Probab. Theory Relat. Fields (2015) 163:89–121
DOI 10.1007/s00440-014-0588-2

Transfer current and pattern fields in spanning trees

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Received: 14 February 2014 / Revised: 25 August 2014 / Published online: 28 October 2014

Abstract When a simply connected domain $D \subset \mathbb{R}^d$ ($d \geq 2$) is approximated in a "good" way by embedded connected weighted graphs, we prove that the transfer current matrix (defined on the edges of the graph viewed as an electrical network) converges, up to a local weight factor, to the differential of Green's function on D. This observation implies that properly rescaled correlations of the spanning tree model and correlations of minimal subconfigurations in the abelian sandpile model have a universal and conformally covariant limit. We further show that, on a periodic approximation of the domain, all pattern fields of the spanning tree model, as well as the minimal-pattern (e.g. zero-height) fields of the sandpile, converge weakly in distribution to Gaussian white noise.

Keywords Transfer current theorem · Spanning trees · Sandpiles · Patterns

Mathematics Subject Classification Primary 82B20 · 60K35; Secondary 39A12 · 60G15

Contents

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1 Introduction

Let $\mathscr{G}$ be a locally finite connected weighted graph, with weight function c (a positive symmetric function over directed edges). It represents both an electrical network with conductances $c(e)$ on each bond e and a (time-homogeneous) random walk X on the vertex set of $\mathscr{G}$ with transition probability $\mathbb{P}(X_{n+1} = y | X_n = x) = c(xy)/\deg_c(x)$, where $\deg_c(x) = \sum_{y \sim x} c(xy)$ is the weighted degree of x. If $e = xy$ is a directed edge and a battery imposes a unit current to flow into x and out of y, the value of the current observed through any bond f is the transfer current $T(e, f)$ between e and f.

It is well-known that the transfer current $T(e, f)$ between two directed edges e and f is equal to the algebraic number of visits of the random walk started at x and stopped when it first hits y, see e.g. [4,7,13]. This relates the transfer current to Green's function G for the random walk (defined in Sect. 2), and we have

$$T(ab, uv) = c(uv)(G(a, u) - G(b, u) - G(a, v) + G(b, v)).$$

Since Green's function is symmetric, the matrix $K(e, e') = \sqrt{c(e)/c(e')}T(e, e')$ is symmetric in both arguments. This symmetry is called the reciprocity law in electrical theory. Further relations between electrical quantities and random walk hitting times and commute times are well-known, see e.g. [13,32] and references therein.

One can give an expression for T involving an eigenbasis of the Laplacian Δ. Let $(f_k)_{k\geq 1}$ be an orthonormal basis of eigenvectors for Δ associated to eigenvalues $\lambda_k \neq 0$. Then, for any two edges $ab, uv \in E$, we have

$$T(ab, uv) = c(uv) \sum_{k\geq 1} \frac{(f_k(u) - f_k(v))(f_k(a) - f_k(b))}{\lambda_k}.$$

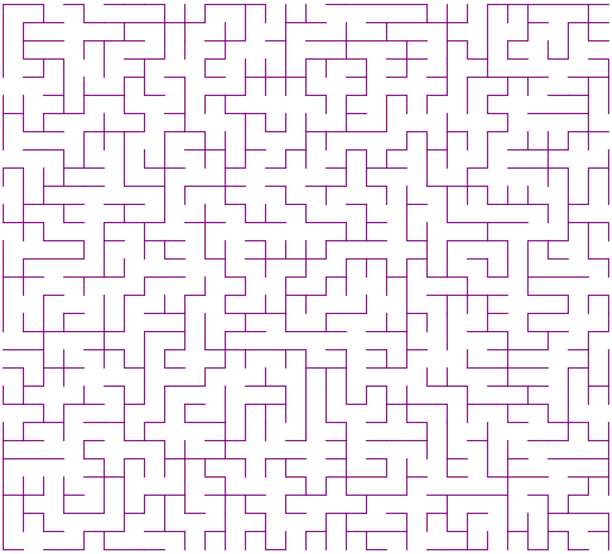

Fig. 1 A (free boundary) uniform spanning tree on a 30×30 square grid

In the case of infinite periodic graphs, the transfer current may be evaluated explicitly by this means.

Because of its relationship to random walk, the transfer current appears in the study of many probabilistic models on graphs. The random spanning tree model $\mathscr{T}$ is the probability measure on spanning trees of $\mathscr{G}$ which assigns a probability proportional to the products of the edge-weights of the tree (see Fig. 1 for a sample).

The transfer current theorem of Burton and Pemantle [7] states that the spanning tree model, viewed as a point process on the set of undirected edges of the graph, is determinantal with kernel T. This means that all local statistics have a closed-form expression in terms of the transfer current of the local edges: for any finite collection of disjoint undirected edges $e_1, \ldots, e_k$ in the graph, we have

$$\mathbb{P}\left(e_1, \ldots, e_k \in \mathscr{T}\right) = \det\left(T\left(e_i, e_j\right)\right)_{1 \leq i, j \leq k}.$$

Any local statistics of the spanning tree is thus a local computation provided we know the value of the transfer current (which depends on the graph in its whole). This is the case for periodic infinite graphs as already pointed out above. Note that the matrix K defined above is also a kernel of this process (since it is the conjugate of T by a diagonal matrix): since it is symmetric, we say that the spanning tree model is a symmetric determinantal process (this implies further properties of the process that we review in Sect. 5.6).

In many cases, although an explicit form of the transfer current is not easy to obtain, one can show that on a large scale, or equivalently in the scaling limit for macroscopically distant points, the transfer current is close to its continuous counterpart. For graphs which approximate a smooth domain in a nice way (we call these good approximations), the transfer current between two macroscopically distant edges tends (up to a local constant depending on the edge weights) to the derivative of Green's function along the directions given by the edges. This is our first main result (see Theorem 1). The proof is an adaptation of the classical arguments from a special case of [9]. The goal is to show the universality of the limit on a large class of graphs in any dimension. Our assumptions on the graph are quite strong (thus the proof is simpler), but these are satisfied by many interesting examples.

Three other models are closely related to the spanning tree model: the two-component spanning forest, the spanning unicycle, and the abelian sandpile model of Dhar [11]. There exist natural couplings between them which enable to express the probability of any event for these models as a spanning tree computation (see [22] for the first two models and see [19,34] for the abelian sandpile; also see [31]). However, local statistics for these models are not necessarily local statistics for the spanning tree and may require the knowledge of the geometry of the spanning tree as a whole. We focus in this paper only on the local statistics that are local statistics for the spanning tree as well. These type of statistics can be computed explicitly and have a well-defined universal limit, expressed in terms of the scaling limit of transfer currents. This is our second main result (Theorems 2 and 3; another application is Theorem 4).

A possible realization of the random process in a finite subregion is called a pattern. Given a collection of countable patterns on disjoint supports, we can define a new point process, or equivalently a random spin field which is the indicator of the occurrence of these patterns. Under sufficiently fast decay of correlations with the distance, pattern fields of determinantal point processes converge to Gaussian white noise (Theorem 5). We use this to study the fields generated by the local events mentioned above in the spanning tree and sandpile models. More precisely, we express local events fields as pattern fields in the spanning tree model and use the determinantal structure of the spanning tree process to show that these fields are Gaussian white noise. This is our third main result (Theorems 6, 7, and 8). For the special case of the zero-height field of the sandpile on $\mathbb{Z}^2$, the result was already obtained by Dürre. The study of pattern fields of determinantal point processes originated in the work of Boutillier [6], who studied them in fluid dimer models on the plane. Gaussian fluctuations for symmetric determinantal point fields were first observed by Soshnikov [38] (this corresponds to patterns consisting of a single point present).

Computing probabilities of local events which are not local for the spanning tree is much harder. In the planar case, it has been addressed by Wilson [40] using methods of Kenyon and Wilson [27].

Describing the scaling limits of spanning trees and related models is challenging. In two dimensions, using conformal invariance, the scaling limit is well understood in terms of Schramm–Loewner evolutions [30]. In dimension three, Kozma [28] has shown the existence of the scaling limit of the loop-erased random walk (that is, the branches of the spanning tree by a theorem of Pemantle [36]) and in high dimension, the scaling limit is also known due to the lace expansion [10]. Furthermore, the geometry of the infinite volume limit has been well-studied [3,4]. However, many natural questions about the geometry of spanning trees on infinite graphs and their scaling limits remain open. The pattern fields we study encode certain information of spanning trees in any dimension, and their scaling limit might tell us something about the scaling limit of spanning trees. Our method is based on the determinantal nature of spanning trees and the study of its kernel, the transfer current. Pattern fields are local quantities, however we give in Sect. 4.2, for future use, some properties which relate the transfer current to global geometric properties of trees.

The paper is organized as follows. In Sect. 2, we introduce some concepts from discrete harmonic analysis: weighted graphs, vector field, derivative, Laplacian, harmonic functions, Green's functions, and transfer current. In Sect. 3 we define and exhibit good approximations for domains in $\mathbb{R}^d$. We prove Theorem 1 which states that on good approximations, the transfer current convergences (up to a local factor) to the derivative (in both variables along the direction of the edges) of Green's function. In Sect. 4 we review some couplings between spanning trees and the other models and relate the probability of events that correspond to local events for the spanning tree, expressed in terms of the transfer current. We derive from this two types of results: on the one hand, the connection between combinatorics of trees and discrete analytic properties of T; on the other hand, we use Theorem 1 to show that certain correlations of the trees and sandpiles have a universal and conformally covariant limit. Finally, we concentrate in Sect. 5 on pattern fields of certain determinantal point processes and show as an application of what precedes (for this, we only need the order of decay of $T(x, y)$ in the limit $|x-y| \to \infty$, not its precise value) that the pattern fields of a random spanning tree and the minimal-pattern (e.g. zero-height) fields of the abelian sandpile converge to Gaussian white noise in the scaling limit, under good approximation of a domain.

2 Discrete harmonic analysis

Let $\mathscr{G} = (V, E, c, \partial V)$ be a finite connected weighted graph with vertex-set V, edge-set E, weight function $c : E \to \mathbb{R}_+^*$, and a boundary-vertex-set ∂V which consists of a (possibly empty) subset of the vertices. We let $V^\circ = V \backslash \partial V$ and call its elements the interior vertices.

Let Ω^0 denote the space of functions over the interior vertices and Ω^1 be the space of 1-forms, that is antisymmetric functions over the set $E^\pm$ of directed edges (two directed edges per element of E). We endow Ω^0 with its canonical scalar product coming from its identification with $\mathbb{R}^{V^\circ}$, that is $\langle f, g\rangle = \sum_{v\in V^\circ} f(v)g(v)$, and endow Ω^1 with the scalar product given by $\langle \alpha, \beta\rangle = 1/2 \sum_{e\in E^\pm} c(e)\alpha(e)\beta(e)$.

In the case where $\mathscr{G}$ is infinite, we now abuse the notations Ω^0 and Ω^1 and suppose that these actually denote the subspaces of elements with finite norm (the ℓ^2 spaces).

We define the map $d : \Omega^0 \to \Omega^1$ by $df(vv') = f(v') - f(v)$ and the map $d^* : \Omega^1 \to \Omega^0$ by $d^*\alpha(v) = \sum_{v'\sim v} c(v'v)\alpha(v'v)$. The maps d and d^* are dual to one another, in the sense that $\langle f, d^*\alpha\rangle = \langle df, \alpha\rangle$ for any $f \in \Omega^0$ and $\alpha \in \Omega^1$.

The Laplacian is defined by $\Delta = d^*d : \Omega^0 \to \Omega^0$, and it is easy to check that for any $f \in \Omega^0$ and $v \in V^\circ$, we have

$$\Delta f(v) = \sum_{v'\sim v} c(vv') \left(f(v) - f(v')\right),$$

where the sum is over neighboring vertices (including boundary vertices). A function f is said to be harmonic at a vertex v whenever $\Delta f(v) = 0$.

The Green function G is the inverse of the Laplacian. Two cases may occur: if the boundary is non empty ($\partial V \neq \emptyset$), Δ is invertible on Ω^0, hence $G = \Delta^{-1}$. In the case when the boundary is empty ($\partial V = \emptyset$), G is the inverse of Δ on the space of mean zero functions. The first one is called the Dirichlet Green function (or wired Green function), the second is called the Neumann Green function (or free Green function).[1] Given a weighted graph $\mathscr{G}$ we will sometimes consider simultaneously both Green's functions by looking at the wired or free boundary conditions.

When the space of harmonic forms, $H = \mathrm{Im}(d) \cap \mathrm{Ker}(d^*)$ is trivial (which is always the case on a finite connected graph: harmonic functions are constant), the space of 1-forms has an orthogonal decomposition (Hodge) as

$$\Omega^1 = \mathrm{Im}(d) \oplus \mathrm{Ker}(d^*),$$

Under this assumption ($H = 0$), when the graph is infinite, Theorem 7.3. of [4] implies that free and wired spanning forests measures coincide.

The orthogonal projection of Ω^1 onto $\mathrm{Im}(d)$ is $P = dGd^*$. Any directed edge xy defines a 1-form $\delta_{xy} - \delta_{yx}$ and we often identify the directed edge with this 1-form. The *transfer current* is the matrix of P in the basis indexed by $E^{\pm}$. For two directed edges xy and uv, it therefore satisfies

$$\begin{aligned} T(xy, uv) &= \sqrt{c(xy)}^{-1} \left\langle xy/\sqrt{c(xy)}, Puv/\sqrt{c(uv)} \right\rangle \sqrt{c(uv)} = c(xy)^{-1} \langle xy, Puv \rangle \\ &= c(uv)\,(G(x,u) - G(y,u) - G(x,v) + G(y,v)). \end{aligned} \tag{1}$$

According to the boundary condition chosen (free or wired), we obtain two different transfer currents.

A vector field on $\mathscr{G}$ is the choice for each vertex $v \in V^\circ$ of an outgoing edge X_v. For any vertex v and directed edge vv', we define $v + (vv')$ to be v'. In particular, $v + X_v$ is the neighbor of v to which the vector X_v points at. We define the derivative of a function f with respect to the vector field X to be the function $v \mapsto \nabla_X f(v) = f(v + X_v) - f(v)$. The derivative of a function f with respect to a vector field X satisfies $\nabla_X f = df(X)$, an equality which is coherent with the one from calculus.

The 1-forms are dual to vector fields and naturally act on them. Another way to see the action of P is

$$\langle \alpha, P\beta \rangle = \langle d^*\alpha, Gd^*\beta \rangle.$$

Under this formulation, the projection can be written as acting on the divergence of vector fields with kernel given by Green's function. The continuum analog of this projection is widely used and is known as the Hodge–Helmholtz projection.

[1] Our choice of the qualifier "Neumann" for this Green function comes from the fact that we can replace the free boundary conditions by Neumann boundary conditions by artificially adding a boundary to the graph. The Laplacian acts on the extended space of functions which take same values at any boundary vertex and its adjacent interior vertex. This corresponds to a vanishing normal derivative on the boundary. This modification of the Laplacian does not change the local times of the random walk at interior points which is sufficient for our needs. In the limit, this "Neumann" Green function actually converges to the elementary solution for the Neumann problem.

3 Scaling limits

3.1 Good approximations of domains

Let $d \geq 2$ and $D \subset \mathbb{R}^d$ be a simply connected open set, with smooth boundary ∂D. We call such a set D a domain. We consider a sequence of connected weighted graphs with boundary-set $\mathcal{G}_n = (V_n, E_n, c_n, \partial V_n)$ embedded in D. By "embedded", we mean that the vertices are points inside D and boundary vertices lie in ∂D. The edges are mapped to smooth non-intersecting segments in such a way that edges between boundary vertices lie inside ∂D. We let Δ_n be the Laplacian on $\mathcal{G}_n$.

We say that a sequence of functions $f_n : V_n^\circ \to \mathbb{R}$, converges uniformly on a compact subset of $K \subset D$ to a function f, if the sequence $(g_n)_{n\geq 1}$ defined by $g_n = \sum_{x \in V_n^\circ} f_n(x)\delta_x$ converges uniformly to f on K.

We say that the sequence $(\mathcal{G}_n)_{n\geq 1}$ is a *good approximation* of D if the following properties hold.

1. (*Approximate mean value property*) For any bounded harmonic function f on V_n, and any ball $B(v, r)$, we have $f(v) - 1/|B(v,r)| \sum_{w \in B(v,r)} f(w) = O(1/r)$.
2. (*Paths approximation*) For any straight line γ in D joining x to y, there exists finite paths γ_n inside $\mathcal{G}_n$ joining two vertices x_n to y_n such that $x_n \to x$, $y_n \to y$, and γ_n uniformly tends to a path γ. Moreover, the discrete length of γ_n is bounded by an absolute constant times the length of γ.

Examples of good approximations include: the lattices $\mathbb{Z}^d$ for $\mathbb{R}^d$ and $\mathcal{G}_n = \mathbb{Z}^d/n \cap D$ for some domain D (the Approximate mean value property follows from Lemma 6 in [20]; the Paths approximation property is clear), and for $d = 2$, isoradial graphs with bounded half-angles, see [25] (see Appendix for a quick review of the definition of isoradial graphs and a justification of why there are good approximations).

The Paths approximation property is rather easy to check on a given graph. It is less so for the Approximate mean value property: see Remark 4 below for a practical way to check it.

3.2 Convergence of the derivative

In the following, $|X(v)|$ denotes the Euclidean length of the edge viewed as an embedded segment. For any vertex v, we denote $B(v, r)$ the discrete ball of center v and radius r in $\mathcal{G}$, and by $|B(v, r)|$ its cardinality.

We say that a compact subset K of D is interior if its distance to the complement of D is strictly positive.

Proposition 1 *Let $(\mathcal{G}_n)_{n\geq 1}$ be a good approximation of a domain D. Consider a vector field X_n. Suppose a sequence of harmonic functions on $\mathcal{G}_n$ converges uniformly on interior compact subsets of D to a harmonic function f. Then the sequence of their rescaled discrete derivatives $|X_n|^{-1}\nabla_{X_n} f$ is uniformly close in the limit $n \to \infty$, on all interior compact subsets of D, to the derivative $\nabla_{X_n} f$ of f.*

Proof Let K be a compact subset of D. Let K' be an interior compact subset such that $K \subset K' \subset D$ and K is at a positive distance r from the exterior of K'. The function f is bounded on K', hence the sequence $(f_n)_{n\geq 1}$ as well. Let $z \in K$ and v be an approximation of z. By the approximate mean value property at v and $v + X_v$, there exists a constant C such that, for any $n \geq 1$, we have

$$\left| \nabla_{X_n} f_n(v) - \frac{1}{|B(v, nr)|} \sum_{w \in V_n^\circ} f_n(w) \left(1_{\{w\in B(v+1,nr)\}} - 1_{\{w\in B(v,nr)\}}\right) \right| \leq \frac{C}{nr}.$$

Now,

$$\sum_{w \in V_n^\circ} f_n(w) \left(1_{\{w\in B(v+1,nr)\}} - 1_{\{w\in B(v,nr)\}}\right) \leq C \left|B\left(v, (n+1)r\right) \setminus B(v, nr)\right|$$
$$\leq C \frac{\left|\partial B\left(v, (n+1)r\right)\right|}{|B(nr)|} = O(C/nr),$$

where we used, in the last equality, the bound of $O(1/R)$ on the surface-area-to-volume ratio of a ball or radius R in $\mathbb{R}^d$. The sequence $|X_n|^{-1}\nabla_{X_n} f_n(v)$ is therefore uniformly bounded on K.

Therefore, one can extract a subsequence $\{n_k\}$, so that $|X_{n_k}|^{-1}\nabla_{X_{n_k}} f_{n_k}$ converges uniformly on compact subsets to some bounded function $h : D \to \mathbb{R}$. We now prove that for any subsequential limit $h = \nabla_{X_{n_k}} f$, which finishes the proof.

By the *Paths approximation* property (and because K is locally convex), for any $u, v \in K$ close enough, and any n, we can take a discrete poly-line segment $\overline{u_n v_n}$, such that $u_n \to u$, $v_n \to v$, and $\overline{u_n v_n}$ converges (in the uniform topology) to the line $\overline{uv}$. We replace the vector field along this line so that it is aligned with it. Along a subsequence $\{n_k\}$, we have

$$f_{n_k}\left(v_{n_k}\right) - f_{n_k}\left(u_{n_k}\right) = \sum_{s \in \overline{u_{n_k} v_{n_k}} \cap V_n} df_{n_k}\left(X_{n_k}\right) = \int_{\overline{u_{n_k} v_{n_k}}} |X_{n_k}|^{-1} \nabla_{X_{n_k}} f_{n_k}(x)\, dx.$$

Taking $n_k \to \infty$, uniform convergence implies

$$f(v) - f(u) = \int_{\overline{uv}} h(x)\, dx.$$

By taking $\overline{uv} = X_u$ at every u, and letting $v \to u$, we have $h = \nabla_X f$. □

Remark 1 Since the derivative of a harmonic function is a harmonic function, Proposition 1 may be applied successively to show that any higher order derivative also converges under the same assumptions.

3.3 Green's function and transfer current

In this paper, we consider $\Delta = -\sum_{i=1}^{d} \partial^2/\partial x_i^2$ to be the positive definite Laplacian. Recall that the Green function with Neumann (resp. Dirichlet) boundary conditions is the (unique up to constant) smooth symmetric kernel over D solution to

$$\Delta_y u(x, y) = \delta_x$$

with boundary condition $\partial u(x, y)/\partial n(y) = 0$ for $y \in \partial D$, where $n(y)$ is the normal vector at the boundary point y (resp. $u(x, y) = 0$ for $y \in \partial D$). On a bounded domain the Neumann Green function is defined up to an additive constant, and we identify it with its action on functions of mean zero.

We now make some further assumptions on our good approximations. First, we suppose that they form an exhausting sequence of some infinite embedded graph $\mathcal{G}_\infty$. We assume that Green's function on $\mathcal{G}_\infty$, denoted by G_0, converges uniformly, upon rescaling, on compact sets (away from the diagonal) to the continuous Green's function g on $\mathbb{R}^d$ with control on the error term, and that the invariance principle holds:

(A1) (*Green's function asymptotics*) $G(z, w) = n^{2-d} g(z, w) + O\left(n^{-d}|z - w|^{-d}\right)$.
(A2) (*Invariance principle*) The random walk on $(\mathcal{G}_n)_{n\geq 1}$ converges in the supremum norm to Brownian motion in D.

These assumptions are satisfied by the following good approximations: the lattices $\mathbb{Z}^d$, see [15,29], and for $d = 2$, isoradial graphs, see [9,25] and Appendix.

Theorem 1 *Suppose the conditions of Proposition* 1 *hold. Then under assumptions* (A1) *and* (A2), *we have*

$$T(e_n, e'_n) = c(e'_n) n^{-d} dg_D|_{(z,w)}(e, e') + o\left(n^{-d}|z - w|^{-d}\right), \tag{2}$$

where g_D is Green's function with the corresponding boundary conditions on D, and dg_D its differential.

Proof Let T_0 denote the transfer current associated with Green's function on the whole space, defined in (1). By Assumption A1, the rescaled Green's function $n^{d-2}G_0$ on the infinite graph converges uniformly on all interior compact subsets to Green's function on the whole plane g. Applying Assumption A1 to (1) and differentiating g twice, we have

$$T_0(e_n, e'_n) = c(e'_n)|e_n||e'_n| n^{-d} \nabla_e^z \nabla_{e'}^w g(z, w) + o\left(n^{-d}|z - w|^{-d}\right). \tag{3}$$

Consider the discrete harmonic function $F = G - G_0$. By the invariance principle (Assumption A2), $n^{d-2}F$ converges to the continuous harmonic function f with boundary values given by $g_D - g$. Applying the same (mean value) argument as in the proof of Proposition 1, the discrete gradient of F is uniformly bounded. Therefore F converges to f, uniformly on interior compact subsets of D. By applying Proposition 1 twice, we obtain the uniform convergence of the rescaled double increment of F to the double derivative of f. Since

$$T(e_n, e_n') = T_0(e_n, e_n') + c(e_n') \left(F(z, w) - F(z + e_n, w) - F\left(z, w + e_n'\right) \right.$$
$$\left. - F\left(z + e_n, w + e_n'\right) \right),$$

combined with (3), this implies

$$T(e_n, e_n') = c(e_n')|e_n||e_n'|n^{-d} \nabla_e^z \nabla_{e'}^w (g + f)(z, w) + o\left(n^{-d}|z - w|^{-d}\right),$$

which finishes the proof. □

Remark 2 When the domain is $D = \mathbb{R}^d$, and we dispose of a two-terms expansion of Green's function (as is the case for $\mathbb{Z}^d$, by [15]), the proof of Theorem 1 is immediate by computation and Taylor expansion.

Remark 3 By using the interpretation of Green's function as the density of time spent in the neighborhood of y when started at x and the explicit Brownian time-space scaling, one can use (A2) to prove a weaker form of (A1), namely that $G(z, w) = n^{2-d} g(z, w) + o(n^{2-d}|z - w|^{2-d})$.

Remark 4 If we suppose that the discrete Laplacian of homogeneous polynomials of degree 2 in $\mathbb{R}[x_1, \ldots, x_d]$ is $2 \sum_{i=1} a_i$, where a_i is the coefficient of x_i^2, then, by using (A.1), the Approximate mean value property may be shown, following the proof of Proposition A.2 in [9]. This assumption also implies that random walk is isotropic (and the variance of the increments is of order of the volume around each vertex) and thus implies (A.2).

As in the setup of [24, Theorem 13], in order to control the behavior of Green's function for points on the boundary, we slightly modify the way we approximate the domain D by ensuring that the approximating graphs have piecewise linear boundaries in the following sense. We consider a sequence $\delta = \delta(n)$ such that $n^{-1}\delta^{-1} = o(1)$, as $n \to \infty$, and an increasing sequence of domains $D^\delta \subset D$ such that D^δ lies within δ of D and such that its boundary is a polytope (polygon, when $d = 2$) with hyperfaces (segments, when $d = 2$) of size δ^{d-1}. D^δ is furthermore assumed to be convex for $d \geq 3$.

We can obtain convergence of the transfer current for points on the boundary whenever we consider, as a replacement of a good approximation to D, a diagonal subsequence of good approximations of the domains D^δ. This follows from the fact that on domains with piecewise linear boundaries, we can use a reflection argument.

Corollary 1 *Let $D \subset \mathbb{R}^d, d \geq 3$ be a convex domain with smooth boundary, or $D \subset \mathbb{R}^2$ be simply connected with smooth boundary. Assume the approximation sequence is chosen as above. Formula* (2) *holds for points on the boundary.*

Proof Reflect the approximation graph of D^δ across the flat piece of the boundary, and glue it with the original graph. The result follows by noting that Green's function on the new graph is a linear combination of Green's function on the original graph. When $d = 2$ the argument works for any simply connected surface (see Proposition 7). □

3.4 Planar graphs

On any planar embedded graph, the conjugate of a harmonic function h on the graph is a harmonic function h^* on the planar dual which is defined by the following discrete Cauchy–Riemann equations: (for any directed edge e, denote by e^* the dual edge directed in such a way that $e \wedge e^* > 0$) $dh^*(e^*) = dh(e)$ for any dual edges e and e^* (where d is the discrete derivative in the dual and primal graph, respectively).

In particular, we have the following.

Lemma 1 *Let G be the Green function of a planar graph $\mathcal{G}$. Let $\tilde{G}$ be the harmonic conjugate of G. For any $e = vv'$, let $e^* = pp'$ be the dual edge to e. We have*

$$\tilde{G}(p, q) - \tilde{G}(p', q) = G^*(p, q) - G^*(p', q),$$

where G^* is the Green function on the dual graph $\mathcal{G}^*$.

Lemma 1 is the discrete analog to the fact that the 'complex' Green function defined as $g_D + ig_N$ where g_D is the Dirichlet Green function and g_N the Neumann Green function, is analytic (away from the diagonal) hence satisfies the Cauchy–Riemann equations.

On the whole plane, we can compute this explicitly for comparison. Start with $g(z, w) = -1/(2\pi) \log |z - w|$. Then by taking the harmonic conjugate (in one of the variables) we obtain $-1/(2\pi)\arg(z - w)$ (it is the same in either variable by a simple geometrical fact about parallel angles being equal). Now taking the harmonic conjugate with respect to the other variable we get $-1/(2\pi) \log |z - w|$ back again, up to an additive constant. By taking differences of Green functions as in the lemma we get equality.

Proof of Lemma 1 Let $f(w) = G(v, w) - G(v', w)$. The function f is harmonic with a defect of harmonicity $+1$ at v and -1 at v'. Its harmonic conjugate is the function $f^*(w) = \tilde{G}(p, q) - \tilde{G}(p', q)$. This is a univalued harmonic function with a defect of harmonicity $+1$ at p and -1 at p' (This may be seen by comparing the harmonic extension of f avoiding edge vv' and the harmonic extension crossing that edge. Since this move is local, when one extends from p to p', one obtains the result). Thus it is equal to $G^*(p, q) - G^*(p', q)$. □

Corollary 2 *Let e and e' be two edges. Then*

$$T(e, e') = T^*(e^*, e'^*),$$

where T is the transfer current on $\mathcal{G}$, and T^* the transfer current on the dual graph $\mathcal{G}^*$.

A purely combinatorial proof of Corollary 2 follows from Equations (4) and (5) in Sect. 4.2 below by noticing that the dual of a two-component spanning forest is a spanning unicycle.

4 Transfer current in statistical physics

In this section, we explain ways in which the transfer current describes models of statistical physics on weighted graphs: point correlations and geometrical properties. Using Theorem 1, we can find the scaling limits of some of these expressions. We also relate analytical properties of T to its combinatorial properties.

4.1 Spanning trees

Let $\mathcal{G}$ be a finite connected weighted graph. A *spanning tree* on $\mathcal{G}$ is a subgraph (V, A), where A is a set of edges, which contains no cycles and is connected. We weight each spanning tree by the product over edges in the tree of their weight and call the corresponding probability measure the *random spanning tree* on $\mathcal{G}$. By the transfer current theorem of Burton and Pemantle [7], the transfer current is a kernel for the random spanning tree measure, that is, for any distinct edges $e_1, \ldots, e_k$, the probability that the random spanning tree contains them is

$$\mathbb{P}(e_1, \ldots, e_k) = \det \left(T(e_i, e_j)\right)_{1 \leq i, j \leq k}.$$

This along with Theorem 1 implies Theorem 2 below.

Given an edge e in $\mathcal{G}$, we let 1_e denote the random variable that takes value 1 if e belongs to the spanning tree, and 0 otherwise (this is an example of a pattern field, defined in Sect. 5 below). For k distinct edges $e_1, \ldots, e_k$, the covariance of the corresponding random variables 1_{e_i} is $\mathrm{Cov}\,(e_1, \ldots, e_k) = \mathbb{E}\left[\left(1_{e_1} - \mathbb{P}\,(e_1)\right) \ldots \left(1_{e_k} - \mathbb{P}\,(e_k)\right)\right]$.

Theorem 2 *Under the assumptions of Theorem 1, the rescaled correlations of the spanning tree model have a universal and conformally covariant limit.*

Proof Standard linear algebra calculation implies (see, e.g. Lemma 21 of [24])

$$\mathrm{Cov}\,(e_1, \ldots, e_k) = \det \left(T(e_i, e_j)\delta_{i \neq j}\right)_{1 \leq i, j \leq k}.$$

Applying Theorem 1, we obtain that, as $n \to \infty$,

$$n^{kd}\mathrm{Cov}\,(e_1, \ldots, e_k) \to \det \left(c\left(e_j\right) dg_D|_{(z_i, z_j)} \left(e_i, e_j\right) \delta_{i \neq j}\right)_{1 \leq i, j \leq k}.$$

Therefore the limit of rescaled correlations exists, and its conformal covariance follows from the conformal covariance of dg_D. □

4.2 Two-component spanning forests and spanning unicycles

A *two-component spanning forest* (2SF) on $\mathcal{G}$ is a subgraph (V, B), where B is a set of edges, which contains no cycles and has exactly two connected components. A *spanning unicycle* (or *cycle-rooted spanning tree*) is a connected subgraph (V, B)

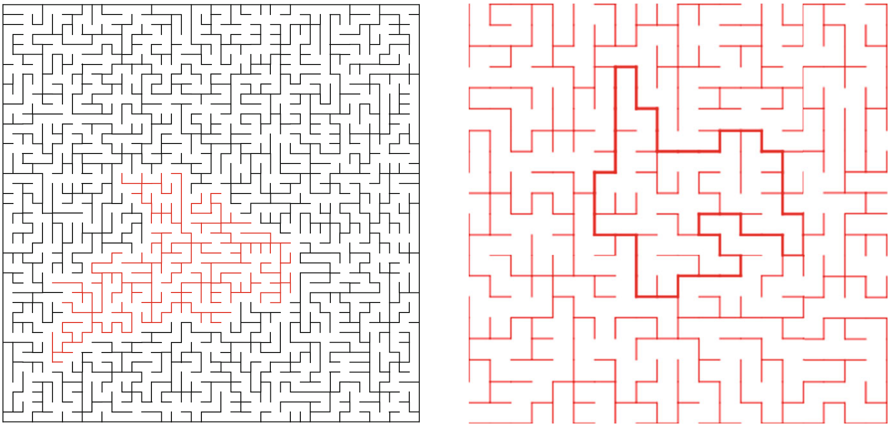

Fig. 2 *Left* A two-component spanning forest wired on the boundary; *Right* a spanning unicycle on a 21×21 square grid conditioned on having a longish cycle

where $|B| = |V|$ (it thus contains a unique cycle). See a picture on Fig. 2. On planar graphs these two notions are dual to one another. In the following we do not need to suppose however (unless otherwise stated) that the graphs are planar.

As for spanning trees, we consider the weight of a subgraph to be the product over edges (of the subgraph) of the edge-weights. We define a probability measure on 2SFs (respectively, random spanning unicycles) by giving a 2SF (respectively, a spanning unicycle) a probability proportional to its weight. This defines the random 2SFs and random spanning unicycles considered hereupon.

A spanning unicycle can be thought of as a spanning tree to which an edge is added. Note however, that the measure obtained from taking a random spanning tree and adding a uniformly random edge is different than the random spanning unicycle defined above, and the Radon–Nikodym derivative is simply the length of the cycle. Nevertheless, we observe that the law of the cycle of the random spanning unicycle conditional on the event that the cycle contains some edge e_1 is the law of the path between the extremities of e_1 in the spanning tree model on the graph where e_1 is removed. The event that the cycle is of a given shape is therefore the union of translates of events for the spanning tree, which can be evaluated explicitly for infinite periodic graphs. In particular for $\mathbb{Z}^2$, using the explicit values of the transfer current computed in [7], we obtain that the probability that the cycle is of length 4 is $-\frac{16}{\pi^3} + \frac{8}{\pi^2} \approx .294$.

Let $\kappa = \sum_{\mathscr{T}} \text{weight}(\mathscr{T})$ be the weighted sum of spanning trees and $\lambda = \sum_{\mathscr{U}} \text{weight}(\mathscr{U})$ the weighted sum of spanning unicycles. In the following, whenever e denotes a directed edge (u, v), we say that u is its head, and v its tail. Furthermore, the symbol $-e$ denotes the directed edge with same support and opposite direction, that is, (v, u).

In the wired case ($\partial V \neq \emptyset$), using the fact (Theorem 3 of [22]) that $\kappa G(x, y)$ is the weighted number of two-component spanning forests such that x and y are disconnected from the boundary, we have for any two directed edges e and e',

$$T(e, e') = \kappa^{-1} \left(N_{e,e'} - N_{e,-e'}\right), \tag{4}$$

where $N_{e,e'}$ is the weighted number of two-component forests such that the extremities of e belong to different components, as well as for the extremities of e', and these are the same for both heads and both tails.

In the free boundary case ($\partial V = \emptyset$), we start with the following result.

Lemma 2 (Lemma 1.2.12 in [21]) *Let U be the set of all spanning unicycles. For any $\theta \in \Omega^1$, we have*

$$\sum_{\mathscr{U} \in U} \theta_\gamma^2 = \kappa \, \langle \theta, (1 - T)\theta \rangle,$$

where $\theta_\gamma = \langle \theta, \delta_\gamma \rangle$ and γ is the choice of a directed representative of the unique cycle of the unicycle $\mathscr{U}$.

Proof It is an immediate corollary of Proposition 7.3 of [5]. The map T is the orthogonal projection on $\mathrm{Im}(d)$, which is the orthogonal complement of $\mathrm{Ker}(d^*)$. Recall from [5] that a fundamental cycle associated to a spanning tree and a directed edge not on the tree, is the directed cycle defined by the union of that edge with the unique path joining the extremities of that edge in the tree (the direction of the cycle is given by the direction of the edge). A basis of $\mathrm{Ker}(d^*)$ is given by the (1-forms corresponding to) fundamental cycles associated to a spanning tree. Hence

$$T = 1 - \kappa^{-1} \sum_{\mathscr{T} \text{ spanning tree}} \sum_{e \notin \mathscr{T}} 1_{C(e,\mathscr{T})},$$

where $C(e, \mathscr{T})$ is the fundamental cycle of e in $\mathscr{T}$. □

Remark 5 Lemma 2 may also be proved by expanding the determinant of the line bundle Laplacian (introduced in [26]) around the trivial connection along θ (that is, expand near $t = 0$ the family of Laplacian determinants associated to the connection $\exp(ti\theta)$). This yields yet another geometric interpretation of the transfer current.

In the planar case, we obtain the following. Taking θ to be the indicator on directed edge e^*, e'^* on the dual graph, we have

$$T^*(e^*, e'^*) = \kappa^{-1} \left(N_{e^*,e'^*} - N_{e^*,-e'^*} \right), \tag{5}$$

where N_{e^*,e'^*} is the number of spanning unicycles, the cycle of which contains e^* and e'^*. In the planar case, the dual of a 2SF is a spanning unicycle. It follows from (4) and (5) that $T(e, e') = T^*(e^*, e'^*)$, which gives another proof of Corollary 2 above.

Lemma 2 gives a way to compute the expected winding of the cycle of the spanning unicycle.

We conclude this subsection by giving two applications of Theorem 2 to the case of planar graphs or graphs embedded in $\mathbb{R}^3$.

For planar graphs with two marked faces f and f', let Z and Z' be the set of east-directed edges crossed by two disjoint dual paths from the outer face to f and f', respectively. Taking θ to be the 1-form indicator of Z and Z', we obtain that the probability that two faces lies inside the cycle of the uniform spanning unicycle on a planar graph is

$$\mathbb{P}(f, f') = -\frac{\kappa}{\lambda} \sum_{e \in Z, e' \in Z'} T(e, e').$$

This formula simplifies to $\mathbb{P}(f, f') = (\kappa/\lambda)\, G^*(f, f')$ by Corollary 2 and was observed in [22]. For any isoradial graphs, the ratio $|V_n|\kappa/\lambda$ converges to a constant τ, see [23] (and [31] for the case of $\mathbb{Z}^2$).

In the case of $\mathbb{Z}^3$, we obtain the following.

Corollary 3 *Consider $\mathscr{G}_n$ to be the subgraph of $\mathbb{Z}^3$ whose vertex-set is $[|1, n|]^3$. Let f be a face in the x, y plane with coordinates (x, y). Let $\mathbf{k}$ be the winding number of the uniform spanning unicycle in $\mathscr{G}_n$ around the tube $f \times [|1, n|]$. Its second moment is given by*

$$\mathbb{E}(\mathbf{k}^2) = \frac{1}{\tau_n n^3} \left(ny - \sum_{y_i, y_j = 1}^{y} \sum_{z, z' = 1}^{n} T(e_i, e'_j) \right),$$

where $\tau_n \to \tau$, and τ is a constant.

Proof Consider a 1-form θ defined to be the indicator of the edges on a directed "curtain" of edges that connect the tube $f \times [1, n]$ to the boundary of $[|1, n|]^3$: more precisely we consider a path π in the xy plane from f to the boundary and the curtain consists of the edges on $\pi \times [1, n]$ oriented consistently. In that way, any oriented simple closed curve γ satisfies $\langle \theta, 1_\gamma \rangle = k$ where k is the algebraic winding number of that curve around the tube.

By Lemma 2 we therefore have

$$\begin{aligned}\mathbb{E}(\mathbf{k}^2) &= \frac{\kappa}{\lambda} \langle \theta, (1 - T)\theta \rangle \\ &= \frac{\kappa}{\lambda} \left(ny - \sum_{y_i, y_j = 1}^{y} \sum_{z, z' = 1}^{n} T(e_i, e'_j) \right),\end{aligned}$$

where $\tau_n = \lambda/(\kappa n^3)$ converges to a constant as shown in [31]. □

Higher dimensional analogs of the previous results can be derived using Lemma 2 and the fact that the ratio τ_n converges by [31].

4.3 Minimal subconfigurations of the abelian sandpile

The *abelian sandpile model* is constructed as the stationary distribution of a certain Markov chain on integer functions over interior vertices of a graph. A sandpile configuration is a particle configuration $\eta : V^\circ \to \mathbb{N}$. If for some $x \in V^\circ$ we have

$\eta(x) \geq \deg x$, then the vertex x may topple, that is send one particle to each of its neighbors (this corresponds to the operation $\eta \mapsto \eta - \Delta(\delta_x)$). Particles that reach the boundary ∂V (referred to as the *sink* in this context) are lost. A sandpile is stable if $\eta(x) < \deg x$ for any $x \in V^\circ \backslash \{s\}$. A bilinear operation $\oplus$ can be defined on the set of stable configurations: addition in $\mathbb{Z}^V$ followed by toppling until stabilization. (The order in which toppling occurs does not matter, it is an abelian network [11].)

The Markov chain on the set of stable sandpiles is the following. At any given time, add (using $\oplus$) a particle to the configuration at a uniformly chosen vertex in V°. It was shown in [11] that the stationary distribution of this chain is unique and is uniform on the set of recurrent states. The *recurrent configurations* of the sandpile are defined as the recurrent states of this Markov chain.

The set of recurrent configurations has a local description given by the burning algorithm [11]. A stable configuration is recurrent if and only if any subconfiguration satisfies that it is unstable regarded as a configuration on the subgraph on which it lives. Using this criterion, Majumdar and Dhar constructed a bijection (called the burning bijection) mapping recurrent configurations of the abelian sandpile to spanning trees [34,35]. It depends on the choice of an ordering of edges around each vertex. See [37] for a theoretical physics perspective on the sandpile as a field theory.

In the case of a weighted graph, we define a measure ν on the set of recurrent configurations which is the pullback of the weighted spanning tree measure under the burning bijection. There is a way to relate ν to the discrete projection of the continuous sandpile (whose dynamics is analog to the one described above for unweighted graphs [16,23]).

Under the burning bijection, any local event for the sandpile therefore translates into an event for the spanning tree, however not necessarily local. An important concept is that of *minimal subconfiguration* (the original definition is from [12] where it is called weakly allowed subconfiguration).

A subconfiguration on a subgraph W is *minimal* if it is part of a recurrent configuration, but by decreasing any of its heights, this is no longer true. In particular, conditional on a minimal subconfiguration, the measure on the outside of W is the sandpile measure on $G \backslash W$, still with sink at ∂V.

The easiest example of a minimal subconfiguration is a single vertex with height 0, or any collection of vertices with 0 height provided none of these vertices are neighbors. There are minimal subconfigurations on any subgraph W of $\mathcal{G}$, see Fig. 3 for a more elaborate example.

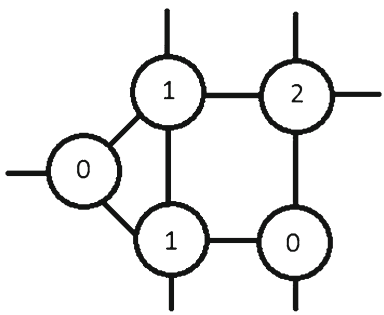

Fig. 3 Example of a minimal subconfiguration

Let $\mathscr{G}_W$ be the graph obtained by wiring all the vertices in $G\backslash W$ with ∂V, and removing all self edges. The following proposition is the generalization of Theorem 1 of [19] to the weighted graph setting.

Proposition 2 *Given any spanning tree $\mathscr{T}$ of $\mathscr{G}_W$, let $\mathscr{E}$ be the edge set*

$$\mathscr{E} = \{(x, y) : x \in W, \{x, y\} \notin \mathscr{T}\}.$$

For any minimal configuration ξ supported on W, there is a spanning tree $\mathscr{T}_0$ of $\mathscr{G}_W$ (given by our choice of burning bijection) such that

$$\mathbb{P}(\eta_W = \xi) = \det(I - T)_{\mathscr{E}_\prime} = \sum_{\mathscr{T}} \frac{\text{weight}\,(\mathscr{T}_\prime)}{\sum_{\mathscr{T}} \text{weight}\,(\mathscr{T})} \det(I - T)_{\mathscr{E}},$$

where the sums are over spanning trees of $\mathscr{G}_W$.

Proof By Lemma 4 in [19], the minimal subconfigurations are mapped (by the burning bijection) to spanning trees for which the projection of the tree on $\mathscr{G}_W$ is again a tree. Therefore, the probability of a minimal configuration is the probability that the small tree is the image $\mathscr{T}_W$ of the subconfiguration on $\mathscr{G}_W$. By the Gibbs property of the weighted spanning tree measure, this is simply the probability that the tree measure on $\mathscr{G}$ does not contain the edges of $\mathscr{E}_W$. Under the choice of orientation, the probability that (for a fixed subconfiguration) we map it to $\mathscr{T}_W$ is proportional to the edge weights in the tree. □

Proposition 2 and Theorem 1 imply the following.

Theorem 3 *Under the hypotheses of Theorem 1, the rescaled correlations of a finite number of macroscopically distant minimal subconfigurations has a universal scaling limit. In particular, the rescaled correlations of zero heights converge to a universal scaling limit (which is conformally covariant in dimension 2).*

4.4 Discrete Gaussian free field current flow

The *discrete Gaussian free field* (DGFF) Γ (with free or wired boundary conditions) is the Gaussian vector with zero mean and covariance matrix given by Green's function (with free or wired boundary conditions).

The current flow of the DGFF, defined by $J = d\Gamma$, is a Gaussian 1-form with zero mean and covariance matrix given by the transfer current, since for any 1-form α, we have

$$\begin{aligned}\mathbb{E}((J\alpha)^2) &= \mathbb{E}((\Gamma d^*\alpha)^2) = \left\langle d^*\alpha, Gd^*\alpha\right\rangle \\ &= \left\langle \alpha, dGd^*\alpha\right\rangle = \langle \alpha, T\alpha\rangle,\end{aligned}$$

where we used the fact that d and d^* are dual.

Whenever the discrete Green function converges to the continuous one, the discrete Gaussian free field converges weakly in distribution to the continuous Gaussian free field. In virtue of Theorem 1, it is also the case for the current flow, which converges to its continuous counterpart.

Theorem 4 *Under the hypotheses of Theorem 1, the flows J_n converge weakly in distribution to $d\Gamma$, where Γ is the GFF and the derivative is in the distributional sense.*

5 Random point fields

5.1 Patterns

A simple point process (point process for short) on a discrete[2] space Ω is a random subset $\mathscr{T} \subset \Omega$, or equivalently, a random configuration of $\{0, 1\}$-valued spins σ_x at each point $x \in \Omega$. We define a *pattern* to be a finite (deterministic) spin configuration which can be represented as a pair of disjoint sets $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1)$, where $\mathbf{x}_0$ is the set of points with 0 spin and $\mathbf{x}_1$ the set of points with spin 1. The *support* of a pattern is the set $\mathbf{x}_0 \cup \mathbf{x}_1$ which we denote by $\{\mathbf{x}\}$. We say that two patterns are disjoint when their supports are disjoint. A mono-pattern is a pattern where all spins have the same value.

Given a collection of patterns X, a point process ξ on Ω induces a point process ξ_X on the disjoint union of the patterns, which we may still view as a point process on Ω (for example, by fixing an ordering of points in Ω and identifying the pattern with its element of least index). This is the *pattern field* associated to X.

Formally, a pattern field can be represented as $\xi_X = \sum_{\mathbf{x} \in X} \delta_{\mathbf{x}}$, which is an element of the dual of $\ell^2(\Omega)$. We will omit the dependence on X if it is clear from the context. For any $f \in \ell^2(\Omega)$, we write $\xi(f) = \langle \xi, f \rangle$ and $\xi(A)$ if $f = 1_A$ is the indicator of a subset $A \subset \Omega$.

5.2 Symmetric and block determinantal processes

Let K be the kernel of a trace-class self-dual map in $\ell^2(\Omega)$ with eigenvalues in $[0, 1]$, that is a symmetric matrix indexed by Ω. For any finite set A, denote K_A the restriction of K onto $\ell^2(A)$. Let $\widehat{I_A}$ be the diagonal matrix with 1s on the entries indexed by A and 0 elsewhere. K defines a symmetric determinantal process, which means that for any pattern $\mathbf{x}$, we have

$$\mathbb{P}(\mathbf{x}) = \det \left(-\widehat{I}_{\mathbf{x}_0} + K \right)_{\{\mathbf{x}\}}. \tag{6}$$

The kernel $I - K$ is a kernel for the complement process $\Omega \backslash \mathscr{T}$.

Although the underlying point process is determinantal, the pattern field is not. However, joint probabilities of patterns can still be written as a large minor of a modified matrix. More precisely,

[2] The result of this section easily adapts to the case of (uncountable) Polish spaces but we restrict to the discrete case to avoid cumbersome technicalities.

$$\mathbb{P}(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}) = \det\left(-\widehat{I}_{\cup_{j=1}^k \mathbf{x}_0^{(j)}} + K\right)_{\cup_{j=1}^k \{\mathbf{x}^{(j)}\}}. \tag{7}$$

Therefore it can be viewed as a block determinantal process, see Sect. 5.3 for more details.

It follows from a short computation that for any finite set $A \subset \Omega$, we have

$$\mathbb{E}\left(z^{\xi(A)}\right) = \det(zK_A + I - K_A) = \prod_{\lambda \in \mathrm{Spec}(K_A)} (z\lambda + (1-\lambda)), \tag{8}$$

which signifies that the number of points in A is a sum of independent Bernoullis. These facts are well-known, see e.g. Chapter 4 of [17] for an introduction to determinantal processes.

Proposition 3 *Let ξ be a symmetric determinantal point field. Let X be a family of disjoint patterns. The number of points observed in $A \cap (\cup_{\mathbf{x}\in X}\{\mathbf{x}\})$, where $A \subset \Omega$ is any finite subset, is a sum of independent Bernoullis.*

Proof Since the patterns are disjoint, for any finite set $A \subset \Omega$, we have $\xi(A \cap (\cup_{\mathbf{x}\in X}\{\mathbf{x}\})) = \xi(A \cap (\cup_{\mathbf{x}}(\mathbf{x}_1^c \cup \mathbf{x}_0)))$. The result follows by applying (7) to obtain (8). □

For any finite set of patterns A, we have

$$\mathrm{Var}(\xi_X(A)) = \sum_{\mathbf{x}\in A}\sum_{\mathbf{y}\in A} \mathrm{Cov}(\mathbf{x}, \mathbf{y}).$$

In the case when the patterns are singletons, A is just a subset of Ω, and

$$\begin{aligned}\mathrm{Cov}(x, y) &= (K(x,x)(1 - K(x,x))1_{\{x=y\}} - K(x,y)^2 1_{\{x\neq y\}}\\ &= (\mathbb{P}(x) - K(x,x)^2)1_{\{x=y\}} - K(x,y)^2 1_{\{x\neq y\}}.\end{aligned}$$

Thus the variance is given by

$$\mathrm{Var}(\xi(A)) = \sum_{x\in A}\left(K\,(x,x) - \sum_{y\in A} K\,(x,y)^2\right).$$

5.3 Fluctuations for block determinantal processes

Before studying the scaling limits of pattern fields, we show (following ideas of Boutillier [6]) a fluctuation result for general block determinantal fields (without assuming the kernels to be symmetric). In combination with the covariance structure studied in the next section, this implies that under the right decay of correlation, pattern fields of determinantal processes converge to Gaussian white noise.

Let $d \geq 1$ and $D \subset \mathbb{R}^d$ be a domain. Let Ω be the space of locally finite particle configurations in D. Denote $\mathscr{F}$ to be the σ-measurable subsets of Ω, generated by the cylinder sets

$$\mathscr{C}_B^n = \{\xi \in \Omega : |\text{ particles of } \xi \text{ that lies in } B| = n\},$$

where $B \subset D$ is a Borel set and $n \in \mathbb{N}$. $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a block determinantal process, if for any $k \in \mathbb{N}$ and $\{x_i\}_{i=1}^k \subset D$,

$$\mathbb{P}\left(\#\left(dx_i\right) = 1, i = 1, \ldots, k\right) = \rho_k\left(x_1, \ldots, x_k\right) dx_1 \cdots dx_k,$$

where the k point correlation function

$$\rho_k\left(x_1, \ldots, x_k\right) = \det\left(A_{x_i x_j}\right)_{i,j=1}^k, \tag{9}$$

and each $A_{x_i x_j}$ is a $m_i \times m_j$ matrix. We will abuse the notation below and let $\mathbb{P}\left(x_1, \ldots, x_k\right)$ denote ρ_k.

Proposition 4 *Let* $(\Omega, \mathscr{F}, \mathbb{P}^n)_{n>0}$ *be a family of block determinantal processes. Assume that* $\det A^n_{x_i x_i}$ *is bounded from below by a strictly positive constant, and* $A^n_{x_i x_j}$ *converges as* $n \to \infty$ *to a limit* $A_{x_i x_j}$ *satisfying* $\left\|A_{x_i x_j}\right\| \leq O\left(|x_i - x_j|^{-d}\right)$ *as* $\left|x_i - x_j\right| \to \infty$*. Then a rescaling–recentering of the associated point field* ξ_n *converges weakly in distribution to a Gaussian field: for any test function* $\varphi \in C_0^\infty(D)$*, we have*

$$\frac{\xi_n(\varphi) - \mathbb{E}(\xi_n(\varphi))}{\operatorname{Var}(\xi_n(\varphi))^{1/2}} \to \mathscr{N}(0, 1).$$

We will prove the proposition by verifying Wick's theorem for the counting field, and shows the finite dimensional distributions converge to that of a Gaussian field (see e.g. [18]). The Wick's theorem states if ξ is a random field such that all of its moments can be expressed in terms of the covariance structure as follows: for any smooth test functions $\varphi_1, \ldots, \varphi_n$,

$$\mathbb{E}\left(\xi\left(\varphi_1\right) \cdots \xi\left(\varphi_n\right)\right) = \begin{cases} \sum_{\text{pairings}} \prod_{i=1}^{n/2} \mathbb{E}\left(\xi\left(\varphi_i\right) \xi\left(\varphi_{\sigma(i)}\right)\right) & n \text{ even}, \\ 0 & n \text{ odd}, \end{cases} \tag{10}$$

then ξ is Gaussian. In practice, by polarization identities it suffices to check the above identity for $\mathbb{E}\left(\xi\left(\varphi\right)^n\right)$.

The basic idea is that when computing higher moments, all the contributions not coming from pair correlations vanish, since the correlations decay fast enough. The argument follows the spirit of [6].

Proof We first verify the Wick formula in the case where the contribution points are macroscopically distant. Let

$$\Xi_k^n(\varphi) = \mathrm{Var}\left(\xi_n(\varphi)\right)^{-k/2} \sum_{\substack{z_1,\ldots,z_k \\ \text{distinct}}} \varphi(z_1)\cdots\varphi(z_k)\,\mathbb{E}_n\left(\prod_{i=1}^{k}\left(1_{z_i} - \mathbb{P}^n(z_i)\right)\right).$$

Note the following exclusion-inclusion formula

$$\begin{aligned}
&\mathbb{E}_n\left[\left(1_{z_1} - \mathbb{P}^n(z_1)\right)\cdots\left(1_{z_k} - \mathbb{P}^n(z_k)\right)\right] \\
&\quad= \sum_{i_1,\ldots,i_p} \prod_{i\notin\{i_1,\ldots,i_p\}} \left(-\mathbb{P}^n(z_i)\right)\mathbb{E}_n\left[\prod_{l=1}^{p} 1_{z_{i_l}}\right] \\
&\quad= \prod_{i=1}^{k}\mathbb{P}^n(z_i) \sum_{C\subset\{1,\ldots,k\}} (-1)^{k-|C|}\det\left(1_{\{i,j\}\in C}B^n_{z_iz_j}\right)_{i,j=1}^{k},
\end{aligned}$$

where $B^n_{z_iz_j} = \left(\mathbb{P}^n(z_i)\right)^{-1}A^n_{z_iz_j}$.

Now expand the determinant, and notice that the off-diagonal entries of the matrix $\left\{1_{\{i,j\}\in C}B^n_{z_iz_j}\right\}$ are small (bounded by $O\left(n^{-d}\right)$). One can rewrite the above expression in terms of sums over permutations fixing no block $B^n_{z_iz_i}$ (or equivalently, as products of cycles), and the support of which intersect each block exactly once. We thus obtain

$$\begin{aligned}
&\mathbb{E}_n\left[\left(1_{z_1} - \mathbb{P}^n(z_1)\right)\cdots\left(1_{z_k} - \mathbb{P}^n(z_k)\right)\right] \\
&\quad= \sum_{C\subset\{1,\ldots,k\}} (-1)^{k-|C|}\prod_{i=1}^{k}\mathbb{P}^n(z_i) \sum_{S \text{ fixing no block}} \mathrm{sgn}(S)\prod_{i=1}^{k}\prod_{\alpha=1}^{m_i} B^n_{(z_i,\alpha),S(z_i,\alpha)} \\
&\quad= \sum_{C\subset\{1,\ldots,k\}} (-1)^{k-|C|}\prod_{i=1}^{k}\mathbb{P}^n(z_i) \sum_{\substack{S\in S_k \\ \text{fix no point}}} \mathrm{sgn}(S)\left(\prod_{i=1}^{k}\sum_{\alpha_i=1}^{m_i} B^n_{(z_i,\alpha_i),\left(z_{S(i)},\alpha_{S(i)}\right)}\right) \\
&\qquad+O\left(n^{-2d}\right),
\end{aligned}$$

where the $O\left(n^{-2d}\right)$ term accounts for the contribution from permutations that has at least two non-fixed point in some block. Indeed, for a particular S that partitions $\{1,\ldots,k\}$ into cycles $\{\gamma_l\}$, one can write

$$\prod_{i=1}^{k}\sum_{\alpha_i=1}^{m_i} B^n_{(z_i,\alpha_i),\left(z_{S(i)},\alpha_{S(i)}\right)} = \prod_{|\gamma_l|=p}\mathrm{Tr}\left(B^n_{z_1z_2}\cdots B^n_{z_pz_1}\right).$$

Let $\{\Gamma_l\}$ denote some partition of $\{1, \ldots, k\}$. Therefore,

$$\Xi_n^{\varepsilon}(\varphi) = \sum_{C \subset \{1,\ldots,k\}} (-1)^{k-|C|} \prod_{i=1}^{k} \mathbb{P}^n(z_i) \sum_{\{\Gamma_l\}} \prod_l \sum_{\substack{\gamma:\, |\gamma|=m \\ \mathrm{supp}\gamma=\Gamma_l}} \mathrm{sgn}\,(\gamma)\,(\mathrm{Var}\,(\xi_n(\varphi)))^{-m/2} \cdot \left(\sum_{\substack{z_1,\ldots,z_m \\ \text{distinct}}} \varphi(z_1)\cdots\varphi(z_m)\,\mathrm{Tr}\left(B^n_{z_1 z_2}\cdots B^n_{z_m z_1}\right) + O\left(n^{-2d}\right) \right).$$

It suffices to show all the contributions from a cycle of length $m \geq 3$ vanish. It is easy to verify that

$$\mathrm{Var}\,(\xi_n(\varphi)) = \sum_{z,z'} \varphi(z)\,\varphi(z')\,\mathbb{P}_n(z,z') - \left(\sum_z \varphi(z)\,\mathbb{P}_n(z)\right)^2 = O\left(n^{2d}\right).$$

Therefore each term arising from a cycle of length $m \geq 3$ is bounded by

$$\begin{aligned} & M n^{-m} \sum_{\substack{z_1,\ldots,z_m \\ \text{distinct}}} \mathrm{Tr}\left(B^n_{z_1 z_2}\cdots B^n_{z_m z_1}\right) \\ & \leq M C_m n^{-m} \sum_{\substack{z_1,\ldots,z_m \\ \text{distinct}}} \left\|B^n_{z_1 z_2}\right\|\cdots\left\|B^n_{z_m z_1}\right\|, \end{aligned} \tag{11}$$

for some $M < \infty$. Since $\mathbb{P}_n(z_i) = O(1)$, the decay rate of $B_{z_i z_j}$ is the same as $A_{z_i z_j}$, which implies that $\sum_{z_2 \neq z_1} \left\|B^n_{z_1 z_2}\right\|^s < \infty$ for any $s > 1$. Apply the following elementary inequality to (11),

$$\prod_{i=1}^{m} a_i \leq \frac{1}{m}\left[(a_1\cdots a_{m-1})^{\frac{m}{m-1}} + \cdots + (a_m a_1\cdots a_{m-2})^{\frac{m}{m-1}}\right],$$

and note that each term can be bounded by $O\left(n^{2-m}\right)$. For instance, the first corresponding term satisfies

$$C \sum_{\substack{z_1,\ldots,z_m \\ \text{distinct}}} \left(\left\|B^n_{z_1 z_2}\right\|\cdots\left\|B^n_{z_m z_1}\right\|\right)^{\frac{m}{m-1}} < \infty,$$

by successively summing over $z_m, z_{m-1}, \ldots, z_2$. Therefore (11) can be bounded by $O\left(n^{2-m}\right)$, which vanishes in the limit for $m \geq 3$.

All the terms contributing to the limit are thus coming from pairings, and we verified Wick's formula (10).

When the z_i's are not necessarily all distinct, we can verify the Wick formula for higher moments by the same argument as in section 3.2.2 in [6], and we omit the computation here. □

5.4 Central limit theorem for pattern fields

Suppose that Ω is infinite and embedded in $\mathbb{R}^d$ in a discrete way (no accumulation points). Let $D \subset \mathbb{R}^d$ be a simply connected domain containing 0. Define $\Omega_n = D \cap \Omega/n$: these sets (when remultiplied by n) form an exhausting sequence of subsets of Ω. We suppose that each Ω_n is equipped with a determinantal process with kernel K_n. We assume that $K_n \to K$ pointwise. This implies that the point process on Ω is the weak limit of the process defined on Ω_n. As we will see, this implies that scaling limits of fields constructed from local events do not feel the influence of the boundary shape of D.

For each n, let $\mathscr{P}_n$ be a collection of patterns in Ω_n. When the choice of $\mathscr{P}_n$ is fixed, we write ξ_n instead of $\xi_{\mathscr{P}_n}$.

Theorem 5 *Assume that $K(x, y) = O(|x - y|^{-d})$. If for any n and $\mathbf{x} \in \mathscr{P}_n$, $\mathbb{P}_n(\mathbf{x})$ is uniformly bounded from below, then as $|A_n| \to \infty$, $(\xi_n(A_n) - \mathbb{E}(\xi_n(A_n)))/\mathrm{Var}(\xi_n(A_n))^{1/2}$ converges in law to a standard Gaussian.*

In the case where the patterns are singletons, the only condition we require is that $\mathrm{Var}(\xi_n(A_n)) \to \infty$, and the result follows from the Lindenberg–Feller theorem as explained in Theorem 4.6.1 of [17].

Proof By pointwise convergence of K_n to K, we have $K_n\left(x_i, x_j\right) = O(n^{-d}\left|x_i - x_j\right|^{-d})$ for any $x_i, x_j \in \Omega_n$. The result now follows from Proposition 4 since the pattern field is block determinantal. □

It follows from Theorem 5 that the rescaled–recentered field constructed from ξ_n converges to a Gaussian field, that can be viewed as a random element of $L^2(D)$. By the lower bound on $\mathbb{P}(x)$, the growth rate of $\mathrm{Var}(\xi_n)$ is $|\mathscr{P}_n|$ (which we will take to be of the order of n^d) and compute the covariance structure of the Gaussian random field obtained as the limit of $n^{-d/2}(\xi_n - \mathbb{E}\xi_n)$.

To characterize the law of a Gaussian field, it suffices to compute $\mathbb{E}(\xi(\varphi)^2)$ for any test function φ (see [18]). If

$$\mathbb{E}(\xi(\varphi)^2) = \iint \varphi(x)\, C(x, y)\, \varphi(y)\, dx\, dy,$$

then ξ is a Gaussian field with correlation kernel C. When $C(x, y) = I\delta(x - y)$, where δ is the Dirac mass and $I > 0$, the field is *Gaussian white noise*, $\mathbb{E}(\xi(\varphi)^2)$ is proportional to $|\varphi|^2_{L^2}$ and I is called the *intensity*.

Let $\mathscr{P}_n$ be an increasing sequence of finite collections of disjoint patterns on Ω_n, such that $|\mathscr{P}_n| = O\left(n^d\right)$. Denote $\mathscr{P} = \cup_n \mathscr{P}_n$. We consider the associated random fields ξ_n.

Assume Ω to be periodic (as an embedded set), that is, suppose it has a finite number of orbits under the action of a rank d group of translations $\Lambda \cong \mathbb{Z}^d$. We also suppose the collection of patterns $\mathscr{P}$ itself to be invariant under Λ, that is, it can be written as the disjoint union of the translates of a finite collection of patterns $\mathscr{P}^0 = \mathscr{P}/\Lambda$.

Theorem 6 *Let Ω be a periodic subset of $\mathbb{R}^d$ as above. Let $D \subset \mathbb{R}^d$ be a simply connected domain containing* 0 *and define $\Omega_n = D \cap \Omega/n$. We suppose that on each of these is defined a determinantal process with kernel K_n satisfying the conditions in Theorem 5. Then, for any periodic set of patterns generated by a pattern $\mathscr{P}^0$, the corresponding field ξ_n satisfies that $n^{-d/2}(\xi_n - \mathbb{E}(\xi_n))$ converges weakly to a Gaussian white noise with intensity*

$$I(\mathscr{P}^0) = \frac{1}{|\mathscr{P}^0|} \sum_{\mathbf{x}\in\mathscr{P}^0} \sum_{\mathbf{y}\in\mathscr{P}} \mathrm{Cov}(\mathbf{x}, \mathbf{y}). \tag{12}$$

Proof As explained above, we already know that the limit exists and is Gaussian. Let us identify the covariance structure. In the following, we make the abuse of notation $\varphi(\mathbf{x})$ to denote the value of $\varphi(z)$ where z is the location (center of mass) of the support of the pattern.

Given any test function $\varphi \in C_0^\infty(D)$, we have

$$\frac{1}{|\mathscr{P}_n|}\mathrm{Var}(\xi_n(\varphi)) = \frac{1}{|\mathscr{P}_n|} \sum_{\mathbf{x}\in\mathscr{P}_n} \sum_{\mathbf{y}\in\mathscr{P}_n} \mathrm{Cov}_n(\mathbf{x}, \mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y}). \tag{13}$$

Consider a small $\delta > 0$ (and write ε for $n^{-d/2}$). We split the previous sum into three contributions as follows

$$= \varepsilon^2 \sum_{\mathbf{x}\in\mathbb{P}_n} \varphi(\mathbf{x}) \Big(\sum_{|\mathbf{x}-\mathbf{y}|>\delta} \varphi(\mathbf{y})\mathrm{Cov}_n\,(\mathbf{x}, \mathbf{y}) + \sum_{|\mathbf{x}-\mathbf{y}|\le\delta} (\varphi(\mathbf{y}) - \varphi(\mathbf{x}))\,\mathrm{Cov}_n\,(\mathbf{x}, \mathbf{y})$$
$$+ \sum_{|\mathbf{x}-\mathbf{y}|\le\delta} \varphi(\mathbf{x})\mathrm{Cov}_n\,(\mathbf{x}, \mathbf{y}) \Big)$$

We will show that the main contribution comes from the third term.

By the assumptions of Theorem 5, we know that there is a matrix K such that for $\mathbf{x} \neq \mathbf{y}$, we have

$$\mathrm{Cov}_n\,(\mathbf{x}, \mathbf{y}) = \det K_{\{\mathbf{x}\}\cup\{\mathbf{y}\}} - \det K_{\{\mathbf{x}\}} \det K_{\{\mathbf{y}\}},$$

which is a finite alternating sum of products that contain at least two off-diagonal block terms. Every such term is bounded by

$$O\left(n^{-d}\mathrm{dist}\,(\mathbf{x}, \mathbf{y})^{-d}\right),$$

where $\mathrm{dist}\,(\mathbf{x}, \mathbf{y}) = \inf_{x\in\mathbf{x}, y\in\mathbf{y}} \|x - y\|$. Therefore $\mathrm{Cov}_n\,(\mathbf{x}, \mathbf{y}) \le Cn^{-2d}\mathrm{dist}\,(\mathbf{x}, \mathbf{y})^{-2d}$.

The first term therefore yields a contribution of $O\left(\varepsilon^2\varepsilon^{-4}\varepsilon^4/\delta^4\right) = O\left(\varepsilon^2/\delta^4\right)$. Due to the smoothness of φ, the second term yields a contribution of

$$O\left(\varepsilon^2\delta \sum_{\mathbf{x},\mathbf{y}\ |\mathbf{x}-\mathbf{y}|\le\delta} \mathrm{Cov}_n(\mathbf{x}, \mathbf{y})\right) = O\,(\delta),$$

since the sum of covariances is absolutely convergent.

The third term yields a contribution of

$$\varepsilon^2 \sum_{\mathbf{x}} \varphi(\mathbf{x})^2 \sum_{\mathbf{y}, |\mathbf{y}-\mathbf{x}| \leq \delta} \mathrm{Cov}_n(\mathbf{x}, \mathbf{y})$$

which, by periodicity, converges, as long as $\delta/\varepsilon \to \infty$, to

$$I(\mathscr{P}) \int_D \varphi(z)^2 |dz|^2, \tag{14}$$

where $I(\mathscr{P}) = \frac{1}{|\mathscr{P}^0|} \sum_{\mathbf{x} \in \mathscr{P}^0} \sum_{\mathbf{y} \in \mathscr{P}} \mathrm{Cov}(\mathbf{x}, \mathbf{y})$.

By choosing $\delta \to 0$ slowly, for example $\delta = \varepsilon^{1/3}$, the two first contributions vanish and (13) thus converges to (14). The process is therefore a Gaussian white noise with intensity $I(\mathscr{P})$. □

In the case of singleton patterns, the intensity can be written as

$$\frac{1}{|\mathscr{P}^0|} \sum_{x \in \mathscr{P}^0} \left(K(x, x) - \sum_{y \in \mathscr{P}} K(x, y)^2 \right).$$

5.5 Interpretation of the intensity

Suppose the determinantal point field is at the same time a Gibbs random field, in the sense that the weight of any configuration T is given by $\prod_{x \in T} w(x)$, for some $w : \Omega \to \mathbb{R}^+$. In that case, the noise intensity can be interpreted as a second derivative of a free energy. Indeed, the partition function is defined by $Z = \sum_T \prod_{x \in T} w(x)$. In particular, for a fixed pattern $\mathbf{x} \subset \Omega$, choose the weight function so that it gives weight $w_0 w$ on $\mathbf{x}$, and w on all $\mathbf{y} \in \mathscr{P}$, $\mathbf{y} \neq \mathbf{x}$. Let $Z(w, w_0)$ be the corresponding partition function. A short computation shows that $\sum_{\mathbf{y} \in \mathscr{P}} \mathrm{Cov}(\mathbf{x}, \mathbf{y}) = -\frac{\partial^2}{\partial w \partial w_0} \log Z(w, w_0)$, which can be interpreted as an electric susceptibility of the network.

5.6 Pattern fields of the spanning tree model

Let $D \subset \mathbb{R}^d$, $d \geq 2$ be a bounded simply connected domain containing 0. Let Υ be an infinite weighted graph embedded in $\mathbb{R}^d$ invariant under a rank d lattice $\Lambda \cong \mathbb{Z}^d$. We define Ω_n to be the edge-set of $\mathscr{G}_n = \Upsilon/n \cap D$.

The spanning tree model on $\mathscr{G}_n$ is a symmetric determinantal process on Ω_n with kernel given by the transfer current (another kernel, which is symmetric itself, is the matrix K defined on page 2 in the introduction).

In this context, a pattern $\mathscr{P}^0$ is a finite set of edges $\{e_1, \ldots, e_l; e_{l+1}, \ldots, e_k\}$ in a fundamental domain of Υ, where the edges $\{e_1, \ldots, e_l\}$ are present and $\{e_{l+1}, \ldots, e_k\}$ are forbidden. Suppose that $\mathscr{P}^0$ lies inside a fundamental domain Υ/Λ, and let $\mathscr{P}_n = \Lambda\mathscr{P}^0$ denote the union of its translates that lie inside Ω_n. We define the corresponding pattern field $\xi_n = \sum_{\mathbf{x}\in\mathscr{P}_n} \delta_{\mathbf{x}} = \sum_{x\in\Lambda} \delta_{\mathscr{P}^0_x}$, where $\mathscr{P}^0_x$ denotes the translate of pattern $\mathscr{P}^0$ by x.

The mean of the pattern field on a finite set may sometimes be computed: when $d = 2$ and Υ is isoradially embedded, the density of edges has the following limit. The probability of an edge on an isoradial graph is $\mathbb{P}(e) = (2/\pi)\theta_e$, where θ_e is the half-angle of that edge by [25] (see Appendix). Hence, the expected number of edges in a finite set A is

$$\mathbb{E}(\xi(A)) = \frac{2}{\pi}\sum_{e\in A}\theta_e.$$

Theorem 7 *Under the assumptions of Theorem 1, each rescaled pattern density field* $n^{-d/2}(\xi_n - \mathbb{E}(\xi_n))$ *converges weakly in distribution to Gaussian white noise. The sum of two pattern fields also converges to Gaussian white noise.*

Proof The spanning tree is a determinantal process, hence the result follows from Theorem 6 provided we have $T(x, y) \le O\left(n^{-2d}|x-y|^{-2d}\right)$, which follows from Theorem 1. The correlation between different patterns follows from a similar calculation, and we omit it here. □

The intensity of the white noise is given by (12). When we study correlation among fields (that is, the cross-term in the covariance structure of the sum of two pattern fields), say between two fields corresponding to patterns $\mathscr{P}^0$ and $\mathscr{P}^1$, which generate two collections $\mathscr{P}^0_n$ and $\mathscr{P}^1_n$, the intensity (which may be negative) is given (computation omitted) by

$$I(\mathscr{P}^0, \mathscr{P}^1) = \lim_{n\to\infty} \frac{1}{\sqrt{|\mathscr{P}^0_n||\mathscr{P}^1_n|}} \sum_{\mathbf{x}\in\Lambda\mathscr{P}^0_n, \mathbf{y}\in\Lambda\mathscr{P}^1_n} \mathrm{Cov}_n(x, y).$$

Let us give the example of an infinite d-regular graph: at each vertex, edges are numbered $1, \ldots, d$. We consider the pattern field ξ^k generated by all edges of type k. The intensities of the joint fields $\langle \xi^i, \xi^j\rangle$ are denoted $I(i, j)$. Since the total number of points in Ω_n is constant (K_n is a projector), for any $x \in \Omega_n$, we have $0 = \mathrm{Cov}_n\left(x, \sum_{y\in\Omega_n} y\right) = \sum_{y\in\Omega_n} \mathrm{Cov}_n(x, y)$. Hence, we obtain

$$\sum_{i,j=1}^{d} I(i, j) = 0. \tag{15}$$

A similar relation is true for the liquid dimers pattern fields in two dimension, see the remark following Theorem 7 in [6].

When $\mathscr{P}_n = \Omega_n$ is the collection of all edges, and $B \subset D$, the non-rescaled covariance of the variables $\xi_n(B)$ (which represents the number of edges inside B) has a limit. (A similar statement is true for other patterns.)

Proposition 5 *Under the assumptions of Theorem 1 and given two disjoint subregions B_1 and B_2 of D, the correlation between the number of edges $\xi_n(B_1)$ and $\xi_n(B_2)$ is*

$$\mathrm{Cov}\,(\xi_n(B_1), \xi_n(B_2)) = -\int_{B_1\times B_2} \left|\frac{\partial^2}{\partial z \partial w} g_D(z,w)\right|^2 |dz|^2 |dw|^2 \,(1+o(1)).$$

Proof This follows from Theorem 1. □

5.7 Minimal-pattern fields of the abelian sandpile

Minimal subconfigurations of the sandpile correspond to local events for the spanning tree model. However, they cannot always be written as simple patterns: the probability that a vertex has height zero is equal to a weighted sum of the probability of (non-disjoint) patterns consisting of a single edge present (and all other missing). We therefore need to deal with the mixture of measures and an argument is provided in the following proposition.

Proposition 6 *Let ξ_n be a random element in $\ell^2(\Omega_n)$ that satisfies a central limit theorem (as a sequence in n), η_n be a random measure on finite subsets of Ω_n. For any $\mathbf{x}_1,\ldots,\mathbf{x}_n \subset \Omega_n$, $\eta_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum c^n_{i_1,\ldots,i_n} \xi_n(x_1^{(i_1)},\ldots,x_n^{(i_n)})$, where $\{c^n_{i_1,\ldots,i_n}\}$ are uniformly bounded, and the sum runs over all the elements of $\mathbf{x}_1,\ldots,\mathbf{x}_n$. Then η_n satisfies a central limit theorem with the same speed.*

Proof The result follows by directly verifying Wick's formula. Consider the kth moment of the rescaled field $(\eta_n - \mathbb{E}\eta_n)/\mathrm{Var}\,(\xi_n)^{-1/2}$ applied to a test function φ:

$$\Xi_k^n(\varphi) = \mathrm{Var}\,(\xi_n(\varphi))^{-k/2} \sum_{z_1,\ldots,z_k} \varphi(z_1)\cdots\varphi(z_k)\,\mathbb{E}\left(\prod_{i=1}^k \left(\eta_n\left(1_{z_i}\right) - \mathbb{E}\eta_n\left(1_{z_i}\right)\right)\right).$$

It suffices to check

$$\lim_{n\to\infty} \Xi_k^n(\varphi) = \begin{cases} (k-1)!!\left(\lim_{n\to\infty}\Xi_2^n(\varphi)\right)^{k/2} & k \text{ even}\\ 0 & k \text{ odd.}\end{cases}$$

This is easily verified using the corresponding result for ξ_n. For instance, when k is even, $\Xi_k^n(\varphi) - (k-1)!!\left(\Xi_2^n(\varphi)\right)^{k/2}$ only contains contributions from the sum of m-point marginal ($m \geq 3$) of η_n. Each of these marginals can be expressed as a finite linear combination of m-point marginals of ξ_n. The central limit theorem for ξ_n implies that the corresponding sum of m-point marginals of ξ_n is $o(1)$. Therefore $\left|\Xi_k^n(\varphi) - (k-1)!!\left(\Xi_2^n(\varphi)\right)^{k/2}\right| = o(1)$. □

We now study the abelian sandpile model on the following class of graphs. Let Υ be an infinite weighted graph embedded in $\mathbb{R}^d$, and invariant under a rank d lattice $\Lambda \cong \mathbb{Z}^d$. This graph is in particular amenable and satisfies the one-end property [33]. The infinite volume limit on sandpiles is thus well-defined [1,19].

Let $D \subset \mathbb{R}^d$ be a domain. Without loss of generality, we may suppose that 0 lies inside the domain D and we define $\Omega_n = D \cap \Upsilon/n$. For a fixed minimal subconfiguration $\mathscr{P}^0$ (that we call minimal-pattern) lying inside the fundamental domain Υ/Λ, let $\mathscr{P}_n = \Lambda\mathscr{P}^0 \cap \Omega_n$ be the collection of its translates lying inside Ω_n. Let ξ_n be the associated random field in Ω_n.

Theorem 8 *Under the assumptions of Theorem 1, and for any minimal-pattern, the corresponding field $n^{-d/2}(\xi_n - \mathbb{E}(\xi_n))$ converges weakly in distribution to Gaussian white noise.*

Proof When Υ is a regular lattice with uniform weights, each spanning tree $\mathscr{T}$ described in Proposition 2 carries the same weight. Therefore we can choose a pattern $\mathscr{P}^1$ for the spanning tree model (on the edges of the graph) such that the minimal-pattern is in bijection with the $\mathscr{P}^1$ event under the burning bijection. The field can therefore be seen as a pattern field of the spanning tree model (associated to $\mathscr{P} = \Lambda\mathscr{P}^1$) (note that the union of two adjacent minimal subconfigurations occurs with zero probability, because of Dhar's burning test). The result is then a corollary of Theorem 7.

Let us deal with the general case. We start by showing that the limit exists and is Gaussian. Recall from Proposition 2 that the probability of a minimal subconfiguration is a linear combination of pattern probabilities for the spanning tree model. By Proposition 6, it suffices to prove the result for a fixed tree pattern $\mathscr{E}$. This follows from Theorem 7.

Let us now identify the covariance structure of the limiting field. As above, it follows from Proposition 2 and Theorem 1 that $\mathrm{Cov}_n(\mathbf{x}, \mathbf{y}) = O\left(n^{-2d}\,|x - y|^{-2d}\right)$. Hence for any $\mathbf{x}$, $\sum_{\mathbf{y}} \mathrm{Cov}_n(\mathbf{x}, \mathbf{y})$ is summable and therefore (as above in the proof of Theorem 6) the limiting field is white noise. □

The intensity of the field is computed as follows. The probability of having two adjacent minimal subconfigurations is zero. The intensity of the white noise which is of the form (12) does not include adjacent patterns. Therefore, each covariance can be replaced (using Proposition 2) by a linear combination of expressions involving the transfer current. It may therefore be evaluated using the explicit formula for the transfer current when this latter is known.

In the special case of the zero height field, one has $\xi_n = \sum_{x \in \Omega_n} \delta_{h_x = 0}$, where h_x is the height of sand at x. The asymptotic density of zero height is known in certain cases. Let $A \subset \Omega_n$ be some set of vertices. We have $\mathbb{E}(\xi_n(A)) = |A|/|A/\Lambda| \sum_{x \in A/\Lambda} \mathbb{P}(h_x = 0) = q|A|/k$, where $q = \sum_{x \in A/\Lambda} \mathbb{P}(h_x = 0)$ and $k = |A/\Lambda|$. In the case of $\mathbb{Z}^2$, the fundamental domain has size $k = 1$ and $q = \mathbb{P}(h_0 = 0) = 2/\pi^2(1 - 2/\pi)$ by using the explicit expression of the transfer current (see [34] for the computation).

6 Questions

1. We have not dealt with non-minimal subconfigurations in the sandpile: do the corresponding fields also have Gaussian fluctuations?

2. Can closed-form numerical expressions be obtained for the different intensities of the pattern fields? Are there algebraic relations between different intensities, generalizing (15)?
3. From the proofs above, one sees that the critical rate of correlation decay in dimension d is r^{-d}, where r is the distance. If the correlation decays faster, the random field fluctuations converge to Gaussian white noise. Boutillier [6] studied the random field associated with liquid dimers on planar graphs, which are in the critical regime, and obtained some long range Gaussian random field in the limit. Are there any natural critical models in higher dimensions?
4. Does the approximate mean value property hold for supercritical percolation cluster and random conductance models, thus allowing one to study the scaling limit of statistical physics models on these random environments? In the random conductance model on $\mathbb{Z}^d$ [2] the Green function is shown to have the same decay r^{2-d} as the continuous Green function. Provided the approximate mean value property is valid, one can show that the transfer current has the decay of r^{-d} which implies that the pattern fields considered above (on this random environment) would have the same fluctuations as in the current paper.
5. What is the distribution of the number of points of a pattern field for a symmetric determinantal process on a finite space?

Acknowledgments We thank Cédric Boutillier, Richard Kenyon, and David Wilson for helpful discussions and feedback. We also thank the anonymous referee for helpful suggestions.

Appendix: Isoradial graphs

In this appendix, we recall why isoradial graphs are good approximations of planar domains which satisfy Assumption (A1) and (A2) (on page 7) about the convergence of the Green functions on the whole plane. Our main reference for this section is [9].

Good approximations

Let us first recall the setting (see e.g. [9,25] and references therein for earlier works on the subject). Let Υ be an infinite planar isoradial graph and for each $\varepsilon > 0$ we denote by Υ^ε a planar isoradial embedding of Υ with mesh size $\varepsilon > 0$ (we may suppose that an isoradial embedding of Υ with mesh size 1 is fixed and that other embeddings are obtained by dilation with respect to a fixed vertex). The planar dual of Υ is also isoradial with same radius and the *diamond graph* is defined to be the graph whose faces are the rhombi $R(e)$ for all edges e, obtained by joining the vertices of an edge and its dual. The angle between an edge e and the side of $R(e)$ at its left is denoted θ_e. It is the half-angle at the origin of edge e of the rhombus $R(e)$, see Fig. 4.

We make the customary assumption (bounded half-angles property) that all angles θ_e of these rhombi are uniformly bounded away from 0 and $\pi/2$. This ensures that for

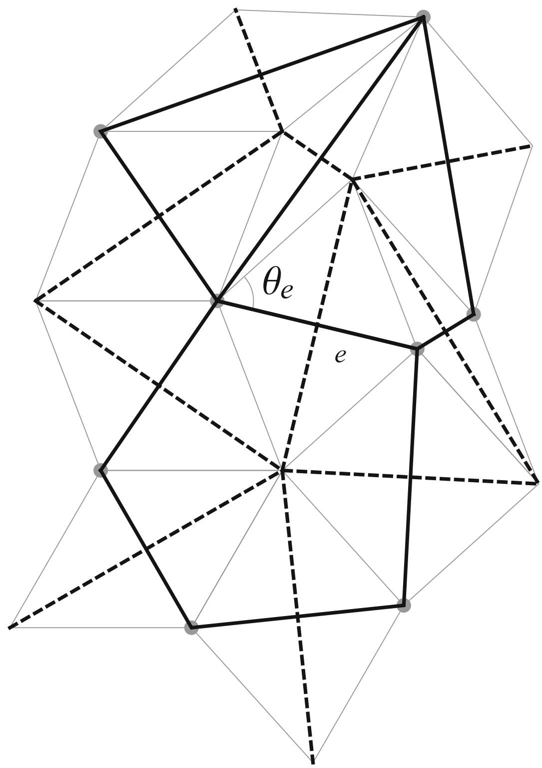

Fig. 4 A portion of an infinite isoradially embedded graph; the graph is represented in *thick lines*; the *dashed edges* represent the dual graph; the *gray lines* form the diamond graph; an edge e of the graph and the corresponding rhombic half-angle θ_e is represented

vertices of Υ^ε, the combinatorial distance in the graph and the Euclidean distance are uniformly related by a factor of ε. Let $D \subset \mathbb{C}$ be a simply-connected compact planar domain. For every $\varepsilon > 0$, consider D^ε to be a finite isoradial subgraph of Υ^ε whose vertices lie in $D \cap \Upsilon^\varepsilon$, which is "simply connected" in the sense that the union of its closed faces is simply connected. A sequence $(D^\varepsilon)_{\varepsilon>0}$ is said to *approximate* D if the Hausdorff distance from D^ε to D is $O(\varepsilon)$.

In the following, we assume that D^ε is endowed with its *critical isoradial conductances*

$$c(e) = \tan \theta_e,$$

where θ_e is the half-angle of the rhombi $R(e)$ (see [25] for a longer definition), and consider the corresponding Laplacian Δ.

It is shown in [9] that isoradial graphs satisfy the *Approximate mean value* property. The *Paths approximation* property follows from the bounded angle property mentioned above. Thus, isoradial graphs are good approximations.

Furthermore, the random walk (biased by weights) on the embedded graph is isotropic. A local time-reparameterization of it converges to Brownian motion. This implies assumption (A2).

Green's functions convergence

It was proved by Kenyon in [25], up to an improvement given by Bü cking in [8], that the Green function on Υ^ε satisfies the following expansion[3]: for any two vertices $v \neq w$, we have

$$G(v, w) = -\frac{1}{2\pi} \log |v - w| + O\left(\frac{\varepsilon^2}{|v - w|^2}\right).$$

This implies that G converges to the Green function on the whole plane $g_{\mathbb{C}}(z, w) = -\frac{1}{2\pi} \log |z - w|$, as $\varepsilon \to 0$.

This is assumption (A1). We have thus recalled why isoradial graphs are good approximations satisfying assumptions (A1) and (A2).

Since the Laplace equation is conformally invariant, for any simply-connected surface with boundary, D, the Neumann and Dirichlet Green function are obtained by their image under a conformal map between the upper half plane $\mathbb{H}$ and D of the Green function on $\mathbb{H}$ with corresponding boundary conditions.

Discrete harmonic functions on isoradial graphs converge to continuous harmonic functions in a very strong sense described in [9]. In particular, the Dirichlet Green function is shown to converge building on Kenyon's asymptotics for the whole plane. We give here the proof for the free boundary Green function. The proof follows from arguments in [9] although the result is not stated explicitly there.

Proposition 7 *For any graph $\mathscr{G}$ approximating a simply connected surface with boundary D in the sense that $\mathscr{G}$ is embedded in D and there exists a system of coordinate patches and conformal maps that map the graph to an isoradial planar graph on these patches, the discrete Neumann Green function converges uniformly on any compact set inside the surface to its continuous counterpart.*

Proof Let us first show it for the unit disk D. Let Υ^ε be an infinite isoradial graph with mesh size ε such that $\mathscr{G}$ is a subgraph of it. Write $G = G_{\Upsilon^\varepsilon} - H$ where H is harmonic with Neumann boundary conditions equal to the normal derivative of G_{Υ^ε}. By C^1 convergence of G_{Υ^ε} when the mesh size goes to zero, the values of the normal derivative converge. Now consider the dual graph of $\mathscr{G}$. The harmonic conjugate H^* of H is univalued since D is simply-connected. Its values on the boundary are determined (up to a constant which we take to be 0) and converge to a limit f^*. By [9] the harmonic function H^* converges to the harmonic extension of f^*. Hence, its dual H converges too, to a function h. The limit of G is therefore $g_{\mathbb{C}} - h$ which is equal to Neumann Green's function g_D^r (up to a constant).

In the case where D is another domain (even non planar), we use a conformal map to bring it back to the previous case. Since convergence is a local result, we may again use the convergence result of [9] and the above argument. □

[3] On $\mathbb{Z}^2$, an all-order expansion is known [15].

Comments on previous related work

Asymptotics of the Green function on isoradial graphs and its derivatives have been well studied in the literature. Among others, we may state [15,39] for asymptotics of the Green function on $\mathbb{Z}^2$and [8,25] for the case of isoradial graphs on the whole plane (stated above). For the convergence of the increment rate of the rescaled Green function over $\varepsilon\mathbb{Z}^2$, see Lemma 17 in [24].

Dürre in [14] showed in the case of $\mathbb{Z}^2$ in the Dirichlet case the analog of our Theorem 1 and also gave the application to the study of zero-height fields in the abelian sandpile (our Theorems 3 and 8 for this particular minimal-pattern in the case of $\mathbb{Z}^2$).

Important results of convergence of discrete harmonic functions and their derivatives (in the Carathéodory topology) in domains with boundary (in particular the Dirichlet Green function) to their continuous counterpart are gathered in [9]. Theorem 1 directly follows from their arguments in the case of isoradial graphs by using C^1 convergence of the wired Green function in each variable. For the free case it follows from Proposition 7 above.

Carathéodory convergence is weaker than Hausdorff convergence, which we suppose. Therefore, the results of [9] applied to the transfer current give the uniform C^1 convergence.

Note that for isoradial graphs there is an explicit formula for the transfer current in terms of path integral which is a linear combination of the Green function explicit formula derived in [25]. We do not use it here because we are interested only in the scaling limit asymptotics which are easier to derive.

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