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Skyrme–Faddeev Instantons on Complex Surfaces

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Abstract: We find stable solutions from various complex surfaces for the σ_2-energy functional. They can be interpreted as instantons for the strong coupling limit of the Skyrme–Faddeev sigma model.

1. Preliminaries

Introduced almost synchronously in differential geometry ([7]) and in nuclear physics ([16]), the σ_2-energy of a C^1 map $\varphi : (M, g) \to (N, h)$ between (semi) Riemannian manifolds is defined in terms of the Hilbert–Schmidt norm as

$$\mathcal{E}_{\sigma_2}(\varphi) = \int_M | \wedge^2 \mathrm{d}\varphi|^2 \nu_g,$$

and it can be seen either as a measure of area elements stretching or a stabilizing term for the soliton solutions in a sigma-model based on the Dirichlet energy $\mathcal{E}_{\sigma_1}(\varphi) = \int_M |\mathrm{d}\varphi|^2 \nu_g$. Also the physics model was originally designed for maps into the three-sphere $\mathbb{S}^3$; a variant for $\mathbb{S}^2$-valued mappings has also drawn a lot of interest ([8]). Notice that the energy density can be calculated as $\sigma_2(\varphi^* h)$, the 2nd elementary symmetric function of the eigenvalues λ_i^2 of $\varphi^* h$ with respect to g.

In this paper we consider the conformally invariant σ_2-variational problem by letting φ be defined on a 4-manifold. Inspired by the terminology for Yang–Mills theory, a map $\varphi : (M^4, g) \to (N, h)$ will be called *instanton solution* for the strongly coupled Skyrme and Faddeev models if it is a stable smooth critical point of $\mathcal{E}_{\sigma_2}$ of finite energy. This type of solution is important in the quantized version of the respective field theory. The only instanton solution found so far is the suspension $\mathbb{S}^4 \to \mathbb{S}^3$ of the of Hopf map described in [18].

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We shall further assume the lowest relevant codomain dimension: N will be a surface. In this case the σ_2-energy equals the so called symplectic Dirichlet energy ([19]) $\mathcal{F}(\varphi) = \frac{1}{2}\int_M |\varphi^*\omega|^2 \nu_g$, where ω is the area 2-form on N. In this set-up the known topological lower bounds for $\mathcal{F}$ ([19,20]) become ineffective so we have to address directly the problem of stability. We succeed in finding instanton solutions on certain (classes of) compact complex surfaces endowed with a conformally Kähler structure and which are non-trivial sphere or torus bundles. It is noteworthy that their stability is due to the geometric property of having area-stable minimal fibres. In addition they are stable as harmonic maps (i.e., critical points of the Dirichlet energy), so they are stable solutions for the full sigma-model defined by the action $\mathcal{E}_{\sigma_1}(\varphi) + \alpha\mathcal{E}_{\sigma_2}(\varphi)$, $\alpha \in \mathbb{R}_+$.

1.1. σ_2-Critical maps. Let $\varphi : (M^4, g) \to (N^2, h)$ be a smooth submersion and λ_1^2 and λ_2^2 be the eigenvalues of φ^*h with respect to g. We call as usual *vertical* and *horizontal*, the distributions on M defined as $\mathcal{V} = \ker \mathrm{d}\varphi$ and $\mathcal{H} = \mathcal{V}^\perp$, respectively; accordingly any vector field X on M decomposes as $X = X^\mathcal{V} + X^\mathcal{H}$, where the superscript indicates the orthogonal projection onto the corresponding subspace. Recall that $B^\mathcal{V}(E, F) = \frac{1}{2}\left[(\nabla_{E^\mathcal{V}} F^\mathcal{V})^\mathcal{H} + (\nabla_{F^\mathcal{V}} E^\mathcal{V})^\mathcal{H}\right]$ $(E, F \in \Gamma(TM))$ is the *second fundamental form* of $\mathcal{V}$ and that $\mu^\mathcal{V} = \frac{1}{\dim \mathcal{V}}$ trace $B^\mathcal{V}$ is its *mean curvature*. If $B^\mathcal{V} = 0$, then $\mathcal{V}$ is called *totally geodesic*, and if $\mu^\mathcal{V} = 0$, then it is called *minimal*.

According to [17], φ is a critical point of $\mathcal{E}_{\sigma_2}$ (in short, σ_2-*critical*) if and only if, at any regular point,

$$-\operatorname{grad}^\mathcal{H}(\ln \lambda_1\lambda_2) + 2\mu^\mathcal{V} = 0, \tag{1}$$

or, equivalently, if and only if ([19])

$$\mathrm{d}\varphi\left((\delta\varphi^*\omega)^\sharp\right) = 0. \tag{2}$$

In particular, a Riemannian or a horizontally homothetic submersion ([2]) with minimal fibres is (harmonic and) σ_2-critical, since in this case all eigenvalues are equal and constant in horizontal directions. To see directly the equivalence between Eqs. (1) and (2), denote $Z_\varphi = (\delta\varphi^*\omega)^\sharp$ and, around a regular point, consider an orthonormal frame of eigenvectors of φ^*h, $\{U_1, U_2, \lambda_1 X_1, \lambda_2 X_2\}$, with U_1 and U_2 vertical, and X_1 and X_2 *basic* vectors (i.e., horizontal and projectable on N); we assume that N is oriented and $\varphi^*\omega(X_1, X_2) = 1$. Then

$$\begin{aligned} Z_\varphi = \lambda_1^2\lambda_2^2\Big(&- [X_1, X_2]^\mathcal{V} - g(2\mu^\mathcal{V} - \operatorname{grad}^\mathcal{H}(\ln \lambda_1\lambda_2), X_2)X_1 \\ &+ g(2\mu^\mathcal{V} - \operatorname{grad}^\mathcal{H}(\ln \lambda_1\lambda_2), X_1)X_2\Big). \end{aligned}$$

In particular, $Z_\varphi^\mathcal{V} = 0$, or equivalently $\varphi^*\omega$ is co-closed, if and only if $\mathcal{H}$ is integrable, and the solutions having this extra-property were shown to be minimizers in their homotopy class ([20]). None of the instanton solutions described below will be of this type (since they are non-trivial fibrations).

2. σ_2-Stable Maps with Area-Stable Fibres

In this section we recall the basic material of the minimal submanifolds theory, cf. [15] (we use the notations in [2, pp. 97]), and inspired by [10] we show that having minimal area-stable fibres forces a submersion with constant σ_2-energy density to be a stable σ_2-critical point.

All manifolds are supposed to be smooth, compact, connected and orientable, and by ν_g we denote the volume element associated to the metric g on a manifold M, so that $\mathrm{Vol}(M) = \int_M \nu_g$. We use the following sign conventions for the curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and $\Delta f = -\operatorname{trace}\nabla \mathrm{d} f$ for the Laplacian on functions.

Definition 1. Let $\psi : (P^p, \kappa) \to (M^m, g)$ be a smooth map of Riemannian manifolds with P compact. The *volume of* ψ is defined as

$$\mathrm{Vol}(\psi) = \int_P \sqrt{\det(\psi^* g)}\nu_\kappa = \int_P \lambda_1 \cdots \lambda_p \nu_\kappa.$$

where the λ_i^2 are the the eigenvalues of $\psi^* g$ with respect to κ.

An isometric immersion ψ is called *minimal* if it is a critical point of the volume functional (this is known to be equivalent with the vanishing of its mean curvature).

Let νP denote the *normal bundle*. The *Jacobi operator* of an isometric minimal immersion is

$$\mathcal{J}_\psi^{\mathrm{Vol}} : \Gamma(\nu P) \to \Gamma(\nu P), \quad \mathcal{J}_\psi^{\mathrm{Vol}}(v) = -\operatorname{trace}(\nabla^\nu)^2 v - \widetilde{R}(v) - S^t \circ S(v),$$

where $\nabla_X^\nu v = (\nabla_X^\psi v)^\perp$ is the connection on νP, $S : \nu P \to \mathrm{Sym}(TP)$, $S_v X = (\nabla_X^\psi v)^{\mathrm{T}}$ is the Weingarten map (shape operator) of ψ, and $\widetilde{R}(v) = \sum_{i=1}^p [R(v, e_i)e_i]^\perp$, $\{e_i\}$ being an orthonormal frame on P. The submanifold P is called *volume-stable* if the second variation of the volume around ψ is positive semi-definite, i.e. for any $v \in \Gamma(\nu P)$,

$$\mathrm{Hess}_\psi^{\mathrm{Vol}}(v, v) := \int_P \langle \mathcal{J}_\psi^{\mathrm{Vol}}(v), v\rangle \nu_{\psi^* g} \geq 0.$$

If P is a surface with this property, we call it *area-stable*.

Let $\varphi : (M^m, g) \to (N^2, h)$ a non-constant submersive map with $m \geq 3$. By the coarea formula [10, Lemma 4.1 (iv)]

$$\int_M \lambda_1\lambda_2 \, \nu_g = \int_{z\in N} \mathrm{Vol}(\varphi^{-1}(z))\nu_h, \tag{3}$$

where in the right hand term we integrate over N the function $\mathrm{Vol}(\varphi^{-1}(.))$ that associates to each $z \in N$ the volume of the corresponding fibre $\varphi^{-1}(z)$. Then the Cauchy–Schwarz inequality yields

$$\mathcal{E}_{\sigma_2}(\varphi) = \int_M \lambda_1^2\lambda_2^2 \, \nu_g \geq \frac{1}{\mathrm{Vol}(M)} \left(\int_{z\in N} \mathrm{Vol}(\varphi^{-1}(z))\nu_h\right)^2. \tag{4}$$

Let φ_t be a smooth variation of $\varphi = \varphi_0$. Denote by $f(t) = \mathcal{E}_{\sigma_2}(\varphi_t)$ and by $g(t) = \frac{1}{\mathrm{Vol}(M)}\left(\int_{z\in N} \mathrm{Vol}(\varphi_t^{-1}(z))\nu_h\right)^2$, so that, by (4), we have $f(t) \geq g(t)$. If $\lambda_1\lambda_2 = cst.$, then we have $f(0) = g(0)$; if, in addition, the fibres are minimal, then $f'(0) = g'(0) = 0$. As in [10] we conclude that $f''(0) \geq g''(0)$. But

$$g''(0) = \frac{2}{\mathrm{Vol}(M)} \left(\int_{z\in N} \mathrm{Vol}(\varphi^{-1}(z))\nu_h\right) \left(\int_{z\in N} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathrm{Vol}(\varphi_t^{-1}(z))\Big|_{t=0} \nu_h\right).$$

By combining with [10, Theorem 5.2], we obtain the following

Proposition 1. *Let $\varphi : (M, g) \to (N^2, h)$ be a submersive σ_2-critical mapping with $\lambda_1\lambda_2 = cst.$ and volume-stable minimal fibres. Then φ is σ_2-stable. If moreover $\lambda_1 = \lambda_2$, then φ is also a stable harmonic map.*

Remark 1. This result supports various extensions beyond the dimensions we are here interested in. (i) A submersive σ_n-critical mapping into (N^n, h) with constant $\sigma_n(\varphi^*h)$ and volume-stable minimal fibres is σ_n-stable. (ii) A homothetic mapping into (N^3, h) with volume-stable minimal fibres is σ_2-stable.

Notice that the applicability of Proposition 1 is restricted to locally trivial fibre bundles (since M is compact). Since the index of a minimal surface in $\mathbb{S}^4$ is at least 2 ([15]), Proposition 1 cannot be applied for mappings defined on the 4-sphere.

3. Instanton Solutions on Complex Surfaces with Vaisman Structures

The complex surfaces that admit Vaisman structures were classified in [3]. They include proper elliptic surfaces, the primary and secondary Kodaira surfaces and the elliptic Hopf surfaces. Projecting along their *canonical foliation* (supposed to be regular) spanned by B and JB provides us with a class of instanton solutions for the σ_2-variational problem.

Let us recall first some facts in l.c.K. geometry that will be useful in the sequel.

Definition 2 ([13]). A Hermitian manifold (M, g, J) is called *locally conformal Kähler* (l.c.K.) if it exists a (globally defined) closed 1-form θ, the *Lee form*, such that $d\Omega = \theta \wedge \Omega$, where $\Omega(X, Y) = g(X, JY)$. If $\dim_{\mathbb{R}} M > 2$, then for all vectors X and Y

$$(\nabla_X J)(Y) = \tfrac{1}{2}\left(\theta(JY)X - \theta(Y)JX + g(X, Y)JB - \Omega(X, Y)B\right), \tag{5}$$

where the *Lee vector field* $B = \frac{2}{2-\dim M} J \operatorname{div} J$ is the dual of θ. If the Lee form is exact, the manifold is *globally conformal Kähler* (g.c.K.). A *Vaisman manifold* [21] is an l.c.K. manifold with parallel Lee form.

Notice two basic l.c.K. identities: $\nabla_B J = 0$ and $\nabla_{JB} J = 0$.

Recall that on a Vaisman manifold B and JB are commuting, Killing and holomorphic vector fields. They generate a foliation $\mathcal{V}$, called the *canonical foliation*. Then, denoting by $\mathcal{H}$ its orthogonal complement, the metric splits as $g = g^{\mathcal{V}} + g^{\mathcal{H}}$ and one can see that a Vaisman metric with unitary Lee vector field conserve this property through a *0-type deformation* ([3]) defined by $g_k = k^2 g^{\mathcal{V}} + k g^{\mathcal{H}}$ $(k > 0)$.

Proposition 2. *The projection to the leaf space (endowed with the metric that renders the projection a Riemannian submersion) from a compact Vaisman four manifold with regular canonical foliation is both stable harmonic and stable σ_2-critical.*

Proof. Let (M, g, J) be a compact Vaisman four-manifold with regular canonical foliation $\mathcal{V}$. One knows ([5,21]) that $\mathcal{V}$ is a totally geodesic foliation by flat 2-tori and that the leaf space $N = M/\mathcal{V}$ is a compact manifold that can be endowed with a Kähler metric such that the projection $\varphi : M \to N$ is a (holomorphic) Riemannian submersion, so clearly a harmonic and σ_2-critical mapping.

Since B is parallel, $|B|$ is constant; we suppose the metric normalized such that $|B| = 1$.

Let F be a fibre of φ (all the fibres are isometric to F since $\mathcal{V}$ is totally geodesic). Then νF is trivial and, since φ is a holomorphic Riemannian submersion, we can choose a global orthonormal frame $\{E_1, E_2 = JE_1\}$ in νF formed by basic vectors. Then any $v \in \Gamma(\nu F)$, can be written as $v = f_1 E_1 + f_2 E_2$ for some smooth functions $f_{1,2} : F \to \mathbb{R}$. Since E_i are basic, $[B, E_i] \in \mathcal{V}$, so $g(\nabla_B E_1, E_2) = g(\nabla_{E_1} B, E_2) = 0$. From (5) we obtain $\nabla_X JB = -\frac{1}{2} JX$ for any $X \perp \mathcal{V}$, and since $[JB, E_i] \in \mathcal{V}$, we have $g(\nabla_{JB} E_1, E_2) = g(\nabla_{E_1} JB, E_2) = -\frac{1}{2}$. By employing these observations we obtain

$$\begin{aligned} &\mathrm{trace}(\nabla^{\nu})^2 v = (-\Delta^F f_1 + JB(f_2) - \tfrac{1}{4} f_1) E_1 + (-\Delta^F f_2 - JB(f_1) - \tfrac{1}{4} f_2) E_2, \\ &\widetilde{R}(v) = (R(v, JB)JB)^{\perp} = \tfrac{1}{4} v, \end{aligned}$$

where Δ^F is the Laplace operator on F. Therefore, for the isometric immersion $\psi : (F, g_F) \to (M, g)$,

$$\mathcal{J}_{\psi}^{\mathrm{Vol}}(v) = (\Delta^F f_1 - JB(f_2)) E_1 + (\Delta^F f_2 + JB(f_1)) E_2.$$

Since JB is Killing, $\mathcal{J}_{\psi}^{\mathrm{Vol}}$ leaves invariant the orthogonal subspaces

$$S_{\lambda_k}^{\psi} = \{f_1 E_1 | \Delta^F f_1 = \lambda_k^F f_1\} \oplus \{f_2 E_2 | \Delta^F f_2 = \lambda_k^F f_2\} \quad (k \in \mathbb{N}), \tag{6}$$

where $0 = \lambda_0^F < \lambda_1^F \leq \cdots \leq \lambda_k^F \leq \ldots$ is the spectrum of Δ^F. Therefore it suffices to show that $\mathcal{J}_{\psi}^{\mathrm{Vol}}$ is positive semi-definite on each $S_{\lambda_k}^{\psi}$. Let $v \in S_{\lambda_k}^{\psi}$. Since JB is Killing, $\int_F JB(f) \nu_{g_F} = 0$ for any function f, so

$$\begin{aligned} \mathrm{Hess}_{\psi}^{\mathrm{Vol}}(v, v) &= \int_F \left\{ \lambda_k^F f_1^2 + |\operatorname{grad}^F f_2|^2 - 2 f_1 JB(f_2) \right\} \nu_{g_F} \\ &\geq \lambda_1^F \int_F f_1^2 \nu_{g_F} + \int_F |\operatorname{grad}^F f_2|^2 \nu_{g_F} - 2 \left(\int_F f_1^2 \nu_{g_F} \right)^{\frac{1}{2}} \left(\int_F |\operatorname{grad}^F f_2|^2 \nu_{g_F} \right)^{\frac{1}{2}}, \end{aligned}$$

where in the second line we have used the Cauchy–Schwarz inequality. Up to a 0-type deformation of the Vaisman metric, we may assume that $\lambda_1^F \geq 1$. This implies $\mathrm{Hess}_{\psi}^{\mathrm{Vol}}(v, v) \geq 0$, so each fibre is a minimal area-stable surface. Applying Proposition 1 yields the conclusion. □

In order to identify a 0-type deformation of the Vaisman metric on the domain having the property that $\lambda_1^F \geq 1$, one can use standard formulae on flat 2-tori ([4]). This is an easy task in the following examples (for more details see [13, Ch. 3]).

Example 1 (*Instanton on a Hopf surface*). Let $\varphi : \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^2(\frac{1}{2})$ be the Hopf map composed with the projection to the second factor, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Endow $\mathbb{S}^1$ with the line element $\frac{1}{4}\mathrm{d}t^2$, and $\mathbb{S}^3$ with the Berger metric $g_{1/2} = \frac{1}{4} g^v + g^h$ (where $g^v + g^h$ is the splitting of the canonical metric induced by the vertical and horizontal distributions associated to the Hopf map). Then with respect to the product metric on the domain and to the canonical metric on the codomain, φ is a Riemannian submersion with totally geodesic fibres, both stable harmonic and stable σ_2-critical according to Proposition 2 (since $B = 2\partial_t$ with $|B| = 1$ and $\lambda_1^F = 4$). This kind of squashed metric has also recently considered in the context of supersymmetric field theories defined on Hopf surfaces [1].

Example 2 (*Instanton on the Kodaira–Thurston surface*). Recall that Nil^3 is defined as the Lie group of all matrices with real entries $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ on which the subgroup with integer entries $\Gamma = \mathbb{Z}^3$ acts by left multiplication. Then the (compact) 3-dimensional Heisenberg nilmanifold is defined as $H_3 = \mathrm{Nil}^3/\Gamma$. The Kodaira–Thurston (compact) surface is defined as $\mathbb{S}^1 \times H_3$ endowed with the (Vaisman) metric $g = \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\mathrm{d}y)^2$. Let $\varphi : \mathbb{S}^1 \times H_3 \to \mathbb{T}^2$, be the projection to the 2-torus $\varphi(t, x, y, z) = (x, y)$. Then φ is a Riemannian submersion with totally geodesic fibres, both stable harmonic and stable σ_2-critical according to Proposition 2 (since $B = \partial_t$ with $|B| = 1$ and $\lambda_1^F = 1$). Moreover, by a similar argument as in Proposition 2, the circle bundle $\pi : H_3 \to \mathbb{T}^2$ is also a stable σ_2-critical map.

Notice that, by a result in [11] (cf. [2, Corollary 4.8.3]), it follows that the corresponding projections from the Berger 3-sphere and from the Heisenberg nilmanifold are stable as harmonic maps.

4. An Instanton Solution on a Hirzebruch Surface

In this section we identify a local σ_2-minimizer defined on the gravitational instanton (i.e. Einstein manifold) discovered by Page [14]. The underlying manifold is a nontrivial $\mathbb{S}^2$-bundle over $\mathbb{S}^2$ (recall that there are 2 inequivalent classes of such sphere bundles) and can be seen as $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$, yet another complex surface with (global) conformally Kähler structure ([6]) called the first Hirzebruch surface; for the explicit description of the Page metric g_0 see Appendix. The fibre bundle projection $\varphi : (\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}, g_0) \to (\mathbb{C}P^1, can)$ is a horizontally homothetic submersion with totally geodesic fibres ([2, pp.244]), so, in particular, a harmonic and σ_2-critical map.

Proposition 3. *The fibration mapping $\varphi : (\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}, g_1) \to (\mathbb{C}P^1, can)$, where g_1 is conformally related to the Page metric and renders φ a Riemannian submersion, is both stable harmonic and stable σ_2-critical.*

Proof. The first Hirzebruch surface $\Sigma_1 := \mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ can be seen as a quotient of $[-a, a] \times \mathbb{S}^3$ where each of the boundary spheres $\{\pm a\} \times \mathbb{S}^3$ is identified with a 2-sphere via the Hopf fibration. Let σ_i $(i = 1, 2, 3)$ denote the invariant 1-forms on $\mathbb{S}^3$ (associated to the Pauli matrices). On the open dense submanifold $(-a, a) \times \mathbb{S}^3$ in Σ_1 we define the metric

$$g_1 = \frac{1}{h(t)^2}\mathrm{d}t^2 + \sigma_1^2 + \sigma_2^2 + \frac{f(t)^2}{h(t)^2}\sigma_3^2,$$

that extends to Σ_1, is conformally related to the Page metric g_0 and renders the map $\varphi : \Sigma_1 \to \mathbb{C}P^1 \cong \mathbb{S}^2(\frac{1}{2})$ (locally given as Hopf map composed to the projection to the second factor) a Riemannian submersion.

Let us consider the orthonormal frame

$$E_1 = \partial_s, \quad E_2 = -\tan s\, \partial_{\phi_1} + \cot s\, \partial_{\phi_2}, \quad E_3 = \tfrac{h(t)}{f(t)}\left(\partial_{\phi_1} + \partial_{\phi_2}\right), \quad E_4 = h(t)\partial_t$$

on $\{(t, \cos s\, e^{i\phi_1}, \sin s\, e^{i\phi_2}) \in (-a, a) \times \mathbb{S}^3 | s \in (0, \frac{\pi}{2}), \phi_{1,2} \in [0, 2\pi]\}$. Notice that $\partial_{\phi_1} + \partial_{\phi_2}$ represents the Reeb vector field X_3 dual to σ_3 and that E_1 and E_2 are basic vector fields.

The vertical foliation of φ is spanned by E_3 and E_4 and is totally geodesic, so the fibres are all isometric. Let $\psi : (F, g_F) \to (\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}, g_1)$ denote the isometric immersion of a fibre F different from $\varphi^{-1}(\pm 1, 0, 0)$, so that $s \notin \{0, \frac{\pi}{2}\}$. Then the normal bundle νF is trivial and we choose the global orthonormal frame in νF given by E_1 and E_2, so that any section of νF can be written as $v = f_1 E_1 + f_2 E_2$ with $f_{1,2}$ smooth functions on F. A direct computation based on the explicit form of the connection yields $\widetilde{R}(v) = \frac{f(t)^2}{h(t)^2} v$ and

$$\mathcal{J}_\psi^{\mathrm{Vol}}(v) = (\Delta^F f_1 + 2X_3(f_2))E_1 + (\Delta^F f_2 - 2X_3(f_1))E_2. \tag{7}$$

We note that (7) can be obtained also by using a generic orthonormal *basic* frame $\{E_1, E_2 = JE_1\}$ in νF and the g.c.K. structure (g_1, J) on Σ_1 with Lee vector field $B = 2f(t)\partial_t$, as in the proof of Proposition 2.

One can directly verify that X_3 is Killing vector field along F, so $\mathcal{J}_\psi^{\mathrm{Vol}}$ leaves invariant the orthogonal subspaces

$$S_{\lambda_k}^\psi = \{f_1 E_1 | \Delta^F f_1 = \lambda_k^F f_1\} \oplus \{f_2 E_2 | \Delta^F f_2 = \lambda_k^F f_2\} \quad (k \in \mathbb{N}),$$

where $0 = \lambda_0^F < \lambda_1^F \leq \cdots \leq \lambda_k^F \leq \ldots$ is the spectrum of Δ^F. Therefore it suffices to show that $\mathcal{J}_\psi^{\mathrm{Vol}}$ is positive semi-definite on each $S_{\lambda_k}^\psi$. Let $v \in S_{\lambda_k}^\psi$. Again using that X_3 is Killing, we have

$$\begin{aligned}
\mathrm{Hess}_\psi^{\mathrm{Vol}}(v, v) &= \int_F \left\{\lambda_k^F f_1^2 + |\operatorname{grad}^F f_2|^2 + 4 f_1 X_3(f_2)\right\} \nu_{g_F} \\
&\geq \lambda_1^F \int_F f_1^2 \nu_{g_F} + \int_F |\operatorname{grad}^F f_2|^2 \nu_{g_F} - 4\left(\int_F f_1^2 \nu_{g_F}\right)^{\frac{1}{2}} \left(\int_F \tfrac{f(t)^2}{h(t)^2} E_3(f_2)^2 \nu_{g_F}\right)^{\frac{1}{2}} \\
&\geq \lambda_1^F \int_F f_1^2 \nu_{g_F} + \int_F |\operatorname{grad}^F f_2|^2 \nu_{g_F} - 4\sqrt{\kappa}\left(\int_F f_1^2 \nu_{g_F}\right)^{\frac{1}{2}} \left(\int_F |\operatorname{grad}^F f_2|^2 \nu_{g_F}\right)^{\frac{1}{2}}
\end{aligned}$$

where in the second line we used the Cauchy–Schwarz inequality and $\kappa = \sup \frac{f(t)^2}{h(t)^2}$. Thus $\mathrm{Hess}_\psi^{\mathrm{Vol}}(v, v)$ is clearly positive if $\lambda_1^F > 4\kappa$.

We are led to find estimates for λ_1^F. All the numerical values appearing below are easy to obtain using for example Mathematica. Since the Ricci curvature tensor of the fibre F reads

$$\begin{aligned}
\mathrm{Ric}^F(E_4, E_4) &= 4\left(1 - \frac{f(t)^2}{h(t)^2}\right) + 2h(t)^2\left(\frac{f'(t)h'(t)}{f(t)h(t)} - \frac{h'(t)^2}{h(t)^2}\right) \\
&\geq c > 2.93,
\end{aligned}$$

and F is a 2-dimensional (so automatically an Einstein) manifold, the classical Lichnerowicz–Obata estimate ([12]) gives $\lambda_1^F \geq 2c > 5.85$. For further reference we mention also that, since the fibres are 2-spheres, the genus $g = 0$ and Yang–Yau estimate ([22]) gives the upper bound $\lambda_1^F \leq 8\pi(g+1)\,\mathrm{Vol}(F)^{-1} < 6.91$.

Noticing that $\kappa \approx 0.27$, we see that $\lambda_1^F > 4\kappa$, so the fibre F is a minimal area-stable surface. Since the fibres are isometric, by Proposition 1 the result follows. □

Since the σ_2-energy is conformally invariant in dimension 4, we obtain

Corollary 1. *The fibration mapping* $\varphi : (\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}, g_0) \to (\mathbb{C}P^1, can)$*, where* g_0 *is the Page metric, is stable* σ_2*-critical.*

Moreover, since complex submanifolds are area minimizers in any Kähler manifold, we obtain also that φ is both stable σ_2-critical and stable harmonic with respect to the conformally related Calabi-extremal Kähler metric on the domain.

5. Appendix

For the convenience of the reader we reproduce here the main ingredients of the Page metric definition, following [9].

Let x_0 be the unique solution in the interval (0, 1) of the equation

$$x^4 + 4x^3 - 6x^2 + 12x - 3 = 0.$$

and $c_0 = \frac{(x_0^2-1)^2}{1+2x_0^2-\frac{x_0^4}{3}}$. Define the functions

$$z(x) = 1 - x^2 + \frac{c_0}{1-x^2}\left(\frac{x^4}{3} - 2x^2 - 1\right), \quad -x_0 < x < x_0,$$

and

$$t(x) = \int_0^x \frac{\mathrm{d}y}{\sqrt{z(y)}}, \quad -x_0 < x < x_0.$$

Set $a = \lim_{x \to x_0} t(x)$ and let $x(t)$ be the inverse function of $t(x)$. Choosing

$$f(t) = \sqrt{z(x(t))}, \quad h(t) = \sqrt{1 - x^2(t)}, \quad -a < t < a,$$

the following equations are satisfied

$$\frac{f''}{f} = -\frac{4}{h^2} + \frac{3f^2}{h^4} + \left(\frac{h'}{h}\right)^2, \quad \frac{h''}{h} = -\frac{f^2}{h^4} + \frac{f'h'}{fh}, \tag{8}$$

with the boundary conditions

$$\begin{aligned} &f(-a) = f(a) = 0, \quad f'(-a) = -f'(a) = 1, \quad f^{(2k)}(-a) = f^{(2k)}(a) = 0 \\ &h(-a) \neq 0, h(a) \neq 0, \quad h^{(2k+1)}(-a) = h^{(2k+1)}(a) = 0, \quad (k \in \mathbb{N}). \end{aligned}$$

Now let σ_i $(i = 1, 2, 3)$ be the invariant 1-forms on $\mathbb{S}^3$ (associated to the Pauli matrices) with the corresponding dual vector fields X_i satisfying

$$[X_1, X_2] = -2X_3, \quad [X_2, X_3] = -2X_1, \quad [X_3, X_1] = -2X_2.$$

The ODE's (8) are equivalent to the fact that

$$g_0 = \mathrm{d}t^2 + h(t)^2(\sigma_1^2 + \sigma_2^2) + f(t)^2\sigma_3^2,$$

is an Einstein metric on the cylinder $(-a, a) \times \mathbb{S}^3$, while the boundary conditions assures that g_0 extends to a metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. Recall also that $JX_1 = X_2$, $JX_3 = f(t)\partial_t$ induces the canonical orthogonal complex structure on $(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}, g_0)$.

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