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On a class of functions that satisfies explicit formulae involving the Möbius function

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Abstract A class of functions that satisfies intriguing explicit formulae of Ramanujan and Titchmarsh involving the zeros of an L-function in the reduced Selberg class of degree one and its associated Möbius function is studied. Moreover, a sufficient and necessary condition for the truth of the Riemann hypothesis due to Riesz is generalized.

Keywords Explicit formulae · Möbius function · Selberg class · L-functions · Riemann zeta-function · Hankel transformations · Special functions

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1 Introduction and results

1.1 Motivation for studying the Möbius function

The Möbius function μ is defined as

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$$\mu(n)=\begin{cases}1, & \text{if } \ n=1,\\ 0, & \text{if } \ p^2|n \text{ for some prime } p,\\ (-1)^k, & \text{if } \ n \text{ is a product of } k \text{ distinct primes.}\end{cases} \tag{1.1}$$

If x denotes a positive real, then the Mertens function M is defined by

$$M(x)=\sum_{n\leqslant x}\mu(n).$$

The interest in studying $\mu(n)$ and $M(x)$ comes from their connection to the distribution of the prime numbers. For instance (see Hardy and Littlewood [18, Sect. 1.1]), the prime number theorem is equivalent to the following statements:

$$M(x)=o(x), \quad \sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0. \tag{1.2}$$

Estimates on Mertens's function date back to the 1880s when Mertens [15] falsely conjectured that $M(x)\leqslant\sqrt{x}$ for all sufficiently large x. Later in 1885, Stieltjes [15] claimed a proof of this conjecture. It was not until 100 years later that Odlyzko and te Riele [37] disproved the Mertens' conjecture. Specifically, they showed the following.

There are explicit constants $C_1>1$ and $C_2<-1$ such that

$$\limsup_{x\to\infty}\frac{M(x)}{\sqrt{x}}\geqslant C_1, \quad \liminf_{x\to\infty}\frac{M(x)}{\sqrt{x}}\leqslant C_2.$$

This means that each of the inequalities $-\sqrt{x}\leq M(x)$ and $M(x)\leq\sqrt{x}$ fails for infinitely many x, or, equivalently, $M(x)=\Omega_{\pm}(\sqrt{x})$. The proof of te Riele and Odlyzko does not provide a specific value of x for which $M(x)\geq\sqrt{x}$, but it is known that there is such an x for $x<10^{156}$. In [6] Best and Trudigan give an alternative disproof of Mertens' conjecture and they show that C_1 can be taken to be 1.6383 and C_2 to be -1.6383. The best unconditional estimate on the Mertens' function is (see Ivić [21, Sect. 12])

$$M(x)\ll x\exp\left(-c_1\log^{\frac{3}{2}}x(\log\log x)^{-\frac{1}{5}}\right),$$

for $c_1>0$; and the bound on the assumption of the Riemann hypothesis is (see Titchmarsh [48, Sect. 14.26])

$$M(x)\ll x^{\frac{1}{2}}\exp\left(\frac{c_2\log x}{\log\log x}\right),$$

for $c_2>0$. The best unconditional Ω result for the Mertens function is

$$M(x)=\Omega_{\pm}(x^{\frac{1}{2}}),$$

and if $\zeta(s)$ has a zero of multiplicity m with $m > 1$ then

$$M(x) = \Omega_{\pm}\left(x^{\frac{1}{2}}(\log x)^{m-1}\right).$$

On the other hand, if the Riemann hypothesis is false, then

$$M(x) = \Omega_{\pm}(x^{\theta-\delta}),$$

where $\theta = \sup_{\rho,\zeta(\rho)=0} \mathrm{Re}(\rho)$ and δ is any positive constant (see Ingham [20]).

1.2 Explicit formulae

An explicit formula is an equation which encapsulates certain arithmetical information and which involves the non-trivial zeros ρ of an L-function.

1.2.1 Ramanujan explicit formula

In 1918 Hardy and Littlewood (see [18, Sect. 2.5] and [48, Sect. 9.8]) published an explicit formula suggested to them by Ramanujan. Under the benign assumption that the non-trivial zeros ρ are all simple, their explicit formula can be stated as follows.

> Let a and b be two positive real numbers such that $ab = \pi$. Let φ and ψ be a pair of suitable cosine reciprocal functions.[1] Let $Z_1(s)$ and $Z_2(s)$ be the Mellin transforms of $\varphi(s)$ and $\psi(s)$, respectively. Then
>
> $$\begin{aligned}\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right) &= \frac{1}{\sqrt{a}}\sum_{\rho} a^{\rho}\frac{Z_1(1-\rho)}{\zeta'(\rho)} \\ &= -\frac{1}{\sqrt{b}}\sum_{\rho} b^{\rho}\frac{Z_2(1-\rho)}{\zeta'(\rho)},\end{aligned} \tag{1.3}$$
>
> provided the series involving ρ are convergent.

If we take $\varphi(x) = \psi(x) = \exp(-x^2)$, then it is easily seen that these functions are cosine reciprocal functions and that

$$Z_1(s) = Z_2(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right).$$

[1] Two functions $f(x)$ and $g(x)$ are cosine reciprocal if

$$\frac{\sqrt{\pi}}{2}f(x) = \int_0^{\infty} g(u)\cos(2ux)du, \quad \frac{\sqrt{\pi}}{2}g(x) = \int_0^{\infty} f(u)\cos(2ux)du.$$

In this case (1.3) becomes

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a^2/n^2}-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-b^2/n^2}=\frac{1}{2\sqrt{a}}\sum_{\rho}a^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}$$
$$=-\frac{1}{2\sqrt{b}}\sum_{\rho}b^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}, \tag{1.4}$$

provided, once again, that the series

$$\sum_{\rho}\alpha^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}$$

is convergent for $\alpha > 0$. Hardy and Littlewood credit Ramanujan for first providing (1.4) and later on for suggesting the generalization (1.3). They do not, however, state the conditions that φ and ψ must satisfy for (1.3) to hold. The arithmetical information is contained in the Möbius function on the left-hand side of (1.3) and (1.4) and the analytic information is encoded in the sums involving the non-trivial zeros on either of the right-hand sides.

In 2013 Dixit [12] gave a one-variable generalization of (1.4). He showed the following result.

If we let a and b be positive reals such that $ab = 1$ and $z \in \mathbb{C}$, then

$$\sqrt{a}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi a^2/n^2}\cos\left(\frac{\sqrt{\pi}az}{n}\right)-\sqrt{b}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi b^2/n^2}\cosh\left(\frac{\sqrt{\pi}bz}{n}\right)$$
$$=-\frac{e^{-\frac{z^2}{8}}}{2\sqrt{\pi b}}\sum_{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{z^2}{4}\right)\pi^{\rho/2}b^{\rho} \tag{1.5}$$

provided the series involving ρ are convergent, and where ${}_1F_1$ denotes the confluent hypergeometric function.

Clearly, if $z = 0$ then (1.5) becomes (1.4).

In [11], Dixit obtained a character analogue of (1.4). To state his result, we recall the following notation of the theory of Dirichlet L-functions. Suppose that χ is a character $\bmod\ q$. The indicator $\mathfrak{h}$ is defined by

$$\mathfrak{h}=\begin{cases}0 & \text{if } \chi \text{ is even},\\ 1 & \text{if } \chi \text{ is odd}.\end{cases} \tag{1.6}$$

The Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi)=\sum_{m=1}^{q}\chi(m)e^{2\pi im/q}.$$

With this in mind, Dixit's second result is as follows:

Let a and b be two positive reals such that $ab = \pi$ and let χ denote a primitive Dirichlet character mod q such that $\chi(-1) = (-1)^{\mathfrak{h}}$. If the non-trivial zeros ρ of $L(s, \chi)$ are all simple then one has

$$a^{\mathfrak{h}+1/2}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1+\mathfrak{h}}}e^{-a^2/(qn^2)} - b^{\mathfrak{h}+1/2}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{1+\mathfrak{h}}}e^{-b^2/(qn^2)}$$
$$= q\frac{\sqrt{\tau(\chi)}}{2\sqrt{a}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\chi)} = -q\frac{\sqrt{\tau(\bar{\chi})}}{2\sqrt{b}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\bar{\chi})} \tag{1.7}$$

provided the series involving ρ are convergent.

Later in [13] one of the authors, Dixit and Zaharescu found the character analogue of (1.5) and in [14] a generalization of (1.5) to Hecke forms.

The transformations in (1.3), (1.4), (1.5) and (1.7) exhibit a transformation of the type $x \to 1/x$, which is an analogue of the Poisson summation formula. These kinds of transformation formulas have broad interest in different branches of mathematics. In this article we establish a class of reciprocal functions, as well as a class of arithmetical functions obtained from a reduced Selberg class, which satisfies the transformation formula mentioned above. At the end of the introduction we provide examples where we obtained the above transformations as special cases. Furthermore, we obtain some new transformations that are not in the literature.

Let us suppose that $A_1 > 0$ and $T > 0$. We define the bracketing condition $\mathcal{B}$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $\zeta(s)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$|\gamma - \gamma'| < \exp(-A_1|\gamma|/\log|\gamma|) + \exp(-A_1|\gamma'|/\log|\gamma'|) \tag{1.8}$$

are included in the same bracket. When a sum over ρ satisfies the bracketing condition $\mathcal{B}$, we will write $\sum_{\rho\in\mathcal{B}} f(\rho)$.

We define the bracketing condition $\mathcal{B}_\chi$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $L(s, \chi)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$|\gamma - \gamma'| < \exp(-A_1|\gamma|/\log|\gamma| + 3) + \exp(-A_1|\gamma'|/\log|\gamma'| + 3) \tag{1.9}$$

are included in the same bracket. Similarly, when a sum over ρ satisfies the bracketing condition $\mathcal{B}_\chi$, we will write $\sum_{\rho\in\mathcal{B}_\chi} f(\rho)$. If we assume that the zeros of $\zeta(s)$ satisfy the bracketing condition $\mathcal{B}$ then one can drop the assumption of convergence of the series in the right-hand sides of (1.3)–(1.5). Likewise, if we assume that the zeros of $L(s, \chi)$ satisfy the bracketing condition $\mathcal{B}_\chi$, then we can drop the assumption of convergence in the right-hand side of (1.7).

The size and the distribution of such bracketings are unknown but their existence is widely accepted. In fact, it is expected that the pairs of zeros $\{\rho, \rho'\}$ that need to be bracketed together in Ramanujan's explicit formula will occur rarely. For results on the correlation of zeros of L-functions, the reader is referred to Montgomery [32], Rudnick and Sarnak [43], Katz and Sarnak [26,27], Murty and Perelli [34], and Murty and Zaharescu [35].

1.2.2 Titchmarsh explicit formula

An explicit formula for the Mertens function was first published in 1951 by Titchmarsh on the assumption of the Riemann hypothesis (see [48, Sect. 14.27]), i.e. let $\rho = \frac{1}{2}+i\gamma$ with $\gamma \in \mathbb{R}$. Specifically,

> On RH and the simplicity of the non-trivial zeros, there exists a sequence T_ν, $\nu \leq T_\nu \leq \nu + 1$, such that
>
> $$M(x) = -2 + \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^\rho}{\rho\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!n\zeta(2n+1)} \tag{1.10}$$
>
> if x is not an integer. If x is an integer, $M(x)$ is to be replaced by
>
> $$M(x) - \tfrac{1}{2}\mu(x).$$

Note that, unlike RH, the assumption that the zeros are all simple is made for convenience. Indeed, this condition can be relaxed and zeros with higher multiplicity can be accommodated at the cost of making the explicit formula much more complicated. Since it is widely believed that all zeros of the Riemann zeta-function are simple we shall operate under this assumption throughout.

In 1991 Bartz (see [3,4]) proved (1.10) unconditionally. A generalization to Cohen–Ramanujan sums of Bartz's results is established in [28] by the first two authors.

1.2.3 Weil explicit formula

The von Mangoldt function is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if} \quad n = p^m \text{ for some } m \in \mathbb{N} \text{ and prime } p, \\ 0, & \text{otherwise.} \end{cases}$$

In 1952 Weil (see [22, Sect. 5.5] and [50]) published an explicit formula for the von Mangoldt function.

> Suppose that f is $\mathcal{C}^\infty$ and compactly supported. Moreover, denote by F its Mellin transform. Then

$$\sum_{\rho} F(\rho) + \sum_{n=1}^{\infty} F(-2n) = F(1) + \sum_{n=1}^{\infty} \Lambda(n) f(n). \tag{1.11}$$

In order to state the main theorems proved in this note, we first need to introduce some further concepts.

1.3 Hankel transformations

Two functions $\varphi(x)$ and $\psi(x)$ are said to be reciprocal under the Hankel transformation of order ν if

$$\varphi(x) = \int_0^{\infty} (ux)^{\frac{1}{2}} J_\nu(ux)\psi(u)du \quad \text{and} \quad \psi(x) = \int_0^{\infty} (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du, \tag{1.12}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν defined by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

The existence of such reciprocity was first shown by Titchmarsh, see [46,47]. In particular he showed the following.

If $\varphi(s)$ is integrable in the sense of Lebesgue and $\nu \geq -\frac{1}{2}$ then

$$\int_0^{a} (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du$$

converges in mean to a function $\psi(x)$ of integrable square in $(0, \infty)$ as $a \to \infty$.

Hankel transformations reduce to Fourier's cosine and sine transforms for $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$, respectively. The Mellin transforms of $\varphi(x)$ and $\psi(x)$ are defined, as usual, by

$$Z_1(s) = \int_0^{\infty} x^{s-1}\varphi(x)dx, \quad Z_2(s) = \int_0^{\infty} x^{s-1}\psi(x)dx. \tag{1.13}$$

Their inverse Mellin transforms are given by

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_1(s)x^{-s}ds, \quad \psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z_2(s)x^{-s}ds. \tag{1.14}$$

The value of c will depend on the nature of the functions φ and ψ.

Definition 1.1 Let $0 < \omega \leq \pi$ and $\alpha < \frac{1}{2}$. If $f(z)$ is such that

(i) $f(z)$ is analytic of $z = re^{i\theta}$ regular in the angle defined by $r > 0$, $|\theta| < \omega$,
(ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\alpha-\epsilon}) & \text{if } \ |z| \text{ is small,} \\ O(e^{-|z|}) & \text{if } \ |z| \text{ is large,} \end{cases} \tag{1.15}$$

for every positive ϵ and uniformly in any angle $|\theta| < \omega$,

then we say that f belongs to the class K and write $f(z) \in K(\omega, \alpha)$.

1.4 The Selberg class

In [44], Selberg introduced a general class $\mathcal{S}$ of L-functions. Let F be an L-function in $\mathcal{S}$ then the completed L-function is defined by

$$\Lambda(s) = Q^s \prod_{i=1}^{d} \Gamma(\alpha_i s + r_i) F(s) \tag{1.16}$$

where $Q > 0$, $\alpha_i > 0$, $r_i \in \mathbb{C}$ with $\mathrm{Re}(r_i) \geq 0$. The degree d_F and conductor q_F are defined by

$$d_F = 2\sum_{j=1}^{d} \alpha_j \quad \text{and} \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{d} \alpha_j^{2\alpha_j}, \tag{1.17}$$

respectively. It is conjectured that the degree d_F and conductor q_F are both integers. For a non-negative integer n, the H-invariants are defined by

$$H_F(n) = 2\sum_{j=1}^{d} \frac{B_n(r_j)}{\alpha_j^{n-1}},$$

where $B_n(x)$ are the familiar n-th Bernoulli polynomials. The first few $B_n(x)$'s are given by

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \ldots.$$

Hence we find that

$$H_F(0) = d_F, \quad H_F(1) = 2\sum (r_j - 1/2), \ldots. \tag{1.18}$$

1.5 Main results

Equipped with these notions our first result is as follows:

Theorem 1.1 *Suppose that F is an element of the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and $H_F(1) = -\nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ defined as above and let x be a positive real. Then there exists a sequence $\{T_l\}$ of positive numbers that satisfies the following.*

(i) *If $q_F = 1$ then*

$$\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty} \sum_{-T_l<\mathrm{Im}(\rho)<T_l} \frac{Z_1(\rho)}{\zeta'(\rho)} x^{\rho} + \sqrt{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 \zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k}. \tag{1.19}$$

(ii) *If $q := q_F \geq 2$ then there exists a primitive Dirichlet character χ mod q with $\chi(-1) = -2\nu$ such that*

$$\sum_{n=1}^{\infty} \mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty} \sum_{-T_l<\mathrm{Im}(\rho)<T_l} \frac{Z_1(\rho)}{L'(\rho, \chi)} x^{\rho} + i^{\frac{1}{2}+\nu} \frac{\sqrt{2\pi}}{\tau(\chi)} \sum_{k=0}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k, \bar{\chi})} \left(\frac{qx}{2\pi}\right)^{-k} + \frac{Z_1(s_0)}{L'(s_0, \chi)} x^{s_0} \tag{1.20}$$

on the assumption that the Riemann hypothesis for Dirichlet L-functions is true and where s_0 denotes a hypothetical Landau–Siegel zero.

Equation (1.19) is reminiscent of the Weil explicit formula except that $\Lambda(n)$ is replaced by $\mu(n)$. Similar formulae due to Berndt [5] and Ferrar (see [16,17], and [47, Sect. 2.9]) for the divisor function $d(n)$ exist as well. Extensions of the Weil explicit formula (1.11) to generalized von Mangoldt functions and other arithmetical functions such as the Liouville λ function can be found in another article by the last two authors [42]. The second result is as follows:

Theorem 1.2 *Suppose that F is an element of the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and $H_F(1) = \nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ defined as above. If a and b are two positive reals such that $ab = 2\pi$, then one has the following.*

(i) *If $q_F = 1$ then*

$$\sqrt{a}\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right) = \frac{1}{\sqrt{a}} \sum_{\rho\in\mathcal{B}} a^{\rho} \frac{Z_1(1-\rho)}{\zeta'(\rho)} = -\frac{1}{\sqrt{b}} \sum_{\rho\in\mathcal{B}} b^{\rho} \frac{Z_2(1-\rho)}{\zeta'(\rho)}. \tag{1.21}$$

(ii) *If $q := q_F \geq 2$ then there exists a primitive Dirichlet character $\chi \mod q$ with $\chi(-1) = -2\nu$ such that*

$$\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)$$
$$=\frac{q^{1/2}\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{Z_1(1-\rho)}{L'(\rho,\chi)}=-\frac{q^{1/2}\sqrt{\tau(\bar{\chi})}}{b^{1/2}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}. \tag{1.22}$$

If one changes (1.15) to the following

$$f(z) = \begin{cases} O(|z|^{-\alpha-\epsilon}) & \text{if} \quad |z| \text{ is small,} \\ O(|z|^{-\beta-\epsilon}) & \text{if} \quad |z| \text{ is large,} \end{cases} \tag{1.23}$$

with $\alpha = 0$ and $\beta > 1$, then Theorem 1.2 would also hold for φ and ψ satisfying the above growth conditions.

One can see the condition $H_F(1) = \nu - \frac{1}{2}$ is necessary. This condition naturally leads us to make the following conjecture.

Conjecture 1.1 *Let F be an element in the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$, $\frac{\pi}{2} < \omega \leq \pi$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal under Hankel transformation of order ν. Then (1.21) holds only when $\nu = -1/2$ and (1.22) holds only when $\nu = \pm 1/2$.*

Remark 1.1 The following special cases are to be noted.

(1) Let $\varphi(x) = \psi(x) = x^{(\nu+1/2)}e^{-\frac{x^2}{2}}$ for $\nu = \pm 1/2$. Clearly, $\varphi, \psi \in K(\omega, a)$. Also

$$Z_1(s) = Z_2(s) = \left(\frac{1}{2}\right)^{(\frac{\nu}{2}-\frac{3}{4})} 2^{\frac{s}{2}}\Gamma\left(\frac{s+\nu+1/2}{2}\right).$$

If we substitute the above values of φ, ψ, Z_1 and Z_2 in (1.22) then we obtain (1.7).

(2) Let $\varphi(x) = e^{-x^2-z^2/2}\cosh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\cos(zx)$. One can see that $\varphi, \psi \in K(\omega, a)$ and that they are reciprocal under cosine transformations, i.e. $\nu = -1/2$. Their Mellin transformations are given by

$$Z_1(s) = \frac{1}{2}e^{-\frac{z^2}{8}}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2},\frac{1}{2};\frac{z^2}{4}\right),$$
$$Z_2(s) = \frac{1}{2}e^{\frac{z^2}{8}}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2},\frac{1}{2};-\frac{z^2}{4}\right).$$

If we substitute the above values of φ, ψ, Z_1 and Z_2 in (1.21) and (1.22) then we obtain (1.5) and [13, Theorem 1.2, part i)], respectively.

(3) Let $\varphi(x) = e^{-x^2-z^2/2}\sinh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\sin(zx)$. One can see that $\varphi, \psi \in K(\omega, a)$ and that they are reciprocal under sine transformations, i.e. $\nu = 1/2$. Their Mellin transformations are given by

$$\Phi(s) = \frac{z}{2}e^{-\frac{z^2}{8}}\Gamma\left(\frac{1+s}{2}\right){}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; \frac{z^2}{4}\right),$$
$$Z_2(s) = \frac{z}{2}e^{\frac{z^2}{8}}\Gamma\left(\frac{1+s}{2}\right){}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; -\frac{z^2}{4}\right).$$

If we substitute the above values of φ, ψ, Φ and Z_2 in (1.22) then we obtain [13, Theorem 1.2, part ii)].

The following corollaries are new transformations in the literature. It is not difficult to find pairs of reciprocal functions and obtain new formulae from (1.21). For instance, one could take the pair of cosine reciprocal functions

$$\varphi(x) = e^{-x}, \quad \psi(x) = \frac{2}{\sqrt{\pi}}\frac{1}{1+x^2},$$

which are in K, and which have Mellin transforms given by

$$Z_1(s) = \Gamma(s), \quad Z_2(s) = \frac{\sqrt{\pi}}{2}\csc\left(\frac{\pi s}{2}\right),$$

valid for $\mathrm{Re}(s) > 0$ and $0 < \mathrm{Re}(s) < 2$, respectively, and obtain the following.

Corollary 1.2 *One has*

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a/n} - 2\sqrt{\frac{b}{\pi}}\sum_{n=1}^{\infty}\frac{n\mu(n)}{n^2+b^2}$$
$$= \frac{1}{a^{1/2}}\sum_{\rho\in\mathcal{B}}a^{\rho}\frac{\Gamma(1-\rho)}{\zeta'(\rho)} = -\frac{1}{2}\sqrt{\frac{\pi}{b}}\sum_{\rho\in\mathcal{B}}\frac{b^{\rho}}{\zeta'(\rho)}\csc\left(\frac{\pi(1-\rho)}{2}\right).$$

However, the symmetry is more striking on the left-hand side when we take a pair of *self-reciprocal* functions. For the coming corollaries a and b will denote two positive real numbers satisfying $ab = 2\pi$ and the non-trivial zeros of $\zeta(s)$ and $L(s,\chi)$ are all assumed to be simple. Here χ denotes the primitive Dirichlet character $\bmod q$.

Corollary 1.3 *Let χ be odd. Then we have*

$$\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\left(\frac{1}{\mathrm{e}^{a\sqrt{2\pi/qn}}-1} - \frac{n}{a}\sqrt{\frac{q}{2\pi}}\right)$$
$$-\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\left(\frac{1}{\mathrm{e}^{b\sqrt{2\pi/qn}}-1} - \frac{n}{b}\sqrt{\frac{q}{2\pi}}\right)$$

$$= \sqrt{\frac{q\tau(\chi)}{2\pi a}} \sum_{\rho \in \mathcal{B}_\chi} \left(\frac{(2\pi)^{1/2} a}{q^{1/2}} \right)^\rho \frac{\Gamma(1-\rho)\zeta(1-\rho)}{L'(\rho,\chi)}. \tag{1.24}$$

Corollary 1.4 *Let χ be even. Then we have the following*

$$\sqrt{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{sech}\left(\sqrt{\frac{\pi}{2}} \frac{a}{n} \right) - \sqrt{b} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{sech}\left(\sqrt{\frac{\pi}{2}} \frac{a}{n} \right)$$
$$= \sqrt{\frac{1}{2\pi a}} \sum_{\rho \in \mathcal{B}} (2^{\frac{3}{2}} \pi^{\frac{1}{2}} a)^\rho \frac{\Gamma(1-\rho)(\zeta(1-\rho, \frac{1}{4}) - \zeta(1-\rho, \frac{3}{4}))}{\zeta'(\rho)} \tag{1.25}$$

as well as

$$\sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \operatorname{sech}\left(\sqrt{\frac{\pi}{2q}} \frac{a}{n} \right) - \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \left(\sqrt{\frac{\pi}{2q}} \frac{b}{n} \right)$$
$$= \sqrt{\frac{q\tau(\chi)}{2\pi a}} \sum_{\rho \in \mathcal{B}_\chi} \left(\frac{2^{\frac{3}{2}} \pi^{\frac{1}{2}} a}{q^{1/2}} \right)^\rho \frac{\Gamma(1-\rho)(\zeta(1-\rho, \frac{1}{4}) - \zeta(1-\rho, \frac{3}{4}))}{L'(\rho,\chi)}, \tag{1.26}$$

where $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function.

Corollary 1.5 *Let $K_\nu(x)$ be the modified Bessel function of second kind. Let $\chi(-1) = -2\nu$. Then for* $\operatorname{Re}(z) > 0$ *we have*

$$\sqrt{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\frac{a^2}{n^2} + z^2 \right)^{-\frac{1}{8}} K_{\frac{1}{4}}\left(z\sqrt{z^2 + \frac{a^2}{n^2}} \right) - \upsilon\sqrt{b} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\frac{b^2}{n^2} + z^2 \right)^{-\upsilon\frac{1}{8}} K_{\frac{1}{4}}\left(z\sqrt{z^2 + \frac{b^2}{n^2}} \right)$$
$$= \frac{1}{\sqrt{2a}} \sum_{\rho \in \mathcal{B}} \left(\frac{a}{2^{1/2}} \right)^\rho \frac{\Gamma(\frac{1-\rho}{2}) K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{\zeta'(\rho)}, \tag{1.27}$$

and for $\nu = \pm 1/2$

$$\frac{a^{1+\nu}\sqrt{\tau(\chi)}}{q^{\frac{1}{2}(\frac{1}{2}+\nu)}} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^{1+\frac{1}{2}+\nu}} \left(z^2 + \frac{a^2}{qn^2} \right)^{\frac{1}{4}(-\nu-1)} K_{\frac{1}{2}(\nu+1)}\left(z\sqrt{z^2 + \frac{a^2}{qn^2}} \right)$$
$$- \frac{a^{1+\nu}\sqrt{\tau(\bar{\chi})}}{q^{\frac{1}{2}(\frac{1}{2}+\nu)}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^{1+\frac{1}{2}+\nu}} \left(z^2 + \frac{b^2}{qn^2} \right)^{\frac{1}{4}(-\nu-1)} K_{\frac{1}{2}(\nu+1)}\left(z\sqrt{z^2 + \frac{b^2}{qn^2}} \right)$$
$$= 2^{\frac{2\nu-1}{4}} \frac{\sqrt{q\tau(\chi)}}{a} \sum_{\rho \in \mathcal{B}_\chi} \left(\frac{a}{(2q)^{1/2}} \right)^\rho \frac{\Gamma(\frac{3}{4} + \frac{1}{2}\nu - \frac{1}{2}\rho) K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho,\chi)}. \tag{1.28}$$

Let us recall that the Weber parabolic cylinder functions $D_n(x)$ are defined by (see Mitra [31])

$$D_n(x) = \frac{\Gamma(\frac{1}{2})2^{\frac{n}{2}}e^{-\frac{1}{4}x^2}}{\Gamma(\frac{1}{2}-\frac{n}{2})}{}_1F_1(-\tfrac{n}{2};\tfrac{1}{2};\tfrac{x^2}{2}) + \frac{\Gamma(-\frac{1}{2})2^{\frac{n}{2}-\frac{1}{2}}e^{-\frac{1}{4}x^2}}{\Gamma(-\frac{n}{2})}{}_1F_1(\tfrac{1}{2}-\tfrac{n}{2};\tfrac{3}{2};\tfrac{x^2}{2})$$

for all reals n and x.

Corollary 1.6 *Let $\chi(-1) = -2\nu$. Then for every $m = 0, 1, 2, \ldots$ we have*

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}D_{4m}\left(\frac{2a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}D_{4m}\left(\frac{2b}{n}\right)\\ &= \frac{2^{2n-1}\sqrt{\pi}}{a^{1/2}}\sum_{\rho\in\mathcal{B}}\frac{a^{\rho}\Gamma(1-\rho)}{\Gamma(\frac{1}{2}(2-4n-\rho))\zeta'(\rho)}{}_2F_1\left(\frac{\frac{1-\rho}{2},\frac{2-\rho}{2}}{\frac{2-4n-\rho}{2}};\frac{1}{2}\right),\end{aligned} \tag{1.29}$$

and for $\chi(-1) = -2\nu$ we have

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}D_{4m+\nu+\frac{1}{2}}\left(\frac{2a}{q^{1/2}n}\right) - \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}D_{4m+\nu++\frac{1}{2}}\left(\frac{2b}{q^{1/2}n}\right)\\ &= 2^{2n-\frac{2}{3+2\nu}}\sqrt{\frac{\pi q\tau(\chi)}{a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)}{\Gamma(\frac{1}{2}(\frac{3}{2}-\nu-4n-\rho))L'(\rho,\chi)}\\ &\quad {}_2F_1\left(\frac{\frac{1-\rho}{2},\frac{2-\rho}{2}}{\frac{1}{2}(\frac{3}{2}-\nu-\rho-4n)};\frac{1}{2}\right),\end{aligned} \tag{1.30}$$

where ${}_2F_1$ is the hypergeometric function.

Corollary 1.7 *One has*

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{a^2}{4n^2}\right)D_{-2}\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{b^2}{4n^2}\right)D_{-2}\left(\frac{b}{n}\right)\\ &= \frac{1}{2^{1/2}a^{1/2}}\sum_{\rho\in\mathcal{B}}(2^{1/2}a)^{\rho}\frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{\zeta'(\rho)}.\end{aligned}$$

If $\nu = \pm\frac{1}{2}$ then

$$\begin{aligned}&\left(\frac{a}{q^{1/2}}\right)^{\nu+1/2}\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{3/2+\nu}}\exp\left(\frac{a^2}{4qn^2}\right)D_{-2\nu-3}\left(\frac{a}{q^{1/2}n}\right)\\ &-\left(\frac{b}{q^{1/2}}\right)^{\nu+1/2}\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{3/2+\nu}}\exp\left(\frac{b^2}{4qn^2}\right)D_{-2\nu-3}\left(\frac{b}{q^{1/2}n}\right)\\ &= \frac{1}{\Gamma(2\nu+3)}\sqrt{\frac{q\tau(\chi)}{2^{\nu+3/2}a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\nu-\rho+\frac{3}{2})\Gamma(\nu+\frac{3}{2}+\frac{\rho}{2})}{L'(\rho,\chi)}.\end{aligned}$$

We remark that [36, p. 266]

$$D_{-1}(z)=\sqrt{\tfrac{\pi}{2}}e^{\frac{1}{4}z^2}\,\mathrm{Erfc}(2^{-1/2}z) \tag{1.31}$$

where Erfc is the complementary error function

$$\mathrm{Erfc}(x)=1-2\pi^{-1/2}\int_0^x e^{-t^2}dt.$$

Corollary 1.8 *For* $\chi(-1)=-1$ *one has*

$$\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\,\mathrm{Erfc}\left(\frac{a}{\sqrt{2qn}}\right)-\sqrt{b\tau(\bar\chi)}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\,\mathrm{Erfc}\left(\frac{b}{\sqrt{2qn}}\right)$$
$$=2\sqrt{\frac{q\tau(\chi)}{a}}\sum_{\rho\in\mathcal{B}}\left(\frac{a}{\sqrt{2q}}\right)^{\rho}\frac{\Gamma(1-\frac{\rho}{2})}{(1-\rho)L'(\rho,\chi)}$$

Straightforward computation shows that

$$\frac{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}{\cosh(x\sqrt{2\pi})}\quad\text{and}\quad\frac{1}{1+2\cosh\left(2x\sqrt{\frac{\pi}{3}}\right)} \tag{1.32}$$

are self-reciprocal Hankel transformations of order $\nu=-1/2$ and

$$\frac{\sinh\left(x\sqrt{\frac{\pi}{2}}\right)}{\cosh(x\sqrt{2\pi})}\quad\text{and}\quad\frac{\sinh\left(x\sqrt{\frac{\pi}{3}}\right)}{2\cosh\left(2x\sqrt{\frac{\pi}{3}}\right)-1} \tag{1.33}$$

are self-reciprocal Hankel transformations of order $\nu=1/2$. In a similar fashion to the above corollaries one can obtain transformation formulas for the functions (1.32) and (1.33). There exist many self-reciprocal Hankel transformations of order $\nu=\pm\frac{1}{2}$ in the literature and a transformation formula can be obtained from each one of them. The functions mentioned in the above Corollaries are well known in literature and have many applications.

Finally, on inspiration coming from (1.4), Hardy and Littlewood [18] found an equivalent condition for the validity of the Riemann hypothesis. This kind of result was first published by Riesz in [41]. The analogues of the conditions for the Dirichlet L-functions and Hecke forms were obtained in [13,14], respectively. The motivation for the coming theorem comes from the following heuristics. Let us suppose that φ and ψ meet the conditions of the previous theorems and that $d_F=q_F=1$. For $y>0$, let us define the functions

$$P_\varphi(y) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right), \quad P_\psi(y) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \psi\left(\frac{y}{n}\right).$$

Now we perform a Maclaurin expansion of φ around $y = 0$ to obtain

$$P_\varphi(y) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=0}^{\infty} \left(\frac{y}{n}\right)^k \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} y^k \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{k+1}} = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \frac{y^k}{\zeta(1+k)},$$

with a similar formula holding for $P_\psi(y)$. The interchange is justified by the fact that φ is in $K(\omega, \alpha)$ so that φ can be written as a convergent Taylor series at 0. The explicit formula (1.21) can be written as

$$a^{\frac{1}{2}} P_\varphi(a) - b^{\frac{1}{2}} P_\psi(b) = -\sum_\rho b^{\rho - \frac{1}{2}} \frac{Z_2(1-\rho)}{\zeta'(\rho)}. \tag{1.34}$$

If we assume the Riemann hypothesis, $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$, and the absolute convergence of

$$\sum_\rho b^{i\gamma} \frac{Z_2(1-\rho)}{\zeta'(\rho)}, \tag{1.35}$$

then the right-hand side of (1.34) is of the form $O(1)$ when $b \to \infty$. Thus, the left-hand side of (1.34) is now $-b^{\frac{1}{2}} P_\psi(b) \ll 1$, or, equivalently

$$\sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} \frac{b^k}{\zeta(1+k)} \ll b^{-\frac{1}{2}}, \tag{1.36}$$

as $b \to \infty$. Seeing how the Riemann hypothesis and the convergence of (1.35) implies the bound (1.36), we will now establish the following theorem which provides an equivalence of the Riemann hypothesis.

Theorem 1.3 *Let us suppose that φ is in $K(\omega, 0)$ and that it is analytic at* 0. *Consider the function*

$$P_\varphi(y) := \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \frac{y^k}{\zeta(1+k)}.$$

One has the following:

(i) *The Riemann hypothesis implies $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$.*
(ii) *If $Z_1(-s)$ has no zeros in the interval $-\frac{1}{2} < \mathrm{Re}(s) \leq 0$, then the estimate $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$ implies the Riemann hypothesis.*

Remark 1.2 If $Z_1(-s)$ had zeros then all the zeros of $\zeta(s)$ would still lie on the critical line except for the zeros that coincide with the zeros of $Z_1(-s)$. In most of the examples that we considered in the corollaries we see that $Z_1(-s)$ has at most finitely many zeros in the region $-\frac{1}{2} < \mathrm{Re}(s) \leq 0$.

2 Preliminary lemmas

We will use the following lemmas to prove our main theorems.

Lemma 2.1 *Let $\varphi, \psi \in L^2(0,\infty)$ be two reciprocal Hankel transforms of order ν. Then*

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}dt, \tag{2.1}$$

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}dt, \tag{2.2}$$

the integrals are mean square integral, $2^{\frac{it}{2}}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})\Phi(\frac{1}{2}+it)$ and $2^{\frac{it}{2}}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})\Psi(\frac{1}{2}+it)$ belong to $L^2(-\infty,\infty)$, and

$$\Phi\left(\frac{1}{2}-it\right) = \Psi\left(\frac{1}{2}+it\right). \tag{2.3}$$

Proof Suppose that φ belongs to $L^2(0,\infty)$. One can see that

$$\int_0^{\infty}\varphi^2(x)dx = \int_{-\infty}^{\infty}\varphi^2(e^x)e^x dx.$$

Hence $F(x) := \varphi(\mathrm{e}^x)\mathrm{e}^{x/2} \in L^2(-\infty,\infty)$. Then from the theory of Fourier transforms, see [47], it follows that

$$Z_1\left(\frac{1}{2}+it\right) = \int_{-\infty}^{\infty} F(x)e^{itx}dx = \int_0^{\infty}\varphi(x)x^{-\frac{1}{2}+it}dx \tag{2.4}$$

exists as a mean square integral for almost all t. Also $Z_1(\frac{1}{2}+it) \in L^2(-\infty,\infty)$ and

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_1\left(\frac{1}{2}+it\right)e^{-ixt}dt. \tag{2.5}$$

The above integral is also a mean square integral. In other words, (2.5) can be written as

$$\varphi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_1\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}dt. \tag{2.6}$$

Similarly, we obtain

$$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_2\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}dt. \tag{2.7}$$

Let us consider two functions Φ and Ψ such that

$$Z_1\left(\frac{1}{2}+it\right)=2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right) \tag{2.8}$$

and

$$Z_2\left(\frac{1}{2}+it\right)=2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right). \tag{2.9}$$

Replacing the above equalities in (2.6) and (2.7) we obtain (2.1) and (2.2). Now we complete the proof by proving (2.3). For all $n\geq -1/2$, $y>0$ and $x>0$ we have

$$\int_0^y \sqrt{ux}J_\nu(ux)du=\frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left(\begin{matrix}\frac{\nu}{2}+\frac{3}{4} \\ \frac{\nu}{2}+\frac{7}{4},\nu+1\end{matrix};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}. \tag{2.10}$$

The right-hand side of (2.10) belongs to $L^2(0,\infty)$ and the Mellin transform is given by

$$\int_0^\infty \frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left(\begin{matrix}\frac{\nu}{2}+\frac{3}{4} \\ \frac{\nu}{2}+\frac{7}{4},\nu+1\end{matrix};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}x^{-\frac{1}{2}+it}dt=\frac{2^{it}y^{\frac{1}{2}-it}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})}{(\frac{1}{2}-it)\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{it}{2})}. \tag{2.11}$$

We also have that $\varphi\in L^2(0,\infty)$ and its Mellin transform is given by (2.4). Hence by an analogue of Plancherel's theorem for Mellin transform (see [47, Theorem 72]) we have

$$\int_0^\infty \varphi(x)\frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left(\begin{matrix}\frac{\nu}{2}+\frac{3}{4} \\ \frac{\nu}{2}+\frac{7}{4},\nu+1\end{matrix};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2^{it} y^{\frac{1}{2}-it} \Gamma(\frac{\nu}{2} + \frac{1}{2} + \frac{it}{2})}{(\frac{1}{2} - it)\Gamma(\frac{\nu}{2} + \frac{1}{2} - \frac{it}{2})} Z_1 \left(\frac{1}{2} - it\right) dt \tag{2.12}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2} + \frac{1}{2} + \frac{it}{2}\right) \Phi\left(\frac{1}{2} - it\right) \frac{y^{\frac{1}{2}-it}}{\frac{1}{2} - it} dt,$$

in the ultimate step we have used (2.8). Now from (1.12) we have

$$\psi(u) = \lim_{a\to\infty} \int_0^a \sqrt{ux} J_\nu(ux)\varphi(x)dx,$$

where the limit converges in the sense of mean square. Therefore, for all $x > 0$, $y > 0$ and $\nu \geq -1/2$ we find that

$$\int_0^y \psi(u)\mathrm{d}u = \lim_{a\to\infty} \int_0^y \int_0^a \sqrt{ux} J_\nu(ux)\varphi(x)dxdu \tag{2.13}$$

$$= \lim_{a\to\infty} \int_0^a \varphi(x) \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left(\begin{matrix} \frac{\nu}{2} + \frac{3}{4} \\ \frac{\nu}{2} + \frac{7}{4}, \nu + 1 \end{matrix} ; -\frac{x^2y^2}{4} \right)}{2^\nu(\nu + 3/2)\Gamma(\nu + 1)} dx$$

$$= \int_0^\infty \varphi(x) \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left(\begin{matrix} \frac{\nu}{2}\upsilon + \frac{3}{4} \\ \frac{\nu}{2} + \upsilon\frac{7}{4}, \nu+1 \end{matrix} ; -\frac{x^2y^2}{4} \right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)} dx.$$

The left-hand side of (2.13) is

$$\int_0^y \psi(u)\mathrm{d}u = \frac{1}{2\pi} \int_0^y \int_{-\infty}^{\infty} 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2} + \frac{1}{2} + \frac{it}{2}\right) \Psi\left(\frac{1}{2} + it\right) u^{-\frac{1}{2}-it} dtdu \tag{2.14}$$

$$= \frac{1}{2\pi} \left(\lim_{X\to\infty} \int_0^y \int_0^X + \lim_{Y\to\infty} \int_0^y \int_{-Y}^0 \right) 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2} + \frac{1}{2} + \frac{it}{2}\right) \Psi\left(\frac{1}{2} + it\right) u^{-\frac{1}{2}-it} dtdu$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right) \Psi\left(\frac{1}{2}+it\right) \frac{y^{\frac{1}{2}-it}}{\frac{1}{2}-it} dt.$$

By (2.13) we see the right-hand sides of (2.12) and (2.14) are equal. Hence from [47, Theorem 32] we conclude that

$$\Phi\left(\frac{1}{2} - it\right) = \Psi\left(\frac{1}{2} + it\right),$$

and this ends the proof. □

Lemma 2.2 *Let φ and ψ be reciprocal functions under Hankel transformation of order ν defined in (1.12). Let $\varphi, \psi \in K(\omega, \alpha)$. Then there exist two regular functions Φ and Ψ such that*

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Phi(s) x^{-s} ds, \tag{2.15}$$

$$\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Psi(s) x^{-s} ds \tag{2.16}$$

for $c > 0$. Moreover, Φ and Ψ satisfy the following:

(1) $\Phi(s) = \Psi(1 - s)$ *for all $s \in \mathbb{C}$,*
(2) $\Psi(s) = O(\mathrm{e}^{(\frac{\pi}{4} - \omega + \epsilon)|t|})$ *for every positive ϵ and uniformly for $\sigma \in \mathbb{R}$.*

Remark 2.1 If φ and ψ satisfy (1.23), then conditions (1) and (2) in Lemma 2.2 hold uniformly for $\alpha < \sigma < \beta$.

Proof Since $\varphi, \psi \in K(\omega, \alpha)$, the right-hand sides of (1.13) are absolutely convergent. Then it follows that $Z_1(s)$ and $Z_2(s)$ are regular in $\alpha < \sigma$. Let

$$Z_1(s) = 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Phi(s), \tag{2.17}$$

and

$$Z_2(s) = 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Psi(s). \tag{2.18}$$

Hence by (2.8) and (2.9) of Lemma (2.1), we deduce that $\Phi(s)$ and $\Psi(s)$ also regular in this region. One can see $\varphi, \psi \in L^2$. Therefore, from (2.3) of Lemma (2.1), $\Psi(s) = \Phi(1-s)$ for $\sigma = 1/2$. Thus, by analytic continuation $\Psi(s) = \Phi(1-s)$ for $\alpha < \sigma$ and hence for all $s \in \mathbb{C}$. Also (2.15) and (2.16) hold for $\alpha < c = \sigma$. Let us consider the line along any radius vector r and angle θ, where $|\theta| < \omega$. Then by Cauchy's theorem we can deform the integral (1.13) to

$$Z_1(\sigma + it) = \int_0^\infty r^{\sigma+it-1} e^{i\theta(\sigma+it)} \varphi(re^{i\theta}) dr,$$

where $\theta, t > 0$. Therefore,

$$|Z_1(\sigma+it)| = e^{-\theta t} \left| \int_0^\infty r^{\sigma+it-1} e^{i\theta(\sigma+it-1)} \varphi(re^{\theta}) dr \right| \le e^{-\theta t} \int_0^\infty r^{\sigma-1} |\varphi(re^{\theta})| dr \ll e^{-|\theta||t|}, \tag{2.19}$$

since $\varphi \in K(\omega, \alpha)$. By Stirling's formula for $\Gamma(\sigma + it)$ in the vertical strip $p \le \sigma \le q$ we have

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi|t|}{2}} \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.20}$$

as $|t| \to \infty$. Now combining (2.17), (2.19) and (2.20) we get

$$\Psi(1-s) = \Phi(s) \ll e^{(\frac{\pi}{4}-|\theta|)|t|} \ll e^{(\frac{\pi}{4}-\omega+\eta)|t|}, \tag{2.21}$$

for every positive η. This proves the Lemma. □

Remark 2.2 For self-reciprocal Hankel transformation functions, similar results of Lemma 2.1 and 2.2 were obtained in [19] for a vertical strip.

The following result is Theorem 3 from Kaczorowski and Perelli [23].

Lemma 2.3 *Let $F \in \mathcal{S}$. Suppose that $d_F = 1$ and $\mathrm{Re}(H_F(1))$ is either 0 or 1. If $q_F = 1$ then $F(s) = \zeta(s)$. If $q_F \ge 2$ then there exists a primitive Dirichlet character χ mod q_F with $\chi(-1) = -(2\,\mathrm{Re}(H_F(1)) + 1)$ such that $F(s) = L(s + i\,\mathrm{Im}(H_F(1)), \chi)$.*

Remark 2.3 It is worthwhile to mention pertinent observations which motivated the authors to study the case $d_F = 1$. The following results are due to Conrey and Ghosh [8] and Kaczorowski and Perelli [23–25].

(1) One has $d_F = 0$ precisely when $F = 1$.
(2) There is no function $F \in S$ with $0 < d_F < 1$.
(3) There is no function $F \in S$ with $1 < d_F < 2$.

The following results due to Montgomery [33], Ramachandra and Balasubramanian [2,39,40] will enable us to prove Theorem 1.1 with $d_F = q_F = 1$ without the assumption of the Riemann hypothesis.

Lemma 2.4 *For any given $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon)$ such that if $T \ge T_0$ then the following holds: between T and $2T$ there exists a t for which*

$$|\zeta(\sigma \pm it)|^{-1} < c_1 t^{\varepsilon}$$

for $-1 \le \sigma \le 2$ with an absolute constant $c_1 > 0$.

For the case where $q_F > 1$, the analogues results are due to Soundararajan [45], Lamzouri [29]. However, this latter depends on the truth of the Riemann hypothesis for Dirichlet L-functions.

Lemma 2.5 *Assume the Riemann hypothesis for Dirichlet L-functions. For any given $\varepsilon > 0$ and primitive Dirichlet character χ mod q there exists a $T_0 = T_0(\varepsilon, q)$ such that if $T \ge T_0$ then the following holds: between T and $2T$ there exists a real number t for which*

$$|L(\sigma \pm it, \chi)|^{-1} < c(q)t^{\varepsilon}$$

for $-1 \le \sigma \le 2$ with an absolute constant $c(q) > 0$.

An intermediate result we will be using is due to Ahlgren et al. [1].

Lemma 2.6 *If χ is a primitive character of conductor N and k is an integer ≥ 2 such that $\chi(-1) = (-1)^k$ then one has*

$$\frac{(k-2)!N^{k-2}\tau(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}} L(k-1, \bar{\chi}) = L'(2-k, \chi). \tag{2.22}$$

3 Proof of Theorem (1.1)

(i) Let F be a Selberg L-function of degree $d_F = 1$ and conductor $q_F = 1$. Then by Lemma (2.3) we see that $F(s) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function. Therefore, there is only one gamma factor in the completed Selberg L-function of F for which $r_j = 0$ and $\lambda_j = 1/2$. From (1.18) we see that $H_F(1) = -1$ when $r_j = 0$ and hence $\nu = 1/2$. Therefore, $\varphi, \psi \in K(\omega, \alpha)$ is a pair of reciprocal sine transformations. Now

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) &= \frac{1}{2\pi i}\sum_{n=1}^{\infty} \mu(n) \int_{\lambda-i\infty}^{\lambda+i\infty} Z_1(s)\left(\frac{x}{n}\right)^s ds \\ &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} Z_1(s)x^s\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)ds. \end{aligned} \tag{3.1}$$

By Lemma (2.2) $Z_1(s) \ll e^{(-\omega+\eta)|t|}$ for every positive η. For $1 < \lambda < 2$ the sum inside the above integral is absolutely convergent. Therefore, the far right-hand side of above equalities is absolutely convergent, which justifies the interchange of the summation and integration. Recall the Dirichlet series valid for $\mathrm{Re}(s) > 1$ of the Möbius function

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

From (2.8) we find that the simple poles of $Z_1(s)$ are at $s = -2k+1$ for $k = 0, 1, 2, \ldots$. For $1 < \lambda < 2$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [c - iT, c + iT, \lambda + iT, \lambda - iT]$ where $T > 0$. Therefore, by residue theorem

$$\frac{1}{2\pi i}\oint_{\Omega}\frac{Z_1(s)}{\zeta(s)}x^s ds = \sum_{-T<\mathrm{Im}(\rho)<T}\lim_{s\to\rho}(s-\rho)\frac{Z_1(s)}{\zeta(s)}x^s = \sum_{-T<\mathrm{Im}(\rho)<T}\frac{Z_1(\rho)}{\zeta'(\rho)}x^{\rho}. \tag{3.2}$$

The functional equation of $\zeta(s)$ is given by

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s). \tag{3.3}$$

From Lemma (2.2) we have

$$Z_1(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma(1-\frac{s}{2})} Z_2(1-s). \tag{3.4}$$

Hence by using (3.3), (3.4) and the duplication of the gamma function we find

$$\int_{c-iT}^{c+iT} \frac{Z_1(s)}{\zeta(s)} x^s ds = \sqrt{2\pi} \int_{c-iT}^{c+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds. \tag{3.5}$$

Now we consider the positive-oriented contour Ω' with sides $[-N-\frac{1}{2}-iT, c-iT]$, $[c-iT, c+iT]$, $[c+iT, -N-\frac{1}{2}+iT]$ and $[-N-\frac{1}{2}+iT, -N-\frac{1}{2}-iT]$. The poles of the integrand of the right-hand side integral of (3.5) are at $k = -1, -2, -3, \ldots$. By the residue theorem we have

$$\frac{\sqrt{2\pi}}{2\pi i} \oint_{\Omega'} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds = \sqrt{2\pi} \sum_{k=1}^{N} \frac{(-1)^k Z_2(1+k)}{(k!)^2 \zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k}. \tag{3.6}$$

Stirling's formula in exact form reads (see [10, p. 47])

$$\Gamma(s) = \sqrt{2\pi} e^{-s} s^{s-\frac{1}{2}} \exp(O(|s|^{-1})). \tag{3.7}$$

Therefore, by Lemma (2.2) and Eq. (3.7) we have

$$\begin{aligned} &\int_{-N-\frac{1}{2}-iT}^{-N-\frac{1}{2}+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds \ll \\ &\int_{-T}^{T} \left(\frac{x}{2\pi}\right)^{-N-\frac{1}{2}} \frac{e^{2(N+1)-2(N+1)\log(\sqrt{t^2+(N+1/2)^2})}}{e^{(\pi+\omega+\eta)|t|}} dt, \end{aligned} \tag{3.8}$$

which tends to zero as $N \to \infty$ for any fixed T. Combining (3.6) and (3.8) we find

$$\begin{aligned} &\sqrt{2\pi} \int_{c-iT}^{c+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds \\ &= \sqrt{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 \zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k} \\ &\quad + \sqrt{2\pi} \left(\int_{-\infty-iT}^{c-iT} + \int_{-\infty+iT}^{c+iT} \right) \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds. \end{aligned} \tag{3.9}$$

Similarly, with (3.8) we have

$$\int_{-\infty\pm iT}^{c\pm iT}\left(\frac{x}{2\pi}\right)^{s}\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}ds \ll \int_{-\infty}^{c}\left(\frac{x}{2\pi}\right)^{\sigma}\frac{e^{1-2\sigma+(2\sigma-1)\log(\sqrt{T^2+\sigma^2})}}{e^{(\pi+\omega+\eta)T}}d\sigma$$
$$\ll \frac{1}{e^{(\pi+\omega+\eta)T}}. \tag{3.10}$$

Now by Lemmas (2.2) and (2.4) we have

$$\int_{c\pm iT}^{\lambda\pm iT}\frac{Z_1(s)}{\zeta(s)}x^s ds \ll T^{\epsilon}e^{(-\omega+\eta)T}, \tag{3.11}$$

where T runs through a sequence $\{T_l\}$ with $T_l > T_0(\epsilon)$. Here ϵ and η are any positive numbers. Now combine (3.1), (3.2), (3.5), (3.6), (3.8) and (3.11) to conclude

$$\sum_{n=1}^{\infty}\mu(n)\varphi\left(\frac{n}{x}\right)=\lim_{l\to\infty}\sum_{-T_l<\mathrm{Im}(\rho)<T_l}\frac{Z_1(\rho)}{\zeta'(\rho)}x^{\rho}+\sqrt{2\pi}\sum_{k=1}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k}.$$

This proves part (i) of Theorem 1.1.

(ii) In this case we consider that F is an L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma (2.3) we find $F(s) = L(s,\chi)$ for some Dirichlet primitive character $\mod q_F$. Therefore, the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $r_j = 1/2$. Since ν is real then $\mathrm{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$. By Lemma (2.2) we know that $\Phi(s)$ is analytic on the whole complex plane. Therefore, the poles of $Z_1(s)$ are at the poles of $\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)$. If $\nu = -1/2$ then $s = 0$ is a pole $Z_1(s)$. For the sake of brevity we will prove the case where χ is a even character $\mod q_F$; that is, when $\nu = 1/2$. The other case is handled in a similar fashion. In this case $Z_1(s)$ is analytic for $\mathrm{Re}(s) > -1$. Arguing as in part (i) we have

$$\sum_{n=1}^{\infty}\mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{Z_1(s)}{L(s,\chi)}x^s ds. \tag{3.12}$$

Consider the positively oriented contour Ω mentioned in part (i). By the residue theorem one can find

$$\frac{1}{2\pi i}\oint_{\Omega}\frac{Z_1(s)}{L(s,\chi)}x^s ds = \frac{Z_1(0)}{L'(0,\chi)} + \sum_{-T<\mathrm{Im}(\rho)<T}\frac{Z_1(\rho)}{L'(\rho,\chi)}x^{\rho}, \tag{3.13}$$

where the ρs denote the non-trivial zeros $L(s,\chi)$, assumed to be simple for notational convenience. If there is a Landau–Siegel zero (see Sect. 14 of [9]) at $s = s_0$ then we would have to add the extra term

$$\operatorname*{res}_{s=0} \frac{Z_1(s)}{L(s,\chi)} x^s = \frac{Z_1(s_0)}{L'(s_0,\chi)} x^{s_0}.$$

We note that this hypothetical zero is real and simple. Moreover, in [7] it was proved that for a conductor q up to 200000 there are no Landau–Siegel zeros. Using the functional equation of Lemma (2.2) and the relation in Lemma (2.6) we find that

$$\frac{Z_1(0)}{L'(0,\chi)} = \frac{\sqrt{2\pi}}{\tau(\chi)} \frac{Z_2(1)}{L(1,\bar{\chi})}. \tag{3.14}$$

Proceeding as in the proof of part (i) we have

$$\begin{aligned}\int_{c-iT}^{c+iT} \frac{Z_1(s)}{L(s,\chi)} x^s ds &= \frac{\sqrt{2\pi}}{\tau(\chi)} \int_{c-iT}^{c+iT} \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})} ds \qquad (3.15)\\ &= \frac{\sqrt{2\pi}}{\tau(\chi)} \sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k,\bar{\chi})} \left(\frac{qx}{2\pi}\right)^{-k}\\ &\quad + \frac{\sqrt{2\pi}}{\tau(\chi)} \left(\int_{-\infty-iT}^{c-iT} + \int_{-\infty+iT}^{c+iT} \right) \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})} ds.\end{aligned}$$

Using Lemma (2.2) and Eq. (3.7) we obtain the bounds for $\int_{-\infty-iT}^{c-iT}$ and $\int_{-\infty+iT}^{c+iT}$ of the form (3.10). Using Lemmas (2.2) and (2.5) we obtain the bound for the horizontal integral of (3.13) which is of the form (3.11). Combining (3.12)–(3.15) we conclude the proof.

4 Proof of Theorem (1.2) and corollaries

(i) By repeating a similar argument as in the previous proof we deduce that if $d_F = q_F = 1$ then $F(s) = \zeta(s)$. This case is already sketched in [18] and the missing ingredient comes from the definition of the K class which allows us to get rid of the far left and horizontal integrals in the path of integration.
(ii) In this case we consider F to be a Selberg L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma (2.3) we find $F(s) = L(s,\chi)$ for some Dirichlet primitive character mod q_F. Therefore, the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $1/2$. Since ν is real we have $\mathrm{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$.
Suppose $H_F = -1$, then $\nu = -1/2$ and χ is an even primitive Dirichlet character mod q_F. Therefore, $\varphi, \psi \in K(\omega,\alpha)$ is a pair of cosine reciprocal functions. For $1 < \lambda < 1+\delta$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [\lambda - iT, \lambda + iT, c + iT, c - iT]$ where $T > 0$. Recall that the functions Z_1 and

Z_2 both have a simple pole at $s = 0$. Hence from (2.17) and (2.18) we find that Φ and Ψ are analytic at $s = 0$. Furthermore, by the residue theorem

$$\frac{1}{2\pi i}\oint_{\Omega} x^{-s} Z_1(s)ds = \operatorname*{res}_{s=0} x^{-s} Z_1(s) = 2^{3/4}\Phi(0),$$

and

$$\frac{1}{2\pi i}\oint_{\Omega} x^{-s} Z_2(s)ds = \operatorname*{res}_{s=0} x^{-s} Z_2(s) = 2^{3/4}\Psi(0).$$

By the use of the bound in Lemma (2.2) and Stirling's formula for $\Gamma(s)$ the integrals along the horizontal lines of the contour Ω tend to zero as $T \to \infty$. Since (2.15) and (2.16) hold for $\lambda > 1$ we have the following cases

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} Z_k(s)ds = \begin{cases} \varphi(x) - 2^{3/4}\Phi(0) & \text{if} \quad k = 1, \\ \psi(x) - 2^{3/4}\Psi(0) & \text{if} \quad k = 2. \end{cases} \tag{4.1}$$

Let $q_F := q$. If χ is an even primitive character of modulus q then $L(s, \chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{q^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\frac{1}{L(s,\bar{\chi})}$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (2.17), (2.18) and the functional equation of Φ and Ψ in Lemma (2.2), then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds. \tag{4.2}$$

By absolute convergence, with $c = \mathrm{Re}(s) < 0$, we may write

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}n}\right)^{-s}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \frac{2^{3/4}\Phi(0)}{L(1,\chi)},\end{aligned}$$

where we have used the case $k = 1$ of (4.1). Similarly, with $\lambda = \mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s\frac{Z_2(1-s)}{L(s,\bar\chi)}ds\\
&=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\tau(\bar\chi)b^s}{(2\pi)^{1/2}q^{s/2}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n^s}Z_2(1-s)ds\\
&=\frac{\tau(\bar\chi)b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{1-\lambda-i\infty}^{1-\lambda+i\infty}\left(\frac{b}{q^{1/2}n}\right)^{-w}Z_2(w)dw\\
&=\frac{\tau(\bar\chi)b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)-\frac{\tau(\bar\chi)b}{(2\pi)^{1/2}q^{1/2}}\frac{2^{3/4}\Psi(0)}{L(1,\bar\chi)},
\end{aligned}$$

by making the change $w = 1 - s$ and using the case $k = 2$ of (4.1). Now, we may use either side of (4.2) to evaluate the residues:

- for the non-trivial zeros ρ of $L(s, \chi)$ which we assume are all simple, we have

$$\sum_{\rho}\operatorname*{res}_{s=\rho}\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s\frac{Z_2(1-s)}{L(s,\bar\chi)}=\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^\rho\frac{Z_2(1-\rho)}{L'(\rho,\bar\chi)};$$

- at $s = 1$ we have a simple pole coming from the $Z_2(1 - s)$ function

$$\operatorname*{res}_{s=1}\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s\frac{Z_2(1-s)}{L(s,\bar\chi)}=-\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\frac{b}{q^{1/2}}\frac{2^{3/4}\Psi(0)}{L(1,\bar\chi)};$$

- at $s = 0$ we have a trivial and simple zero of $L(s, \bar\chi)$ and we know that $Z_2(1 - s)$ is analytic and non-zero, so

$$\operatorname*{res}_{s=0}\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s\frac{Z_2(1-s)}{L(s,\bar\chi)}=\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\frac{Z_2(1)}{L'(0,\bar\chi)}=\frac{2^{3/4}\Phi(0)}{L(1,\chi)},$$

where we have used Lemma (2.6) with $N = q$ and $k = 2$ in the last equality. Consequently, by the residue theorem we have

$$\begin{aligned}
&\frac{\tau(\bar\chi)b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)-\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right)\\
&=\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^\rho\frac{Z_2(1-\rho)}{L'(\rho,\bar\chi)}.
\end{aligned}$$

Multiplying both sides by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $q^{1/2} = \sqrt{\tau(\chi)\tau(\bar{\chi})}$ we have the desired result for even characters

$$\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)$$
$$= -q^{1/2}\frac{\sqrt{\tau(\bar{\chi})}}{b^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}. \tag{4.3}$$

We note that if we had used the other side of (4.2) instead, then the result would have been

$$\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)$$
$$= q^{1/2}\frac{\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{Z_1(1-\rho)}{L'(\rho,\chi)}. \tag{4.4}$$

We denote by $\rho = \beta + i\gamma$ a non-trivial zero of $L(s,\bar{\chi})$ and we choose $T > 0$ to tend to infinity through values such that $|T-\gamma| > \exp(-A_1|\gamma|/\log|\gamma| + 3)$ for every ordinate γ of a zero of $L(s,\chi)$. Using

$$\log|L(s,\chi)| \geqslant \sum_{|t-\gamma|\leqslant 1}\log|t-\gamma| + O(\log(qt))$$

yields

$$\log|L(\sigma+iT,\chi)| \geqslant -\sum_{|T-\gamma|\leqslant 1}A_1\gamma/\log\gamma + O(\log qT) > -A_\chi T, \tag{4.5}$$

where $A_\chi < \omega$ if A_1 is small enough, and $T > T_0$. Since the main technique behind the proofs of explicit formulae is contour integration, this will enable us to make unwanted horizontal integrals tend to zero as $T \to \infty$ through the above values. To prove that indeed these horizontal integrals tend to zero as $T \to \infty$ for the chosen values we note that from (4.5) we obtain

$$\frac{1}{|L(1-s,\chi)|} \ll \exp(A_\chi T)$$

where $A_\chi < \omega$. Then by Lemma (2.2) and Stirling's formula for $\Gamma(s)$ one gets

$$\frac{1}{2\pi i}\int_{\lambda-iT}^{c-iT}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds \ll \exp\left((A_\chi-\omega+\epsilon)|t|\right) \to 0$$

for each $\epsilon > 0$. This could alternatively be proved by using Remark 2.1. The other horizontal integral is dealt with similarly.

Let us now consider $H_F = 0$, then $\nu = 1/2$ and χ is an odd primitive Dirichlet character $\bmod\ q_F$. Therefore, $\varphi, \psi \in K(\omega, 0, -\delta)$ is a pair of sine reciprocal functions. Note Z_1 and Z_2 are both analytic at $s = 1$ hence Φ and Ψ both analytic at $s = 1$. Then by the functional equation in (2.2) we see Φ and Ψ are both analytic at $s = 0$. Therefore, both Z_1 and Z_2 are analytic at $s = 0$. Similarly, as (4.1) we find

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} Z_k(s)\mathrm{d}s = \begin{cases} \varphi(x) & \text{if } k = 1, \\ \psi(x) & \text{if } k = 2. \end{cases} \tag{4.6}$$

Let $q := q_F$. If χ is an odd, primitive and non-principal character of $\bmod\ q$ then $L(s, \chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{iq^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{1}{L(s,\bar{\chi})},$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (2.8), (2.9) and the functional equation of Φ and Ψ in Lemma (2.2), then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds.$$

By absolute convergence with $\mathrm{Re}(s) = c$ we can change summation and integration to obtain

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds \\
&= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}n}\right)^{-s}Z_1(s)ds \\
&= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right),
\end{aligned} \tag{4.7}$$

where in ultimate step we have used (4.6) with $k = 1$. Moreover, also by absolute convergence with $\mathrm{Re}(s) = \lambda$, we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds\\
&=\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\left(\frac{b}{q^{1/2}}\right)^{s}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^s}Z_2(1-s)ds\\
&=\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{1-\lambda-i\infty}^{1-\lambda+i\infty}\left(\frac{b}{q^{1/2}n}\right)^{-w}Z_2(w)dw\\
&=\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right),
\end{aligned}
$$

where we have made the change $w = 1 - s$. A similar reasoning as the one we used for even primitive characters shows that the contribution from the horizontal integrals of this contour will tend to zero as well. Next, we compute the residues

- for the non-trivial zeros ρ one has

$$
\sum_{\rho}\operatorname*{res}_{s=\rho}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}=\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
$$

By the residue theorem one has

$$
\begin{aligned}
&\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)-\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right)\\
&=\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
\end{aligned}
$$

Multiplying by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $\sqrt{\tau(\chi)\tau(\bar{\chi})} = iq^{1/2}$ one has

$$
\begin{aligned}
&\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right)-\sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)\\
&=-\frac{q^{1/2}}{b^{1/2}}\sqrt{\tau(\bar{\chi})}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)}{L'(\rho,\bar{\chi})}Z_2(1-\rho), \qquad (4.8)
\end{aligned}
$$

and this proves the theorem.

Proof of Corollary 1.3 By taking $\nu = \frac{1}{2}$ so that $\chi(-1) = -2\frac{1}{2} = -1$, and choosing

$$\varphi(x) = \frac{1}{e^{\sqrt{2\pi}x} - 1} - \frac{1}{\sqrt{2\pi}x},$$

we have

$$\begin{aligned}\psi(x) &= \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(ux)\varphi(u)du \\ &= -\frac{1}{2} - \frac{1}{\sqrt{2\pi}x} + \frac{1}{2}\coth\left(\sqrt{\frac{\pi}{2}}x\right) = \frac{1}{e^{\sqrt{2\pi}x} - 1} - \frac{1}{\sqrt{2\pi}x} = \varphi(x).\end{aligned}$$

We note that $\varphi, \psi \in K$. The Mellin transform is given (see Sect. 9.12 of [47] and Eq. (2.7.1) of [48])

$$Z_i(s) = \int_0^\infty x^{s-1}\left(\frac{1}{e^{\sqrt{2\pi}x} - 1} - \frac{1}{\sqrt{2\pi}x}\right)dx = (2\pi)^{-\frac{1}{2}s}\Gamma(s)\zeta(s),$$

for $0 < \mathrm{Re}(s) < 1$ and $i = 1, 2$. We note that

$$Z_i(1-\rho) = (2\pi)^{-\frac{1}{2}(1-\rho)}\Gamma(1-\rho)\zeta(1-\rho).$$

By plugging these into (1.22) we obtain

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n}\left(\frac{1}{e^{a\sqrt{2\pi/q}n} - 1} - \frac{n}{a}\sqrt{\frac{q}{2\pi}}\right) \\ &- \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^\infty \frac{\bar{\chi}(n)\mu(n)}{n}\left(\frac{1}{e^{b\sqrt{2\pi/q}n} - 1} - \frac{n}{b}\sqrt{\frac{q}{2\pi}}\right) \\ &= \sqrt{\frac{q\tau(\chi)}{2\pi a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{(2\pi)^{1/2}a}{q^{1/2}}\right)^\rho \frac{\Gamma(1-\rho)\zeta(1-\rho)}{L'(\rho,\chi)},\end{aligned}$$

as it was to be shown. □

Proof of Corollary 1.4 First take χ to be even, i.e. $1 = \chi(-1) = -2\nu$ so that $\nu = -\frac{1}{2}$. Choose $\varphi(x) = \mathrm{sech}(\frac{1}{\sqrt{2}}\sqrt{\pi}x)$. We verify that this is cosine reciprocal by noting that

$$\psi(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_{-\frac{1}{2}}(ux)\varphi(u)du = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(ux)\varphi(u)du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(ux)\,\mathrm{sech}(\tfrac{1}{\sqrt{2}}\sqrt{\pi}u)du = \mathrm{sech}(\tfrac{1}{\sqrt{2}}\sqrt{\pi}x) = \varphi(x),$$

and that $\varphi, \psi \in K$. The Mellin transform is given (see entry 6.1 of [36]) by

$$Z_i(s) = 2^{1-\frac{3}{2}s}\pi^{-\frac{s}{2}}\Gamma(s)(\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4}))$$

for $\mathrm{Re}(s) > 0$ and $i = 1, 2$. Plugging this into (1.22) we obtain

$$\begin{aligned}
&\sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n}\,\mathrm{sech}\left(\sqrt{\frac{\pi}{2q}}\frac{a}{n}\right) - \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n}\left(\sqrt{\frac{\pi}{2q}}\frac{b}{n}\right)\\
&= \sqrt{\frac{q\tau(\chi)}{2\pi a}} \sum_{\rho\in\mathcal{B}_\chi} \left(\frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}a}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)(\zeta(1-\rho, \frac{1}{4}) - \zeta(1-\rho, \frac{3}{4}))}{L'(\rho, \chi)}.
\end{aligned}$$

Next, take the same choice of φ and plug it into (1.21) so that

$$\begin{aligned}
&\sqrt{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\,\mathrm{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right) - \sqrt{b} \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\,\mathrm{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right)\\
&= \sqrt{\frac{1}{2\pi a}} \sum_{\rho\in\mathcal{B}} (2^{\frac{3}{2}}\pi^{\frac{1}{2}}a)^{\rho} \frac{\Gamma(1-\rho)(\zeta(1-\rho, \frac{1}{4}) - \zeta(1-\rho, \frac{3}{4}))}{\zeta'(\rho)}
\end{aligned}$$

and this ends the proof. □

Proof of Corollary 1.4 In [38] it is shown that for $\mathrm{Re}(a) > 0$ one has

$$x^{\frac{1}{2}+\mu}(x^2 + a^2)^{\frac{1}{4}(-\mu-1)} K_{\frac{1}{2}(\mu+1)}(a\sqrt{x^2 + a^2})$$

is Hankel reciprocal with respect to μ and that the Mellin transform is given by

$$\begin{aligned}
\phi_\mu(s) &= \int_0^\infty x^{s+\mu-\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{4}(\mu+1)} K_{\frac{1}{2}(\mu+1)}(a\sqrt{x^2 + a^2})dx\\
&= 2^{\frac{1}{2}s+\frac{1}{2}\mu-\frac{3}{4}}\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\mu + \tfrac{1}{4})K_{-\frac{1}{2}(s-\frac{1}{2})}(a^2).
\end{aligned}$$

If we take $\mu = -\frac{1}{2}$, i.e. if we deal with cosine reciprocity, then

$$(x^2 + a^2)^{-\frac{1}{8}} K_{\frac{1}{4}}(a\sqrt{a^2 + x^2})$$

is cosine reciprocal and $\varphi, \psi \in K$. Thus,

$$Z_1(1-\rho) = \phi_{-\frac{1}{2}}(1-\rho) = 2^{\frac{1}{2}(1-\rho)-1}\Gamma(\tfrac{1}{2}(1-\rho))K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2).$$

Plugging these back into (1.21) gives us

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{a^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{a^2}{n^2}}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{b^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{b^2}{n^2}}\right)$$
$$= \frac{1}{\sqrt{2a}}\sum_{\rho\in\mathcal{B}}\left(\frac{a}{2^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1-\rho}{2})K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{\zeta'(\rho)}$$

and (1.22) gives us

$$\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\left(\frac{a^2}{qn^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{a^2}{qn^2}}\right)$$
$$-\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\left(\frac{b^2}{qn^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{b^2}{qn^2}}\right)$$
$$=\sqrt{\frac{q\tau(\chi)}{2a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{q^{1/2}2^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1-\rho}{2})K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho,\chi)}.$$

If we take $\mu = \frac{1}{2}$ then the same procedure on ϕ gives

$$Z_1(1-\rho) = \phi_{\frac{1}{2}}(1-\rho) = 2^{-\frac{1}{2}\rho}\Gamma(1-\tfrac{1}{2}\rho)K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2).$$

Therefore, (1.22) yields

$$\frac{a}{q^{1/2}}\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}\left(\frac{a^2}{qn^2}+z^2\right)^{-\frac{3}{8}}K_{\frac{3}{4}}\left(z\sqrt{z^2+\frac{a^2}{qn^2}}\right)$$
$$-\frac{b}{q^{1/2}}\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^2}\left(\frac{b^2}{qn^2}+z^2\right)^{-\frac{3}{8}}K_{\frac{3}{4}}\left(z\sqrt{z^2+\frac{b^2}{qn^2}}\right)$$
$$=\sqrt{\frac{q\tau(\chi)}{a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{2^{1/2}q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\frac{1}{2}\rho)K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho,\chi)}.$$

Combining both cases yields the Corollary. □

Proof of Corollary 1.6 For $\mathrm{Re}(s) > 0$ the Mellin transform of the Weber parabolic cylinder function is given by entry 13.48 of [36]

$$\int_0^\infty x^{s-1} D_n(x)dx = 2^{\frac{n-2}{2}} \sqrt{\pi} \frac{\Gamma(s)}{\Gamma(\frac{1}{2}(1-n+s))} {}_2F_1\left(\begin{matrix} \frac{s}{2}, \frac{1+s}{2} \\ \frac{1}{2}(1-n+s) \end{matrix}; \frac{1}{2} \right),$$

where ${}_2F_1$ is the hypergeometric function. For $m = 0, 1, 2, \ldots$ it is shown in [49] that $D_{4m}(2x) = \varphi(x)$ is cosine reciprocal and $\varphi, \psi \in K$. Thus, (1.21) yields

$$\begin{aligned} &\sqrt{a} \sum_{n=1}^\infty \frac{\mu(n)}{n} D_{4m}\left(\frac{2a}{n}\right) - \sqrt{b} \sum_{n=1}^\infty \frac{\mu(n)}{n} D_{4m}\left(\frac{2b}{n}\right) \\ &= \frac{2^{2n-1}\sqrt{\pi}}{a^{1/2}} \sum_{\rho \in \mathcal{B}} \frac{a^\rho}{\Gamma(\frac{1}{2}(2-4n-\rho))\zeta'(\rho)} {}_2F_1\left(\begin{matrix} \frac{1-\rho}{2}, \frac{2-\rho}{2} \\ \frac{1}{2}(2-4n-\rho) \end{matrix}; \frac{1}{2} \right), \end{aligned}$$

and for even characters (1.22) yields

$$\begin{aligned} &\sqrt{a\tau(\chi)} \sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n} D_{4m}\left(\frac{2a}{q^{1/2}n}\right) - \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^\infty \frac{\bar{\chi}(n)\mu(n)}{n} D_{4m}\left(\frac{2b}{q^{1/2}n}\right) \\ &= 2^{2n-1} \sqrt{\frac{\pi q \tau(\chi)}{a}} \sum_{\rho \in \mathcal{B}_\chi} \left(\frac{a}{q^{1/2}}\right)^\rho \frac{\Gamma(1-\rho)}{\Gamma(\frac{1}{2}(2-4n-\nu\rho))L'(\rho,\chi)} {}_2F_1\left(\begin{matrix} \frac{1-\rho}{2}, \frac{2-\rho}{2} \\ \frac{1}{2}(2-4n-\rho) \end{matrix}; \frac{1}{2} \right). \end{aligned}$$

Moreover, it is also shown in [49] that $D_{4m+1}(2x)$ is sine reciprocal for $m = 0, 1, 2, \ldots$. Thus, for odd characters (1.22) yields

$$\begin{aligned} &\sqrt{a\tau(\chi)} \sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n} D_{4m+1}\left(\frac{2a}{q^{1/2}n}\right) - \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^\infty \frac{\bar{\chi}(n)\mu(n)}{n} D_{4m+1}\left(\frac{2b}{q^{1/2}n}\right) \\ &= 2^{\frac{4n-1}{2}} \sqrt{\frac{\pi q \tau(\chi)}{a}} \sum_{\rho \in \mathcal{B}_\chi} \left(\frac{a}{q^{1/2}}\right)^\rho \frac{\Gamma(1-\rho)}{\Gamma(\frac{1}{2}(1-\rho-4n))L'(\rho,\chi)} {}_2F_1\left(\begin{matrix} \frac{1-\rho}{2}, \frac{2-\rho}{2} \\ \frac{1}{2}(1-\rho-4n) \end{matrix}; \frac{1}{2} \right). \end{aligned}$$

Putting these two results together yield the statement of the corollary. □

Proof of Corollary 1.6 In [49] it is shown that

$$x^{\nu+1/2} \mathrm{e}^{x^2/4} D_{-2\nu-3}(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\nu(xy) y^{\nu+1/2} e^{y^2/4} D_{-2\nu-3}(y)dy$$

for $\mathrm{Re}(\nu) > -1$, and that

$$f(s) = \int_0^\infty x^{s-1} \mathrm{e}^{x^2/4} D_n(x)dx = \frac{\Gamma(s)\Gamma(-\frac{1}{2}n - \frac{1}{2}s)}{2^{n/2+s/2+1}\Gamma(-n)},$$

for $0 < \mathrm{Re}(s) < \mathrm{Re}(-n)$. Next, take $\nu = -\frac{1}{2}$ so that we have $\varphi(x) = \mathrm{e}^{x^2/4} D_{-2}(x) = \psi(x) \in K$, and

$$Z_1(s) = f_{-2}(s) = \frac{\Gamma(s)\Gamma(1-\frac{1}{2}s)}{2^{s/2}\Gamma(2)}, \quad Z_1(1-\rho) = \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{2^{1/2}2^{-\rho/2}\Gamma(2)}.$$

Replace it in (1.21) to get

$$\begin{aligned}
&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{a^2}{4n^2}\right)D_{-2}\left(\frac{a}{n}\right)-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{b^2}{4n^2}\right)D_{-2}\left(\frac{b}{n}\right)\\
&=\frac{1}{2^{1/2}a^{1/2}}\sum_{\rho\in\mathcal{B}}(2^{1/2}a)^{\rho}\frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{\zeta'(\rho)}.
\end{aligned}$$

Replacing the above in (1.22) gives us

$$\begin{aligned}
&\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\exp\left(\frac{a^2}{4qn^2}\right)D_{-2}\left(\frac{a}{q^{1/2}n}\right)\\
&-\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\exp\left(\frac{b^2}{4qn^2}\right)D_{-2}\left(\frac{b}{q^{1/2}n}\right)\\
&=\sqrt{\frac{q\tau(\chi)}{2a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1}{2}-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{L'(\rho,\chi)}.
\end{aligned}$$

Finally, taking instead $\nu = \frac{1}{2}$ so that $\varphi(x) = xe^{x^2/4} D_{-4}(x) = \psi(x) \in K$ as well as

$$Z_1(s) = f_{-4}(s+1) = \frac{\Gamma(s+1)\Gamma(2-\frac{1}{2}s-\frac{1}{2})}{2^{-1/2+s/2}\Gamma(4)} \quad Z_1(1-\rho) = \frac{\Gamma(2-\rho)\Gamma(1+\frac{1}{2}\rho)}{2^{-\rho/2}\Gamma(4)}.$$

Replacing this in (1.22) yields

$$\begin{aligned}
&\frac{a}{q^{1/2}}\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^2}\exp\left(\frac{a^2}{4qn^2}\right)D_{-4}\left(\frac{a}{q^{1/2}n}\right)\\
&-\frac{b}{q^{1/2}}\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^2}\exp\left(\frac{b^2}{4qn^2}\right)D_{-4}\left(\frac{b}{q^{1/2}n}\right)\\
&=\frac{1}{6}\sqrt{\frac{q\tau(\chi)}{a}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(2-\rho)\Gamma(1+\frac{1}{2}\rho)}{L'(\rho,\chi)},
\end{aligned}$$

and this ends the proof. □

Proof of Corollary 1.8 From [49] we know that

$$x^{\nu-\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{-2\nu}(x)=\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)y^{\nu-\frac{1}{2}}e^{-\frac{1}{4}y^2}D_{-2\nu}(y)dy.$$

Taking $\nu=\frac{1}{2}$ we see that $e^{-\frac{1}{4}x^2}D_{-1}(x)$ is sine reciprocal. So we set $\varphi(x)=\psi(x)=e^{-\frac{1}{4}x^2}D_{-1}(x)\in K$. Recalling from (1.31) that

$$D_{-1}(x)=\sqrt{\tfrac{\pi}{2}}\mathrm{e}^{\frac{1}{4}x^2}\operatorname{Erfc}(2^{-\frac{1}{2}}x)$$

and using entry 13.5 of [36], which says that

$$\int_0^\infty x^{s-1}\mathrm{e}^{b^2x^2}\operatorname{Erfc}(ax)dx=\pi^{-\frac{1}{2}}s^{-1}a^{-s}\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right){}_2F_1\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2}\\1+\frac{1}{2}s\end{matrix};\frac{b^2}{a^2}\right)$$

for $b<a$ and $\mathrm{Re}(s)>0$, we see that the Mellin transform is given by

$$\begin{aligned}Z_1(s)&=\int_0^\infty x^{s-1}\varphi(x)dx=\int_0^\infty x^{s-1}e^{-\frac{1}{4}x^2}D_{-1}(x)dx=\sqrt{\tfrac{\pi}{2}}\int_0^\infty x^{s-1}\operatorname{Erfc}(2^{-\frac{1}{2}}x)dx\\&=\sqrt{\tfrac{\pi}{2}}\pi^{-\frac{1}{2}}s^{-1}(2^{\frac{1}{2}s})\Gamma\left(\frac{1}{2}+\frac{1}{2}s\right){}_2F_1\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2}\\1+\frac{1}{2}s\end{matrix};0\right)=\frac{2^{s/2-1/2}\Gamma(\frac{1}{2}+\frac{s}{2})}{s}.\end{aligned}$$

Replacing this in (1.22) yields

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^\infty\frac{\chi(n)\mu(n)}{n}\exp\left(-\frac{a^2}{4qn^2}\right)D_{-1}\left(\frac{a}{q^{1/2}n}\right)\\&-\sqrt{b\tau(\bar\chi)}\sum_{n=1}^\infty\frac{\bar\chi(n)\mu(n)}{n}\exp\left(-\frac{b^2}{4qn^2}\right)D_{-1}\left(\frac{b}{q^{1/2}n}\right)\\&=2\frac{q^{1/2}\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{2^{1/2}q^{1/2}}\right)^\rho\frac{\Gamma(1-\frac{\rho}{2})}{1-\rho}\frac{1}{L'(\rho,\chi)},\end{aligned}$$

as it was to be shown. □

5 Proof of Theorem (1.3)

A similar argument as in the beginning of the proof of Theorem 1.1 yields $F(s)=\zeta(s)$ and $\nu=-1/2$. Therefore, $Z_1(s)$ is meromorphic with simple poles at $s=0,-2,-4,\ldots$. Thus, for $0<c<1$ we define

$$W(x) := \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{Z_1(-s)}{\zeta(1+s)} x^s ds.$$

By using the fact that $c > 0$ we can write

$$W(x) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} Z_1(-s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+s}} x^s ds = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} Z_1(-s) \left(\frac{n}{x}\right)^{-s} ds.$$

The change of variable $w = -s$ yields

$$\begin{aligned} W(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int\limits_{-c-i\infty}^{-c+i\infty} Z_1(w) \left(\frac{x}{n}\right)^{-w} dw = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \varphi\left(\frac{x}{n}\right) - \operatorname*{res}_{w=0} Z_1(w) \left(\frac{x}{n}\right)^{-w} \right\} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \varphi\left(\frac{x}{n}\right) - 2^{3/4}\Phi(0) \right\} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{x}{n}\right) = P_\varphi(x), \end{aligned}$$

where in the second line we have used the fact that $-1 < -c < 0$ and the prime number theorem on the fourth line. By the theory of Mellin transforms we obtain

$$\Upsilon(s) := \int\limits_{0}^{\infty} P_\varphi(x) x^{-s-1} dx = \frac{Z_1(-s)}{\zeta(1+s)}. \tag{5.1}$$

Therefore, multiplying both sides by s we have that

$$s\zeta(1+s)\Upsilon(s) = sZ_1(-s), \tag{5.2}$$

for $0 < \mathrm{Re}(s) < 1$. Now we will study (5.1) for $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. To do this, we split the integral representation of $\Upsilon(s)$ at x =1 and apply the bound $P_\varphi(x) \ll x^{-\frac{1}{2}+\delta}$ for any $\delta > 0$ as $x \to \infty$ so that

$$\Upsilon(s) = \int\limits_{0}^{1} P_\varphi(x) x^{-s-1} dx + \int\limits_{1}^{\infty} P_\varphi(x) x^{-s-1} dx = O(1) + O\left(\int\limits_{1}^{\infty} x^{-\frac{1}{2}v+\delta} x^{-\sigma-1} dx\right) = O(1).$$

Thus, one can now see that the application of the bound $P_\varphi(x) \ll x^{-\frac{1}{2}+\delta}$ makes the integral analytic on the interval $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. We reason as follows. Since the simple pole of $\zeta(1+s)$ and $Z_1(-s)$ is annihilated by the zero of s at $s = 0$ we see that the left-hand side of (5.2) is analytic. Since (5.2) holds for $0 < \mathrm{Re}(s) < 1$, by the theory of analytic continuation, it also holds on $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. If $Z_1(-s)$ does not have any zeros in the interval $-\frac{1}{2} < \mathrm{Re}(s) \le 0$, then the left-hand side of (5.2) is non-zero in $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. However, since $\Upsilon(s)$ has been shown to be analytic in this interval when the bound on $P_\varphi(x)$ is applied, this implies that $\zeta(1+s)$ does not have zeros in $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. This implies the Riemann hypothesis.

If $Z_1(-s)$ actually had zeros then all the zeros of the Riemann zeta-function would still lie on the critical line except for the zeros that coincide with the zeros of $Z_1(-s)$.

Let us now prove that the Riemann hypothesis implies the bound $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$. We recall a formulation of the Riemann hypothesis involving Mertens's function due to Littlewood [30] which says that

$$M(x) \ll x^{\frac{1}{2}+\varepsilon}.$$

An application of partial summation allows us to transform this into

$$M(\nu, n) := \sum_{m=\nu}^{n} \frac{\mu(m)}{m} \ll_\varepsilon \nu^{-\frac{1}{2}+\varepsilon} \tag{5.3}$$

uniformly in n. Recalling the definition of P_φ we have

$$P_\varphi(y) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\varphi\left(\frac{y}{n}\right) = \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu}^{\infty}\right)\frac{\mu(n)}{n}\varphi\left(\frac{y}{n}\right) =: P_{\varphi,1}(y) + P_{\varphi,2}(y),$$

where $\nu = \lfloor y^{1-\varepsilon} \rfloor$. We handle each sum separately. For the first sum

$$P_{\varphi,1}(y) = \sum_{n=1}^{\nu-1} \frac{\mu(n)}{n}\varphi\left(\frac{y}{n}\right) \ll \sum_{n=1}^{\nu-1} \frac{e^{-y/n}}{n},$$

since $\varphi \in K(\omega, 0)$ and where we have used the asymptotic of φ for large y. Therefore,

$$P_{\varphi,1}(y) \ll ye^{-y}. \tag{5.4}$$

For the second sum, we have

$$\begin{aligned} P_{\varphi,2}(y) &= \sum_{n=\nu}^{\infty} \frac{\mu(n)}{n}\varphi\left(\frac{y}{n}\right) = \sum_{n=\nu}^{\infty} M(\nu, n)\left\{\varphi\left(\frac{y}{n}\right) - \varphi\left(\frac{y}{n+1}\right)\right\} \\ &= \sum_{n=\nu}^{\infty} M(\nu, n)\left\{-\frac{y}{\lambda_n^2}\varphi'\left(\frac{y}{\lambda_n}\right)\right\}, \\ &\ll \nu^{-\frac{1}{2}+\varepsilon}\left(\sum_{n=\nu}^{\beta-1} + \sum_{n=\beta}^{\infty}\right)\left|\frac{y}{\lambda_n^2}\varphi'\left(\frac{y}{\lambda_n}\right)\right| \\ &=: P_{\varphi,3}(y) + P_{\varphi,4}(y) \end{aligned} \tag{5.5}$$

where in the last line we have used the mean value theorem with $a = n < \lambda_n (= c) < n + 1 = b$ and where

$$P_{\varphi,3}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\nu}^{\beta-1} \left| \frac{y}{\lambda_n^2} \varphi' \left(\frac{y}{\lambda_n} \right) \right|, \quad P_{\varphi,4}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\beta}^{\infty} \left| \frac{y}{\lambda_n^2} \varphi' \left(\frac{y}{\lambda_n} \right) \right|$$

with $\beta = \lfloor y^{1+\varepsilon} \rfloor$. We start with $P_{\varphi,4}(y)$ first. By the definition of the class K and by Cauchy's integral formula we see that

$$\varphi' \left(\frac{y}{\lambda_n} \right) \ll e^{-y/\lambda_n}$$

for $\lambda_n \geq \beta$. Thus,

$$P_{\varphi,4}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\beta}^{\infty} \left| \frac{y}{\lambda_n^2} e^{-y/\lambda_n} \right| \ll \nu^{-\frac{1}{2}+\varepsilon} e^{-y/\beta} \beta^{-1+(\delta+\varepsilon)} y \sum_{n=\beta}^{\infty} \frac{1}{\lambda_n^{1+\varepsilon}} \ll y^{-\frac{1}{2}+\varepsilon'}. \tag{5.6}$$

For the sum $P_{\varphi,3}$ we reason as follows. First, φ is analytic, thus φ' is continuous in a compact interval containing $I(\varepsilon, y) = (y^{-\varepsilon}, y^{\varepsilon}) \subset [0, y]$. Therefore, there exists a point $c \in I(\varepsilon, y)$ such that

$$\varphi'(c) = \max_{[0,y]} \varphi'(x).$$

The value of c is independent of y. To see this, note that $\varphi'(x) \ll \mathrm{e}^{-x}$ when $x \to \infty$. Then we can find a positive real number C independent of y such that $\varphi'(c) > \varphi'(y)$ for all $y > C$. Therefore,

$$P_{\varphi,3}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \frac{y}{\nu^2} \varphi'(c) \sum_{n=\nu}^{\beta-1} 1 \ll y^{-\frac{1}{2}+\varepsilon''}. \tag{5.7}$$

Putting together (5.4)–(5.7) we see that the Riemann hypothesis implies the bound $P_{\varphi}(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$.

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