Lett Math Phys (2015) 105:1689–1701
DOI 10.1007/s11005-015-0796-4

Poisson–Lie T-Duality and Courant Algebroids

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Received: 22 February 2015 / Revised: 4 August 2015 / Accepted: 4 August 2015
Published online: 19 August 2015 –

Abstract. Poisson–Lie T-duality is explained using the language of Courant algebroids.

Mathematics Subject Classification. 53D18, 37K05.

Keywords. T-duality, Courant algebroids.

1. Introduction

This note explains Poisson–Lie T-duality from the point of view of Courant algebroids. It is a compilation of my letters [9] to Alan Weinstein written in 1998–1999, which circulated in the "Poisson community" (including, among others, Anton Alekseev, Paul Bressler, Yvette Kosmann-Schwarzbach and Ping Xu) for some time.

During 16 years since the letters were written, the basic technical tools (e.g., reduction of Courant algebroids) were rediscovered and works linking (Abelian) T-duality and Courant algebroids appeared, notably the paper by Cavalcanti and Gualtieri [3]. I still decided to write my account and include some details missing in [9]. Perhaps the most important reason is that I introduced exact Courant algebroids while trying to understand Poisson–Lie T-duality, and I believe that this duality, first introduced in [6], which generalized the usual Abelian T-duality, is essential for understanding of both Courant algebroids and of the world of T-dualities.

This note summarizes the first four letters of [9]. In particular, it does not deal with differential graded symplectic geometry and its link with Courant algebroids, which is discussed in the remaining letters. While it is certainly relevant for Poisson–Lie T-duality, I decided to exclude it to keep the focus on one thing, and also because I already wrote about it in [10].

Supported in part by the grant MODFLAT of the European Research Council and the NCCR SwissMAP of the Swiss National Science Foundation.

2. Exact Courant Algebroids

Courant algebroids and Dirac structures were introduced by Liu, Weinstein and Xu in [8].

DEFINITION 1. A Courant algebroid (CA) is a vector bundle $E \to M$ equipped with a non-degenerate quadratic form $\langle\, ,\, \rangle$, with a bundle map

$$a: E \to TM$$

(the anchor map) and with a $\mathbb{R}$-bilinear map

$$[\, ,\,]: \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

satisfying

A: $[s,[t,u]] = [[s,t],u] + [t,[s,u]]$ for any $s,t,u \in \Gamma(E)$
B: $a([s,t]) = [a(s), a(t)]$ for any $s,t \in \Gamma(E)$
C: $[s, ft] = f[s,t] + (a(s)f)t$ for any $s,t \in \Gamma(E),\ f \in C^\infty(M)$
D: $a(s)\langle t,u\rangle = \langle [s,t],u\rangle + \langle t,[s,u]\rangle$
E: $[s,s] = a^t\big(\mathrm{d}\langle s,s\rangle/2\big)$, where $a^t: T^*M \to E^* \xrightarrow{\langle,\rangle} E$ is the transpose of a.

A Dirac structure in E is a subbundle $L \subset E$ which is Lagrangian w.r.t $\langle,\rangle$ (i.e., $L^\perp = L$) such that $\Gamma(L)$ is closed under $[,]$.

Remark 1. This definition from [9] is somewhat simpler than the (equivalent) original definition of Liu–Wenstein–Xu [8], who used the skew-symmetric part of $[,]$.

Axiom E can be replaced by the more innocent looking

$$\langle s, [t,t]\rangle = \langle [s,t], t\rangle\,.$$

Axioms A–D are equivalent to the following: every section $s \in \Gamma(E)$ induces a vector field Z_s on E over $a(s)$, such that the flow of Z_s preserves all the structure; the bracket $[s,s']$ is the Lie derivative of s' under this flow. The map $s \mapsto Z_s$ is $\mathbb{R}$-linear. We have $[Z_s, Z_{s'}] = Z_{[s,s']}$ (as follows from axiom A).

EXAMPLE 1. ([8]) If M is a point then E is a Lie algebra with invariant non-degenerate quadratic form $\langle\, ,\, \rangle$.

EXAMPLE 2. ([8]) If M is a manifold then $E = (T \oplus T^*)M$, with

$$\langle (u,\alpha), (v,\beta)\rangle = \alpha(v) + \beta(u), \tag{1a}$$

$$a(u,\alpha) = u, \tag{1b}$$

$$[(u,\alpha),(v,\beta)] = ([u,v], L_u\beta - i_v \mathrm{d}\alpha) \tag{1c}$$

is the standard CA over M. In this case, $Z_{(u,0)}$ is the natural lift of u to (the natural bundle) $(T\oplus T^*)M$, and $Z_{(0,\alpha)}$ is the vertical vector field with value $-i_v d\alpha$ at $(v,\beta)\in(T\oplus T^*)M$.

If $L\subset E$ is a Lagrangian vector subbundle of a CA (i.e., if $L^\perp=L$), we can measure the non-involutivity of L (i.e., its failure to be a Dirac structure) by

$$\mathcal{F}_L:\bigwedge^2 L\to E/L\cong L^*,\quad \mathcal{F}_L(s,t)=[s,t] \bmod L\quad (\forall s,t\in\Gamma(L))$$

where the isomorphism $E/L\cong L^*$ is given by $\langle,\rangle$, or equivalently by

$$\mathcal{H}_L\in\Gamma(\wedge^3 L^*),\quad \mathcal{H}_L(s,t,u)=\langle[s,t],u\rangle\quad(\forall s,t,u\in\Gamma(L))$$

(the fact that $\mathcal{F}_L$ and $\mathcal{H}_L$ are well defined is readily verified; even though $\mathcal{F}_L$ and $\mathcal{H}_L$ are really the same object, it will be convenient to have a separate notation). L is a Dirac structure iff $\mathcal{F}_L=0$ (or $\mathcal{H}_L=0$).

If $E\to M$ is a CA with anchor map a then $a\circ a^t=0$ (as follows from axioms E and B), i.e.,

$$0\to T^*M\xrightarrow{a^t}E\xrightarrow{a}TM\to 0 \tag{2}$$

is a chain complex.

DEFINITION 2. A Courant algebroid $E\to M$ is exact if (2) is an exact sequence.

The simplest example of an exact CA is the standard CA; as we shall see below, every exact CA is locally standard.

EXAMPLE 3. Let D be a Lie group and $G\subset D$ a closed subgroup. Let the Lie algebra $\mathfrak{d}$ of D be equipped with a Ad-invariant non-degenerate quadratic form $\langle\,,\,\rangle_{\mathfrak{d}}$ such that $\mathfrak{g}^\perp=\mathfrak{g}$. Then $E=\mathfrak{d}\times D/G$ is an exact Courant algebroid over D/G. For constant sections of E, the bracket is the Lie bracket on $\mathfrak{d}$ and the anchor is the action of $\mathfrak{d}$ on the homogeneous space D/G.

If $\mathfrak{h}\subset\mathfrak{d}$ is another Lagrangian Lie subalgebra, i.e., if $\mathfrak{h}$ is a Dirac structure in $\mathfrak{d}$, then $\mathfrak{h}\times D/G\subset\mathfrak{d}\times D/G$ is a Dirac structure.

Exact CAs can be classified in the following way. If E is an exact CA then there is a Lagrangian subbundle $L\subset A$ such that $a|_L:L\to TM$ is an isomorphism (as can be seen by a partition of unity argument). We shall call such a subbundle $L\subset E$ a connection in E. Equivalently, a connection can be described as a splitting $\sigma:TM\to E$ of the exact sequence (2), such that its image L is Lagrangian. Connections form an affine space over $\Omega^2(M)$ (if $\tau\in\Omega^2(M)$ then $(\tau+\sigma)(v):=\sigma(v)+a^t(i_v\tau)$).

If L is a connection then one can easily see that

$$H(u,v,w) := \langle [\sigma(u), \sigma(v)], \sigma(w)\rangle$$

defines a closed 3-form $H \in \Omega^3(M)$ (if we identify TM with L then $H = \mathcal{H}_L$); the 3-form H is called the curvature of the connection L. If we use $\sigma \oplus a^t$ to identify E with $TM \oplus T^*M$ then its anchor a and pairing $\langle,\rangle$ are as in the standard CA, and the bracket is

$$[(u,\alpha),(v,\beta)] = ([u,v], L_u\beta - i_v \mathrm{d}\alpha + H(u,v,\cdot))\,. \tag{3}$$

On the other hand, for any closed H the bracket (3) makes $TM \oplus T^*M$ to an exact Courant algebroid. If we change σ by a 2-form $\tau \in \Omega^2(M)$ then H gets replaced by $H + d\tau$. As a result, we have the following theorem:

THEOREM 1. (classification of exact CAs). *Exact Courant algebroids over M are classified by $H^3(M,\mathbb{R})$; exact Courant algebroids with a chosen connection are classified by closed* 3*-forms H, with the bracket given by* (3).

If E is the exact CA given by (3), and $L \subset E$ a Dirac structure, then on each integral leaf $N \subset M$ of $a(L) \subset TM$ we have a 2-form α_N satisfying $\mathrm{d}\alpha_N = H|_N$; the integral leaves and the 2-forms determine L uniquely.

3. Exact CAs and Two-Dimensional Variational Problems

Let Σ be an oriented surface, M a manifold and $\omega \in \Omega^2(M)$ a 2-form. Let us consider the functional S on maps $f: \Sigma \to M$ given by

$$S[f] = \int_\Sigma f^*\omega. \tag{4}$$

We shall consider more general functionals in Remark 4 below; recall, however, that any local functional can be replaced by (4) if we replace M by an appropriate jet space (the de Donder–Weyl (=multisymplectic) method).

A map $f: \Sigma \to M$ is critical w.r.t. S iff

$$f^*(i_u \mathrm{d}\omega) = 0$$

for every vector field u on M. If $\tau \in \Omega^2(M)$ is closed then ω and $\omega + \tau$ give equivalent variational problems. More generally, if $H \in \Omega^3_{cl}(M)$ is a closed 3-form, we can consider maps $f: \Sigma \to M$ satisfying

$$f^*(i_u H) = 0 \tag{5}$$

and call them critical (or H-critical). As H is locally of the form $d\omega$, we can still see this equation as a solution of a variational problem.

Remark 2. From quantum point of view, to make the path integral formally meaningful, one needs to upgrade H to a Cheeger–Simons differential character, or equivalently to a class in the smooth Deligne cohomology [5].

Let us now consider the exact CA $E \to M$ with connection $L \subset E$ such that its curvature is H. (We can set $E = (T \oplus T^*)M$ with the bracket (3) and $L = TM$; if $H = \mathrm{d}\omega$, we can equivalently take the standard CA and set L to be the graph of $\omega : TM \to T^*M$.) For any map $f : \Sigma \to M$, let

$$\tilde{T} f : T\Sigma \to L$$

be the lift of the tangent map $Tf : T\Sigma \to TM$ given by $a \circ \tilde{T} f = Tf$. The map $\tilde{T} f$ can be used to pull back sections of $\bigwedge L^*$ to differential forms on Σ; this pullback will be denoted by $f^{\tilde{*}}$.

The Euler–Lagrange Equation (5) can be rephrased as a 'zero-curvature condition'.

PROPOSITION 1. *A map* $f : \Sigma \to M$ *is critical iff*

$$f^{\tilde{*}}(\mathcal{F}_L) = 0 \ \left(\in \Omega^2\big(\Sigma, f^*(E/L)\big)\right).$$

The importance of E is that its sections, rather than just vector fields on M, can be interpreted as symmetries and give rise to conservation laws.

THEOREM 2. ("Noether") *If* $s \in \Gamma(E)$ *is such that the flow of* Z_s *preserves* L *then for any critical map* $f : \Sigma \to M$ *the 1-form* $f^{\tilde{*}}\langle s, \cdot\rangle \in \Omega^1(\Sigma)$ *satisfies*

$$\mathrm{d}\big(f^{\tilde{*}}\langle s, \cdot\rangle\big) = 0.$$

Proof. We identify E with $(T \oplus T^*)M$ using the connection L; the bracket on E is then (3) and $L = TM$. If $s = (u, \alpha)$ then $f^{\tilde{*}}\langle s, \cdot\rangle = f^*\alpha$. The flow of Z_s preserves L iff

$$\mathrm{d}\alpha + i_u H = 0.$$

If f is critical then $f^*(i_u H) = 0$. We thus get $\mathrm{d}\big(f^{\tilde{*}}\langle s, \cdot\rangle\big) = f^*(\mathrm{d}\alpha) = 0$. □

The main theme of this paper is the study of symmetries that in place of closed 1-forms give rise to flat connection. The fact that Euler–Lagrange equations can be seen as a zero-curvature condition (Proposition 1) will play an important role. To explain it, we will need equivariant exact CAs, which we introduce in the following section.

Remark 3. If we considered one-dimensional variational problems instead of two-dimensional, then exact CAs would be replaced by Lie algebroids $A \to M$ which are extensions $0 \to \mathbb{R} \to A \to TM \to 0$. A splitting of this extension, i.e., a connection in A, gives rise to a closed 2-form (the curvature of the connection). To make formal sense of the path integral, we would rather need a principal $U(1)$-bundle $P \to M$ with a connection; in this case $A = (TP)/U(1)$.

In the case of two-dimensional problems principal $U(1)$-bundles are replaced by $U(1)$-gerbes (as observed by Brylinski [1], reinterpreting Gawęzki's approach via smooth Deligne cohomology [5]). Exact CAs are thus closely related to $U(1)$-gerbes.

Remark 4. If Σ is a surface with pseudo-conformal structure, with local light-like coordinates t_+, t_-, and $r \in \Gamma(T^{*\otimes 2}M)$ is a tensor field on M, let us consider the standard σ-model action functional on maps $f : \Sigma \to M$,

$$S_r[f] = \int_\Sigma r\Big(\frac{\partial f}{\partial t_+}, \frac{\partial f}{\partial t_-}\Big)\, \mathrm{d}t_+\, \mathrm{d}t_-.$$

Let E be the standard CA and $R \subset E$ be the graph of $TM \to T^*M$, $v \mapsto r(v, \cdot)$ (then $R^\perp$ is the graph of $v \mapsto -r(\cdot, v)$). For any $f : \Sigma \to M$ let us lift $Tf : T\Sigma \to TM$ to $\tilde{T}f : T\Sigma \to E$ by requiring $\tilde{T}f(\partial_{t_+}) \in R$ and $\tilde{T}f(\partial_{t_-}) \in R^\perp$. In this case, Noether theorem says:

If R is invariant under the flow of Z_s for some $s \in \Gamma(E)$, and if $f : \Sigma \to M$ is critical for the functional S_r, then $\mathrm{d}\big(f^{\tilde{*}}\langle s, \cdot\rangle\big) = 0$.

Proposition 1 becomes trickier and we do not state it here (see [10, Section 5], where it is formulated in terms of differential graded manifolds). A similar picture can be found for any Lagrangian density depending on the first derivatives of f.

4. Equivariant Courant Algebroids and Their Reduction

Let $\mathfrak{g}$ be a Lie algebra, $E \to M$ a Courant algebroid, and $\rho : \mathfrak{g} \to \Gamma(E)$ a $[,]$-preserving linear map. The functions $\langle \rho(\xi), \rho(\eta)\rangle \in C^\infty(M)$, $\xi, \eta \in \mathfrak{g}$, are constant on the integral leaves of E, as

$$0 = \rho([\xi, \eta] + [\eta, \xi]) = [\rho(\xi), \rho(\eta)] + [\rho(\eta), \rho(\xi)] = a^t \mathrm{d}\, \langle \rho(\xi), \rho(\eta)\rangle\,.$$

The resulting (possibly degenerate) pairing $\langle \rho(\xi), \rho(\eta)\rangle$ on $\mathfrak{g}$ is ad-invariant. This leads us to the following definition.

DEFINITION 3. Let $\mathfrak{g}$ be a Lie algebra and $\langle , \rangle_{\mathfrak{g}}$ an invariant symmetric bilinear pairing on $\mathfrak{g}$ (possibly degenerate). If E is a CA, a representation of $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$ in E is a linear map $\rho : \mathfrak{g} \to \Gamma(E)$ such that $\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]$ and $\langle \rho(\xi), \rho(\eta)\rangle = \langle \xi, \eta\rangle_{\mathfrak{g}}$.

A representation of $\mathfrak{g}$ in E gives us an action of $\mathfrak{g}$ on E by $Z_{\rho(\xi)}$'s. If G is a connected Lie group with the Lie algebra $\mathfrak{g}$, and the action of $\mathfrak{g}$ on E gives rise to an action of G on E, we shall say that E is a G-equivariant CA.

Remark 5. If exact CAs $E\to M$ are seen as approximations of $U(1)$-gerbes over M then the "gerby" version of a G-equivariant exact CA $E\to M$ is a multiplicative gerbe over G acting on a gerbe over M. In this context, it is best to replace exact CAs $E\to M$ with principal $\mathbb{R}[2]$-bundles $X\to T[1]M$ in the category of differential graded manifolds. Multiplicative gerbes over G are approximated by central extensions of the group $T[1]G$ by $\mathbb{R}[2]$, and such extensions are classified by invariant symmetric bilinear forms on $\mathfrak{g}$. See [10, section 3] for some details.

It is easy to see that $\mathfrak{g}$-invariant sections of E, orthogonal to the image of $\mathfrak{g}$ in E, are closed under the bracket $[,]$. This gives us, after we mod out by the sections which are in the kernel of $\langle,\rangle$, the following theorem.

THEOREM 3. (reduction of CAs) *Let $E\to M$ be a G-equivariant CA. Suppose that the action of G on M is free and proper. For any $x\in M$ let*

$$(E_{/G})_x=(\rho_x(\mathfrak{g}))^{\perp}/(\rho_x(\mathfrak{g})^{\perp}\cap\rho_x(\mathfrak{g}))=(\rho_x(\mathfrak{g}))^{\perp}/\rho_x(\mathfrak{g}'),$$

where $\mathfrak{g}'\subset\mathfrak{g}$ is the kernel of $\langle,\rangle_{\mathfrak{g}}$. After factorization by the action of G, $E_{/G}$ becomes a vector bundle over M/G. The CA structure on E descends to a CA structure on $E_{/G}$. If E is exact and $\langle,\rangle_{\mathfrak{g}}=0$ (and thus $\mathfrak{g}'=\mathfrak{g}$) then $E_{/G}$ is exact.

Remark 6. This reduction procedure was rediscovered and generalized in [2].

If $M\to M/G$ is a principal G-bundle (i.e., if G acts freely and properly on M), let its Pontryagin class be

$$\left[\langle F,F\rangle_{\mathfrak{g}}\right]\in H^4(M,\mathbb{R})$$

where F is the curvature of a connection on the principal bundle.

THEOREM 4. (classification of equivariant exact CAs) *If G acts freely and properly on M then M admits a G-equivariant exact CA iff the Pontryagin class of the principal G-bundle $M\to M/G$ vanishes.*

There is a natural free and transitive action of the group $H^3(M/G,\mathbb{R})$ on the set of isomorphism classes of G-equivariant CAs $E\to M$, where the class of a closed 3-form $\gamma\in\Omega^3_{cl}(M/G)$ acts by modifying the bracket on $E\to M$ via

$$[s,t]_{new}=[s,t]+a^t\left(p^*\gamma(a(s),a(t),\cdot)\right).$$

Proof. Let us choose a connection $A\in\Omega^1(M,\mathfrak{g})$ on the principal G-bundle $M\to M/G$. If $E\to M$ is a G-equivariant exact CA, we can choose a G-invariant Lagrangian splitting $E\cong(T\oplus T^*)M$ (i.e., a connection $L\subset E$), such that

$$\rho(\xi)=\left(\xi_M,\frac{1}{2}\langle\xi,A\rangle_{\mathfrak{g}}\right)\in\Gamma\left((T\oplus T^*)M\right)\quad(\forall\xi\in\mathfrak{g}),\tag{6}$$

where $\xi_M = a(\rho(\xi))$ is the vector field on M given by $\xi \in \mathfrak{g}$. The bracket on $E \cong (T \oplus T^*)M$ is now given by (3) for some G-invariant closed 3-form $H \in \Omega^3_{cl}(M)$.

G-invariance of the splitting $E \cong (T \oplus T^*)M$ means that for every vector field u on M and any $\xi \in \mathfrak{g}$ the section $\left[\rho(\xi), (u, 0)\right]$ of E is a vector field (i.e., its 1-form part is zero). Equivalently,

$$i_{\xi_M} H = -\frac{1}{2} \langle \xi, \mathrm{d}A \rangle_{\mathfrak{g}} . \tag{7}$$

Equation (7) also ensures that ρ is a representation of $\mathfrak{g}$ in E.

The Chern–Simons 3-form $\mathsf{cs} = \langle A, \mathrm{d}A \rangle_{\mathfrak{g}} + \frac{1}{3} \langle [A, A], A \rangle_{\mathfrak{g}}$ satisfies (up to a factor) the same equation

$$i_{\xi_M} \mathsf{cs} = \langle \xi, \mathrm{d}A \rangle_{\mathfrak{g}},$$

and $\mathrm{d}\,\mathsf{cs} = p^* \langle F, F \rangle_{\mathfrak{g}}$, where $p : M \to M/G$ is the projection and F the curvature of A. As a result, the general solution of (7) is

$$H = p^*\theta - \mathsf{cs}/2, \;\; \mathrm{d}\theta = \frac{1}{2} \langle F, F \rangle_{\mathfrak{g}} .$$

If we change the splitting of E by a 2-form $\tau \in \Omega^2(M/G)$ then H gets replaced by $H + p^*\mathrm{d}\tau$, i.e., θ by $\theta + \mathrm{d}\tau$. G-equivariant CAs over M are thus classified by solutions of $\mathrm{d}\theta = \frac{1}{2} \langle F, F \rangle_{\mathfrak{g}}$ modulo exact 3-forms. □

As an application of the reduction procedure, let us now describe a construction/classification of transitive CAs, i.e., of CAs with surjective anchors. If $\tilde{E} \to N$ is a transitive CA with anchor $\tilde{a} : \tilde{E} \to TN$ then $B := \tilde{E} / \operatorname{Im} \tilde{a}^t$ is a transitive Lie algebroid with an invariant inner product on the bundle of vertical Lie algebras.

THEOREM 5. (exact equivariant vs. transitive CAs) *If $M \to N = M/D$ is a principal D-bundle and $\langle , \rangle_{\mathfrak{d}}$ is non-degenerate then the reduction by D gives an equivalence between D-equivariant exact CAs $E \to M$ and transitive CAs $\tilde{E} \to N$ such that $\tilde{E} / \operatorname{Im} \tilde{a}^t = (TM)/D$.*

Proof. If $E \to M$ is a D-equivariant CA then the fact that $\tilde{E} := E_{/D}$ is transitive and $\tilde{E} / \operatorname{Im} \tilde{a}^t = (TM)/D$ follows from the definition of $E_{/D}$.

If $\tilde{E} \to N$ is a transitive CA such that $\tilde{E} / \operatorname{Im} \tilde{a}^t = (TM)/D$ then we can (re)construct a D-equivariant exact $E \to M$ as follows. Let

$$E := p^* \tilde{E} \oplus \mathfrak{d},$$

with the following structure: the anchor

$$a_E : p^* \tilde{E} \oplus \mathfrak{d} \to TM$$

is the sum of the projection $p^* \tilde{E} \to p^* B = TM$ and of the natural map $\mathfrak{d} \times M \to TM$, the pairing $\langle , \rangle_E$ is the direct sum of the pairings on $\tilde{E}$ and on $\mathfrak{d}$, and the bracket is given by

$$[p^*s, p^*t]_E = p^*[s,t]_{\tilde{E}}, \;\; [\xi,\eta]_E = [\xi,\eta]_{\mathfrak{d}}, \;\; [p^*s,\xi]_E = 0$$

for all $s,t \in \Gamma(\tilde{E})$, $\xi,\eta \in \mathfrak{d}$. □

Remark 7. If $B \to N$ is an arbitrary transitive Lie algebroid with invariant inner product on its vertical Lie algebras then transitive CAs $\tilde{E} \to N$ such that $\tilde{E}/\tilde{a}^t = B$ exist iff the Pontryagin class of B vanishes; in this case $H^3(N,\mathbb{R})$ acts freely and transitively on their isomorphism classes (isomorphisms which are identity on B). If $B = (TM)/D$ as above then this result follows from Theorems 4 and 5. For general B, it can be proven by a direct calculation; we do not need this result here, so we refer the reader to [9, no. 4]. This result was rediscovered and extended to regular CAs in [4].

5. Reduction and Curvature

In this section, $\mathfrak{d}$ is a Lie algebra with a non-degenerate invariant symmetric pairing $\langle,\rangle_{\mathfrak{d}}$ and D a connected Lie group with Lie algebra $\mathfrak{d}$.

Let D act freely and properly on a manifold M and let $E \to M$ be an equivariant CA. Let $E_{/D} \to M/D$ be the reduction of E; we have $p_D^* E_{/D} = \rho(\mathfrak{d})^{\perp} \subset E$, where $p_D : M \to M/D$ is the projection.

Let $L_D \subset E_{/D}$ be a Lagrangian subbundle. Then

$$L := p_D^* L_D \subset \rho(\mathfrak{d})^{\perp} \subset E$$

is a D-invariant Lagrangian subbundle of $\rho(\mathfrak{d})^{\perp}$; any D-invariant Lagrangian subbundle of $\rho(\mathfrak{d})^{\perp}$ is of this form.

We define

$$\mathcal{F}_L : \bigwedge^2 L \to (\rho(\mathfrak{d})^{\perp}/L) \cong L^*, \quad \mathcal{F}_L(s,t) = [s,t] \bmod L \quad (\forall s,t \in \Gamma(L))$$

and

$$\mathcal{H}_L \in \Gamma(\wedge^3 L^*), \quad \mathcal{H}_L(s,t,u) = \langle [s,t], u \rangle \quad (\forall s,t,u \in \Gamma(L))$$

(a quick inspection shows that $\mathcal{F}_L$ and $\mathcal{H}_L$ are well defined). Notice (using D-invariant sections in the definition of $\mathcal{F}_L$ and $\mathcal{H}_L$) that

$$\mathcal{F}_L = p_D^* \mathcal{F}_{L_D} \text{ and } \mathcal{H}_L = p_D^* \mathcal{H}_{L_D}.$$

Let now $G \subset D$ be a Lagrangian subgroup (i.e., its Lie algebra $\mathfrak{g}$ is a Lagrangian subspace of $\mathfrak{d}$, or equivalently $(\mathfrak{d},\mathfrak{g})$ is a Manin pair). Let us consider the reduced CA $E_{/G} \to M/G$. We have a natural identification $E_{/G} \cong \rho(\mathfrak{d})^{\perp}/G$ (as $\rho(\mathfrak{g})^{\perp} = \rho(\mathfrak{g}) \oplus \rho(\mathfrak{d})^{\perp}$ and thus $\rho(\mathfrak{g})^{\perp}/\rho(\mathfrak{g}) \cong \rho(\mathfrak{d})^{\perp}$). Let us define a Lagrangian subbundle $L_G \subset E_{/G}$ to be $L_G = L/G$ (i.e., $L = p_G^* L_G$). Using G-invariant sections of L, we get

$$\mathcal{F}_L = p_G^* \mathcal{F}_{L_G} \text{ and } \mathcal{H}_L = p_G^* \mathcal{H}_{L_G}, \tag{8}$$

where $p_G : M \to M/G$ is the projection. As a result, we have

PROPOSITION 2. *$p_D^*\mathcal{F}_{L_D} = p_G^*\mathcal{F}_{L_G}$ and $p_D^*\mathcal{H}_{L_D} = p_G^*\mathcal{H}_{L_G}$. In particular, $L_G \subset E_{/G}$ is a Dirac structure iff $L_D \subset E_{/D}$ is a Dirac structure.*

Let us now suppose in addition that E is exact (which implies that $E_{/G}$ is exact) and that its anchor $a: E \to TM$ is injective on $L \subset E$. Let

$$V := a(L) \subset TM.$$

Notice that

$$\operatorname{rank} V = \operatorname{rank} L = \dim M - \frac{1}{2}\dim \mathfrak{d} = \dim M/G.$$

The non-involutivity of the distribution $V \subset TM$ is measured by its curvature

$$F_V : \bigwedge^2 V \to TM/V, \quad F_V(u,v) = [u,v] \bmod V \quad (\forall u, v \in \Gamma(V)).$$

From definitions, we get that

$$F_V\big(a(s), a(t)\big) = a\big(\mathcal{F}_L(s,t)\big) \quad \forall s, t \in \Gamma(L). \tag{9}$$

6. Non-Abelian Conservation Laws and Poisson–Lie T-Duality

Poisson–Lie T-duality is a geometric version of “non-Abelian Noether theorem”, where a symmetry gives rise to a flat connection instead of a closed 1-form, and also an equivalence between two (or more) variational problems. It was introduced in [6]. The idea of exact CAs was extracted from this T-duality; the following is a “coordinate-free” interpretation of Poisson–Lie T-duality in terms of exact CAs.

Let us use the setup and notation of the previous section: $E \to M$ is a D-equivariant exact CA (the action of D on M is free and proper), $L \subset \rho(\mathfrak{d})^\perp$ is a Lagrangian D-invariant subbundle such that the anchor a is injective on L, and $G \subset D$ is a Lagrangian subgroup.

The Lagrangian subbundle $L_G \subset E_{/G}$ is a connection iff $V := a(L) \subset TM$ is transverse to the fibers of the projection $M \to M/G$. Supposing this (or removing the points where transversality fails), let

$$H_G \in \Omega^3_{cl}(M/G)$$

be the curvature of the connection $L_G \subset E_{/G}$ and let $A_G \in \Omega^1(M, \mathfrak{g})$ be the connection on the principal G-bundle $p_G : M \to M/G$ whose horizontal bundle is $V \subset TM$.

THEOREM 6. (“non-Abelian Noether”) *If $f : \Sigma \to M/G$ is H_G-critical then f^*A_G is a flat connection on the principal G-bundle $f^*M \to \Sigma$.*

Proof. It follows immediately from Proposition 1 and relations (8) and (9) □

Motivated by Proposition 1, we shall say that a map $\phi:\Sigma\to M$ is L-critical if the tangent map $T\phi:T\Sigma\to TM$ can be lifted to a vector bundle map $\tilde{T}\phi:T\Sigma\to L$ (i.e., if the image of $T\phi$ is in $V=a(L)$) and if $\phi^{\tilde{*}}\mathcal{F}_L=0$ $(\in\Omega^2(\Sigma,\phi^*L^*))$. Notice that the action of D on M sends L-critical maps $\Sigma\to M$ to L-critical maps; by the following theorem, such maps are equivalent to H_G-critical maps $\Sigma\to M/G$.

THEOREM 7. (Poisson–Lie T-duality) *If $\phi:\Sigma\to M$ is L-critical then $p_G\circ\phi:\Sigma\to M/G$ is H_G-critical. If Σ is 1-connected and $f:\Sigma\to M/G$ is H_G-critical then there is a L-critical map $\phi:\Sigma\to M$ such that $f=p_G\circ\phi$; the map ϕ is unique up to the action of G.*

If $G'\subset D$ is another Lagrangian subgroup then lifting H_G-critical maps to L-critical and projecting them to M/G' gives us an equivalence between H_G-critical maps $\Sigma\to M/G$ and $H_{G'}$-critical maps $\Sigma\to M/G'$.

Proof. If $\phi:\Sigma\to M$ is L-critical then H_G-criticality of $p_G\circ f$ follows from Proposition 1 and from (8). If Σ is 1-connected and $f:\Sigma\to M/G$ is H_G-critical then by Theorem 6 there is a map $\phi:\Sigma\to M$ such that $p_G\circ\phi=f$ and such that the image of $T\phi$ is in V (ϕ is unique up to the action of G). Relation (8) and H_G-criticality of f then imply that ϕ is L-critical. □

The name "Poisson–Lie T-duality" comes from the case when $\mathfrak{g}\cap\mathfrak{g}'=0$, i.e., when $(\mathfrak{d},\mathfrak{g},\mathfrak{g}')$ is a Manin triple and G and G' a dual pair of Poisson–Lie groups.

Remark 8. If we start with a half-dimensional subbundle $R_D\subset E_{/D}$ in place of L_D (we do not suppose that R_D is Lagrangian) we obtain a subbundle $R_G\subset E_{/G}$. When we locally trivialize the exact CA $E_{/G}$, we get a σ-model as described in Remark 4. Theorems 6 and 7 remain valid (after the appropriate modification). It was for this type of models that Poisson–Lie T-duality was originally formulated in [6] (sans the language of CAs). We do not give details, as it is easier to pass to equivalent σ-models given by 2-forms/closed 3-forms.

This picture was used in [3] in the case of Abelian D (without discussing critical maps).

7. Dirac Structures and Boundary Conditions (D-Branes)

Let us return to variational problems. Let M be a manifold, $N\subset M$ a submanifold, and let us choose forms $\omega\in\Omega^2(M)$, $\alpha_N\in\Omega^1(N)$. If Σ is a surface, let us consider the action functional

$$S[f]=\int_\Sigma f^*\omega+\int_{\partial\Sigma} f^*\alpha_N$$

defined on maps $f:\Sigma\to M$ mapping $\partial\Sigma$ to N. The Euler–Lagrange equation for critical f's is now

$$f^* i_u \mathrm{d}\omega = 0 \text{ on } \Sigma, \quad f^* i_v(\omega|_N + \mathrm{d}\alpha_N) = 0 \text{ on } \partial\Sigma$$

for every vector fields u on M and v on N.

More invariantly and generally, we choose a closed 3-form $H \in \Omega^3_{cl}(M)$ and a 2-form $\beta_N \in \Omega^2(N)$ such that $\mathrm{d}\beta_N = H|_N$. Locally, then we can find ω and α_N such that $H = \mathrm{d}\omega$ and $\beta_N = \omega|_N + \mathrm{d}\alpha_N$. The Euler–Lagrange relations now say

$$f^* i_u H = 0 \text{ on } \Sigma, \quad f^* i_v \beta_N = 0 \text{ on } \partial\Sigma. \tag{10}$$

Remark 9. For quantization, to make formal sense of the path integral, the pair (H, β_N) should be extended to a relative Cheeger–Simons differential character. This fact was discussed in the case of the WZW model in [7] and full generality in [11].

As in Section 3 let $L \subset E$ be the exact CA with connection whose curvature is H. Let $C \subset E$ be a Dirac structure. On any leaf $N \subset M$ of the distribution $a(C) \subset TM$, we then have a 2-form $\beta_N \in \Omega^2(N)$ such that $d\beta_N = H|_N$. We can thus use the Dirac structure C to impose a boundary condition; the map f is required to send each component of $\partial\Sigma$ to a leaf N, and critical maps are given by the Euler–Lagrange Equation (10).

Proposition 1 has now the following form.

PROPOSITION 3. *A map $f: \Sigma \to M$ is critical with the boundary condition given by C iff*

$$\tilde{f^*}(\mathcal{F}_L) = 0 \quad \Big(\in \Omega^2\big(\Sigma, f^*(E/L)\big)\Big)$$

and

$$(\tilde{T} f)\big(T(\partial\Sigma)\big) \subset C.$$

Let us now describe Dirac structures (and thus boundary conditions) compatible with Poisson–Lie T-duality. We shall use the setup of Section 6: a free and proper action of D on M, a D-equivariant exact CA $E \to M$, a Lagrangian D-invariant subbundle $L \subset \rho(\mathfrak{d})^\perp$ such that a is injective on L, a Lagrangian subgroup $G \subset D$. This data give us the connection L_G in the exact CA $E_{/G} \to M/G$ with curvature $H_G \in \Omega^3_{cl}(M/G)$.

We can now use Proposition 2 to describe Dirac structures (or boundary conditions) in M/G compatible with Poisson–Lie T-duality. We start with a Dirac structure $C_D \subset E_{/D}$; by Proposition 2 (using "C" in place of "L") it gives us a Dirac structure $C_G \subset E_{/G}$ and a D-invariant subbundle $C \subset \rho(\mathfrak{d})^\perp \subset E$.

If $f: \Sigma \to M/G$ is a H_G-critical map satisfying the boundary condition given by C_G then its lift $\phi: \Sigma \to M$ (see Theorem 7) will satisfy

$$(\tilde{T}\phi)(T(\partial\Sigma)\big) \subset C$$

and thus, if $G' \subset D$ is another Lagrangian subgroup, the map $p_{G'} \circ \phi : \Sigma \to M/G'$ will be $H_{G'}$-critical (Theorem 7) and will satisfy the boundary condition given by $C_{G'} \subset E_{/G'}$.

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