definition of the (presumable) A_∞-functor

$$c_j\equiv c_{\gamma,h,j}:\mathscr{B}_{\gamma,h}\to\mathscr{F}^\sharp(M)$$

for odd j with $1\le j\le\ell$. On objects, c_j takes $(\widetilde{L},\widetilde{E})$ to (L,E), as already mentioned in Sect. 4.3. As for morphisms, consider objects $\widetilde{L}_i\equiv(\widetilde{L}_0,\widetilde{E}_i)$ for $i=0,1$ and the corresponding $L_i \equiv (L_i, E_i)$. Note that for each of the summands $CF(\widetilde{L}_0, \widetilde{L}_1)^{o_j} \subset CF(\widetilde{L}_0, \widetilde{L}_1)$ appearing in the splitting (4), we have a canonical isomorphism of vector spaces

$$CF(\widetilde{L}_0, \widetilde{L}_1)^{o_j} \cong CF(L_0, L_1).$$

This is compatible with gradings, because the indices of corresponding intersection points y and $\widetilde{y} = (o_j, y)$ satisfy $i(\widetilde{y}) = i(y)$ whenever j is odd (cf. Lemma 4.3). We define the first-order component of c_j on morphisms to be the composition

$$c_j^1 : CF(\widetilde{L}_0, \widetilde{L}_1) \to CF(\widetilde{L}_0, \widetilde{L}_1)^{o_j} \cong CF(L_0, L_1),$$

where the first map is the projection onto morphisms of type o_j. The higher-order components c_j^d for $d \geq 2$ are defined to be identically zero.

Proposition 4.6 (Cf. [6, Prop. 4.2.3]) *The functors $c_j : \mathscr{B}_{\gamma,h} \to \mathscr{F}^\sharp(M)$ are A_∞-quasi-isomorphisms. Moreover, c_i and c_j are homotopic for any two odd i, j with $1 \leq i, j \leq \ell$.*

Proof The proof is an adaptation of the proof of Proposition 4.2.3 in [6] that takes into account gradings and signs. We have already noted that the maps c_j^1 are compatible with gradings. Using the arguments in [6] together with Lemma 4.5, one sees that the differentials of $CF(\widetilde{L}_0, \widetilde{L}_1)$ and $CF(L_0, L_1)$ are related as described in [6, Remark 4.2.2], which implies that the complexes are quasi-isomorphic. Then, one checks that the $\{c_j^d\}$ really define A_∞-functors c_j with a common homotopy inverse $e \equiv e_{\gamma,h} : \mathscr{F}^\sharp(M) \to \mathscr{B}_{\gamma,h}$. □

4.7 Exact triangles from cobordisms

The construction of the exact triangles associated with a Lagrangian cobordism now follows the scheme in [6, Sections 4.3, 4.4]. Since even to outline this would require quite a bit of additional notation, we content ourselves with saying that all relevant statement carry over with minor modifications. The bridge between $\mathscr{F}^\sharp(M)$ and the world of cobordisms is now provided by the A_∞-quasi-isomorphisms

$$\mathscr{B}_{\gamma,h} \underset{e}{\overset{c_j}{\rightleftarrows}} \mathscr{F}^\sharp(M)$$

from Proposition 4.6. To state the upshot of all this, let $L \equiv (L, E)$ and $L_1 \equiv (L_1, E_1), \ldots, L_k \equiv (L_k, E_k)$ be Lagrangian branes in M carrying local systems, and let

$$V : L \rightsquigarrow (L_1, \ldots, L_k)$$

be a Lagrangian cobordism equipped with a brane structure and a local system which restrict to the given ones on the ends.

Proposition 4.7 *There exist $\mathscr{F}^\sharp(M)$-modules $\mathcal{M}_V^1, \ldots, \mathcal{M}_V^k$ such that $\mathcal{M}_V^1 =$ $\mathrm{Yon}(L_1)$ and such that there are exact triangles*

$$\mathcal{M}_V^{j-1} \to \mathrm{Yon}(L_j) \to \mathcal{M}_V^j \to \mathcal{M}_V^{j-1}[1]$$

in $\mathrm{mod}(\mathscr{F}^\sharp(M))$ *for* $j = 2, \ldots, k$. *Moreover, there is an* A_∞*-quasi-isomorphism* $\mathrm{Yon}(L) \to \mathcal{M}_V^k$.

Here, $\mathrm{Yon} : \mathscr{F}^\sharp(M) \to \mathrm{mod}(\mathscr{F}^\sharp(M))$ denotes the Yoneda embedding of $\mathscr{F}^\sharp(M)$ into the A_∞-category $\mathrm{mod}(\mathscr{F}^\sharp(M))$ of A_∞-modules over itself. The Proposition summarizes what would be the analogues of Corollary 4.3.3 and Proposition 4.4.1 of [6] in our setting (and the $\mathcal{M}_V^j$ are the analogues of the $\mathcal{M}_{V,\gamma_j,h_j}$ there). Note that the directions of the arrows in the exact triangles are reversed compared to [6] because we use cohomological conventions.

Theorem 4.1 is an immediate consequence of Proposition 4.7.

4.8 Cobordism and Grothendieck groups

Denote by $\mathscr{F}(M)$ the full subcategory of $\mathscr{F}^\sharp(M)$ consisting of Lagrangian branes with trivial rank-one local systems. Let $\Omega_{\mathrm{Lag}}(M)$ and $\Omega_{\mathrm{Lag}}^\sharp(M)$ be the Lagrangian cobordism group defined as in Sects. 2.3 and 2.4 with respect to $\mathscr{L} = \mathrm{Ob}\,\mathscr{F}(M)$, $\mathscr{L}^\sharp = \mathrm{Ob}\,\mathscr{F}^\sharp(M)$ and relation subgroup $R,\ R^\sharp$ induced by appropriate Lagrangians cobordisms between these generators. Denote by $K_0(D\mathscr{F}(M))$ and $K_0(D\mathscr{F}^\sharp(M))$ be the Grothendieck groups of the derived Fukaya categories.

Proposition 4.8 *There exist canonical surjective group homomorphisms*

$$\Theta : \Omega_{\mathrm{Lag}}(M) \to K_0(D\mathscr{F}(M)) \text{ and } \Theta : \Omega_{\mathrm{Lag}}^\sharp(M) \to K_0(D\mathscr{F}^\sharp(M))$$

induced by $L \mapsto L$ *resp.* $(L, E) \mapsto (L, E)$.

The statement for $\Omega_{\mathrm{Lag}}(M) \to K_0(D\mathscr{F}(M))$ follows immediately from Theorem 4.1 (together with Lemma 9.4), while that for $\Omega_{\mathrm{Lag}}^\sharp(M) \to K_0(D\mathscr{F}^\sharp(M))$ needs in addition Proposition 10.1, which implies that the relations in $\Omega_{\mathrm{Lag}}^\sharp(M)$ coming from short exact sequences of local systems are respected.

5 Lagrangian surgery and cobordisms in $\widetilde{T}^2$

Surgering transversely intersecting Lagrangians L_0, L_1 means cutting out small neighborhoods in L_0 and L_1 of each intersection point and gluing in Lagrangian handles, such as to produce a new Lagrangian submanifold $L_0 \# L_1$ (see [16]). The local model is the surgery of the Lagrangian subspaces $\mathbb{R}^n$ and $i\mathbb{R}^n$ of $\mathbb{C}^n$. Biran–Cornea [5] describe how to construct a Lagrangian cobordism

$$\mathbb{R}^n \# i\mathbb{R}^n \rightsquigarrow (\mathbb{R}^n, i\mathbb{R}^n).$$

In general, one can produce cobordisms $L_0 \# L_1 \rightsquigarrow (L_0, L_1)$ by gluing in this local model via Darboux charts. We describe this construction for $n = 1$ in Sects. 5.1 and

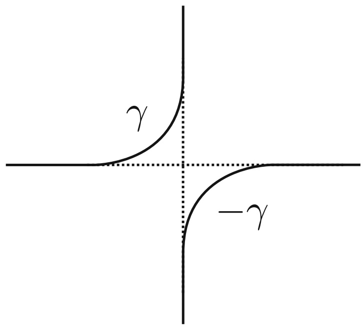

Fig. 6 $\mathbb{R}\#_\gamma i\mathbb{R}$

5.2. Then, we address some specifics of the cobordisms coming from surgery of curves in T^2.

Throughout this section, we identify $\mathbb{R}^2 \cong \mathbb{C}$ in the usual way, so that Lagrangian cobordisms now live in $\widetilde{M} = \mathbb{C} \times M$.

5.1 Model cobordisms in dimension 1

To surger $\mathbb{R}$ and $i\mathbb{R} \subset \mathbb{C}$, cut out two neighborhoods of $0 \in \mathbb{C}$ and connect the ends on $\mathbb{R}$ thus created with the ends on $i\mathbb{R}$ by two curve segments. More formally, choose a smooth curve $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma = a + ib$, such that

- $\gamma(t) = t$ for $t \in (-\infty, -\varepsilon]$,
- $\gamma(t) = it$ for $t \in [\varepsilon, \infty)$,
- $a'(t),\ b'(t) > 0$ for $t \in (-\varepsilon, \varepsilon)$

for some $\varepsilon > 0$. Then, define $\mathbb{R}\#_\gamma i\mathbb{R} = \{\pm\gamma(t) \mid t \in \mathbb{R}\}$, as shown in Fig. 6.

To construct the corresponding Lagrangian cobordism

$$V_\gamma : \mathbb{R}\#_\gamma i\mathbb{R} \rightsquigarrow (\mathbb{R}, i\mathbb{R}),$$

consider the embedding $\psi_\gamma : \mathbb{R} \times S^1 \to \mathbb{C}^2$, $(t, (x_1, x_2)) \mapsto (\gamma(t) \cdot x_1, \gamma(t) \cdot x_2)$, where $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. As a first step, we set

$$V'_\gamma = \psi_\gamma(U)$$

for $U = \big\{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid 0 \le x_1, -2\varepsilon \le tx_1 \le 2\varepsilon\big\} \subset \mathbb{R} \times S^1$. By checking where ψ_γ takes the boundary components of U, one sees that V'_γ is a manifold with boundary

$$\partial V'_\gamma = \{-2\varepsilon\} \times \mathbb{R} \cup \{2\varepsilon i\} \times i\mathbb{R} \cup \{0\} \times \mathbb{R}\#_\gamma i\mathbb{R} \subset \mathbb{C} \times \mathbb{C}.$$

To complete the construction of V_γ, we extend the part of $\partial V'_\gamma$ lying over $0 \in \mathbb{C}$ to a cylindrical end as explained in the proof of Lemma 6.1.1 of [5] (the other two ends are already cylindrical).

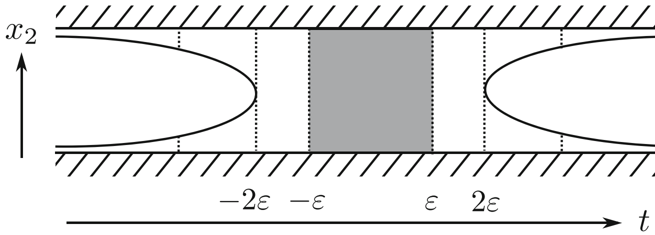

Fig. 7 V_γ schematically

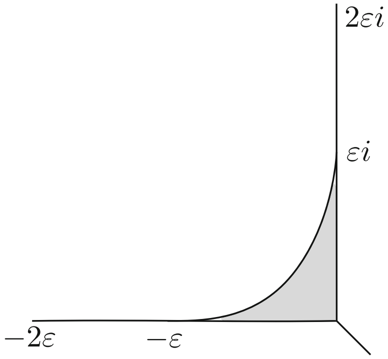

Fig. 8 Projection of V_γ to $\mathbb{C}$

Figure 7 shows a schematic picture of V_γ that should be viewed in light of the parametrization $\psi_\gamma : U \to V'_\gamma$. To understand the labels, note that the U can be identified with a subset of $\mathbb{R} \times [-1, 1]$ via $(t, (x_1, x_2)) \mapsto (t, x_2)$. The striped regions represent the cylindrical end we attached at the very end of the construction, while the region between them represents V'_γ. The two curved lines correspond to the negative ends $\mathbb{R}$ and $i\mathbb{R}$ of the cobordism, and the horizontal ones to the two components of its positive end $\mathbb{R}\#i\mathbb{R}$.

Figure 8 shows the projection of V_γ under $\pi : \mathbb{C}^2 \to \mathbb{C}$ (the projection to the first factor). In this picture, the line segment in the lower right is the image of the cylindrical end attached at the very end, the rest is the image of V'_γ.

We will refer to the following subsets of V_γ as its *fat*, respectively, its *thin* part:

$$V_\gamma^f = \psi_\gamma\big(\{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid 0 \le x_1, -\varepsilon \le t \le \varepsilon\}\big),$$
$$V_\gamma^t = \psi_\gamma\big(\{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid x_1 = 0,\ x_2 = \pm 1\}\big).$$

In Fig. 7, V_γ^f is represented by the shaded square, and V_γ^t is represented by the union of the two horizontal lines. Note that V_γ^f is precisely the part of V_γ lying over the "fat" part of $\pi(V_\gamma)$ (the shaded region in Fig. 8), whereas V_γ^t projects via π to $0 \in \mathbb{C}$. Moreover, note that V_γ deformation retracts onto $V_\gamma^f \cup V_\gamma^t$.

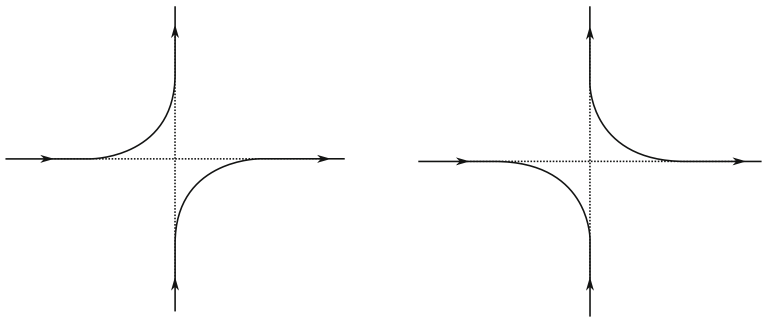

Fig. 9 Surgering compatibly and non-compatibly with given orientations

5.2 Surgery in surfaces and cobordisms

Let now L_0 and L_1 be two (possibly disconnected) Lagrangians in a surface Σ intersecting transversely in m points $q_0, \dots, q_{m-1}$, fix small neighborhoods $U_j \subset \Sigma$ of the q_j and Darboux charts $\phi_j : U_j \to \mathbb{C}$ such that $\phi_j(L_0 \cap U_j) \subset \mathbb{R}$ and $\phi_j(L_1 \cap U_j) \subset i\mathbb{R}$. Moreover, choose curves $\gamma_j : \mathbb{R} \to \mathbb{C}$ as in the previous section. The surgered manifold $L_0 \# L_1$ corresponding to these data is obtained by gluing in the local surgery models corresponding to the γ_j using the charts ϕ_j.

To construct the corresponding cobordism $V : L_0 \# L_1 \rightsquigarrow (L_0, L_1)$, we first define subsets of $\mathbb{C}$ as follows:

$$I_0 = \{x \in \mathbb{R} \mid -2\varepsilon \leq x \leq 0\},$$
$$I_1 = \{iy \in i\mathbb{R} \mid 0 \leq y \leq 2\varepsilon\},$$
$$I_2 = \{x - ix \in \mathbb{C} \mid 0 \leq x \leq 2\varepsilon\}.$$

Moreover, we set $\tilde{L}_i = L_i \setminus \bigcup_{j=0}^{m-1} (L_i \cap \phi_j^{-1}(B_{2\varepsilon}(0)))$ for $i = 0, 1$, and $\tilde{V}_{\gamma_j} = V_{\gamma_j} \cap (\mathbb{C}\tilde{B}_{2\varepsilon}(0))$ for $j = 0, \dots, m-1$, where the V_{γ_j} are the model cobordisms from the previous section. Then, we set

$$V = (I_0 \times \tilde{L}_0) \cup (I_1 \times \tilde{L}_1) \cup (I_2 \times (\tilde{L}_0 \cup \tilde{L}_1)) \cup \bigcup_{j=0}^{m-1} \left(\mathrm{id} \times \phi_j^{-1}\right) (\tilde{V}_{\gamma_j}),$$

Remark 5.1 For this to work, it was necessary to choose *all* charts ϕ_j such that $L_0 \cap U_j$ was mapped to $\mathbb{R}$ and $L_1 \cap U_j$ to $i\mathbb{R}$. This reflects that one needs to select one of the L_i whose corresponding cylindrical end lies over $\mathbb{R}$, and the other for which it lies over $i\mathbb{R}$. Of course, one can swap $\mathbb{R}$ and $i\mathbb{R}$ by a $\frac{\pi}{2}$-rotation, so that this requirement does not impose any restriction at first. However, if one wants to glue in all local surgery models in a way compatible with given orientations of the L_i (which implies that the resulting cobordism is orientable), then the local topological type of the surgery at one intersection point determines that at all others (Fig. 9).

As in the local case, we define *fat* and *thin* parts of V by

$$V^f = \bigcup_{j=0}^{m-1} \left(\mathrm{id} \times \phi_j^{-1}\right)\left(\tilde{V}^f_{\gamma_j}\right),$$
$$V^t = \bigcup_{j=0}^{m-1} \left(\mathrm{id} \times \phi_j^{-1}\right)\left(\tilde{V}^t_{\gamma_j}\right) \;\cup\; \{0\} \times (\tilde{L}_0 \cup \tilde{L}_1),$$

where $\tilde{V}^f_{\gamma_j} = V^f_{\gamma_j} \cap (\mathbb{C}\tilde{B}_{2\varepsilon}(0))$ and $\tilde{V}^t_{\gamma_j} = V^t_{\gamma_j} \cap (\mathbb{C}\tilde{B}_{2\varepsilon}(0))$. One can deformation retract V onto $V^f \cup V^t$ by retracting the glued-in $\tilde{V}_{\gamma_j}$ onto their fat and thin parts, and the remaining cylindrical parts of V onto $\{0\} \times (\tilde{L}_0 \cup \tilde{L}_1)$. From this description, one sees readily that V is homotopy equivalent to a one-dimensional CW complex with m cells of dimension 0 and $2m$ cells of dimension 1, where $m = \# L_0 \cap L_1$.

5.3 Cobordisms in $\widetilde{T}^2$: Maslov class and grading

We identify the tangent bundle of $\widetilde{T}^2 = \mathbb{C} \times T^2$ with $\widetilde{T}^2 \times \mathbb{R}^4$ in the obvious way, so that the Gauß map of a Lagrangian in $\widetilde{T}^2$ takes values in the Lagrangian Grassmannian $\mathrm{Gr}(\mathbb{R}^4)$.

In the following, we call a Lagrangian $L \subset T^2$ *straight* if it lifts to a straight line in the universal cover. The cobordisms of relevance for us come from surgering oriented straight Lagrangians in T^2 in a way that is compatible with the orientations in the sense of Remark 5.1. (This ensures not only orientability of the resulting cobordisms, but also that $L_0 \# L_1$ has no contractible components).

Lemma 5.2 *Let L_0, L_1 be oriented straight Lagrangians in T^2 of different slopes, let $L_0 \# L_1$ be the result of an orientation-compatible surgery, and let $V : L_0 \# L_1 \rightsquigarrow (L_0, L_1)$ be the resulting cobordism. Then, the Gauß map $\Gamma_V : V \to \mathrm{Gr}(\mathbb{R}^4)$ is null-homotopic. In particular, V has vanishing Maslov class.*

Proof We will show that Γ_V is homotopic to a map which factors through a map defined on a contractible domain; this statement immediately implies the assertion. Note that we need to only show this for $\Gamma_V|_{V^f \cup V^t}$, since V deformation retracts onto $V^f \cup V^t$. Moreover, we can assume that the local surgeries at all intersection points q_i are defined with respect to the same local model, and thus, that V is obtained by gluing in the same V_γ at every q_i; namely, isotopies of the curves γ_j (with respect to which the surgery is defined) to one given curve γ induce a Lagrangian isotopy of the resulting cobordisms.

Under these assumptions, all maps $\Gamma_V \circ (\mathrm{id} \times \phi_j^{-1}) : V^f_\gamma \to \mathrm{Gr}(\mathbb{R}^4)$ are equal to one and the same map $\tilde{\Gamma} : V^f_\gamma \to \mathrm{Gr}(\mathbb{R}^4)$. Since Γ_V is constant on every connected component of $(V^f \cup V^t) \backslash V^f$, it factors through $\tilde{\Gamma}$, whose domain V^f_γ is contractible. □

A grading of a Lagrangian $L \subset \widetilde{T}^2$ is a function $\alpha : L \to \mathbb{R}$ lifting the composition $\det^2 \circ \Gamma_L : L \to \mathrm{Gr}(\mathbb{R}^4) \to S^1$ (cf. Sect. 3.1.1), where $\Gamma_L : L \to \mathrm{Gr}(\mathbb{R}^4)$ denotes the

Gauß map. Note that every Lagrangian cobordism $V : L_0 \# L_1 \rightsquigarrow (L_0, L_1)$ coming from surgery of two linear Lagrangians in T^2 admits a grading $\alpha_V : V \to \mathbb{R}$ because its Γ_V is null-homotopic by Lemma 5.2. The next lemma tells how the restrictions of α_V to the ends are related.

Lemma 5.3 *The restrictions of the grading $\alpha_V : V \to \mathbb{R}$ to the ends satisfy $\alpha_V|_{L_0} < \alpha_V|_{L_0 \# L_1} < \alpha_V|_{L_1}$.*

Here, we assume the positive end $L_0 \# L_1$ has been "linearized" by a Hamiltonian isotopy, so that $\alpha_V|_{L_0 \# L_1}$ is constant. The lemma can be proven easily by examining the local models from Sect. 5.1.

6 Preliminaries on $\Omega_{\mathrm{Lag}}(T^2)$

The Lagrangian cobordism group $\Omega_{\mathrm{Lag}}(T^2)$ we study is defined as in Sect. 2.3 with $\mathscr{L}(T^2) = \mathrm{Ob}\,\mathscr{F}(T^2)$, that is, $\Omega_{\mathrm{Lag}}(T^2)$ has generators non-contractible curves equipped with brane structures, and relations coming from cobordisms of vanishing Maslov class equipped with compatible brane structures. In this section, we prove as much as possible about $\Omega_{\mathrm{Lag}}(T^2)$ as we can at this point. The result, Proposition 6.2, will later be upgraded to Theorem 1.2.

6.1 Notation for curves

Let $m, n \in \mathbb{Z}$ such that $\gcd(m, n) = 1$, and let $x \in \mathbb{R}$. If $(m, n) \neq (\pm 1, 0)$, we define

$$L_{(m,n),x} \subset T^2$$

to be the oriented straight curve of slope $(m, n) \in \mathbb{Z}^2 \cong H_1(T^2; \mathbb{Z})$ passing through the point $(x, 0) \in T^2$. We define $L_{(\pm 1,0),x}$ to be the straight horizontal curve through $(0, x) \in T^2$ oriented such that it represents $(\pm 1, 0) \in H_1(T^2; \mathbb{Z})$. When $x = 0$, we abbreviate

$$L_{(m,n)} \equiv L_{(m,n),0}.$$

See Fig. 10 for an illustration. We view these curves as objects of $\mathscr{F}(T^2)$ by equipping them with the standard grading and the bounding Pin structure, which we refer to as the *standard brane structure*; note that the orientation induced on $L_{(m,n)}$ by the standard grading agrees with the orientation it had a priori (cf. Sect. 3.1).

6.2 Cylinders

A cylinder in T^2 is a smooth map $u : C = [0, 1] \times S^1 \to T^2$. Given such a cylinder u, we equip its boundary curves $L^i = u(\{i\} \times S^1) \subset T^2$, $i = 0, 1$, with the boundary orientation induced from the standard orientation of $[0, 1] \times S^1$, and with the standard brane structure. Note that the area $\int_C u^*\omega \in \mathbb{R}$ of u is determined up to an integer by its boundary, as can be seen by considering the cover of T^2 corresponding to the subgroup of $\pi_1(T^2)$ generated by u. This justifies the wording in the following lemma.

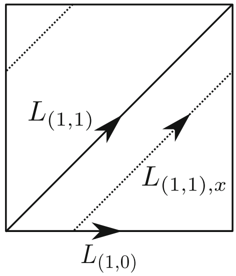

Fig. 10 Notation for curves

Lemma 6.1 *There exists a canonical group homomorphism* $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2)$ *taking* $x \in \mathbb{R}/\mathbb{Z}$ *to* $[L^1]-[L^0] \in \Omega_{\mathrm{Lag}}(T^2)$*, where* L^0, L^1 *are any two non-contractible curves that bound a cylinder of area* x.

Proof Our task was to prove that the map described is well defined (i.e., independence of the chosen cylinder), and that it is a group homomorphism. Note that we may always replace curves by the straight representatives of their Hamiltonian isotopy classes, as Hamiltonian isotopies change neither areas nor cobordism classes.

Our first claim is that the cobordism class of the boundary of a cylinder of area $x \in \mathbb{R}/\mathbb{Z}$ with boundary circles representing the class $(0, \pm 1) \in H_1(T^2; \mathbb{Z})$ is well defined, i.e., that in $\Omega_{\mathrm{Lag}}(T^2)$, we have

$$[L_{(0,1),x+y}] - [L_{(0,1),y}] = [L_{(0,1),x}] - [L_{(0,1)}] \tag{5}$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$. Before we justify that, note that subtracting $[L_{(0,-1)}]$ from both sides of (5) and rearranging yields

$$\Big([L_{(0,1),x+y}] - [L_{(0,1)}]\Big) = \Big([L_{(0,1),x}] - [L_{(0,1)}]\Big) + \Big([L_{(0,1),y}] - [L_{(0,1)}]\Big), \tag{6}$$

which shows that our map (if well defined) is a group homomorphism.

To obtain (5), observe that surgering $L_{(1,0)}$ and $L_{(0,1),x+y}$ has the same result as surgering $L_{(1,0),-x}$ and $L_{(0,1),y}$ up to Hamiltonian isotopy (namely $L_{(1,1),\frac{1}{2}+x+y}$ in both cases, which is a right shift by $x+y$ of what one would get when surgering $L_{(1,0)}$ and $L_{(0,1)}$). This observation tells us that $[L_{(1,0)}] + [L_{(0,1),x+y}] = [L_{(1,0),-x}] + [L_{(0,1),y}]$, or equivalently that

$$[L_{(0,1),x+y}] - [L_{(0,1),y}] = [L_{(1,0),-x}] - [L_{(1,0)}] \tag{7}$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$. The identity (5) follows from that as the right-hand side of (7) is independent of y.

To complete the proof of well definedness, we have to show that for any two general curves L^0 and L^1 bounding a cylinder of area x (after reversing the orientation on L^0), we have $[L^1]-[L^0] = [L_{(0,1),x}]-[L_{(0,1)}]$. One way of seeing this is by thinking about how to build the L^i using iterated surgery of copies of $L_{(1,0)}$ and translated copies of $L_{(0,1)}$. For example, if L^0 can be constructed by iteratively surgering m copies of $L_{(1,0)}$ and n copies of $L_{(0,1)}$, then L^1 can be constructed by iteratively surgering m copies of $L_{(1,0)}$ and n copies of $L_{(0,1),\frac{x}{n}}$. From this, we obtain

$$[L^1] - [L^0] = n\Big([L_{(0,1),\frac{x}{n}}] - [L_{(0,1)}]\Big) = [L_{(0,1),x}] - [L_{(0,1)}],$$

where the second equality follows from (6). [Alternatively, one can directly deduce from (5) and (7) that the boundaries of all "vertical" and all "horizontal" cylinders of area x represent the same class, and then get the general case by applying suitable maps $A \in \mathrm{SL}(2, \mathbb{Z})$]. □

6.3 An exact sequence

Denote by $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2)$ the map described in the previous subsection, and by $\eta : \Omega_{\mathrm{Lag}}(T^2) \to H_1(T^2; \mathbb{Z})$ the canonical map given by $[L]_\Omega \mapsto [L]_{H_1}$. The following proposition collects what we can say about $\Omega_{\mathrm{Lag}}(T^2)$ at this point.

Proposition 6.2 *The sequence of group homorphisms*

$$\mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2) \to H_1(T^2; \mathbb{Z}) \to 0$$

is exact.

Proof It is clear that the canonical map $\eta : \Omega_{\mathrm{Lag}}(T^2) \to H_1(T^2; \mathbb{Z})$ is surjective and that $\operatorname{im}\zeta \subseteq \ker\eta$. To prove $\ker\eta \subseteq \operatorname{im}\zeta$, consider the map

$$\mathbb{R}/\mathbb{Z} \oplus H_1(T^2; \mathbb{Z}) \to \Omega_{\mathrm{Lag}}(T^2),\ (x, (m, n)) \mapsto \zeta(x) + m[L_{(1,0)}] + n[L_{(0,1)}], \quad (8)$$

where we identify $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ in the obvious way. Observe that $\Omega_{\mathrm{Lag}}(T^2)$ is generated by $[L_{(1,0)}]$ and the elements of the family $[L_{(0,1),x}]_{x\in\mathbb{R}/\mathbb{Z}}$, because every Lagrangian can be obtained by iteratively surgering these. An alternative set of generators of $\Omega_{\mathrm{Lag}}(T^2)$ is given by $[L_{(1,0)}], [L_{(0,1)}]$ and the family $\zeta(x) = [L_{(0,1),x}] - [L_{(0,1)}]$, $x \in \mathbb{R}/\mathbb{Z}$, which shows that the map (8) is surjective. Together with the fact that the composition $H_1(T^2; \mathbb{Z}) \hookrightarrow \mathbb{R}/\mathbb{Z} \oplus H_1(T^2; \mathbb{Z}) \to \Omega_{\mathrm{Lag}}(T^2)$ is a section of η, this implies that $\ker\eta \subseteq \operatorname{im}\zeta$. □

In order to upgrade Proposition 6.2 to Theorem 1.2, it remains to prove the injectivity of $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2)$. It is in fact easy to rule out the existence of a Lagrangian cobordism $L \rightsquigarrow L'$ for isotopic but not Hamiltonian isotopic curves: The existence of such a cobordism would imply that $HF(N, L) \cong HF(N, L')$ for any other curve N (as a consequence of spelling out what Proposition 4.7 says), contradicting that we have $HF(L, L) \cong \mathbb{Z}^2$ and $HF(L, L') = 0$. However, there might be more complicated relations leading to the identity $[L] - [L'] = 0 \in \Omega_{\mathrm{Lag}}(T^2)$. The fact that we cannot rule these out directly is one reason for the detour via homological mirror symmetry we will take in the next section.

7 Homological mirror symmetry for T^2

7.1 Abouzaid–Smith's mirror functor

Abouzaid–Smith prove in [3] that the split-closed derived Fukaya category $D^\pi \mathscr{F}^\#(T^2)$ is equivalent to the derived category $D^b(X)$ of the *Tate curve* X, which is an elliptic curve over the Novikov field Λ given by a specific Weierstrass equation (see [3]). To state the precise result, let $P_0 \in X$ be a base point, denote by $\mathcal{O}(nP_0)$ the line bundle corresponding to the divisor nP_0, and by $\mathcal{O}_{P_0}$ the skyscraper sheaf with one-dimensional stalk supported at P_0.

Theorem 7.1 *There exists an equivalence of triangulated categories*

$$\Phi : D^b(X) \simeq D^\pi \mathscr{F}^\sharp(T^2)$$

which takes $\mathcal{O}(nP_0)$ to $L_{(1,-n)}$ for every $n \in \mathbb{Z}$, and $\mathcal{O}_{P_0}$ to $L_{(0,-1),\frac{1}{2}}$.

The proof is based on parts of Polishchuk–Zaslow's computations in [18], who work over an elliptic curve defined over $\mathbb{C}$. To make the connection to the current setting, one studies X by means of its analytification

$$X^{an} = \Lambda^*/q^{\mathbb{Z}},$$

the quotient of $\Lambda^* = \Lambda\backslash\{0\}$ by the discrete subgroup generated by $q \in \Lambda$. X^{an} can be given the structure of a rigid-analytic space over Λ (we refer the reader to [8] for general background on rigid-analytic geometry, and to [8,21] for specific information on the Tate curve). The complex-analytic θ functions appearing in [18] can be interpreted as formal power series and hence functions on Λ^*, which give rise to sections of rigid-analytic vector bundles over X^{an}, analogously to the complex case. A rigid-analytic GAGA principle says that the categories of coherent algebraic sheaves on X and of coherent analytic sheaves on X^{an} are equivalent, which allows to translate back to algebraic geometry. We will drop the notational distinction between X and X^{an} in the following.

To outline Abouzaid–Smith's proof of Theorem 7.1, consider on the algebraic side the full subcategory $\Gamma\mathscr{A}^\vee \subset D^b_\infty(X)$ consisting of the line bundles $\mathcal{O}(nP_0)$, $n \in \mathbb{Z}$ (where $D^b_\infty(X)$ is a dg-enhancement of $D^b(X)$, see [3]); on the symplectic side, consider the full A_∞-subcategory $\Gamma\mathscr{A} \subset \mathscr{F}^\sharp(T^2)$ with objects the Lagrangians $L_{(1,n)}$, $n \in \mathbb{Z}$. Both collections of objects split-generate on their respective sides. Polishchuk–Zaslow's computations [18] imply that the cohomological categories $H^0(\Gamma\mathscr{A}^\vee)$ and $H^0(\Gamma\mathscr{A})$ are equivalent by a functor taking $\mathcal{O}(nP_0)$ to $L_{(1,-n)}$. Then, a deformation theoretic result by Polishchuk [17] says that $H^0(\Gamma\mathscr{A}^\vee) \simeq H^0(\Gamma\mathscr{A})$ can be equipped with an essentially unique non-formal A_∞-structure. As the A_∞-structures on $\Gamma\mathscr{A}^\vee$ and $\Gamma\mathscr{A}$ are both non-formal, Abouzaid–Smith conclude the existence of an A_∞-quasi-equivalence between $\Gamma\mathscr{A}^\vee$ and $\Gamma\mathscr{A}$, which extends to an A_∞-quasi-equivalence between the respective split closures. The triangulated equivalence of Theorem 7.1 is then obtained by taking cohomology.

Remark 7.2 Abouzaid–Smith [3] really prove Theorem 7.1 for $\mathscr{F}(T^2)$, the version of the Fukaya category without local systems. However, the proof in [3] goes through for both $\mathscr{F}(T^2)$ and $\mathscr{F}^\sharp(T^2)$, because the $L_{(1,n)}$ split-generate both versions. One can prove this, for example, with the machinery from Sect. 4, by iteratively surgering the $L_{(1,n)}$ and equipping the resulting cobordisms with suitable local systems. Cf. Remark 6.7 in [3], which indicates that the effects of allowing appropriate non-trivial local systems and of taking the split closure of $D\mathscr{F}(T^2)$ are equivalent.

7.2 Recovery of the mirror functor

The triangulated equivalence $\Phi : D^b(X) \to D^\pi\mathscr{F}^\sharp(T^2)$ of Theorem 7.1 is established using a deformation theoretic argument, and a priori it is not clear how precisely Φ acts on arbitrary objects. It is not even obvious that the mirror object of every indecomposable sheaf is a Lagrangian with a local system, as opposed to some "abstract" object introduced when passing to the split closure. In contrast, the equivalence constructed by Polishchuk–Zaslow [18] is given by explicit formulae, but it is not clear how compatible it is with the triangulated structure.

The aim here was to partially recover the effect of Φ on objects. We denote by $\mathcal{V}(r, d)$ the set of isomorphism classes of indecomposable vector bundles of rank r and degree d on X, and we identify $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ in the standard way. As usual, we include objects of Coh X in $D^b(X)$ by viewing them as complexes concentrated in degree 0.

Proposition 7.3 *(i) Let $\mathscr{S}$ be an indecomposable skyscraper sheaf on X with stalk of rank h. Then, $\Phi(\mathscr{S})$ is isomorphic to a Lagrangian of slope $(0, -1) \in H_1(T^2; \mathbb{Z})$ equipped with a local system of rank h.*

(ii) Let $E \in \mathcal{V}(r, d)$ and set $h = \gcd(r, d)$. Then, $\Phi(E)$ is isomorphic to a Lagrangian of slope $\frac{1}{h}(r, -d) \in H_1(T^2; \mathbb{Z})$, equipped with an indecomposable local system of rank h.

Every object of $D^b(X)$ is isomorphic to a direct sum of shifted copies of vector bundles and skyscraper sheaves (see, e.g., Corollary 3.15 in [11]). Since moreover the shift functor of $D^\pi\mathscr{F}^\sharp(T^2)$ just shifts the grading of Lagrangian branes, Theorem 7.1 and Proposition 7.3 together imply the following statement.

Corollary 7.4 *Every object of $D^\pi\mathscr{F}^\sharp(T^2)$ is, up to isomorphism, a direct sum of objects of $\mathscr{F}^\sharp(T^2)$, i.e., of Lagrangian branes with local systems.*

Corollary 7.5 *The inclusion $D\mathscr{F}^\sharp(T^2) \hookrightarrow D^\pi\mathscr{F}^\sharp(T^2)$ is an equivalence.*

The proof of Proposition 7.3 will occupy the rest of this section.

7.3 Proof of Proposition 7.3(i) for $h = 1$

We start with some preliminary discussion. Every point $Q \in X$ can be written as $Q = [-q^x M]$ with $x \in \mathbb{R}/\mathbb{Z}$ and $M \in S^1\Lambda$ that are uniquely determined.[2] From now on, we fix the base point of X to be

[2] Recall that by slight abuse of notation we write X for the analytification $X^{an} = \Lambda^*/q^{\mathbb{Z}}$.

$$P_0 = [-q^{1/2}].$$

This choice determines an equivalence $\Phi : D^b(X) \to D^\pi \mathscr{F}^\sharp(T^2)$ as described in Theorem 7.1, i.e., such that Φ takes $\mathcal{O}(nP_0)$ to $L_{(1,-n)}$.

We denote by $O = [q^0]$ the neutral element for the natural group structure on X induced by multiplication on Λ^*. Since X is an elliptic curve, it also carries an "elliptic curve group structure" with O as neutral element, see [10, Section IV.4]. In fact, both group structures must coincide (cf. [8, p.127]), and we denote the operation in this group by $\oplus$. It is related to the operation in the divisor group by

$$Q \oplus Q' = Q'' \quad \Longleftrightarrow \quad Q + Q' \sim Q'' + O, \tag{9}$$

where $\sim$ denotes linear equivalence of divisors (again, see [10, Section IV.4]).

To determine the mirror images of the one-dimensional skyscraper sheaves, we will use that (almost) every such skyscraper sheaf can be obtained as a direct summand of a cone on a morphism $\mathcal{O}(-2P_0) \to \mathcal{O}_X$. To see this, note that

$$\mathrm{Hom}_{D^b(X)}(\mathcal{O}(-2P_0), \mathcal{O}_X) \cong H^0(X; \mathcal{O}(2P_0)),$$

and hence the cone of every morphism $\mathcal{O}(-2P_0) \to \mathcal{O}_X$ is of the form $\mathcal{O}_D$ for a divisor D belonging to the linear system $|2P_0|$. Observe that, by (9), $|2P_0|$ is the set of all divisors $D = Q + Q'$ such that $Q \oplus Q' = P_0 \oplus P_0$, hence the set of divisors of the form

$$D = [-q^x M] + [-q^{-x} M^{-1}]$$

with $x \in \mathbb{R}/\mathbb{Z}$ and $M \in S^1\Lambda$. Whenever the points $[-q^x M]$ and $[-q^{-x} M^{-1}]$ are distinct, which is the case unless $x \in \{0, \frac{1}{2}\}$ and $M \in \{\pm q^0\}$, the sheaf $\mathcal{O}_D$ corresponding to such D is a direct sum of the corresponding one-dimensional skyscraper sheaves, that is,

$$\mathcal{O}_D = \mathcal{O}_{[-q^x M]} \oplus \mathcal{O}_{[-q^{-x} M^{-1}]}.$$

(And $\mathcal{O}_D = \mathcal{O}_{2[q^x M]}$ in the four cases in which $[-q^x M] = [-q^{-x} M^{-1}]$).

The space $\mathrm{Hom}_{D^b(X)}(\mathcal{O}(-2P_0), \mathcal{O}_X) \cong H^0(X; \mathcal{O}(2P_0))$ has a preferred basis given by the theta functions $\theta^0, \theta^1 : \Lambda^* \to \Lambda$ defined by

$$\theta^0(w) = \sum_{n \in \mathbb{Z}} w^{2n} q^{n^2},$$

$$\theta^1(w) = \sum_{n \in \mathbb{Z}} w^{2n+1} q^{\left(n+\frac{1}{2}\right)^2}.$$

To understand this, observe that these functions satisfy the functional equation $\theta^i(qw) = q^{-1} w^{-2}$ and therefore can be considered as sections of the line bundle

$$\Lambda^* \times \Lambda / \big((w, \xi) \sim (qw, q^{-1} w^{-2} \xi)\big)$$

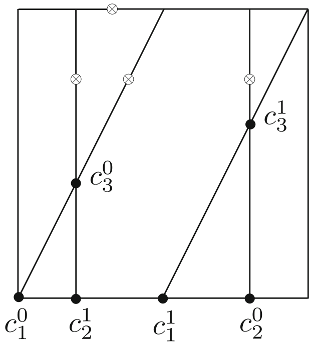

Fig. 11 The intersection points generating the various morphism spaces

over X, which is $\mathcal{O}(2P_0)$. (In fact, the reason for choosing the base point to be $P_0 = [-q^{1/2}]$ was to ensure that $H^0(X; \mathcal{O}(2P_0))$ is spanned by these "standard" theta functions). We refer to [18, Section 2.3] or [9] for the description of vector bundles on elliptic curves (over $\mathbb{C}$) via "multipliers," to [8, Section 4.7] for information on vector bundles over rigid-analytic spaces, and to [15] for background on theta functions.

Now, we turn to the symplectic side and study the cones on morphisms

$$c_1 \in \mathrm{Hom}_{D^\pi \mathscr{F}^\sharp(T^2)}(L_{(1,2)}, L_{(1,0)})$$

in $D^\pi \mathscr{F}^\sharp(T^2)$. This morphism space is spanned by the two intersection points

$$c_1^0, c_1^1 \in L_{(1,2)} \cap L_{(1,0)},$$

see Fig. 11 (more precisely, c_1^0, c_1^1 are the identity homomorphisms between the fibers of the respective local systems over these intersection points, which in our model for local systems are all equal to Λ).

Recall that $L_{(1,2)}$ and $L_{(1,0)}$ are the mirror images of $\mathcal{O}(-2P_0)$ and $\mathcal{O}_X$. Since the restriction of Φ to the full subcategory of $D^b(X)$ consisting of the $\mathcal{O}(nP_0)$ is essentially the Polishchuk–Zaslow functor [18], the corresponding isomorphism between morphism spaces is determined by

$$\mathrm{Hom}_{D^b(X)}(\mathcal{O}(-2P_0), \mathcal{O}_X) \ni \theta^i \leftrightarrow c_1^i \in \mathrm{Hom}_{D\mathscr{F}^\sharp(T^2)}(L_{(1,2)}, L_{(1,0)})$$

for $i = 0, 1$.

Lemma 7.6 *Let $c_1 = \sigma^0 c_1^0 + \sigma^1 c_1^1 : L_{(1,2)} \to L_{(1,0)}$ be a nonzero morphism and let $s = \sigma^0\theta^0 + \sigma^1\theta^1$ be the corresponding section of $\mathcal{O}(2P_0)$. If s vanishes at two distinct points $[-q^x M] \neq [-q^{-x}M^{-1}]$, then*

$$\mathrm{Cone}(c_1) \cong \left(L_{(0,-1),x}, E_M^1\right) \oplus \left(L_{(0,-1),-x}, E_{M^{-1}}^1\right).$$

If s has a double zero, then $\mathrm{Cone}(c_1) \cong (L_{(0,-1),x}, E_M^2)$ *for $x \in \{0, \frac{1}{2}\}$, $M \in \{\pm q^0\}$, and where E_M^2 is the unique non-trivial extension of E_M^1 by itself.*

The lemma will be proven in the next subsection. Before that, we finish proving Proposition 7.3(i) for $h = 1$. Since Φ takes cones to cones, we can infer from Lemma 7.6 and the discussion preceding it that

$$\Phi\left(\mathcal{O}_{[-q^x M]} \oplus \mathcal{O}_{[-q^{-x} M^{-1}]}\right) \cong \left(L_{(0,-1),x}, E^1_M\right) \oplus \left(L_{(0,-1),-x}, E^1_{M^{-1}}\right)$$

whenever $x \notin \{0, \frac{1}{2}\}$ and $M \notin \{\pm 1\}$. Since there are no other ways of writing the object on the right-hand side as a direct sum, we conclude that the mirror images of $\mathcal{O}_{[-q^x M]}$ and $\mathcal{O}_{[-q^{-x} M^{-1}]}$ are $(L_{(0,-1),x}, E^1_M)$ and $(L_{(0,-1),-x}, E^1_{M^{-1}})$. For the remaining four skyscrapers $\mathcal{O}_{[-q^x M]}$ with $x \in \{0, \frac{1}{2}\}$ and $M \in \{\pm 1\}$, a similar argument shows that they are mirror to the $(L_{(0,-1),x}, E^1_M)$. Since every skyscraper sheaf with stalk of rank 1 is of the form $\mathscr{S} = \mathcal{O}_{[-q^x M]}$, this concludes the proof. □

7.4 Proof of Lemma 7.6

Set $Y_0 := L_{(1,2)}$ and $Y_1 := L_{(1,0)}$ and let $c_1 = \sigma^0 c_1^0 + \sigma^1 c_0^1 \in \hom^0(Y_0, Y_1)$ be a nonzero morphism. We first rephrase what needs to be proven: Our task is to show that there exists an object Y_2 as described in the statement of the lemma, together with morphisms $c_2 \in \hom^0(Y_1, Y_2)$ and $c_3 \in \hom^1(Y_2, Y_0)$, such that the triangle

$$Y_0 \xrightarrow{[c_1]} Y_1 \xrightarrow{[c_2]} Y_2 \xrightarrow{[c_3]} Y_0[1] \tag{10}$$

is exact in $H(\mathscr{F}^\sharp(T^2))$. We will use Lemma 3.7 in [19] to prove this, which says that it suffices to show that

$$\begin{aligned} \mu^2(c_3, c_2) &= 0, \\ \mu^2(c_1, c_3) &= 0, \\ \mu^3(c_1, c_3, c_2) &= e_{Y_1}, \end{aligned} \tag{11}$$

where e_{Y_1} is a chain representing the identity in $\mathrm{Hom}^0(Y_1, Y_1)$, and to show the acyclity of the complex $\big(\hom(X, Y_2)[1] \oplus \hom(X, Y_0)[1] \oplus \hom(X, Y_1), \partial\big)$ for every test object X (see [19] for the description of the differential ∂). We will in fact only verify parts of this criterion, and then argue that this is already sufficient.

We first consider the case that c_1 is such that the corresponding section $s = \sigma^0\theta^0 + \sigma^1\theta^1$ of $\mathcal{O}(2P_0)$ vanishes at distinct points $[-q^x M], [-q^{-x} M^{-1}] \in X$, as opposed to having a double zero. Set

$$Y_2 = \left(L_{(0,-1),x}, E^1_M\right) \oplus \left(L_{(0,-1),-x}, E^1_{M^{-1}}\right), \tag{12}$$

and denote by c_3^0, c_3^1 the intersection points generating the space $\hom^1(Y_2, Y_0)$, as depicted in Fig. 11.

Step 1 $\mu^2(c_1, c_3^0) = 0$ *is equivalent to* $s([-q^{-x} M^{-1}]) = 0$*, and* $\mu^2(c_1, c_3^1) = 0$ *is equivalent to* $s([-q^x M]) = 0$.

Proof We will only show the statement for c_3^0, the other one being completely analogous. The signs with which the polygons encountered in the computation of $\mu^2(c_1, c_3^0)$ contribute will be determined according to the recipe described in [20, Section 7] or [13, Section 2] (replacing $Spin$ by Pin). In particular, we think of the bounding Pin structures on our Lagrangians as double covers which are trivialized except over one point where the two sheets are interchanged; these points are indicated by $\otimes$ in Fig. 11.

The triangles contributing to $\mu^2(c_1, c_3^0)$ are precisely the images of the family of triangles Δ_n, $n \in \mathbb{Z}$, in the universal cover $\mathbb{R}^2 \xrightarrow{\pi} T^2$ described as follows: Δ_n has one vertex at $(x, 2x)$, one at (x, n), and one at $(\frac{1}{2}n, n)$, see Fig. 12. The first two points project to c_3^0 and c_2^0, respectively, while the third one projects to c_1^0 or to c_1^1, according to whether n is even or odd. The area of Δ_n is

$$A_n = \left(\frac{1}{2}(n - 2x)\right)^2.$$

According to the recipe in [20, Section 7], the sign with which $\pi(\Delta_n)$ contributes to $\mu^2(c_1, c_3^0)$ is $(-1)^{s_n+1}$ in the case at hand, where s_n is the number of lifted $\otimes$ symbols encountered when travelling around the edges of Δ_n. One sees easily that s_n has the same parity as n, and therefore, the sought-for sign is $(-1)^{n+1}$.

The remaining ingredient for the computation of $\mu^2(c_1, c_3^0)$ is the parallel transport around the edges of $\pi(\Delta_n)$. Since the local systems on Y_0 and Y_1 are trivial, this is equal to the parallel transport in the local system E_M^1 along the vertical segment of (oriented) length $2x - n$ (the orientation of $L_{(0,-1)}$ points downwards); we denote this parallel transport map by M^{2x-n}.

Assembling all these ingredients, we obtain

$$\mu^2(c_1, c_3^0) = \left(\sigma^0 \sum_{n\in\mathbb{Z}} -M^{2x-2n} q^{(n-x)^2} + \sigma^1 \sum_{n\in\mathbb{Z}} M^{2x-(2n+1)} q^{(n+1/2-x)^2}\right) c_2^0.$$

Some basic arithmetic shows that the vanishing of this is equivalent to

$$\sigma^0 \sum_{n\in\mathbb{Z}} (-q^{-x} M^{-1})^{2n} q^{n^2} + \sigma^1 \sum_{n\in\mathbb{Z}} (-q^{-x} M^{-1})^{2n+1} q^{(n+\frac{1}{2})^2} = 0,$$

which says nothing but $s([-q^{-x} M^{-1}]) = 0$. □

Step 2 *There exists a morphism* $\widetilde{c}_1 \in \hom^0(Y_0, Y_1)$ *such that* $\operatorname{Cone}(\widetilde{c}_1) \cong Y_2$, *where* Y_2 *is as defined in* (12).

Proof There exists a cobordism

$$V : L_{(1,0)} \rightsquigarrow (L_{(0,-1),x}, L_{(0,-1),-x}, L_{(1,2)})$$

obtained from surgering $L_{(1,2)}$ with $L_{(0,-1),x}$ and $L_{(0,-1),-x}$, which results in $L_{(1,0)}$. It can be equipped with a local system restricting to E_M^1 and $E_{M^{-1}}^1$ on the two negative

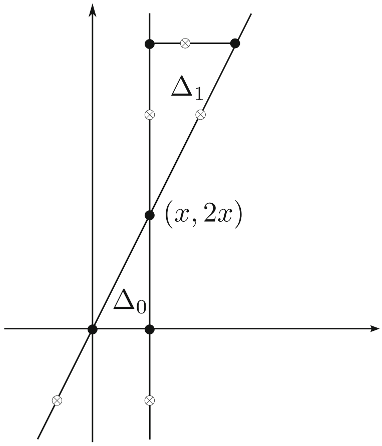

Fig. 12 Triangles in the universal cover contributing to $\mu^2(c_1, c_3^0)$

ends corresponding to $L_{(0,-1),x}$ and $L_{(0,-1),-x}$, and to trivial local systems on the other ends. Moreover, it can be equipped with a brane structure restricting to the standard brane structures of these Lagrangians as described in Sect. 6 (to verify this for the grading, use Lemma 5.3). From this, one can conclude the existence of a $\widetilde{c}_1$ with $\mathrm{Cone}(\widetilde{c}_1) \cong Y_2$ by "bending" the negative end of V corresponding to $L_{(1,2)}$ such as to become positive, and then using a slight generalization of Theorem 4.1 for cobordisms with multiple ends on both sides. □

Step 3 *We have* $\widetilde{c}_1 = c_1$ *up to a nonzero factor.*

Proof Since $\mathrm{Cone}(\widetilde{c}_1) \cong Y_2$, we infer that there is some nonzero morphism $c_3 = \eta^0 c_3^0 + \eta^1 c_3^1 \in \hom^1(Y_2, Y_0)$ such that $\mu^2(\widetilde{c}_1, c_3) = \eta^0 \mu^2(\widetilde{c}_1, c_3^0) + \eta^1 \mu^2(\widetilde{c}_1, c_3^1) = 0$, again by [19, Lemma 3.7]. This is equivalent to the individual vanishing of $\eta^0 \mu^2$ $(\widetilde{c}_1, c_3^0)$ and $\eta^1 \mu^2(\widetilde{c}_1, c_3^1)$, because the first is a multiple of c_2^0, while the second is a multiple of c_2^1. Since at least one of η^0 and η^1 is nonzero, we conclude that $\mu^2(\widetilde{c}_1, c_3^0) = 0$ or $\mu^2(\widetilde{c}_1, c_3^1) = 0$.

Consider now the section $\widetilde{s} = \widetilde{\sigma}^0 \theta^0 + \widetilde{\sigma}^1 \theta^1$ of $\mathcal{O}(2P_0)$ corresponding to $\widetilde{c}_1 = \widetilde{\sigma}^0 c_1^0 + \widetilde{\sigma}^1 c_1^1$. Assuming that $\mu^2(\widetilde{c}_1, c_3^0) = 0$, we conclude that $\widetilde{s}([-q^{-x} M^{-1}]) = 0$ by the result of Step 1, and hence also $\widetilde{s}([-q^x M]) = 0$, as $\widetilde{s}$ in a section of $\mathcal{O}(2P_0)$. (Assuming $\mu^2(\widetilde{c}_1, c_3^1) = 0$ would have led to the same conclusion).

But this implies that $\widetilde{s} = s$ up to a nonzero factor, and hence $\widetilde{c}_1 = c_1$ up to a nonzero factor. □

Combing the results of Steps 2 and 3, we conclude that

$$\mathrm{Cone}(c_1) \cong Y_2,$$

as required. This finishes the prove of the lemma in the case that s has two distinct zeros.

We still have to argue that the cones of the four remaining morphisms, for which the corresponding sections $s \in H^0(X; \mathcal{O}(2P_0))$ have a double zero, are as claimed.

By a similar cobordism argument as in the proof Step 2, we can infer that for every $x \in \{0, \frac{1}{2}\}$ and every $M \in \{\pm 1\}$, there exists a morphism $c_1 \in \hom^0(Y_0, Y_1)$ and an automorphism a of $(L_{(0,-1),x}, E^1_M)$ such that $\mathrm{Cone}(c_1) \cong \mathrm{Cone}(a)$. These morphisms c_1 must be the four remaining ones, because we have already found the cones of all others to be different from what we get here. Moreover, their cones must be indecomposable, since the same is true on the algebraic side. But the only indecomposable cone on an automorphism of $(L_{(0,-1),x}, E^1_M)$ is $(L_{(0,-1),x}, E^2_M)$. □

7.5 Proof of Proposition 7.3

Proposition 7.3(i) for $h = 1$ has already been proven. We break the rest of the proof up into several steps. The common strategy is to exhibit for a given sheaf $\mathcal{F}$ a cone decomposition in $D^b(X)$ with linearization consisting of objects for which we already know Proposition 7.3 holds, and with the property that $\mathcal{F}$ is *determined* by the cone decomposition.[3] We then construct, using iterated surgery, an object of $D^\pi \mathscr{F}^\sharp(T^2)$ which has a cone decomposition mirror to the one of $\mathcal{F}$ and therefore must be mirror to $\mathcal{F}$.

Step 1 *Proposition 7.3(ii) holds for all line bundles $\mathcal{E} \in \mathcal{V}(1, d)$, $d \in \mathbb{Z}$.*

Proof We can write $\mathcal{E} = \mathcal{O}((d+1)P_0 - Q)$ for some $Q \in X$, and there is an exact triangle

$$\mathcal{O}_Q[-1] \to \mathcal{E} \to \mathcal{O}((d+1)P_0) \to \mathcal{O}_Q$$

in $D^b(X)$. Since $\mathrm{Hom}_{D^b(X)}(\mathcal{O}((d+1)P_0), \mathcal{O}_Q)$ is one-dimensional, $\mathcal{E}$ is in fact the *only* indecomposable object fitting into an exact triangle with the other two middle fixed. To find its mirror object, it suffices therefore to exhibit an exact triangle in $D^\pi \mathscr{F}^\sharp(T^2)$ in which two rightmost objects are mirror to $\mathcal{O}((d+1)P_0)$ and $\mathcal{O}_Q$.

We know already that these mirror objects are $L_{(1,-(d+1))}$ and $(L_{(0,-1),x}, E^1_M)$ for certain $x \in \mathbb{R}/\mathbb{Z}$ and $M \in S^1\Lambda$. Now, there exists some $x' \in \mathbb{R}/\mathbb{Z}$ such that surgering $L_{(0,-1),x}$ and $L_{(1,-d),x'}$ produces $L_{(1,-(d+1))}$, up to Hamiltonian isotopy. After equipping the corresponding cobordism $V : L_{(1,-(d+1))} \rightsquigarrow (L_{(0,-1),x}, L_{(1,-d),x'})$ with an appropriate brane structure and a local system , Theorem 4.1 yields an exact triangle

$$(L_{(0,-1),x}, E^1_M)[-1] \to (L_{(1,-d),x'}, E^1_{M^{-1}}) \to L_{(1,-(d+1))} \to (L_{(0,-1),x}, E^1_M)$$

We conclude that $\Phi(\mathcal{E}) = (L_{(1,-d),x'}, E^1_{M^{-1}})$ up to isomorphism, as required. □

Step 2 *Proposition 7.3 holds for every $\mathcal{E} \in \mathcal{V}(r, d)$ whenever* $\gcd(r, d) = 1$.

Proof Theorem 11.1 implies that there exists an integer n such that $\mathcal{E}$ fits into an exact triangle

$$\mathcal{L}[-1] \to (\mathcal{L}')^{\oplus r-1} \to \mathcal{E} \to \mathcal{L}$$

[3] Here and in the following, we should often add "up to isomorphism" to be really precise.

with $\mathcal{L} = (\det \mathcal{E})((r-1)n)$ and $\mathcal{L}' = \mathcal{O}_X(-n)$, where $-(k)$ denotes tensoring by the k^{th} power of the hyperplane bundle. Hence, $\mathcal{E}$ admits an iterated cone decomposition with linearization $(\mathcal{L}, \mathcal{L}', \dots, \mathcal{L}')$, where $\mathcal{L}'$ appears $r-1$ times (cf. Lemma 9.3).

We claim that $\mathcal{E}$ is actually the only indecomposable object of $D^b(X)$ admitting a cone decomposition with this linearization. Suppose that $\widetilde{\mathcal{E}}$ is another indecomposable object with the same property. It follows that $[\widetilde{\mathcal{E}}] = [\mathcal{E}] \in K_0(X)$ by Lemma 9.4, and hence $\det \widetilde{\mathcal{E}} = \det \mathcal{E}$ and $\operatorname{rk} \widetilde{\mathcal{E}} = \operatorname{rk} \mathcal{E}$, because $(\det, \operatorname{rk}) : K_0(X) \to \operatorname{Pic}(X) \oplus \mathbb{Z}$ is an isomorphism (see [10]). Since $\widetilde{\mathcal{E}}$ is indecomposable, we conclude that $\widetilde{\mathcal{E}} \in \mathcal{V}(r, d)$. By Theorem 11.3, the condition $\gcd(r, d) = 1$ implies that $\det : \mathcal{V}(r, d) \to \mathcal{V}(1, d) \subset \operatorname{Pic}(X)$ is bijective, and hence $\widetilde{\mathcal{E}} = \mathcal{E}$.

To find $\Phi(\mathcal{E})$, it is therefore sufficient to construct an object of $D^\pi \mathscr{F}^\sharp(T^2)$ admitting a cone decomposition with linearization mirror to $(\mathcal{L}, \mathcal{L}', \dots, \mathcal{L}')$. The line bundles there have degrees $\deg \mathcal{L} = d + 3n(r-1)$ and $\deg \mathcal{L}' = -3n$, since the hyperplane bundle has degree 3. We conclude, using the previous step, that their mirror objects are of the form

$$\Phi(\mathcal{L}) = (L, E) \quad \text{and} \quad \Phi(\mathcal{L}') = (L', E'),$$

with Lagrangians L and L' of slopes $(1, -d-3n(r-1))$ and $(1, 3n)$ that are equipped with rank-one local systems E and E'. Starting with L and iteratively surgering $r-1$ times with L' leads to a sequence of Lagrangians $N_0 = L, N_1, \dots, N_{r-1}$ of slopes

$$[N_j] = (1 + j, -d - 3n(r - 1 - j)) \in H_1(T^2; \mathbb{Z}), \quad j = 0, \dots, r-1.$$

(Note that N_j might have multiple components, and a priori these might have slopes equal to that of L', which would be problematic. But the only way this can happen is that N_j has $1 + j$ components of slope $(1, (-d - 3n(r - 1 - j))/(1 + j))$; to exclude this being equal to $(1, 3n) = [L']$, we assume that n has been chosen such that $-d < 3nr$. This is no restriction, since any sufficiently big n leads to an exact triangle as above.)

To this sequence of surgeries corresponds a sequence of cobordisms

$$V_j : N_j \rightsquigarrow (L, \underbrace{L', \dots, L'}_{j \text{ times}}), \quad j = 0, \dots, r-1.$$

These can be equipped with brane structures and rank-one local systems that extend those on the negative ends and determine a brane structure and a rank-one local system E_j on the positive end. Theorem 4.1 applied to V_{r-1} says that there is an iterated cone decomposition of (N_{r-1}, E_{r-1}) with linearization

$$((L, E), (L', E'), \dots, (L', E')).$$

This is mirror to the one for $\mathcal{E}$, and we infer $\Phi(\mathcal{E}) \cong (N_{r-1}, E_{r-1})$. That proves the claimed statement, since N_{r-1} has slope $(r, -d)$. □

Step 3 *Proposition 7.3 holds for all $\mathcal{E} \in \mathcal{V}(hr, hd)$ with $\gcd(r, d) = 1$ and $h \geq 1$, and for all indecomposable skyscraper sheaves with stalks of rank $h \geq 1$.*

Proof By Lemma 11.5, every indecomposable vector bundle $\mathcal{E}$ on X is of the form $E_{\mathcal{L}_1}(hr, hd) \otimes \mathcal{L}_0$ for some (r, d) with $\gcd(r, d) = 1$, $h \in \mathbb{N}$, and certain line bundles $\mathcal{L}_0, \mathcal{L}_1$ of degrees zero and one (see Appendix 3 for the notation). Moreover, for every $h \geq 1$ and every $Q \in X$, there is a unique indecomposable skyscraper sheaf $\mathcal{O}_{hQ}$ with stalk of rank h supported at Q, and every indecomposable skyscraper sheaf is of this form.

Denote by $\mathcal{Y}_h$ either the vector bundle $E_{\mathcal{L}_1}(hr, hd) \otimes \mathcal{L}_0$ for fixed $r, d, \mathcal{L}_0$ and $\mathcal{L}_1$, or the skyscraper sheaf $\mathcal{O}_{hQ}$ for fixed Q. We will prove that for any such choice, the mirror objects of the $\mathcal{Y}_h$, $h \geq 1$, are of the form (L, E^h), with L a Lagrangian of slope (r, d), or $(0, -1)$, and E^h an indecomposable local system of rank h over L such that there exists a short exact sequence

$$0 \to E^1 \to E^{h+1} \to E^h \to 0$$

of local systems on L (which in this case is equivalent to saying that the unique eigenvalue of the monodromy is the same for all of them). The proof will be by induction on h. The claim for $h = 1$ is what was proven in the previous steps.

In both of the above cases and for every $h \in \mathbb{N}$, there is an exact triangle

$$\mathcal{Y}_h[-1] \to \mathcal{Y}_1 \to \mathcal{Y}_{h+1} \to \mathcal{Y}_h$$

in $D^b(X)$. For $\mathcal{Y}_h = E_{\mathcal{L}_1}(hr, hd) \otimes \mathcal{L}_0$, this comes from the short exact sequence obtained by tensoring the short exact sequence in Theorem 11.2(i) with $E_{\mathcal{L}_1}(hr, hd) \otimes \mathcal{L}_0$ (also cf. 11.4). For $\mathcal{Y}_h = \mathcal{O}_{hQ}$, it comes from the short exact sequence $0 \to \mathcal{O}_Q \to \mathcal{O}_{(h+1)Q} \to \mathcal{O}_{hQ} \to 0$. Moreover, we have

$$\dim \operatorname{Ext}^1(\mathcal{Y}_h, \mathcal{Y}_1) = 1$$

in both cases (to see this in the first case, use Serre duality and [4, Lemma 22]). Hence, $\mathcal{Y}_{h+1}$ is the only indecomposable object of $D^b(X)$ that can arise as a cone on a morphism $\mathcal{Y}_h[-1] \to \mathcal{Y}_1$.

By inductive assumption, the mirror images of $\mathcal{Y}_1$ and $\mathcal{Y}_h$ share the same Lagrangian brane that is equipped with indecomposable local systems E^1, E^h of ranks 1 and h, and whose monodromies have the same eigenvalue (because there exists a morphism of local systems $E^1 \to E^h$). There is a unique indecomposable local system E^{h+1} of rank $h + 1$ fitting into a short exact sequence $0 \to E^1 \to E^{h+1} \to E^h \to 0$ of local systems on L (namely, the unique indecomposable local system whose monodromy has this eigenvalue). Hence, there exists an exact triangle

$$(L, E^h)[-1] \to (L, E^1) \to (L, E^{h+1}) \to (L, E^h)$$

in $D\mathscr{F}^\sharp(T^2)$ by Proposition 10.1. We conclude that $\Phi(\mathcal{Y}_{h+1}) = (L, E^{h+1})$ by the same argument as in the previous steps. This is as required. $\square$

This ends the proof of Proposition 7.3. $\square$

8 Proofs of the main theorems

We will first prove Theorem 1.3 and then deduce Theorems 1.1 and 1.2.

8.1 Proof of Theorem 1.3

The cobordism group $\Omega^{\sharp}_{\mathrm{Lag}}(T^2)$ appearing in Theorem 1.3 is defined as described in Sect. 2.4 with $\mathscr{L}^{\sharp}(T^2) = \mathrm{Ob}\,\mathscr{F}^{\sharp}(T^2)$, i.e., it has as generators Lagrangian branes with local systems and relations coming from cobordisms with vanishing Maslov class carrying compatible local systems and gradings, as well as additional relations induced by short exact sequences of local systems.

Let X be the Tate curve mirror to T^2 and denote by

$$K_0(X) = K_0(D^b(X))$$

the Grothendieck group of its derived category of coherent sheaves. The mirror functor $\Phi : D^b(X) \to D^{\pi}\mathscr{F}^{\sharp}(T^2)$ of Theorem 7.1 induces an isomorphism of Grothendieck groups as it is an equivalence of triangulated categories. Since moreover the inclusion $D\mathscr{F}^{\sharp}(T^2) \hookrightarrow D^{\pi}\mathscr{F}^{\sharp}(T^2)$ is an equivalence by Corollary 7.5, and hence also induces an isomorphism of Grothendieck groups, we obtain an isomorphism

$$K_0(D\mathscr{F}^{\sharp}(T^2)) \xrightarrow{\cong} K_0(X).$$

Denote by $\underline{\mathrm{Coh}}\,X \subset D^bX$ the full subcategory consisting of direct sums of shifted sheaves; as mentioned before, $\underline{\mathrm{Coh}}\,X$ is equivalent to D^bX. Consider the map

$$F_{\Phi} : \langle D^b(X)\rangle \to \langle \mathscr{L}^{\sharp}(T^2)\rangle$$

induced by Φ, in the sense that it first replaces an arbitrary object of D^bX by an isomorphic object of $\underline{\mathrm{Coh}}\,X$, and then applies the mirror functor to get a sum of Lagrangian branes with local systems. We will prove that F_{Φ} descends to a well-defined group homomorphism $K_0(X) \to \Omega^{\sharp}_{\mathrm{Lag}}(T^2)$, thus completing the diagram

$$\begin{tikzcd} \Omega^{\sharp}_{\mathrm{Lag}}(T^2) \arrow[rr, "\Theta^{\sharp}"] & & K_0(D\mathscr{F}^{\sharp}(T^2)) \arrow[dl, "\cong"] \\ & K_0(X) \arrow[ul, dashed] & \end{tikzcd} \tag{13}$$

where $\Theta^{\sharp} : \Omega^{\sharp}_{\mathrm{Lag}}(T^2) \to K_0(D\mathscr{F}^{\sharp}(T^2))$ is the canonical surjective homomorphism from Proposition 4.8. Once well definedness is proven, it is clear by construction that the composition of the two lower homomorphisms in (13) provides a left-inverse for $\Theta^{\sharp}$, showing in particular that $\Theta^{\sharp}$ is injective. Given that we already have surjectivity, this will conclude the proof of Theorem 1.3.

8.1.1 Well definedness

Denote by R the set of K_0-relations among objects of $\langle \underline{\mathrm{Coh}}\, X\rangle$, i.e., the kernel of $\langle \underline{\mathrm{Coh}}\, X\rangle \to K_0(X)$. It follows from Proposition 9.5 that the inclusion $\underline{\mathrm{Coh}}\, X \hookrightarrow D^b X$ induces an isomorphism $\langle \underline{\mathrm{Coh}}\, X\rangle/R \cong K_0(X)$. To prove that F_Φ induces a map $K_0(X) \to \Omega^\sharp_{\mathrm{Lag}}(T^2)$, we must therefore show that F_Φ takes R to $R^\sharp$, the kernel of $\langle \mathscr{L}^\sharp(T^2)\rangle \to \Omega^\sharp_{\mathrm{Lag}}(T^2)$. The essential part is to prove that this is true for the subset $R_0 = R \cap \langle \mathrm{Ind}\,\mathrm{Coh}\, X\rangle$, the set of K_0-relations among indecomposable objects of $\mathrm{Coh}\, X$.

Proposition 8.1 *R_0 is generated by relations coming from short exact sequences of the following types*

- $0 \to \mathcal{F} \to \mathcal{G} \to 0$ *with indecomposable* $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}\, X$*;*
- $0 \to \mathcal{O}(D - Q) \to \mathcal{O}(D) \to \mathcal{O}_Q \to 0$ *with D a divisor and $Q \in X$;*
- $0 \to (\mathcal{O}_X^{\oplus r-1})(-n) \to \mathcal{E} \to (\det \mathcal{E})((r-1)n) \to 0$ *with $\mathcal{E} \in \mathcal{V}(r, d)$ such that* $\gcd(r, d) = 1$*, and $n \in \mathbb{Z}$ such that $\mathcal{E}(n)$ is generated by global sections;*
- $0 \to \mathcal{Y}_1 \to \mathcal{Y}_{h+1} \to \mathcal{Y}_h \to 0$ *for $h \geq 1$.*

Here, $\mathcal{Y}_h$ denotes either a vector bundle of the form $E_{\mathcal{L}_0}(rh, dh) \otimes \mathcal{L}_1$ for some fixed r, d with $\gcd(r, d) = 1$ *and fixed line bundles $\mathcal{L}_0, \mathcal{L}_1$ of degree 0 and 1, respectively, or an indecomposable skyscraper sheaf $\mathscr{O}_{hq}$ for some $q \in X$. (The SESs of the second type are the obvious ones; those of the third type are as in Theorem 11.1; and those of the fourth type are as mentioned in Step 3 of the proof of Proposition 7.3).*

Proof Denote by $S \subset \langle \mathrm{Ind}\,\mathrm{Coh}\, X\rangle$ the subgroup generated by the relations induced by the short exact sequences of the types stated, and by $[\![\mathcal{F}]\!]$ the class of $\mathcal{F} \in \mathrm{Ind}\,\mathrm{Coh}\, X$ in $\langle \mathrm{Ind}\,\mathrm{Coh}\, X\rangle/S$. We will show that

$$[\![\mathcal{F}]\!] = [\![\det \mathcal{F}]\!] + (\mathrm{rk}\, \mathcal{F} - 1)[\![\mathcal{O}_X]\!]. \tag{14}$$

This implies immediately that the canonical map $\langle \mathrm{Ind}\,\mathrm{Coh}\, X\rangle/S \to K_0(X)$, which is easily seen to be surjective using Proposition 9.5, is also injective: Namely, since $(\det, \mathrm{rk}) : K_0(X) \to \mathrm{Pic}(X) \oplus \mathbb{Z}$ is an isomorphism, the equality $[\mathcal{F}] = [\mathcal{G}]$ in $K_0(X)$ is equivalent to $\det \mathcal{F} = \det \mathcal{G}$ and $\mathrm{rk}\, \mathcal{F} = \mathrm{rk}\, \mathcal{G}$, from which $[\![\mathcal{F}]\!] = [\![\mathcal{G}]\!]$ follows by (14). This implies that the described relations generate all of R_0.

We first claim that every sheaf can be written as a sum or difference in line bundles in $\langle \mathrm{Ind}\,\mathrm{Coh}\, X\rangle/S$. Now, an indecomposable sheaf on X is either a vector bundle of the form $E_{\mathcal{L}_0}(rh, dh) \otimes \mathcal{L}_1$ for certain r, d with $\gcd(r, d) = 1$ and $h \geq 1$, or a skyscraper sheaf $\mathscr{O}_{hq}$ for some $q \in X$ and $h \geq 1$. Using the relations of the last type, we can inductively reduce to considering the case $h = 1$, i.e., the case of vector bundles $\mathcal{E} \in \mathcal{V}(r, d)$ with $\gcd(r, d) = 1$ or of skyscraper sheaves $\mathcal{O}_Q$. Now, the relations of second and third types show that these satisfy

$$[\![\mathcal{E}]\!] = (r-1)[\![\mathcal{O}_X(-n)]\!] + [\![\det \mathcal{E}((r-1)n)]\!], \quad [\![\mathcal{O}_Q]\!] = [\![\mathcal{O}(D)]\!] - [\![\mathcal{O}(D - Q)]\!],$$

and since the classes on the right-hand sides are those of lines bundles, our claim is shown.

We now prove that

$$[\![\mathcal{L}]\!] + [\![\mathcal{L}']\!] = [\![\mathcal{L} \otimes \mathcal{L}']\!] + [\![\mathcal{O}_X]\!], \quad [\![\mathcal{L}]\!] - [\![\mathcal{L}']\!] = [\![\mathcal{L} \otimes \mathcal{L}'^{-1}]\!] \tag{15}$$

for all lines bundles $\mathcal{L}$ and $\mathcal{L}'$. Observe that every line bundle is of the form $\mathcal{O}(D)$ for a divisor $D = \sum_{i=1}^{n} P_i - \sum_{j=1}^{m} Q_j$. Using the relations of the second type in the statement of the lemma, one obtains inductively that

$$[\![\mathcal{O}(D)]\!] = [\![\mathcal{O}_X]\!] + \sum_{i=1}^{n} [\![\mathcal{O}_{P_i}]\!] - \sum_{j=1}^{m} [\![\mathcal{O}_{Q_j}]\!].$$

The identities in (15) follow from this because $\mathcal{O}(D) \otimes \mathcal{O}(D') = \mathcal{O}(D + D')$ and $\mathcal{O}(D) \otimes \mathcal{O}(D')^{-1} = \mathcal{O}(D - D')$.

Equation (14) now follows easily for all $\mathcal{F} \in \operatorname{Ind}\operatorname{Coh} X$. As we have seen, we can write $[\![\mathcal{F}]\!] = \sum_{i=1}^{k} s_i [\![\mathcal{L}_i]\!]$ with line bundles $\mathcal{L}_i$ and $s_i \in \{\pm 1\}$, and then $\det \mathcal{F} = \bigotimes_{i=1}^{k} \mathcal{L}_i^{s_i}$ and $\operatorname{rk} \mathcal{F} = \sum_{i=1}^{k} s_i$. Applying (15) inductively yields (14). □

Generators of R_0 of the first type listed in Proposition 8.1 identify isomorphic indecomposable sheaves; their mirror images are Hamiltonian isotopic curves with isomorphic local systems, and we know that there exists a cobordism between these. Cobordisms corresponding to the generators of R_0 of the other types were constructed in the proof of Proposition 7.3. This implies that F_Φ takes R_0 to $R^\sharp$.

The remaining relations in R come from taking direct sums and shifting, that is, we can write $R = R_0 + R_1$ where R_1 is generated by elements of the form $\mathcal{F} \oplus \mathcal{G} - (\mathcal{F} + \mathcal{G})$ and $\mathcal{F}[1] + \mathcal{F}$. That F_Φ takes these generators, and hence, all of R_1 to $R^\sharp$ is clear by definition for those of type "direct sum," and follows for those of type "shift" from the fact that $(L[1], L)$ is null-cobordant for every Lagrangian brane L: A null-cobordism is given by $V = \gamma \times L$ equipped with a suitable grading, where $\gamma \subset \mathbb{R}^2$ is a curve with two negative ends.

This finishes the proof of Theorem 1.3. □

8.2 Proof of Theorem 1.1

Consider the diagram

$$\begin{array}{ccc} \Omega_{\mathrm{Lag}}(T^2) & \longrightarrow & \Omega^\sharp_{\mathrm{Lag}}(T^2) \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta^\sharp} \\ K_0(D\mathscr{F}(T^2)) & \longrightarrow & K_0(D\mathscr{F}^\sharp(T^2)) \end{array}$$

in which the upper arrow is the canonical map induced by $\mathscr{L}(T^2) \hookrightarrow \mathscr{L}^\sharp(T^2)$, and where the lower arrow is the map induced by the inclusion $D\mathscr{F}(T^2) \hookrightarrow D\mathscr{F}^\sharp(T^2)$. The square commutes because both compositions take the class of any given Lagrangian L in $\Omega_{\mathrm{Lag}}(T^2)$ to the class of L in $K_0(D\mathscr{F}^\sharp(T^2))$. Note that the upper horizontal map is injective, as it is a section of the group homomorphism $\Omega^{\sharp}_{\mathrm{Lag}}(T^2) \to \Omega_{\mathrm{Lag}}(T^2)$ induced by $(L, E) \mapsto \mathrm{rk}(E)L$ (that this is well defined is immediate from the definition of the relations in both groups, see Sect. 2). Since $\Theta^{\sharp}$ is already known to be an isomorphism, this implies that Θ is an isomorphism and hence concludes the proof of Theorem 1.1. $\square$

8.3 Proof of Theorem 1.2

In view of Proposition 6.2, we are left with showing that the map $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\mathrm{Lag}}(T^2)$ defined in Sect. 6.2 is injective. To see this, consider the composition

$$\mathbb{R}/\mathbb{Z} \xrightarrow{\zeta} \Omega_{\mathrm{Lag}}(T^2) \xrightarrow{\Theta} K_0(D\mathscr{F}(T^2)) \to K_0(D\mathscr{F}^{\sharp}(T^2)) \to K_0(X).$$

Recall that $\zeta(x) = [L_{(0,1),x}] - [L_{(0,1)}] = [L_{(0,-1)}] - [L_{(0,-1),x}]$. By what we know about the action of the mirror functor (see the proof of Proposition 7.3), the composition hence takes $x \mapsto [\mathscr{O}_{[-q^0]}] - [\mathscr{O}_{[-q^x]}] \in K_0(X)$, which is zero if and only if $x = 0 \in \mathbb{R}/\mathbb{Z}$. So the entire composition is injective, and therefore ζ is. $\square$

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9 Appendix 1: Iterated cone decompositions and K_0

9.1 Triangulated categories

A *triangulated category* $\mathscr{D}$ is an additive category equipped with an additive autoequivalence $S : \mathscr{D} \to \mathscr{D}$ called the *shift functor*, and a set of *exact triangles* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S(X)$. These data are required to satisfy a list of axioms, for which we refer to [22]. The most relevant for us is that every morphism $f : X \to Y$ in $\mathscr{D}$ can be completed to such an exact triangle. The object Z is then determined up to isomorphism, and we call it a *cone* on the morphism f. Moreover, we write $X[n] = S^n X$ and $f[n] = S^n(f)$ for the effect of iterates of the shift functor S on objects and morphisms. This is in reminiscence of the homotopy category of complexes $K(\mathscr{A})$ over an Abelian category $\mathscr{A}$, which is the prototypical example of a triangulated category.

9.2 Generation and iterated cone decompositions

Given a full subcategory of a triangulated category $\mathscr{D}$ with objects a collection $\{X_i \mid i \in I\}$, one can consider the full subcategory consisting of all objects that are cones on morphisms between the X_i. Iterating this construction, i.e., including in each step all cones on morphisms between the previously constructed objects, one ends up with the subcategory of $\mathscr{D}$ *generated* by the X_i.

In the other direction, one can ask whether and how an object X of $\mathscr{D}$ can be constructed as an iterated cone on morphisms between other objects. The following notion is useful to formalize this.

Definition 9.1 Let $\mathscr{D}$ be a triangulated category and let $X \in \mathscr{D}$. An *iterated cone decomposition of* X is a sequence of exact triangles

$$C_{i-1}[-1] \to X_i \to C_i \to C_{i-1}, \quad i = 1, \ldots, k,$$

with objects $C_0, C_1, \ldots, C_k \in \mathscr{D}$ that satisfy $C_0 = 0$ and $C_k = X$. The tuple $(X_1, \ldots, X_k)$ is called the *linearization* of the cone decomposition.

Remark 9.2 This definition is adapted to the cohomological conventions we use and therefore differs from the one in [6, Section 2.6], where homological conventions are used.

Iterated cone decompositions can themselves be iterated and are well behaved with respect to that in the sense of the following lemma (cf. the composition in Biran–Cornea's category $T^S\mathscr{D}$ of iterated cone decompositions [6, Section 2.6]).

Lemma 9.3 *Suppose that $X \in \mathscr{D}$ admits an iterated cone decomposition with linearization $(X_1, \ldots, X_k)$, and that one of the X_h, $1 \leq h \leq k$ admits an iterated cone decomposition with linearization $(X_h^1, \ldots, X_h^\ell)$. Then, X admits an iterated cone decomposition with linearization*

$$(X_1, \ldots, X_{h-1}, X_h^1, \ldots, X_h^\ell, X_{h+1}, \ldots, X_k).$$

9.3 Grothendieck groups

Let $\mathscr{D}$ be a triangulated category. The Grothendieck group $K_0(\mathscr{D})$ is defined as the quotient $K_0(\mathscr{D}) = \langle \mathrm{Ob}\,\mathscr{D}\rangle / R$ of the free Abelian group generated by the objects of $\mathscr{D}$ by the subgroup R generated by all expressions $X - Y + Z \in \langle \mathrm{Ob}\,\mathscr{D}\rangle$ such that there exists an exact triangle $X \to Y \to Z \to X[1]$. The following lemma is straightforward.

Lemma 9.4 *Suppose that $X \in \mathscr{D}$ admits an iterated cone decomposition with linearization $(X_1, \ldots, X_k)$. Then, $[X] = [X_1] + \cdots + [X_k]$ in $K_0(\mathscr{D})$.*

One can also define the Grothendieck group $K_0(\mathscr{A})$ of an Abelian category $\mathscr{A}$, by starting with the free Abelian group on objects and imposing a relation $[A]+[C] = [B]$ for every short exact sequence $0 \to A \to B \to C \to 0$. Recall the canonical inclusion $\mathscr{A} \hookrightarrow D^b(\mathscr{A})$, which on objects is given by viewing $X \in \mathrm{Ob}\,\mathscr{A}$ as a complex concentrated in degree zero. The following statement is well known.

Proposition 9.5 *The canonical inclusion $\mathscr{A} \hookrightarrow D^b(\mathscr{A})$ induces an isomorphism $K_0(\mathscr{A}) \cong K_0(D^b\mathscr{A})$.*

10 Appendix 2: Exact triangles from SESs of local systems

Consider a symplectic manifold (M, ω) for which we can define the Fukaya category $\mathscr{F}^\sharp(M)$ with gradings and signs as outlined in Sect. 3 such that the objects are Lagrangian branes with local systems. There appears to be no reference in the literature for the following statement.

Proposition 10.1 *Let L be a Lagrangian brane and let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence of local systems on L. Then, there exists an exact triangle*

$$(L, E'')[-1] \to (L, E') \to (L, E) \to (L, E'')$$

in $D\mathscr{F}^\sharp(M)$.

We view local systems as assignments of vector spaces and parallel transport maps as described in Sect. 3.1.3. By a short exact sequence of local systems $0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$, we mean a family of short exact sequences of vector spaces

$$0 \to E'_x \xrightarrow{i_x} E_x \xrightarrow{p_x} E''_x \to 0$$

for every $x \in L$ such that the i_x and p_x define morphisms of local systems, i.e., such that they commute with parallel transport maps. For the following proof, we choose a splitting

$$0 \to E'_x \underset{q_x}{\overset{i_x}{\rightleftarrows}} E_x \underset{j_x}{\overset{p_x}{\rightleftarrows}} E''_x \to 0$$

for every $x \in L$, that is, maps $q_x : E_x \to E'_x$ and $j_x : E''_x \to E_x$ such that $q_x \circ i_x = \mathrm{id}_{E'_x}$, $p_x \circ j_x = \mathrm{id}_{E''_x}$ and $i_x \circ q_x + j_x \circ p_x = \mathrm{id}_{E_x}$. Note that the j_x and q_x do generally not define morphisms of local systems, unless the short exact sequence splits globally.

In the proof of Proposition 10.1, it will be convenient to model the relevant morphism spaces in $\mathscr{F}^\sharp(M)$ as spaces of Morse cochains with coefficients in local systems. This is possible under certain conditions on M and L, the fundamental example being that M is a cotangent bundle and L is an exact Lagrangian (from which the case of interest in this paper follows immediately by a covering argument).

We will adapt a construction used in [2] (which goes back to [7]) and consider an A_∞-category $\mathscr{M}(L)$ defined as follows: The objects of $\mathscr{M}(L)$ are all those objects of $\mathscr{F}^\sharp(M)$ whose underlying Lagrangian is L. For the morphism spaces, we fix a Morse function $f : L \to \mathbb{R}$ and define

$$\hom^i_{\mathscr{M}(L)}(E_0, E_1) = \bigoplus_{\substack{x \in \mathrm{Crit}\, f \\ |x|=i}} \mathrm{Hom}(E_{0,x}, E_{1,x}),$$

where $|\cdot|$ denotes the Morse index; here, we denote objects of $\mathscr{M}(L)$ simply by their local system. The A_∞-operations $\mu^d_{\mathscr{M}(L)}$ are defined by considering rigid perturbed gradient flow trees with vertices at critical points and summing up parallel transport maps in the relevant local systems along the edges of these trees. See [2] for the description of the relevant moduli spaces.

By adapting the arguments in [2], one can show (under certain conditions, as indicated above) that there is an A_∞-quasi-isomorphism

$$\mathscr{F}^\sharp(L) \to \mathscr{M}(L),$$

where $\mathscr{F}^\sharp(L)$ denotes the full A_∞-subcategory of $\mathscr{F}^\sharp(M)$ consisting of objects whose underlying Lagrangian is L. It will therefore suffice to prove that there exists an exact triangle of the claimed form in $H^0(Tw\mathscr{M}(L))$, where $Tw\mathscr{M}(L)$ denotes the category of twisted complexes over $\mathscr{M}(L)$, which we use to model the triangulated closure of that category (see [19, Section (3l)]).

Proof of Proposition 10.1 We assume that the Morse function $f : L \to \mathbb{R}$ defining the morphism spaces in $\mathscr{M}(L)$ has a single local minimum $x_0 \in L$. As for notation, we write $\pi'_\gamma, \pi_\gamma, \pi''_\gamma$ for the parallel transport in E', E, E'' along a path γ, and we denote by $\overline{\gamma}$ the path obtained by reversing γ.

Let $c_2 \in \hom^0(E', E) = \mathrm{Hom}(E'_{x_0}, E_{x_0})$ and $c_3 \in \hom^0(E, E'') = \mathrm{Hom}(E_{x_0}, E''_{x_0})$ be the morphisms in $\mathscr{M}(L)$ determined by the short exact sequence, that is, $c_2 = i_{x_0}$ and $c_3 = p_{x_0}$. Then, define $c_1 \in \hom^1(E'', E') = \bigoplus_y \mathrm{Hom}(E''_y, E'_y)$ by

$$c_1 = \bigoplus_y \sum_\gamma \pm \pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}} \circ j_y,$$

where the first sum runs over all critical points y of f with Morse index 1, and the second over all gradient flow lines of f from x_0 to y, and where the sign $\pm$ is associated with γ as indicated above. In fact, we view c_1 as living in $\hom^0(E''[-1], E')$, and c_3 as living in $\hom^1(E, E''[-1])$.

We claim that these morphisms fit into an exact triangle

$$E''[-1] \xrightarrow{c_1} E' \xrightarrow{c_2} E \xrightarrow{c_3} E''$$

in $H^0(Tw\mathscr{M}(L))$. In this model, the cone $C = \mathrm{Cone}(c_1)$ is the twisted complex

$$C = \left(E'' \oplus E', \delta = \begin{pmatrix} 0 & 0 \\ -c_1 & 0 \end{pmatrix} \right),$$

see [19, Section 3(p)] (we suppress the shift of c_1). It comes together with morphisms $p_C = (e_{E''}, 0) \in \hom_{Tw}(C, E'')$ and $i_C = (0, e_{E'})^T \in \hom_{Tw}(E', C)$, where $e_{E''} = \mathrm{id}_{E''_{x_0}}$ and $e_{E'} = \mathrm{id}_{E'_{x_0}}$ denote the chain-level identity morphisms of E'', E' in $\mathscr{M}(L)$.

To prove our claim, we will make use of Lemma 3.27 in [19], which gives a criterion for exactness of triangles in $H^0(Tw\mathscr{M}(L))$. According to that, we have to find a cocyle $b \in \hom_{Tw}(E, C)$ such that $[b]$ is an isomorphism and $[\mu^2_{Tw}(p_C, b)] = [c_3]$, $[\mu^2_{Tw}(b, c_2)] = [i_C]$ in $H^0(Tw\mathscr{M}(L))$. (Again, we suppress some shifts.)

We claim that $b = (b'', b')$ with $b'' = p_{x_0} \in \hom^0(E, E'')$, $b' = q_{x_0} \in \hom^0(E, E')$ satisfies these requirements. The necessary computations are straightforward. We start by verifying that b is a cocycle, i.e., that $\mu^1_{Tw}(b) = 0$. Unravelling the definition of μ^1_{Tw}, cf. [19, Section (3l)], we obtain

$$\mu^1_{Tw}(b) = \begin{pmatrix} \mu^1(b'') \\ \mu^1(b') - \mu^2(c_1, b'') \end{pmatrix},$$

where the μ^d's are those of $\mathscr{M}(L)$. Now, $\mu^1(b'') = \mu^1(p_{x_0})$ vanishes because the p_x form a morphism of local systems. To compute $\mu^2(c_1, b'')$, note that for every critical point y, there is a unique perturbed Y-shaped gradient tree with outgoing edges converging to x_0 and y that contributes to the count, and that the incoming edge of this tree also converges to y (recall that x_0 is the unique local minimum). In view of this and recalling the definition of c_1, we obtain

$$\begin{aligned}
\mu^2(c_1, b'') &= \bigoplus_y \sum_\gamma \pm \pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}} \circ j_y \circ p_y \\
&= \bigoplus_y \sum_\gamma \pm \pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}} \circ \left(\mathrm{id}_{E_y} - i_y \circ q_y\right) \\
&= \bigoplus_y \left(\sum_\gamma \pm \pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}} - \sum_\gamma \pm q_y \right) \\
&= \bigoplus_y \sum_\gamma \pm \pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}}.
\end{aligned}$$

Here, we use that the p_x give a morphism of local systems, i.e., commute with parallel transport maps (which makes the additional parallel transport maps disappear that would appear in the first line). Moreover, we use that $\pi'_\gamma \circ q_{x_0} \circ \pi_{\overline{\gamma}} \circ i_y = \pi'_\gamma \circ q_{x_0} \circ i_{x_0} \circ \pi_{\overline{\gamma}} = \mathrm{id}_{E'_y}$, and that $\sum_\gamma \pm q_y = 0$ as x_0 is a cocyle in the usual Morse complex. The result of the computation is equal to $\mu^1(b')$, which shows that also the second component of $\mu^1_{Tw}(b)$ vanishes.

It remains to check that $[\mu^2_{Tw}(p_C, b)] = [c_3]$, $[\mu^2_{Tw}(b, c_2)] = [i_C]$ in $H^0(Tw\mathscr{M}(L))$. After unravelling again definitions, the first identity follows immediately (on the chain level). As for the second, we obtain

$$\mu^2_{Tw}(b, c_2) = \begin{pmatrix} \mu^2(b'', c_2) \\ \mu^2(b', c_2) - \mu^3(c_1, b'', c_2) \end{pmatrix}.$$

The first component is $\mu^2(b'', c_2) = p_{x_0} i_{x_0} = 0$, as required. As for the second component, we have $\mu^2(b', c_2) = q_{x_0} i_{x_0} = \mathrm{id}_{E'_{x_0}} = i_C$, and hence, we are done if we can show that $\mu^3(c_1, b'', c_2) = 0$. Recall that $b'' = p_{x_0}$ and $c_2 = i_{x_0}$, and that $p_{x_0} i_{x_0} = 0$; this together with the fact that the i_x commute with parallel transport maps suffices to conclude that the term vanishes. Hence, the second required identity also holds on the chain level.

11 Appendix 3: Vector bundles on elliptic curves

For the convenience of the reader, we collect here a couple of facts from Atiyah's classification [4] of vector bundles over an elliptic curve X, which are used in Sects. 7 and 8. We denote by $\mathcal{V}(r, d)$ the set of isomorphism classes of vector bundles on X of rank r and degree d.

Theorem 11.1 (Cf. Theorem 3 in [4]) *There exists an integer $N(r, d)$ such that for every $n \geq N(r, d)$, every $\mathcal{E} \in \mathcal{V}(r, d)$ fits into a short exact sequence*

$$0 \to \mathcal{O}_X^{\oplus(r-1)}(-n) \to \mathcal{E} \to \det \mathcal{E}((r-1)n) \to 0.$$

(Here, $-(k)$ denotes tensoring by the k^{th} power of the hyperplane bundle.)

Theorem 11.2 (Cf. Theorem 5 in [4]) *(i) There is a unique $\mathcal{F}_r \in \mathcal{V}(r, 0)$ such that $H^0(X; \mathcal{F}_r) \neq 0$, and there is a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0$ for every $r > 1$. (ii) Every $\mathcal{E} \in \mathcal{V}(r, 0)$ is of the form $\mathcal{F}_r \otimes \mathcal{L}$ for a unique $\mathcal{L} \in \mathcal{V}(1, 0)$.*

Theorem 11.3 (Cf. Theorems 6 and 7 in [4]) *The choice of a line bundle $\mathcal{L} \in \mathcal{V}(1, 1)$ determines natural 1–1 correspondences $\alpha_{r,d} : \mathcal{V}(h, 0) \to \mathcal{V}(r, d)$, where $h = \gcd(r, d)$. These are such that $\det \alpha_{r,d}(\mathcal{E}) = \det \mathcal{E} \otimes \mathcal{L}^{\otimes d}$, which implies that $\det : \mathcal{V}(r, d) \to \mathcal{V}(1, d)$ is an h-1 map.*

Given $\mathcal{L} \in \mathcal{V}(1, 1)$ and the corresponding map $\alpha_{r,d} : \mathcal{V}(h, 0) \to \mathcal{V}(r, d)$ from Theorem 11.3, we write $E_{\mathcal{L}}(r, d) := \alpha_{r,d}(\mathcal{F}_h)$.

Lemma 11.4 (Cf. Lemma 24 in [4]) *Suppose that $\gcd(r, d) = 1$. Then, $E_{\mathcal{L}}(r, d) \otimes \mathcal{F}_h \cong E_{\mathcal{L}}(hr, hd)$.*

Lemma 11.5 (Cf. Lemma 26 in [4]) *Let $\mathcal{L}_1 \in \mathcal{V}(1, 1)$. Then, for every $\mathcal{E} \in \mathcal{V}(r, d)$, there exists some $\mathcal{L}_0 \in \mathcal{V}(1, 0)$ such that $\mathcal{E} = E_{\mathcal{L}_1}(r, d) \otimes \mathcal{L}_0$.*

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