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Almost Kähler Ricci Flows and Einstein and Lagrange–Finsler Structures on Lie Algebroids

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Abstract. In this work we investigate Ricci flows of almost Kähler structures on Lie algebroids when the fundamental geometric objects are completely determined by (semi) Riemannian metrics, or (effective) regular generating Lagrange/Finsler functions. There are constructed canonical almost symplectic connections for which the geometric flows can be represented as gradient ones and characterized by nonholonomic deformations of Grigory Perelman's functionals. The first goal of this paper is to define such thermodynamical type values and derive almost Kähler–Ricci geometric evolution equations. The second goal is to study how fixed Lie algebroid, i.e. Ricci soliton, configurations can be constructed for Riemannian manifolds and/or (co) tangent bundles endowed with nonholonomic distributions modelling (generalized) Einstein or Finsler–Cartan spaces. Finally, some examples of generic off-diagonal solutions for Lie algebroid-type Ricci solitons and (effective) Einstein and Lagrange–Finsler algebroids are provided.

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1. Introduction

Various theories of geometric flows have been studied intensively in the past decade. The most popular is the Ricci flow theory [1–3] finally elaborated by Perelman [4–6] in the form which allowed proofs of the Thurston and Poincaré conjectures (details and proofs are given in [7–9]). The main results are related to the evolution of Riemannian and Kähler metrics and symplectic curvature flows; see additional references in [10,11].

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It should be also emphasized that the Ricci flow theory has an increasing impact on physical mathematics. In a more general context, we were interested to study Ricci flow evolution models for non-Riemannian geometries, for instance, of nonholonomic manifolds endowed with compatible metric and nonlinear connection structures [12], metric compatible and noncompatible Riemann–Finsler–Lagrange spaces [13,14], noncommutative geometries and generalizations [15], fractional and/or diffusion Ricci flows [16], etc.

The purpose of this article is to formulate a geometric approach for the almost Kähler Ricci flows on/of Lie algebroids when the fundamental geometric objects are completely defined by a regular Lagrange, L, generating function, or a (generalized) Einstein metric $\mathbf{g}$ with conventional integer $(n+m)$-dimensional splitting. Similar constructions can be performed for certain effective (analogous) models and/or, in particular, for Finsler, F, fundamental functions.[1]

Here is an outline of the work. In Sect. 2, we survey the geometry of Lie algebroids and prolongations endowed with nontrivial N-connection structure and formulate an approach to the almost Kähler geometry on nonholonomic Lie algebroids. The Ricci flow theory for almost symplectic geometries determined by canonical metric compatible algebroid connections is studied in Sect. 3. Finally, we provide a series of examples of almost Kähler Ricci soliton solutions, related to (modified) gravity models and Lagrange–Finsler Lie algebroid mechanics in Sect. 4. The Appendix contains some necessary and important coefficient-type formulas and proofs.

2. Almost Kähler Lie Algebroids and N-Connections

In this section, we recall the basic definitions on nonholonomic Lie algebroids endowed with N-connection structure, see [18,19], and develop the approach for almost Kähler Lie algebroids.

2.1. Preliminaries

To begin, let us fix a real manifold V of dimension $n \geq 2$ and necessary smooth class and consider a conventional horizontal (h) and vertical (v) splitting determined by a Whitney sum for its tangent bundle TV,

$$\mathbf{N}: \; TV = hTV \oplus vTV. \tag{1}$$

Definition 2.1. A $h{-}v$ splitting $\mathbf{N}$ (1) defines a nonlinear connection, N-connection, structure.

We shall use boldface symbols to emphasize that the geometric objects on a space $\mathbf{V} = (V, \mathbf{N}, \mathbf{g}), \mathbf{TV}$, or $\mathbf{TM} = (TM, \mathbf{N}, L)$, are adapted to an

[1] In this paper, a nonholonomic manifold $(V, \mathcal{N})$, [$\dim V = n + m$ with finite $n, m \geq 2$] is modelled as a (semi) Riemannian one of necessary smooth class and endowed with a non-integrable (nonholonomic, equivalently, anholonomic) distribution $\mathcal{N}$. For geometric models of Lagrange–Finsler spaces, we can work with nonholonomic vector/tangent bundles, when $V = TM$ is the total space of the tangent bundle on a real manifold $M, \dim M = n$; see details and references in [17].

N-splitting (1).[2] In global form, the concept of N-connection was formulated by Ehresmann [20] in 1955. E. Cartan used such a geometric object in his works on Finsler geometry beginning 1935 (see [17] for references and an alternative definition of N-connections via exact sequences), but in coordinate form, $\mathbf{N} = N_i^a(x,y)\mathrm{d}x^i \otimes \partial/\partial y^a$. On (semi) Riemannian manifolds, an $h{-}v$ splitting of type (1) can be defined by a corresponding class of nonholonomic frames.

Let us consider a nonholonomic distribution on V defined by a generating function $\mathcal{L}(u)$, with nondegenerate Hessian $\tilde{h}_{ab} = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial y^a \partial y^b}$, $\det|\tilde{h}_{ab}| \neq 0$. If $V = TM$, and $n = m$, we can consider $\mathcal{L} = L(x,y)$ as a regular Lagrangian defining a Lagrange space [21]. For $L = F^2(x,y)$, where $F(x,\xi y) = \xi F(x,y), \xi > 0$, we get a homogeneous Finsler generating function (additional conditions are imposed on F for different models of Finsler-like geometries); see Sect. 4.4. We shall work with arbitrary $\mathcal{L}$ considering the Finsler configurations as certain particular ones distinguished by homogeneity conditions and additional assumptions.

Theorem 2.1. *The Euler–Lagrange equations* $\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial y^i} - \frac{\partial\mathcal{L}}{\partial x^i} = 0$, *where* $y^i = \mathrm{d}x^i/\mathrm{d}\tau$, *for* $x^i(\tau)$ *depending on real parameter* τ, *are equivalent to the "nonlinear" geodesic equations* $\frac{\mathrm{d}x^a}{\mathrm{d}\tau} + 2\tilde{G}^a(x,y) = 0$, *i.e. to the paths of a canonical semispray* $S = y^i\frac{\partial\mathcal{L}}{\partial x^i} - 2\tilde{G}^a\frac{\partial}{\partial y^a}$, *when* $\tilde{G}^a = \frac{1}{4}\ \tilde{h}^{a\ n+i}(\frac{\partial^2\mathcal{L}}{\partial y^{n+i}\partial x^k}y^{n+k} - \frac{\partial\mathcal{L}}{\partial x^i})$, *where* $\tilde{h}^{ab}$ *is inverse to* $\tilde{h}_{ab}$, *and the set of coefficients* $\tilde{N}_i^a = \frac{\partial\tilde{G}^a}{\partial y^{n+i}}$ *defines an N-connection structure* (1).

An explicit proof consists from straightforward computations. We put "tilde" on some geometric objects to emphasize that they are generated by $\mathcal{L}$.

Corollary 2.1. *Any prescribed generating function* $\mathcal{L}$ *on* $\mathbf{V}$ *defines a canonical distinguished metric, d-metric* $\tilde{\mathbf{g}}$, *which is adapted to the N-connection splitting determined by data* $\tilde{h}_{ab}$ *and* $\tilde{N}_i^a$, *i.e.* $\tilde{\mathbf{g}} = h\tilde{g} + v\tilde{g}$, *when*

[2] *Notational Remarks:* All constructions in this work can be performed in coordinate-free form. Nevertheless, to find/generate explicit examples of solutions of systems of partial differential equations, PDEs, it is necessary to consider some adapted frame and coordinate systems. Working with different classes of geometries and spaces, it is more convenient to treat indices also as abstract labels. This simplifies the formulas and suggests ideas on how certain proofs are possible when local constructions have an "obvious" global extension. For instance, index-type formulas are largely used in G. Perelman's works and for applications in mathematical relativity and/or geometric mechanics.

For a conventional $n+m$ splitting, the local coordinates $u = (x,y)$ can be labelled in the form $u^\alpha = (x^i, y^a)$, where $i,j,k,\ldots = 1,2,\ldots n$ and $a,b,c\ldots = n+1, n+2, \ldots, n+m$, and x^i and y^a are, respectively, the h and v coordinates. A general local base is written $e_{\alpha'} = (e_{i'}, e_{a'})$ for some frame transforms $e_{\alpha'} = e^\alpha_{\alpha'}(u)\partial_\alpha$, where $\partial_\alpha = \partial/\partial u^\alpha = (\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a)$, and define corresponding dual transforms with inverse matrices $e^{\alpha'}_\alpha(u)$, when $e^{\alpha'} = e^{\alpha'}_\alpha(u)\mathrm{d}u^\alpha$. Calligraphic symbols will be considered for algebroid configurations. The left "up–low" indices will be considered as abstract labels without summation rules. Finally, we note that the Einstein summation rule on "low–up" repeating indices will be applied if the contrary is not stated.

$$\tilde{\mathbf{g}} = \tilde{g}_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + \tilde{h}_{ab}\tilde{\mathbf{e}}^a \otimes \tilde{\mathbf{e}}^b, \quad \tilde{g}_{ij} = \tilde{h}_{n+i\ n+j}, \tag{2}$$

$$\tilde{\mathbf{e}}_\alpha = (\tilde{\mathbf{e}}_i = \partial_i - \tilde{N}_i^a \partial_a, e_a = \partial_a), \quad \tilde{\mathbf{e}}^\alpha = (e^i = \mathrm{d}x^i, \tilde{\mathbf{e}}^a = \mathrm{d}y^a + \tilde{N}_i^a \mathrm{d}x^i) \tag{3}$$

The proof is a similar to that for the Sasaki lift from M to TM; see [22,23].

2.2. Distinguished Lie Algebroids and Prolongations

In Refs. [18,19], we introduced

Definition 2.2. A Lie distinguished algebroid (d-algebroid) $\mathcal{E} = (\mathbf{E}, \lfloor \cdot,\cdot \rfloor, \rho)$ over a manifold M is defined by (1) an N-connection structure, $\mathbf{N} : TE = hE \oplus vE$, and (2) a Lie algebroid structure determined by (2a) a real *vector bundle* $\tau : \mathbf{E} \to M$ together with (2b) a *Lie bracket* $\lfloor \cdot,\cdot \rfloor$ on the spaces of global sections $\mathrm{Sec}(\tau)$ of map τ and (2c) an *anchor* map $\rho : \mathbf{E} \to TM$, i.e. a bundle map over identity and constructed such that for the homomorphism $\rho : \mathrm{Sec}(\tau) \to \mathcal{X}(M)$ of $C^\infty(M)$-modules $\mathcal{X}$, this map satisfies the condition

$$\lfloor X, fY \rfloor = f \lfloor X, Y \rfloor + \rho(X)(f)Y, \quad \forall X, Y \in \mathrm{Sec}(\tau) \text{ and } f \in C^\infty(M).$$

If $\mathbf{N}$ is integrable (i.e. with trivial N-connection structure), a Lie d-algebroid is just a *Lie algebroid* $\mathcal{E} = (E, \lfloor \cdot,\cdot \rfloor, \rho)$. For "non-boldface" constructions, see details in Refs. [24–30]. The anchor map ρ is equivalent to a homomorphism between the Lie algebras $(\mathrm{Sec}(\tau), \lfloor \cdot,\cdot \rfloor)$ and $(\mathcal{X}(M), \lfloor \cdot,\cdot \rfloor)$.[3]

Example 2.1. (Nonholonomic Lie algebroids) If $\mathbf{E} = TV$ for a nonholonomic manifold $\mathbf{V} = (V, \mathbf{N})$ with N-splitting (1), the values like $\mathcal{X}(M)$ are considered for $M \to hV$ and sections τ are modelled on vV. We shall construct Ricci soliton configurations, or Einstein manifolds, with Lie algebroid symmetry in Sect. 4.

Let us extend the concept of prolongation Lie algebroid [27–30] (in our case) to include N-connections. We consider a Lie d-algebroid $\mathcal{E} = (\mathbf{E}, \lfloor \cdot,\cdot \rfloor, \rho)$ and a fibration $\pi : \mathbf{P} \to M$ both defined over the same manifold M. In general, $\mathbf{E}, \mathbf{P}$ and $\mathbf{TM}$, or $\mathbf{TV}$, may be enabled with different N-connection structures. The local coordinates are $u^{\underline{\alpha}} = (x^i, y^A) \in P$ when $\{e_a\}$ will be used for a local basis of sections of $\mathbf{E}$. If $\mathbf{E} = \mathbf{TV}$ we write $u^{\underline{\alpha}} = (x^i, y^I)$. In our constructions, we can consider that $\mathbf{P} = \mathbf{E}$. The anchor map $\rho : \mathbf{E} \to \mathbf{TM}$

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[3] Locally, the properties of a Lie d-algebroid $\mathcal{E}$ are determined by the functions $\rho_a^i(x^k)$ and $C_{ab}^e(x^k)$, where $x = \{x^k\}$ are local coordinates on a chart U on M (or on hV if a nonholonomic manifold is considered), with $\rho(e_a) = \rho_a^i(x)\mathbf{e}_i$ and $\lfloor e_a, e_b \rfloor = \mathbf{C}_{ab}^f(x)e_f$, satisfying the following equations $\rho_a^i \mathbf{e}_i \rho_b^j - \rho_b^i \mathbf{e}_i \rho_a^j = \rho_f^j C_{ab}^f$ and $\sum_{\mathrm{cycl}\ (a,b,f)} \left(\rho_a^i \partial_i C_{be}^f + C_{be}^d C_{ad}^f \right) = 0$. Boldface operators are defined by N-coefficients similarly to (3). For trivial N-elongated partial derivatives and differentials, we can use local coordinate frames when $\mathbf{e}_i \to \partial_i, e_a = \partial_a, e^i = \mathrm{d}x^i, \mathbf{e}^a \to \mathrm{d}y^a$.

The exterior differential on $\mathcal{E}$ with nonholonomic $\mathbf{E}$ can be defined in standard form with E. We introduce on $\mathcal{E}$ the operator $d : \mathrm{Sec}(\bigwedge^k \tau^*) \to \mathrm{Sec}(\bigwedge^{k+1} \tau^*), d^2 = 0$, where $\bigwedge$ is the antisymmetric product operator. The contributions of an N-connection can be seen from such formulas for a smooth formula $f : M \to \mathbb{R}, \mathrm{d}f(X) = \rho(X)f$, for $X \in \mathrm{Sec}(\tau)$, when $\mathrm{d}x^i = \rho_a^i \mathbf{e}^a$ and $\mathrm{d}\mathbf{e}^f = -\frac{1}{2} C_{ab}^f \mathbf{e}^a \wedge \mathbf{e}^b$. With respect to any section X on M, we can define the Lie derivative $\mathcal{L}_X = i_X \circ d + d \circ i_X :\ \mathrm{Sec}(\bigwedge^k \tau^*) \to \mathrm{Sec}(\bigwedge^k \tau^*)$, using the cohomology operator d and its inverse i_X.

and the tangent map $\mathbf{T}\pi : \mathbf{TP} \to \mathbf{TM}$ are all defined to be h–v adapted for nonholonomic bundles and/or fibered structures. These structures can be used to construct the subset

$$\mathcal{T}_s^E\mathbf{P} := \{(b, v) \in \mathbf{E}_x \times T_x\mathbf{P}; \rho(b) = T_p\pi(v); p \in \mathbf{P}_x, \pi(p) = x \in M\} \quad (4)$$

and prove this result:

Theorem 2.2. (Definition) *The the prolongation* $\mathcal{T}^{\mathbf{E}}\mathbf{P} := \bigcup_{s\in S} \mathcal{T}_s^{\mathbf{E}}\mathbf{P}$ *of a nonholonomic* $\mathbf{E}$ *over* π *is another Lie d-algebroid* (*the construction* (4) *can be considered for any set of charts covering such spaces*).

Similarly to holonomic configurations with trivial N-connection structure, the prolongation Lie d-algebroid $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ is called the (nonholonomic) $\mathbf{E}$-tangent bundle to π, which is also a nonholonomic vector bundle over $\mathbf{P}$. The corresponding projection $\tau_{\mathbf{P}}^{\mathbf{E}}$ is just onto the first factor, $\tau_{\mathbf{P}}^{\mathbf{E}}(b, v) = b$ being adapted to the N-connection structures. The elements of $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ are parameterized by N-adapted triples $(p, b, v) \in \mathcal{T}^{\mathbf{E}}\mathbf{P} \to (b, v) \in \mathcal{T}^{\mathbf{E}}\mathbf{P}$ if that will not result in ambiguities. The N-adapted anchor $\rho^\pi : \mathcal{T}^{\mathbf{E}}\mathbf{P} \to \mathcal{T}\mathbf{P}$ is given by maps $\rho^\pi\ (p, b, v) = v$, i.e. projection onto the third factor when the h-components transforms into other h-components, etc. For more special cases, we can define the projection onto the second factor (i.e. a morphism of Lie d-algebroids over π), $\mathcal{T}\pi : \mathcal{T}^{\mathbf{E}}\mathbf{P} \to \mathbf{E}$, when $\mathcal{T}\pi(p, b, v) = b$. An element $(p_1, b_1, v_1) \in \mathcal{T}^{\mathbf{E}}\mathbf{P}$ is vertical if $\mathcal{T}\pi\ (p_1, b_1, v_1) = b_1 = 0$, i.e. such elements are of type $(p, 0, v)$ when v is a π-vertical vector (tangent to $\mathbf{P}$ at point p).

To understand the consequences of Theorem 2.2, let us consider some local constructions. In coefficient form, any element of a prolongation Lie d-algebroid $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ can be parameterized as $\overline{z} = (p, b, v) \in \mathcal{T}^{\mathbf{E}}\mathbf{P}$, for $b = z^a e_a$ and $v = \rho_a^i z^a \mathbf{e}_i + v^A\partial_A$, for $\partial/\partial y^A$, can be decomposed as $\overline{z} = z^a\mathcal{X}_a + v^A\mathcal{V}_A$. The couple $(\mathcal{X}_a, \mathcal{V}_A)$, with vertical $\mathcal{V}_A$, defines a local basis of sections of $\mathcal{T}^{\mathbf{E}}\mathbf{P}$. For such bases, we can write $\mathcal{X}_a = \mathcal{X}_a(p) = \left(e_a(\pi(p)), \rho_a^i\mathbf{e}_{i|p}\right)$ and $\mathcal{V}_A = \left(0, \partial_{A|p}\right)$, where partial derivatives and their N-elongations are taken in a point $p \in S_x$. It is also possible to elaborate on prolongation Lie d-algebroids an N-adapted exterior differential calculus. The anchor map $\rho^\pi(Z) = \rho_a^i Z^a\mathbf{e}_i + V^A\partial_A$ is an N-elongated operator acting on sections Z with associated decompositions of type $\overline{z}$. The corresponding Lie brackets are

$$\lfloor \mathcal{X}_a, \mathcal{X}_b \rfloor^\pi = C_{ab}^f\mathcal{X}_f, \quad \lfloor \mathcal{X}_a, \mathcal{V}_B \rfloor^\pi = 0, \quad \lfloor \mathcal{V}_A, \mathcal{V}_B \rfloor^\pi = 0.$$

Denoting by $\left(\mathcal{X}^a, \mathcal{V}^B\right)$ the dual bases to $(\mathcal{X}_a, \mathcal{V}_A)$, we can elaborate a differential calculus for N-adapted differential forms using

$$\mathrm{d}x^i = \rho_a^i\mathcal{X}^a, \quad \text{for } \mathrm{d}\mathcal{X}^f = -\frac{1}{2}C_{ab}^f\mathcal{X}^a \wedge \mathcal{X}^b, \qquad \text{and} \qquad \mathrm{d}y^A = \mathcal{V}^A, \quad \text{for } \mathrm{d}\mathcal{V}^A = 0. \quad (5)$$

N-connections can be introduced on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ similarly to (1).

Definition 2.3. On a prolongation Lie d-algebroid, an N-connection is defined by an h–v splitting,

$$\mathcal{N} : \mathcal{T}^{\mathbf{E}}\mathbf{P} = h\mathcal{T}^{\mathbf{E}}\mathbf{P} \oplus v\mathcal{T}^{\mathbf{E}}\mathbf{P}. \quad (6)$$

We can consider $\mathcal{N} : \mathcal{T}^E\mathbf{P} \rightarrow \mathcal{T}^E\mathbf{P}$, with $\mathcal{N}^2 = \mathrm{id}$, as a nonholonomic vector bundle, and Lie d-algebroid, morphism defining an almost product structure on ${}^P\pi : T\mathbf{P} \rightarrow \mathbf{P}$, for a smooth map on $T\mathbf{P}\backslash\{0\}$, where $\{0\}$ denotes the set of null sections. Any N-connection $\mathcal{N}$ induces h- and v-projectors for every element $\overline{z} = (p, b, v) \in \mathcal{T}^{\mathbf{E}}\mathbf{P}$, when $h(\overline{z}) = {}^hz$ and $v(z) = {}^vz$, for $h = \frac{1}{2}(\mathrm{id} + \mathcal{N})$ and $v = \frac{1}{2}(\mathrm{id} - \mathcal{N})$. The respective h- and v-subspaces are $h\mathcal{T}^{\mathbf{E}}\mathbf{P} = \ker(\mathrm{id} - \mathcal{N})$ and $v\mathcal{T}^{\mathbf{E}}\mathbf{P} = \ker(\mathrm{id} + \mathcal{N})$.

Let us analyze some local constructions related to $\mathcal{N}$-connection structures for prolongation Lie d-algebroids. Locally, N-connections are determined respectively by their coefficients $\mathbf{N} = \{N_i^A\}$ and $\mathcal{N} = \{\mathcal{N}_a^A\}$, when

$$\mathbf{N} = N_i^A(x^k, y^B)\mathrm{d}x^i \otimes \partial_A \quad \text{and} \quad \mathcal{N} = \mathcal{N}_a^A\mathcal{X}^a \otimes \mathcal{V}_A. \tag{7}$$

Such structures on $T\mathbf{P}$ and $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ are compatible if $\mathcal{N}_a^A = N_i^A\rho_a^i$. Using $\mathcal{N}_a^A$, we can generate sections $\delta_a := \mathcal{X}_a - \mathcal{N}_a^A\mathcal{V}_A$ as a local basis of $h\mathcal{T}^{\mathbf{E}}\mathbf{P}$.

Corollary 2.2. *Any N-connection $\mathcal{N}_a^A$ on $\mathcal{T}^E\mathbf{P}$ determines an N-adapted frame structure*

$$\mathbf{e}_{\overline{\alpha}} := \{\delta_a = \mathcal{X}_a - \mathcal{N}_a^C\mathcal{V}_C, \mathcal{V}_A\}, \tag{8}$$

and its dual

$$\mathbf{e}^{\overline{\beta}} := \{\mathcal{X}^a, \delta^B = \mathcal{V}^B + \mathcal{N}_c^B\mathcal{X}^c\}. \tag{9}$$

Proof. In the above formulas, the "overlined" small Greek indices split $\overline{\alpha} = (a, A)$ if an arbitrary vector bundle $\mathbf{P}$ is considered, or $\overline{\alpha} = (a, b)$ for $\mathbf{P} = \mathbf{E}$. A proof follows from an explicit construction of such N-adapted frames. Then the formulas can be considered for arbitrary frames of references. □

The N-adapted bases (8) satisfy the relations $\mathbf{e}_{\overline{\alpha}}\ \mathbf{e}_{\overline{\beta}} - \mathbf{e}_{\overline{\beta}}\mathbf{e}_{\overline{\alpha}} = W_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}\mathbf{e}_{\overline{\gamma}}$, with nontrivial anholonomy coefficients $W_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}} = \{C_{ab}^f, \Omega_{ab}^C, \partial_B\mathcal{N}_c^C\}$. The corresponding generalized Lie brackets are

$$\lfloor\delta_a, \delta_b\rfloor^\pi = C_{ab}^f\delta_f + \Omega_{ab}^C\mathcal{V}_C, \quad \lfloor\delta_a, \mathcal{V}_B\rfloor^\pi = (\partial_B\mathcal{N}_a^C)\mathcal{V}_C, \quad \lfloor\mathcal{V}_A, \mathcal{V}_B\rfloor^\pi = 0. \tag{10}$$

Definition 2.4. The curvature of N-connection $\mathcal{N}_a^A$ is by definition the Neigenhuis tensor ${}^hN$ of the operator h,

$$\begin{aligned} {}^hN(\cdot,\cdot) &= \lfloor h\cdot, h\cdot\rfloor^\pi - h\lfloor h\cdot, \cdot\rfloor^\pi - h\lfloor \cdot, h\cdot\rfloor^\pi + h^2\lfloor h\cdot, h\cdot\rfloor^\pi \\ &= -\frac{1}{2}\Omega_{ab}^C\mathcal{X}^a \wedge \mathcal{X}^b \otimes \mathcal{V}_C, \end{aligned}$$

where

$$\Omega_{ab}^C = \delta_b\mathcal{N}_a^C - \delta_a\mathcal{N}_b^C + C_{ab}^f\mathcal{N}_f^C. \tag{11}$$

It should be noted that the above formulas for $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ mimic (on sections of E for $\mathbf{P} = \mathbf{E}$) the geometry of tangent bundles and/or nonholonomic manifolds of even dimension, endowed with N-connection structure. It is applied in modern classical and quantum gravity, with various modifications, and nonholonomic Ricci flow theory; see Refs. [12,15,17]. If $\mathbf{P} \neq \mathbf{E}$ we model nonholonomic vector bundle and generalized Riemann geometries on sections of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$.

2.3. Canonical Structures on Lie d-Algebroids

Almost Kähler Lie algebroid geometries can be modelled on prolongation of Lie d-algebroids.

2.3.1. d-Connections and d-Metrics on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$.

Definition 2.5. A distinguished connection, d-connection, $\mathcal{D} = (h\mathcal{D}, v\mathcal{D})$, on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ is a linear connection preserving under parallelism the splitting (6).

The formulas for distinguished torsion and curvature can be generalized for prolongation of Lie d-algebroids.

Definition 2.6. The torsion and curvature for any d-connection $\mathcal{D}$ on $\mathcal{T}^{E}\mathbf{P}$ are defined, respectively, by

$$\mathcal{T}(\overline{x},\overline{y}) := \mathcal{D}_{\overline{x}}\overline{y} - \mathcal{D}_{\overline{y}}\overline{x} + \lfloor \overline{x},\overline{y} \rfloor^{\pi}, \quad \text{and}$$
$$\mathcal{R}(\overline{x},\overline{y})\overline{z} := \left(\mathcal{D}_{\overline{x}}\mathcal{D}_{\overline{y}} - \mathcal{D}_{\overline{y}}\mathcal{D}_{\overline{x}} - \mathcal{D}_{\lfloor \overline{x},\overline{y} \rfloor^{\pi}}\right)\overline{z}.$$

Let us consider sections $\overline{x},\overline{y},\overline{z}$ of $\mathcal{T}^{E}\mathbf{P}$ when, for instance $\overline{z} = z^{\overline{\alpha}}\mathbf{e}_{\overline{\alpha}} = z^{a}\delta_{a} + z^{A}\mathcal{V}_{A}$, or $\overline{z} = h\overline{z} + v\overline{z}$. Using the rules of absolute differentiation (5) for N-adapted bases $\mathbf{e}_{\overline{\alpha}} := \{\delta_a, \mathcal{V}_A\}$ and $\mathbf{e}^{\overline{\beta}} := \{\mathcal{X}^{\alpha}, \delta^{B}\}$ and associating to $\mathcal{D}$ a d-connection 1-form $\mathbf{\Gamma}^{\overline{\gamma}}_{\overline{\alpha}} := \mathbf{\Gamma}^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}}\mathbf{e}^{\overline{\beta}}$, we can compute the N-adapted coefficients of the torsion $\mathcal{T} = \{\mathbf{T}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\}$ and curvature $\mathcal{R} = \{\mathbf{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\overline{\delta}}\}$ 2-forms; see details in Appendix.

We can introduce a metric structure as a nondegenerate symmetric second rank tensor $\overline{\mathbf{g}} = \{\mathbf{g}_{\overline{\alpha}\overline{\beta}}\}$.

Proposition 2.1. (Definition) *Any metric structure on $\mathcal{T}^{E}\mathbf{P}$ can be represented in N-adapted form as a d-metric $\overline{\mathbf{g}} = h\mathbf{g} \oplus v\mathbf{g}$.*

Proof. It follows from an explicit construction adapted to a Whitney sum (6) and with respect to N-adapted frames; see formula (A.3). □

If a d-metric $\overline{\mathbf{g}}$ and a d-connection $\mathcal{D}$ are independent geometric structures, such values are characterized (additionally to $\mathcal{T}$ and $\mathcal{R}$) by a nonmetricity field $\mathcal{Q}(\overline{y}) := \mathcal{D}_{\overline{y}}\overline{\mathbf{g}}$, with N-adapted coefficients $\mathbf{Q}^{\overline{\gamma}}_{\overline{\alpha}\overline{\beta}} = \mathcal{D}^{\overline{\gamma}}\mathbf{g}_{\overline{\alpha}\overline{\beta}}$.

Proposition 2.2. (Definition) *The data $(\overline{\mathbf{g}}, \mathcal{D})$ are metric compatible if $\mathcal{Q} = \mathcal{D}\overline{\mathbf{g}} = 0$ holds in N-adapted form for any h- and v-components.*

Proof. It follows from a straightforward computation when the coefficients of d-metric $\mathbf{g}_{\overline{\alpha}\overline{\beta}}$ (A.3) are introduced into $\mathcal{D}_{\overline{y}}\overline{\mathbf{g}} = 0$, for $\overline{y} = y^{\overline{\alpha}}\mathbf{e}_{\overline{\alpha}} = y^{a}\delta_{a} + y^{A}\mathcal{V}_{A}$. Such a condition splits into respective conditions for h–v components which with respect to N-adapted frames are $\mathcal{D}_{f}\mathbf{g}_{ab} = 0, \mathcal{D}_{A}\mathbf{g}_{ab} = 0, \mathcal{D}_{f}\mathbf{g}_{AB} = 0, \mathcal{D}_{C}\mathbf{g}_{AB} = 0$. □

2.3.2. The Canonical d-Connection. On $\mathcal{T}^{\mathbf{E}}\mathbf{P}$, we can define two important metric compatible linear connection structures completely determined by $\mathbf{g}_{\overline{\alpha}\overline{\beta}}$. The first one is the standard torsionless Levi-Civita connection ∇ (which is not adapted to the N-connection splitting) and the second one is a Lie d-algebroid generalization, $\widehat{\mathbf{D}} \rightarrow \widehat{\mathcal{D}} = h\widehat{\mathcal{D}} + v\widehat{\mathcal{D}}$, when the h–v splitting is determined by (6).

Theorem 2.3. *There is a canonical d-connection* $\widehat{\mathcal{D}}$ *completely defined by data* $(\mathcal{N}, \mathbf{g}_{\overline{\alpha}\overline{\beta}})$ *for which* $\widehat{\mathcal{D}}\overline{\mathbf{g}} = 0$ *and h- and v-torsions of* $\widehat{\mathcal{T}}$ *are prescribed, respectively, to be with N-adapted coefficients* $\widehat{T}^a{}_{bf} = C^a{}_{bf}$ *and* $\widehat{T}^A{}_{BC} = 0$.

Proof. See proof in Sect. A.2. □

For trivial algebroid structures, $\widehat{\mathcal{D}} \to \widehat{\mathbf{D}}$ as on usual nonholonomic manifolds and/or vector bundles.

Remark 2.1.
- There is a canonical distortion relation $\widehat{\mathcal{D}} = \overline{\nabla} + \widehat{\mathcal{Z}}$, when both linear connections $\widehat{\mathcal{D}}$ and $\overline{\nabla}$ (the last one is the Levi-Civita connection) and the distorting d-tensor $\widehat{\mathcal{Z}}$ are defined by the same data $(\mathcal{N}, \mathbf{g}_{\overline{\alpha}\overline{\beta}})$. In explicit form, the N-adapted coefficients of such values are computed following formulas (A.5) and (A.6).
- We note that $h\widehat{\mathcal{T}}^\alpha = 0$ for $\widehat{\mathbf{D}}$ on $\mathbf{TM}$ but $h\widehat{\mathcal{T}}^\alpha \neq 0$ for $\widehat{\mathcal{D}}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$. The formulas for $\widehat{L}^a_{bf}$ in (A.5) contain additional terms with $C^a{}_{bf}$ which results in nontrivial $\widehat{T}^a{}_{bf} = C^a{}_{bf}$ and additional terms in N-adapted coefficients $\mathbf{R}^a{}_{ebf}$ and $\mathbf{R}^A{}_{Bbf}$ of curvature (A.2).

To generate exact solutions, it may be more convenient to work with an auxiliary d-connection ${}^c\widehat{\mathcal{D}} := \nabla + {}^c\widehat{\mathcal{Z}}$ for which $h\ {}^c\widehat{\mathcal{T}}^\alpha = 0$ and $v\ {}^c\widehat{\mathcal{T}}^\alpha = 0$. In N-adapted form, this results in ${}^c\widehat{T}^a_{bf} = 0$ and ${}^c\widehat{T}^A_{BC} = 0$ but, in general, $\widehat{\mathcal{R}} \neq {}^c\widehat{\mathcal{R}}$ and $\widehat{\mathcal{T}} \neq {}^c\widehat{\mathcal{T}}$. The nontrivial Lie d-algebroid structure is encoded in ${}^c\widehat{\mathcal{Z}}$ via structure functions ρ^i_a and $C^a{}_{bf}$ and N-elongated frames (8) and (9). The N-adapted coefficients of ${}^c\widehat{\mathcal{D}}$ are computed ${}^c\widehat{L}^a_{bf} = \frac{1}{2}\mathbf{g}^{ae}\left(\delta_f \mathbf{g}_{be} + \delta_b \mathbf{g}_{fe} - \delta_e \mathbf{g}_{bf}\right)$ and ${}^c\widehat{L}^A_{Bf} = \widehat{L}^A_{Bf}$, ${}^c\widehat{B}^\alpha_{\beta C} = \widehat{B}^\alpha_{\beta C}$, ${}^c\widehat{B}^A_{BC} = \widehat{B}^A_{BC}$ are those from (A.5).

2.4. Almost Symplectic Geometric Data

2.4.1. Semi-Spray Configurations and N-Connections. We show how a canonical almost symplectic structure can be generated on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ by any regular effective Lagrange function $\mathcal{L}$.

Lemma 2.1. *Prescribing any (effective) Lagrangian* $\mathcal{L}$, *we can construct a canonical N-connection* ${}^q\mathcal{N} := -\mathcal{L}i_q S$ *defined by a semi-spray* $q = y^a \mathcal{X}_a + q^A \mathcal{V}_A$ *and Lie derivative* $\mathcal{L}i_q$ *acting on any* $X \in Sec(T\mathbf{E})$ *following formula* ${}^q\mathcal{N}(X) = -\lfloor q, SX \rfloor^\pi + S\lfloor q, X\rfloor^\pi$.

Proof. We use the semi-spray formula $Sq = \triangle$ with the operators S and $\triangle$ from (A.9) and compute

$$ {}^q\mathcal{N}(\mathcal{X}_a) = -\lfloor q, S(\mathcal{X}_a)\rfloor^\pi + S\lfloor q, \mathcal{X}_a\rfloor^\pi = \mathcal{X}_a + (\partial_a q^b + y^f C^b_{fa})\mathcal{V}_b. $$

For ${}^q\mathcal{N}(\mathcal{V}_a) = -\mathcal{V}_a$ and ${}^q\mathcal{N}(\mathcal{X}_a) = \mathcal{X}_a - 2\ {}^q\mathcal{N}^f_a(x,y)\mathcal{V}_f$, we define the N-connection coefficients $\mathcal{N}^f_\alpha = -\frac{1}{2}(\partial_a q^f + y^b C^f_{ba})$; see formulas (7). □

We can formulate on Lie d-algebroids the analog of Theorem 2.1:

Theorem 2.4. *Any effective regular Lagrangian $\mathcal{L} \in C^{\infty}(\mathbf{E})$ defines a canonical N-connection on prolongation Lie algebroid $\mathcal{T}^{\mathbf{E}}\mathbf{E}$,*

$$\widetilde{\mathcal{N}} = \left\{ \widetilde{\mathcal{N}}_a^f = -\frac{1}{2}(\partial_a \varphi^f + y^b C_{ba}^f) \right\}, \tag{12}$$

determined by semi-spray configurations encoding the solutions of the Euler–Lagrange equations (A.12).

Proof. It is a straightforward consequence of the above lemma and (A.11). To generate N-connections, we can use sections $\Gamma_{\mathcal{L}} = y^a \mathcal{X}_a + \varphi^a \mathcal{V}_a$, with $q^e = \varphi^e(x^i, y^b)$ (A.11). Let us consider the value $S = y^a \rho_a^i \frac{\partial \mathcal{L}}{\partial x^i} - 2\tilde{G}^A \frac{\partial}{\partial y^A}$, where $\tilde{G}^A$ is associated with the "nonlinear" geodesic equations for sections, $\frac{\mathrm{d}y^A}{\mathrm{d}\tau} + 2\tilde{G}^A = 0$, depending on real parameter τ, and define an N-connection structure $\tilde{N}_a^F = \frac{\partial \tilde{G}^F}{\partial y^a}$ of type (7). For $\mathbf{P} = \mathbf{E}$ such sections can be related to the integral curves of the Euler–Lagrange equations (A.12) if we chose the sections $\tilde{G}^F \to \tilde{G}^f$ and $\varphi^f(x^i, y^b)$ in such forms that $\widetilde{\mathcal{N}}_a^f = -\frac{1}{2}(\partial_a \varphi^f + y^b C_{ba}^f) = \frac{\partial \tilde{G}^f}{\partial y^a}$. The constructions can be performed on any chart covering such spaces, i.e. we can prove that the coefficients (12) define an N-connection structure (6). □

Proposition 2.3. *Any metric structure on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ can be represented in N-adapted form as a d-metric $\overline{\mathbf{g}} = h\tilde{\mathbf{g}} \oplus v\tilde{\mathbf{g}}$ constructed as a formal Sasaki lift determined by an effective regular generating function $\mathcal{L}$.*

Proof. Sasaki lifts are used for extending certain metric structures from a base manifold, for instance, to the total space of a tangent bundle; see details in [22]. For vector bundles, the formulas (2) provide a typical example of such a construction using canonical N-connection structure $\tilde{\mathbf{N}}$. The method can be generalized for prolongation Lie d-algebroids. At the first step, we use the canonical N-connection $\widetilde{\mathcal{N}} = \{\widetilde{\mathcal{N}}_a^f\}$ (12) and construct N-adapted frames of type (8) and (9), respectively,

$$\widetilde{\mathbf{e}}_{\overline{\alpha}} := \{\widetilde{\delta}_a = \mathcal{X}_a - \widetilde{\mathcal{N}}_a^f \mathcal{V}_f, \mathcal{V}_b\} \quad \text{and} \quad \widetilde{\mathbf{e}}^{\overline{\beta}} := \{\mathcal{X}^a, \widetilde{\delta}^b = \mathcal{V}^b + \widetilde{\mathcal{N}}_f^b \mathcal{X}^f\}. \tag{13}$$

Then (the second step) we define a canonical d-metric

$$\widetilde{\mathbf{g}} := \widetilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}} \mathbf{e}^{\overline{\beta}} \otimes \mathbf{e}^{\overline{\beta}} = \tilde{g}_{ab}\, \mathcal{X}^a \otimes \mathcal{X}^b + \tilde{g}_{ab} \widetilde{\delta}^a \otimes \widetilde{\delta}^b \tag{14}$$

using the Hessian (A.10). Considering an arbitrary d-metric $\overline{\mathbf{g}} = \{\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'}\}$ (A.3), we can find a regular $\mathcal{L}$ and certain frame transforms $\mathbf{e}_{\overline{\gamma}'} = e^{\overline{\gamma}}_{\ \overline{\gamma}'} \tilde{\mathbf{e}}_{\overline{\gamma}}$ when $\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\overline{\beta}}_{\ \overline{\beta}'} \tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}$. We can work equivalently both with $\overline{\mathbf{g}}$ and/or $\tilde{\mathbf{g}}$ if a nonholonomic distribution $\mathcal{L}$ is prescribed and the vielbein coefficients $e^{\overline{\gamma}}_{\ \overline{\gamma}'}$ are defined as solutions of the corresponding algebraic quadratic system of equations for some chosen data $\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'}$ and $\tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}$. □

2.4.2. Riemann–Lagrange Almost Symplectic Structures. Let us consider canonical data defined, respectively, by N-elongated frames $\widetilde{\mathbf{e}}_{\overline{\alpha}} = (\widetilde{\mathbf{e}}_a = \delta_a, \mathcal{V}_A)$ (13), N-connection $\widetilde{\mathcal{N}}$ (12) and d-metric $\overline{\mathbf{g}} = \widetilde{\mathbf{g}}$ (14).

Proposition 2.4. (Definition) *For any regular effective Lagrange structure* $\mathcal{L}$, *we can define a canonical almost complex structure on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *following formulas* $\widetilde{\mathcal{J}}(\widetilde{\mathbf{e}}_a) = -\mathcal{V}_{m+a}$ *and* $\widetilde{\mathcal{J}}(\mathcal{V}_{m+a}) = \widetilde{\mathbf{e}}_a$, *when* $\widetilde{\mathcal{J}} \circ \widetilde{\mathcal{J}} = -\mathbb{I}$.

Proof. It follows from an explicit construction of a d-tensor field that

$$\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}^{\overline{\alpha}}_{\overline{\beta}} \widetilde{\mathbf{e}}_{\overline{\alpha}} \otimes \widetilde{\mathbf{e}}^{\overline{\alpha}} = -\mathcal{V}_{m+a} \otimes \mathcal{X}^a + \widetilde{\mathbf{e}}_a \otimes \widetilde{\delta}^a.$$

Using vielbeins $e^{\overline{\alpha}}_{\ \overline{\alpha}'}$ and their duals $e_{\overline{\beta}}^{\ \overline{\beta}'}$, we can compute the coefficients of $\widetilde{\mathcal{J}}$ with respect to any $\widetilde{e}_{\overline{\alpha}}$ and $\widetilde{e}^{\overline{\alpha}}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, when $\widetilde{\mathcal{J}}^{\overline{\alpha}}_{\ \overline{\beta}} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e_{\overline{\beta}}^{\ \overline{\beta}'} \widetilde{\mathcal{J}}^{\overline{\alpha}'}_{\ \overline{\beta}'}$. □

In general, we can define an almost complex structure $\mathcal{J}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ for an arbitrary N-connection $\mathcal{N}$ (6) by using N-adapted bases (8) and (9) which are not necessarily induced by an effective Lagrange function $\mathcal{L}$. This allows us to generate almost Hermitian models and not almost Kähler ones.

Definition 2.7. The Nijenhuis tensor field for any almost complex structure $\widetilde{\mathcal{J}}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ determined by an N-connection $\mathcal{N}$ (equivalently, the curvature of N-connection $\mathcal{N}$) is by definition

$$^{\mathcal{J}}\mathbf{\Omega}(\overline{x},\overline{y}) := -[\overline{x},\overline{y}]+[\mathcal{J}\overline{x},\mathcal{J}\overline{y}]-\mathcal{J}[\mathcal{J}\overline{x},\overline{y}]-\mathcal{J}[\overline{x},\mathcal{J}\overline{y}], \tag{15}$$

for any sections $\overline{x},\overline{y}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$.

We can introduce an arbitrary almost symplectic structure as a 2-form on a prolongation Lie d-algebroid.

Definition 2.8. An almost symplectic structure on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ is defined by a nondegenerate 2-form $\theta = \frac{1}{2}\theta_{\overline{\alpha}\overline{\beta}}(x^i, y^B) e^{\overline{\alpha}} \wedge e^{\overline{\beta}}$.

Using frame transforms, we can prove

Proposition 2.5. *For any* θ *on* $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ *when* $h\theta(\overline{x},\overline{y}) := \theta(h\overline{x}, h\overline{y}), v\theta(\overline{x},\overline{y}) := \theta(v\overline{x}, v\overline{y})$, *there is a unique N-connection* $\mathcal{N} = \{\mathcal{N}^A_a\}$ (7) *when*

$$\theta = (h\overline{x}, v\overline{y}) = 0 \quad \textit{and} \quad \theta = h\theta + v\theta. \tag{16}$$

Proof. In N-adapted form,

$$\theta = \frac{1}{2}\theta_{ab}(x^i, y^C)\mathcal{X}^a \wedge \mathcal{X}^b + \frac{1}{2}\theta_{AB}(x^i, y^C)\delta^A \wedge \delta^B, \tag{17}$$

where the first term is for $h\theta$ and the second term is $v\theta$, i.e. we get the second formula in (16). □

Definition 2.9.
- (a) An almost Hermitian model of a prolongation Lie d-algebroid $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ equipped with an N-connection structure $\mathcal{N}$ is defined by a triple $\mathcal{H}^{\mathbf{E}}\mathbf{E} = (\mathcal{T}^{\mathbf{E}}\mathbf{E}, \theta, \mathcal{J})$, where $\theta(\overline{x},\overline{y}) := \mathbf{g}(\mathcal{J}\overline{x},\overline{y})$.
- (b) A Hermitian prolongation Lie d-algebroid $\mathcal{H}^{\mathbf{E}}\mathbf{E}$ is almost Kähler, denoted $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, if and only if $\mathrm{d}\boldsymbol{\theta} = 0$.

For effective regular Lagrange configurations, we can formulate:

Theorem 2.5. *Having chosen a generating function* $\mathcal{L}$, *we can model equivalently a prolongation Lie d-algebroid* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *as an almost Kähler geometry, i.e.* $\mathcal{H}^{\mathbf{E}}\mathbf{E} = \mathcal{K}^{\mathbf{E}}\mathbf{E}$.

Proof. For the canonical geometric data $(\overline{\mathbf{g}} = \tilde{\mathbf{g}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{J}})$, we define the symplectic form $\tilde{\theta}(\overline{x}, \overline{y}) := \tilde{\mathbf{g}}(\mathcal{J}\overline{x}, \overline{y})$ for any sections $\overline{x}, \overline{y}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$. In local N-adapted form, $\tilde{\theta} = \tilde{g}_{ab}\delta^a \wedge \mathcal{X}^b$. Let us consider the form $\tilde{\omega} := \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^{m+a}}\mathcal{X}^a$. Using Proposition 2.5 and N-connection $\widetilde{\mathcal{N}}$ (12), we prove that $\tilde{\theta} = d\tilde{\omega}$, which means that $d\tilde{\theta} = dd\tilde{\omega} = 0$. The constructions can be redefined in arbitrary frames, $\theta_{\overline{\alpha}'\overline{\beta}'} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\overline{\beta}}_{\ \overline{\beta}'} \tilde{\theta}_{\overline{\alpha}\overline{\beta}}$, for a 2-form of type (17),

$$\tilde{\theta} = \frac{1}{2}\tilde{\theta}_{ab}(x^i, y^C)\mathcal{X}^a \wedge \mathcal{X}^b + \frac{1}{2}\tilde{\theta}_{AB}(x^i, y^C)\tilde{\delta}^A \wedge \tilde{\delta}^B. \tag{18}$$

□

2.4.3. N-Adapted Symplectic Connections. Let us consider how d-connection structures can be defined on $\mathcal{H}^{\mathbf{E}}\mathbf{E}$ and/or $\mathcal{K}^{\mathbf{E}}\mathbf{E}$.

Definition 2.10. An almost symplectic d-connection ${}^{\theta}\mathcal{D}$ for a model $\mathcal{H}^{\mathbf{E}}\mathbf{E}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, or (equivalently) a d-connection compatible with an almost symplectic structure θ, is defined such that this linear connection is N-adapted, i.e. a d-connection, and ${}^{\theta}\mathcal{D}_{\overline{x}}\theta = 0$, for any section $\overline{x}$ of $\mathcal{T}^{\mathbf{E}}\mathbf{E}$.

Lemma 2.2. *We can always fix a d-connection* ${}^{\circ}\mathcal{D}$ *on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *and then construct an almost symplectic* ${}^{\theta}\mathcal{D}$.

Proof. Let us consider a θ in N-adapted form (17). Introducing

$$\begin{aligned} {}^{\circ}\mathcal{D} &= \{h\ {}^{\circ}\mathcal{D} = ({}_{h}^{\circ}\mathcal{D}_a, {}_{v}^{\circ}\mathcal{D}_a);\ v\ {}^{\circ}\mathcal{D} = ({}_{h}^{\circ}\mathcal{D}_A, {}_{v}^{\circ}\mathcal{D}_A)\} \\ &= \{{}^{\circ}\mathbf{\Gamma}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = ({}^{\circ}\mathbf{L}^a_{bf},\ {}^{\circ}\mathbf{L}^A_{Bf};\ {}^{\circ}\mathbf{B}^a_{bC},\ {}^{\circ}\mathbf{B}^A_{BC})\}, \end{aligned}$$

we can verify that

$$\begin{aligned} {}^{\theta}\mathcal{D} &= \{h\ {}^{\theta}\mathcal{D} = ({}_{h}^{\theta}\mathcal{D}_a, {}_{v}^{\theta}\mathcal{D}_a);\ v\ {}^{\theta}\mathcal{D} = ({}_{h}^{\theta}\mathcal{D}_A, {}_{v}^{\theta}\mathcal{D}_A)\} \\ &= \{{}^{\theta}\mathbf{\Gamma}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = ({}^{\theta}\mathbf{L}^a_{bf},\ {}^{\theta}\mathbf{L}^A_{Bf};\ {}^{\theta}\mathbf{B}^a_{bC},\ {}^{\theta}\mathbf{B}^A_{BC})\},\ \text{with} \end{aligned}$$

$$\begin{aligned} {}^{\theta}\mathbf{L}^a_{bf} &= {}^{\circ}\mathbf{L}^a_{bf} + \frac{1}{2}\theta^{ae}\,{}_{h}^{\circ}\mathcal{D}_f\theta_{be}, & {}^{\theta}\mathbf{L}^A_{Bf} &= {}^{\circ}\mathbf{L}^A_{Bf} + \frac{1}{2}\theta^{AE}\,{}_{v}^{\circ}\mathcal{D}_f\theta_{EB}, \\ {}^{\theta}\mathbf{B}^a_{bC} &= {}^{\circ}\mathbf{B}^a_{bC} + \frac{1}{2}\theta^{ae}\,{}_{h}^{\circ}\mathcal{D}_C\theta_{be}, & {}^{\theta}\mathbf{B}^A_{BC} &= {}^{\theta}\mathbf{B}^A_{BC} + \frac{1}{2}\theta^{AE}\,{}_{v}^{\circ}\mathcal{D}_C\theta_{EB}, \end{aligned} \tag{19}$$

satisfies the conditions ${}_{h}^{\theta}\mathcal{D}_a\theta_{be} = 0, {}_{v}^{\theta}\mathcal{D}_a\theta_{AB} = 0, {}_{h}^{\theta}\mathcal{D}_A\theta_{be} = 0, {}_{v}^{\theta}\mathcal{D}_A\theta_{AB} = 0$, which are h- and v-projections of ${}^{\theta}\mathcal{D}_{\overline{x}}\theta = 0$ from Definition 4.1. □

Let us introduce the operators

$$\Theta^{ab}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \theta_{cd}\theta^{ab}) \quad \text{and} \quad \Theta^{AB}_{CD} = \frac{1}{2}(\delta^A_C\delta^B_D - \theta_{CD}\theta^{AB}). \tag{20}$$

Theorem 2.6. *The set of d-connections* ${}^{s}\mathbf{\Gamma}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = ({}^{s}\mathbf{L}^a_{bf},\ {}^{s}\mathbf{L}^A_{Bf};\ {}^{s}\mathbf{B}^a_{bC},\ {}^{s}\mathbf{B}^A_{BC})$ *labeled by an abstract left index "s", compatible with an almost symplectic structure* θ (17), *is parameterized by*

$$\begin{aligned} {}^{s}\mathbf{L}^a_{bc} &= {}^{\theta}\mathbf{L}^a_{bc} + \Theta^{da}_{be}\ \mathbf{Y}^e_{dc}, & {}^{s}\mathbf{L}^A_{Bc} &= {}^{\theta}\mathbf{L}^A_{Bc} + \Theta^{CA}_{BE}\ \mathbf{Y}^E_{Cc}, \\ {}^{s}\mathbf{B}^a_{bC} &= {}^{\theta}\mathbf{B}^a_{bC} + \Theta^{ea}_{bf}\ \mathbf{Y}^f_{eC}, & {}^{s}\mathbf{B}^A_{BC} &= {}^{\theta}\mathbf{B}^A_{BC} + \Theta^{EA}_{BF}\ \mathbf{Y}^F_{EC}, \end{aligned} \tag{21}$$

where the N-adapted coefficients are given by (19), *the* Θ*-operators are those from* (20) *and* $\mathbf{Y}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = (\mathbf{Y}^e_{dc}, \mathbf{Y}^E_{Cc}, \mathbf{Y}^f_{eC}, \mathbf{Y}^F_{EC})$ *are arbitrary d-tensor fields.*

Proof. It follows from straightforward N-adapted computations. □

Remark 2.2. The Lie algebroid structure functions C^d_{bf} in (A.5) can be considered as an example of d-tensor fields $\mathbf{Y}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ in (21). On $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, we can work as on $\mathbf{TM}$, but for different classes of nonholonomic distributions for sections. The d-connections $\widehat{\mathcal{D}}$, ${}^{\theta}\mathcal{D}$ can be constructed for correspondingly defined N-connection structures $\mathcal{N}, \widetilde{\mathcal{N}}$ when the main geometric properties are similar to some geometric models with $\widehat{\mathbf{D}}$, ${}^{\theta}\mathbf{D}$ and certain $\mathbf{N}, \widetilde{\mathbf{N}}$. The nonholonomic frame structures on Lie d-algebroids are different from those on nonholonomic tangent bundles because in the first case the vierbein fields encode the anchor- and Lie-type structure functions.

We can select a subclass of metric and/or almost symplectic compatible d-connections on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ which are completely defined by $\mathbf{g}$ and prescribed by an effective Lagrange structure $\mathcal{L}(x,y)$.

Theorem 2.7. *On $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, there is a unique normal d-connection* ${}^{n}\mathcal{D} =$

$$\begin{aligned}&\{h\ {}^{n}\mathcal{D} = ({}^{n}_{h}\mathcal{D}_a = \widehat{\mathcal{D}}_a, {}^{n}_{v}\mathcal{D}_a = \widehat{\mathcal{D}}_a); v\ {}^{n}\mathcal{D} = ({}^{n}_{h}\mathcal{D}_A = \widehat{\mathcal{D}}_A, {}^{n}_{v}\mathcal{D}_A = \widehat{\mathcal{D}}_A)\}\\ &= \{\ {}^{n}\mathbf{\Gamma}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = (\ {}^{n}\mathbf{L}^{a}_{bf} = \widehat{\mathbf{L}}^{a}_{bf},\ {}^{n}\mathbf{L}^{A=m+a}_{B=m+b\ \ f} = \widehat{\mathbf{L}}^{a}_{bf};\\ &\quad {}^{n}\mathbf{B}^{a=A-m}_{b=B-m\ \ C} = \widehat{\mathbf{B}}^{A}_{BC},\ {}^{n}\mathbf{B}^{A}_{BC} = \widehat{\mathbf{B}}^{A}_{BC})\},\end{aligned} \tag{22}$$

which is metric compatible, $\widehat{\mathcal{D}}_a\widetilde{\mathbf{g}}_{bc} = 0$ and $\widehat{\mathcal{D}}_A\widetilde{\mathbf{g}}_{BC} = 0$, and completely defined by $\overline{\mathbf{g}} = \tilde{\mathbf{g}}$ and a fixed $\mathcal{L}(x,y)$.

Proof. We provide a proof constructing such a normal d-connection in explicit form as an example when ${}^{n}\mathcal{D} = \widetilde{\mathcal{D}}$ generalizes the concept of Cartan d-connection in Lagrange–Finsler geometry on $\mathbf{TM}$. Such a d-connection is completely defined by couples of h- and v-components $\widetilde{\mathcal{D}} = (\widetilde{\mathcal{D}}_a, \widetilde{\mathcal{D}}_A)$, i.e. $\widetilde{\mathbf{\Gamma}}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = (\widehat{\mathbf{L}}^{a}_{bf}, \widehat{\mathbf{B}}^{A}_{BC})$. Let us choose

$$\begin{aligned}\widehat{\mathbf{L}}^{a}_{bf} &= \frac{1}{2}\widetilde{\mathbf{g}}^{ae}\ (\delta_f\widetilde{\mathbf{g}}_{be} + \delta_b\widetilde{\mathbf{g}}_{fe} - \delta_e\widetilde{\mathbf{g}}_{bf}),\\ \widehat{\mathbf{B}}^{A}_{BC} &= \frac{1}{2}\widetilde{\mathbf{g}}^{AD}\ (\mathcal{V}_C\widetilde{\mathbf{g}}_{BD} + \mathcal{V}_B\widetilde{\mathbf{g}}_{CD} - \mathcal{V}_D\widetilde{\mathbf{g}}_{BC}),\end{aligned} \tag{23}$$

where the N-elongated derivatives are taken in the form (13) and $\widetilde{\mathbf{g}}_{ab} = \widetilde{\mathbf{g}}_{A=m+a\ B=m+b}$ are generated by canonical values, using the Hessian (A.10) and (14) induced by a regular $\mathcal{L}(x,y)$; we can prove that this d-connection is unique and satisfies the conditions of the theorem. Via frame transforms, we can consider any metric structure $\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}$. □

Finally, we formulate this very important result for our purposes:

Theorem 2.8. *The normal d-connection ${}^{n}\mathcal{D} = \widetilde{\mathcal{D}}$ defines a unique almost symplectic d-connection, $\widetilde{\mathcal{D}} \equiv {}^{\theta}\widetilde{\mathcal{D}}$, see Definition 2.10, which is N-adapted and compatible to the canonical almost symplectic form $\tilde{\theta}$ (18), i.e. ${}^{\theta}\widetilde{\mathcal{D}}\tilde{\theta}=0$ and $\widetilde{\mathbf{T}}^{a}{}_{cb} = \widetilde{\mathbf{T}}^{A}{}_{CB} = 0$; see torsion coefficients (A.16).*

Proof. Using the coefficients (23), we can check that such a normal d-connection satisfies the conditions of this theorem. □

Conclusion 2.1. *Prescribing an effective generating function* $\mathcal{L}$ *on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, *we can transform this prolongation Lie d-algebroid into a canonical almost Kähler one,* $\mathcal{K}^{\mathbf{E}}\mathbf{E}$. *It is possible to work equivalently with any geometric data*

$$\left[\overline{\mathbf{g}}, \mathcal{N}, \widehat{\mathcal{D}} = \overline{\nabla} + \widehat{\mathcal{Z}}\right] \approx \left[\tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}\right] \approx \left[\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot,\cdot),\ {}^{\theta}\widetilde{\mathcal{D}}\right].$$

The Lie algebroid structure functions (ρ^i_a, C^f_{ab}) *are encoded into nonholonomic distributions on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *determining such equivalent prolongation Lie d-algebroid configurations.*

2.5. Almost Kähler Einstein and Lagrange Lie d-Algebroids

We can formulate analogues of Einstein equations for different classes of d-connections on prolongation Lie d-algebroids and almost Kähler models.

Corollary 2.3. (Definition) *The Ricci tensor of a d-connection* $\mathcal{D}$ *on a* $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ *endowed with a d-metric structure* $\overline{\mathbf{g}}$ *is defined following the formula* $\mathcal{R}ic = \{\mathbf{R}_{\overline{\alpha}\overline{\beta}} := \mathbf{R}^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\}$, *see the coefficients* (A.2) *for Riemannian d-tensor* $\mathcal{R}^{\overline{\alpha}}{}_{\overline{\beta}} = \{\mathbf{R}^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}\overline{\delta}}\}$, *and characterized by N-adapted coefficients*

$$\mathbf{R}_{\overline{\alpha}\overline{\beta}} = \{\mathbf{R}_{ab} \coloneqq \mathbf{R}^{c}{}_{abc},\ \mathbf{R}_{aA} \coloneqq -\mathbf{R}^{c}{}_{acA},\ \mathbf{R}_{Aa} \coloneqq \mathbf{R}^{B}{}_{AaB},\ \mathbf{R}_{AB} \coloneqq \mathbf{R}^{C}{}_{ABC}\}. \tag{24}$$

Proof. The formulas for h–v components (24) are respective contractions of the coefficients (A.2). □

The scalar curvature ${}^{s}\mathbf{R}$ of $\mathcal{D}$ is by definition

$${}^{s}\mathbf{R} := \mathbf{g}^{\overline{\alpha}\overline{\beta}}\mathbf{R}_{\overline{\alpha}\overline{\beta}} = \mathbf{g}^{ab}\mathbf{R}_{ab} + \mathbf{g}^{AB}\mathbf{R}_{AB}. \tag{25}$$

Using (24) and (25), we compute the Einstein d-tensor $\mathbf{E}_{\overline{\alpha}\overline{\beta}} := \mathbf{R}_{\overline{\alpha}\overline{\beta}} - \frac{1}{2}\mathbf{g}_{\overline{\alpha}\overline{\beta}}\ {}^{s}\mathbf{R}$ of $\mathcal{D}$. Such a tensor can be used for modeling effective gravity theories on sections of $\mathcal{T}^{\mathbf{E}}\mathbf{P}$ with nonholonomic frame structure [18,19].

3. Almost Kähler–Ricci Lie Algebroid Evolution

The goal of this section is to prove that N-adapted Ricci flow theories for almost Kähler models of prolongation Lie algebroids, $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, can be formulated as models of generalized gradient nonholonomic flows. We extend the Grisha Perelman's geometric thermodynamic functional approach [4–6] and show how modified R. Hamilton type equations [1,2] can be derived for the almost Kä hler evolution of Lie d-algebroids.

Symplectic and almost symplectic geometric flows have been studied in modern Ricci flow theory; see a series of examples and reviews of results in [8–11]. Our approach is very different from those with "pure" complex and/or symplectic forms and connections.

3.1. Perelman's Functionals in Almost Kähler Variables

There are both conceptual and technical difficulties which do not allow us to formulate a generalized Ricci flow theory for non-Riemannian geometries with independent metric and linear connection structures, or their almost symplectic analogs. Nevertheless, unified geometric evolution theories can be constructed for certain classes of nonholonomic manifolds and Lagrange–Finsler spaces [12–14], both with metric compatible and noncompatible N-adapted connections if the fundamental geometric objects are determined in unique forms by distortion relations completely determined by a metric and/or almost symplectic structure.

Remark 3.1. Proofs of theorems for almost Kähler models on prolongation Lie d-algebroids on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and/or $\mathcal{K}^{\mathbf{E}}\mathbf{E}$ can be obtained by N-adapted nonholonomic deformations of geometric constructions for the standard Levi-Civita connection $\overline{\nabla}$, when $\left[\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}\right] \approx \left[\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot,\cdot),\ {}^{\theta}\widetilde{\mathcal{D}} = \overline{\nabla} + \widetilde{\mathcal{Z}}\right]$. The functions determining nonholonomic distributions, N-connection coefficients and Lie algebroid structure functions are considered to be of the same smooth class as the coefficients of $\overline{\mathbf{g}}$ and $\overline{\nabla}$.

The theory of Lagrange–Ricci flows on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ is formulated for evolving nonholonomic dynamical systems on the space of equivalent geometric data $\left(\mathcal{L} : \overline{\mathbf{g}}, \overline{\nabla}\right)$, when the functionals ${}_{\shortmid}\mathcal{F}$ and ${}_{\shortmid}\mathcal{W}$ are postulated to be of Lyapunov type; see formulas (29) and (30). Ricci solitonic configurations are defined as "fixed" on τ points of the corresponding dynamical systems. The "stationary" variational conditions depend on what type of the Ricci tensor we use, for instance, the one for the connections $\overline{\nabla}$ or $\widehat{\mathcal{D}}$. We can elaborate N-adapted almost Kähler scenarios if the Perelman's functionals are redefined in terms of geometric data $(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}})$ and the derived flow equations are considered in N-adapted variables. Both approaches are equivalent if the distortion relations are considered for the same family of metrics, $\overline{\mathbf{g}}(\tau) = \widetilde{\mathbf{g}}(\tau)$ correspondingly computed for a set $\mathcal{L}(\tau)$.

Lemma 3.1. *For the scalar curvature and Ricci tensor determined by the distortion relation*

$$\widetilde{\mathcal{D}} = {}^{\theta}\widetilde{\mathcal{D}} = \overline{\boldsymbol{\nabla}} + \widetilde{\mathcal{Z}}, \tag{26}$$

the Perelman's functionals are defined equivalently in almost Kähler canonical N-adapted variables,

$$\widetilde{\mathcal{F}}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \breve{f}) = \int_{\overline{\mathcal{V}}} ({}^{s}\widetilde{\mathbf{R}} + |h\widetilde{\mathcal{D}}\breve{f}|^2 + |\ v\widetilde{\mathcal{D}}\breve{f}|)^2) e^{-\breve{f}}\ \mathrm{d}v, \tag{27}$$

$$\widetilde{\mathcal{W}}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \breve{f}, \breve{\tau}) = \int_{\overline{\mathcal{V}}} [\breve{\tau}({}^{s}\widetilde{\mathbf{R}} + |h\widetilde{\mathcal{D}}\breve{f}| + |v\widetilde{\mathcal{D}}\breve{f}|)^2 + \breve{f} - 2m]\breve{\mu}\ \mathrm{d}v, \tag{28}$$

where the new scaling function $\breve{f}$ is introduced for $\int_{\overline{\mathcal{V}}} \breve{\mu}\mathrm{d}v = 1$ with volume element $\mathrm{d}v$, $\breve{\mu} = (4\pi\breve{\tau})^{-m} e^{-\breve{f}}$ *and* $\breve{\tau} > 0$, *in which* $\breve{\tau} = {}^{h}\tau = {}^{v}\tau$ *for a couple of possible h- and v-flow parameters,* $\breve{\tau} = ({}^{h}\tau,\ {}^{v}\tau)$.

Proof. On $\mathcal{T}^E\mathbf{P}$, evolution models can be formulated as in standard theory for Riemann metrics [1,2,4–6], but for a family of geometric data $\left(\overline{\mathbf{g}}(\tau), \overline{\nabla}(\tau)\right)$ induced by a family of regular $\mathcal{L}(\tau) \in C^\infty(P)$ with a flow parameter $\tau \in [-\epsilon, \epsilon] \subset \mathbb{R}$, when $\epsilon > 0$ is taken sufficiently small. For $P = E$, we can postulate on the space of $Sec(E)$, for $\pi : E \to M$, $\dim E = n + m \geq 3$ and $\dim M = n \geq 2$, the (Perelman's) functionals

$$
{}_{\shortmid}\mathcal{F}(\overline{\mathbf{g}}, \overline{\nabla}, f, \tau) = \int\limits_{\overline{\mathcal{V}}} \left({}_{\shortmid}R + \left|\overline{\nabla} f\right|^2 \right) e^{-f} \, \mathrm{d}v, \tag{29}
$$

$$
{}_{\shortmid}\mathcal{W}(\overline{\mathbf{g}}, \overline{\nabla}, f, \tau) = \int\limits_{\overline{\mathcal{V}}} \left[\tau \left({}_{\shortmid}R + \left|\overline{\nabla} f\right| \right)^2 + f - 2m) \right] \mu \, \mathrm{d}v, \tag{30}
$$

where the volume form dv and scalar curvature ${}_{\shortmid}R$ of $\overline{\nabla}$ are computed for sets of off-diagonal metrics $g_{\overline{\alpha}\overline{\beta}}$ (A.4) with Euclidean signature. The integration is taken over compact regions $\overline{\mathcal{V}} \subset \mathcal{T}^{\mathbf{E}}\mathbf{E}, \dim \mathcal{V} = 2m$, corresponding to sections over a $U \subset M$. We can fix $\int_{\overline{\mathcal{V}}} \mathrm{d}v = 1$, with $\mu = (4\pi\tau)^{-m} e^{-f}$ and a real parameter $\tau > 0$. We introduce a new function $\breve{f}$ instead of f. The scalar functions are redefined in such a form that the "sub-integral" formula (29) under the distortion of the Ricci tensor (A.19) is rewritten in terms of geometric objects derived for the canonical d-connection, $({}_{\shortmid}R + \left|\overline{\nabla} f\right|^2)e^{-f} = ({}^s\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}}\breve{f}|^2)e^{-\breve{f}} \; + \widetilde{\Phi}$. We obtain the N-adapted functional (27). For the second functional (30), we rescale $\tau \to \breve{\tau}$ and write

$$
\left[\tau({}_{\shortmid}R + \left|\overline{\nabla} f\right|)^2 + f - 2m\right] \mu = [\breve{\tau}(\; {}^s\widetilde{\mathbf{R}} + |h\widetilde{\mathcal{D}}\breve{f}| + | \; v\widetilde{\mathcal{D}}\breve{f}|)^2 + \breve{f} - 2m]\breve{\mu} + \widetilde{\Phi}_1,
$$

for some $\widetilde{\Phi}$ and $\widetilde{\Phi}_1$ for which $\int_{\overline{\mathcal{V}}} \widetilde{\Phi} \mathrm{d}v = 0$ and $\int_{\overline{\mathcal{V}}} \widetilde{\Phi}_1 \mathrm{d}v = 0$. Finally, we get the formula (28). □

In the rest of this section, we shall only sketch the key points for proofs of theorems when the geometric constructions are straightforward consequences of those presented for the Levi-Civita connection in [4–9] and extended to nonholonomic configurations in [12–16]. For our models, we consider operators which up to frame transforms are defined by $\mathcal{L}$ via $\overline{\nabla}$ on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$. Following Remark 3.1 for distortions completely defined by $\widetilde{\mathbf{g}}$, we can study canonical Cartan and almost Kähler Ricci flows on prolongation Lie algebroids as nonholonomic deformations of the Riemannian evolution.

Using $\tilde{\theta}(\cdot, \cdot) := \tilde{\mathbf{g}}(\mathcal{J}\cdot, \cdot)$, we can define the canonical (almost symplectic) Laplacian operator, $\widetilde{\Delta} := \widetilde{\mathcal{D}}\widetilde{\mathcal{D}}$, and (the Levi-Civita) Laplace operator, $\overline{\Delta} = \overline{\nabla}\overline{\nabla}$, and consider a parameter $\tau(\chi), \partial\tau/\partial\chi = -1$. For simplicity, we shall not include normalized terms or values of type $\int_{\overline{\mathcal{V}}} \widetilde{\Phi}_1 \mathrm{d}v = 0$ if those values can be generated, or transformed into gradient type ones, via nonholonomic deformations. Inverting the distortion relations (26), we can compute

$$
\begin{aligned}
\overline{\Delta} &= \widetilde{\Delta} + {}^Z\widetilde{\Delta}, \quad {}^Z\widetilde{\Delta} = \widetilde{\mathcal{Z}}_{\overline{\alpha}}\widetilde{\mathcal{Z}}^{\overline{\alpha}} - [\widetilde{\mathcal{D}}_{\overline{\alpha}}(\widetilde{\mathcal{Z}}^{\overline{\alpha}}) + \widetilde{\mathcal{Z}}_{\overline{\alpha}}\widetilde{\mathcal{D}}^{\overline{\alpha}}]; \\
\overline{R}_{\;\overline{\beta}\overline{\gamma}} &= \widetilde{\mathbf{R}}_{\;\overline{\beta}\overline{\gamma}} + \widetilde{\mathbf{Z}}ic_{\overline{\beta}\overline{\gamma}}, \; {}_{\shortmid}R = {}^s\widetilde{\mathbf{R}} + \widetilde{\mathbf{g}}^{\overline{\beta}\overline{\gamma}}\widetilde{\mathbf{Z}}ic_{\overline{\beta}\overline{\gamma}} = {}^s\widetilde{\mathbf{R}} + {}^s\widetilde{\mathbf{Z}},
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&{}^{s}\widetilde{\mathbf{Z}} = \widetilde{\mathbf{g}}^{\overline{\beta}\overline{\gamma}}\ \widetilde{\mathbf{Z}}ic_{\overline{\beta}\overline{\gamma}} = \ h\widetilde{Z} + v\widetilde{Z},\ h\widetilde{Z} = \widetilde{\mathbf{g}}^{ab}\ \widetilde{\mathbf{Z}}ic_{ab},\ v\widetilde{Z} = \widetilde{\mathbf{g}}^{AB}\ \widetilde{\mathbf{Z}}ic_{AB};\\
&{}^{s}\overline{R} = \ h\overline{R} + v\overline{R},\ \ h\overline{R} := \widetilde{\mathbf{g}}^{ab}\ \overline{R}_{ab},\ v\overline{R} = \widetilde{\mathbf{g}}^{AB}\overline{R}_{AB},
\end{aligned}
$$

where the terms with left up label "Z" are determined by $\widetilde{\mathcal{Z}}$ (for instance, $\widetilde{\mathbf{Z}}ic_{\overline{\beta}\overline{\gamma}}$ are components of respective deformations of the Ricci d-tensor). For convenience, the capital indices $A, B, C\ldots$ are used for distinguishing v-components and even the prolongation Lie algebroid is constructed for $\mathbf{P} = \mathbf{E}$.

3.2. N-Adapted Almost Symplectic Evolution Equations

Let us consider the symmetrization and anti-symmetrization operators, for instance, $\mathbf{R}_{(\alpha\beta)} := \frac{1}{2}(\mathbf{R}_{\alpha\beta} + \mathbf{R}_{\beta\alpha})$ and $\mathbf{R}_{[\alpha\beta]} := \frac{1}{2}(\mathbf{R}_{\alpha\beta} - \mathbf{R}_{\beta\alpha})$. Using deformations of type (31) of corresponding geometric values in the proof, for instance, given in for Proposition 1.5.3 of [7], we obtain

Theorem 3.1. (a) *The N-adapted Ricci flows for the Cartan d-connection $\widetilde{\mathcal{D}}$ preserving a symmetric metric structure $\tilde{\mathbf{g}}$ on $\mathcal{T}^{E}\mathbf{E}$ can be characterized by this system of geometric flow equations:*

$$
\frac{\partial \widetilde{\mathbf{g}}_{ab}}{\partial \chi} = -(\widetilde{\mathbf{R}}_{ab}\ + \widetilde{\mathbf{Z}}ic_{ab}),\ \frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -(\widetilde{\mathbf{R}}_{AB} + \widetilde{\mathbf{Z}}ic_{AB}), \tag{32}
$$

$$
\widetilde{\mathbf{R}}\ _{aA} = -\widetilde{\mathbf{Z}}ic_{aA},\ \widetilde{\mathbf{R}}\ _{Aa} = \widetilde{\mathbf{Z}}ic_{Aa}, \tag{33}
$$

$$
\frac{\partial \widetilde{f}}{\partial \chi} = -(\widetilde{\Delta} +\ {}^{Z}\widetilde{\Delta})\widetilde{f} + \left|\left(\widetilde{\mathcal{D}} - \widetilde{\mathcal{Z}}\right)\widetilde{f}\right|^{2} -\ {}^{s}\widetilde{\mathbf{R}} -\ {}^{s}\widetilde{\mathbf{Z}}, \tag{34}
$$

and the property that

$$
\begin{aligned}
\frac{\partial}{\partial \chi}\mathcal{F}(\widetilde{\mathbf{g}}, \widetilde{\mathcal{D}}, \widetilde{f}) = &\int_{\widetilde{\mathcal{V}}} [|\widetilde{\mathbf{R}}_{ab}\ + \widetilde{\mathbf{Z}}ic_{ab} + (\widetilde{\mathcal{D}}_{a} - \widetilde{\mathcal{Z}}_{a})(\widetilde{\mathcal{D}}_{b} - \widetilde{\mathcal{Z}}_{b})\widetilde{f}|^{2}\\
&+|\widetilde{\mathbf{R}}_{AB}\ + \widetilde{\mathbf{Z}}ic_{AB} + (\widetilde{\mathcal{D}}_{A} - \widetilde{\mathcal{Z}}_{A})(\widetilde{\mathcal{D}}_{B} - \widetilde{\mathcal{Z}}_{B})\widetilde{f}|^{2}]e^{-\widetilde{f}}\mathrm{d}v,\\
\int_{\widetilde{\mathcal{V}}} e^{-\widetilde{f}}\mathrm{d}v = \mathit{const.}&
\end{aligned}
$$

(b) *In almost symplectic variables with $\tilde{\theta} =\ \tilde{g}_{ab}\delta^{a} \wedge X^{b}$ and almost Kähler d-algebroids $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, and for redefined scaling function $\widetilde{f}$, up to normalizing terms, the h- and v-evolution equations are written in equivalent form as*

$$
\frac{\partial \widetilde{\theta}_{ab}}{\partial \chi} = -\widetilde{\mathbf{R}}_{[ab]},\ \frac{\partial \widetilde{\theta}_{AB}}{\partial \chi} = -\widetilde{\mathbf{R}}_{[AB]}. \tag{35}
$$

For different classes of distortions of type (31), we can redefine the scaling functions from the above lemma and write the evolution equations (32) in the form (35) for symplectic variables with (18). On $\mathcal{T}^{E}\mathbf{E}$, the corresponding system of Ricci flow evolution equations can be written for $\widehat{\mathcal{D}}$,

$$
\begin{aligned}
&\frac{\partial \widetilde{\mathbf{g}}_{ab}}{\partial \chi} = -2\widehat{\mathbf{R}}_{ab},\quad \frac{\partial \widetilde{\mathbf{g}}_{AB}}{\partial \chi} = -2\widehat{\mathbf{R}}_{AB}, \\
&\widehat{\mathbf{R}}\ _{aA} = 0,\quad \widehat{\mathbf{R}}\ _{Aa} = 0,\quad \frac{\partial \widehat{f}}{\partial \chi} = -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathcal{D}}\widehat{f}\right|^{2} -\ {}^{s}\widehat{\mathbf{R}},
\end{aligned} \tag{36}
$$

which can be derived from the functional $\widehat{\mathcal{F}}(\widetilde{\mathbf{g}}, \widehat{\mathcal{D}}, \widehat{f}) = \int_{\overline{\mathcal{V}}}(\ {}^{s}\widehat{\mathbf{R}} + |\widehat{\mathcal{D}}\widehat{f}|^2)$ $e^{-\widehat{f}}\ dv$. We note that the conditions of type $\widehat{\mathbf{R}}_{\alpha A} = 0$ and $\widehat{\mathbf{R}}_{A\alpha} = 0$ must be imposed in order to model N-adapted evolution scenarios only with symmetric metrics. In general, a nonholonomically constrained evolution can result in nonsymmetric metrics.

Corollary 3.1. *The geometric almost Kähler d-algebroid evolution defined in Theorem 3.1 is characterized by corresponding flows (for all time $\tau \in [0, \tau_0)$) of N-adapted frames, $\tilde{\mathbf{e}}_{\overline{\alpha}}(\tau) = \tilde{\mathbf{e}}_{\overline{\alpha}}^{\ \overline{\alpha}'}(\tau, x^i, y^C)\partial_{\overline{\alpha}'}$, which up to frame/coordinate transforms are defined by the coefficients*

$$\tilde{\mathbf{e}}_{\overline{\alpha}}^{\ \overline{\alpha}'}(\tau, x^i, y^C) = \begin{bmatrix} e_a^{\ a'}(\tau, x^i, y^C) & \tilde{N}_b^B(\tau, x^i, y^C)\ e_B^{\ a'}(\tau, x^i, y^C) \\ 0 & e_A^{\ A'}(\tau, x^i, y^C) \end{bmatrix},$$

$$\tilde{\mathbf{e}}^{\overline{\alpha}}_{\ \overline{\alpha}'}(\tau, x^i, y^C) \ = \begin{bmatrix} e^a_{\ a'} = \delta^a_{a'} & e^b_{\ \underline{i}} = -\tilde{N}_b^B(\tau, x^i, y^C)\ \ \delta^b_{a'} \\ e^a_{\ A'} = 0 & e^A_{\ A'} = \delta^A_{A'} \end{bmatrix},$$

with $\tilde{g}_{ab}(\tau) = e_a^{\ a'}(\tau, x^i, y^C)\ e_b^{\ b'}(\tau, x^i, y^C)\eta_{a'b'}$ and $\tilde{g}_{AB}(\tau) = e_A^{\ A'}(\tau, x^i, y^C)$ $e_B^{\ B'}(\tau, x^i, y^C)\eta_{A'B'}$, where $\eta_{a'b'} = \mathrm{diag}[1, \ldots, 1]$ and $\eta_{A'B'} = \mathrm{diag}[1, \ldots, 1]$, to fix a Riemannian signature of $\tilde{\mathbf{g}}^{[0]}_{\alpha\beta}(x^i, y^C)$, are given by equations $\frac{\partial}{\partial\tau}\tilde{\mathbf{e}}^{\overline{\alpha}}_{\ \overline{\alpha}'} = \tilde{\mathbf{g}}^{\overline{\alpha}\overline{\beta}}\ \widetilde{\mathbf{R}}_{\overline{\beta}\overline{\gamma}}\ \ \tilde{\mathbf{e}}^{\overline{\gamma}}_{\ \overline{\alpha}'}$ if we prescribe that the geometric constructions are derived by the Cartan d-connection.

The proof of this corollary for $\mathcal{K}^{\mathbf{E}}\mathbf{E}$ is similar to those presented in N-adapted forms for nonholonomic Ricci flows and/or Finsler–Ricci evolution, or on $\mathcal{T}^E\mathbf{E}$, see [12–15]. All constructions depend on the type of d-connection we chose for our considerations.

3.3. Functionals for Entropy and Thermodynamics $\mathcal{K}^{\mathbf{E}}\mathbf{E}$

For three-dimensional Ricci flows of Riemannian metrics, the value ${}_{|}\mathcal{W}$ (30) was introduced by Perelman [4] as a "minus entropy" functional. We can consider that $\widetilde{\mathcal{W}}$ (28) has a similar interpretation, but in almost symplectic variables and on prolongation Lie d-algebroids. The main equations stated by Theorem 3.1 for $\widetilde{\mathcal{F}}$ (27) can be proven in equivalent form.

Theorem 3.2. *The Ricci flow evolution equations with symmetric metrics and respective almost symplectic forms on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and, correspondingly, $\mathcal{K}^{\mathbf{E}}\mathbf{E}$, see (32), (33) and (35), and functions $\widehat{f}(\chi)$ and $\widehat{\tau}(\chi)$ being solutions of*

$$\frac{\partial \tilde{f}}{\partial \chi} = -(\widetilde{\Delta} + \ {}^{Z}\widetilde{\Delta})\tilde{f} + \left|(\widetilde{\mathcal{D}}_a - \widetilde{\mathcal{Z}}_a)\tilde{f}\right|^2 - \ \ {}^{s}\widetilde{\mathbf{R}} \ + \frac{2m}{\hat{\tau}},\ \frac{\partial \tilde{\tau}}{\partial \chi} = -1,$$

can be derived for a functional $\widetilde{\mathcal{W}}$ satisfying the condition

$$\frac{\partial}{\partial \chi}\tilde{\mathcal{W}}(\widetilde{\mathbf{g}}(\chi), \tilde{f}(\chi), \tilde{\tau}(\chi)) = 2\int_{\mathcal{V}} \tilde{\tau}[|\widetilde{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} - \widetilde{\mathbf{Z}}ic_{\overline{\alpha}\overline{\beta}} + (\widetilde{\mathcal{D}}_{\overline{\alpha}} - \widetilde{\mathcal{Z}}_{\overline{\alpha}})(\widetilde{\mathcal{D}}_{\overline{\beta}} - \widetilde{\mathcal{Z}}_{\overline{\beta}})\tilde{f}$$
$$-\frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}|^2](4\pi\tilde{\tau})^{-m}e^{-\tilde{f}}\ dv,$$

for $\int_{\mathcal{V}} e^{-\tilde{f}}dv = const$. Such a functional is N-adapted and nondecreasing if it is both h- and v-nondecreasing.

Proof. For the Levi-Civita connection on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, the proof is similar to that in Proposition 1.5.8 in [7] containing the details of the original result from [4]. Using N-adapted deformations, the geometric constructions are performed in almost symplectic variables on $\mathcal{K}^{\mathbf{E}}\mathbf{E}$. □

Let us remember some main concepts from statistical thermodynamics. It is considered a partition function $Z = \int \exp(-\beta E) d\omega(E)$ for a canonical ensemble at temperature β^{-1}. Such a temperature is defined by the measure determined by the density of states $\omega(E)$. We can provide a statistical analogy computing respective thermodynamical values. In standard form, we introduce $\langle E \rangle := -\partial \log Z / \partial \beta$, the entropy $S := \beta \langle E \rangle + \log Z$ and the fluctuation $\sigma := \left\langle (E - \langle E \rangle)^2 \right\rangle = \partial^2 \log Z / \partial \beta^2$. The original idea of Perelman [4] was to use such values for characterizing Ricci flows of Riemannian metrics. The constructions can be elaborated in N-adapted form for geometric flows subjected to non-integrable constraints on various spaces endowed with nonholonomic distributions of commutative and noncommutative type, Lie algebroids, etc [12–16].

Theorem 3.3. *The N-adapted metric compatible* (*with symmetric metrics*) *Ricci on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *are characterized by* (a) *canonical thermodynamic values*

$$\left\langle \widehat{E} \right\rangle = -\widehat{\tau}^2 \int_{\mathcal{V}} \left({}^s\widehat{\mathbf{R}} + |\widehat{\mathcal{D}} \widehat{f}|^2 - \frac{m}{\widehat{\tau}} \right) \widehat{\mu} \, dv,$$

$$\widehat{S} = - \int_{\mathcal{V}} [\widehat{\tau}({}^s\widehat{\mathbf{R}} + |\widehat{\mathcal{D}} \widehat{f}|^2) + \widehat{f} - 2m] \widehat{\mu} \, dv,$$

$$\widehat{\sigma} = 2\widehat{\tau}^4 \int_{\mathcal{V}} \left[|\widehat{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} - \widehat{\mathbf{Z}}ic_{\overline{\alpha}\overline{\beta}} + (\widetilde{\mathcal{D}}_{\overline{\alpha}} - \widehat{\mathcal{Z}}_{\alpha})(\widehat{\mathcal{D}}_{\overline{\beta}} - \widehat{\mathcal{Z}}_{\overline{\beta}}) \widehat{f} - \frac{1}{2\tilde{\tau}} \mathbf{g}_{\overline{\alpha}\overline{\beta}}|^2 \right] \widehat{\mu} \, dv$$

(b) *and/or by effective Lagrange and/or almost Kähler Ricci flows*

$$\left\langle \tilde{E} \right\rangle = -\tilde{\tau}^2 \int_{\overline{\mathcal{V}}} \left({}^s\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}} \tilde{f}|^2 - \frac{m}{\tilde{\tau}} \right) \tilde{\mu} \, dv,$$

$$\tilde{S} = - \int_{\overline{\mathcal{V}}} [\tilde{\tau}({}^s\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}} \tilde{f}|^2) + \tilde{f} - 2m] \tilde{\mu} \, dv,$$

$$\tilde{\sigma} = 2\tilde{\tau}^4 \int_{\overline{\mathcal{V}}} [|\widetilde{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} + \widetilde{\mathcal{D}}_{\overline{\alpha}} \widetilde{\mathcal{D}}_{\overline{\beta}} \tilde{f} - \frac{1}{2\tilde{\tau}} \tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}|^2] \tilde{\mu} \, dv,$$

where all values are constructed equivalently in Cartan and/or almost symplectic variables on $\mathcal{K}^{\mathbf{E}}\mathbf{E}$.

Proof. Similar proofs in coordinate and/or N-adapted forms are given in [7, 12–16]. We have to use the corresponding partition function $\tilde{Z} = \exp\left\{ \int_{\overline{\mathcal{V}}} [-\tilde{f} + m] \, \tilde{\mu} dv \right\}$ for computations on $\mathcal{K}^{\mathbf{E}}\mathbf{E}$. The formulas in the conditions of theorem depend on the type of d-connection, $\nabla \to \widehat{\mathcal{D}}$, or $\nabla \to \widetilde{\mathcal{D}}$, which is chosen for nonholonomic deformations. Corresponding rescaling $\breve{f} \to \tilde{f}$, or $\widehat{f}$, and $\breve{\tau} \to \tilde{\tau}$, or $\widehat{\tau}$, have to be considered. □

Finally, we note that Ricci flows with different d-connections are characterized by different thermodynamical values and stationary configurations.

4. Ricci Solitons with Lie Algebroid Symmetries

In this section, we shall construct in explicit form some examples of exact solutions for Ricci soliton Lie d-algebroid configurations. The fist class of models describes generalized Einstein spaces with nonholonomic (for instance, almost symplectic) variables and the second one is determined by Lagrange–Finsler generating functions.

4.1. Preliminaries on Lie d-Algebroid Solitons

Lie d-algebroid Ricci solitons can be viewed as fixed points of generalized Ricci flows with a functional $\widetilde{\mathcal{W}}$ (28) satisfying the conditions of Theorem 3.2. Such nonholonomically constrained dynamical systems correspond to self-similar solutions describing N-adapted geometric evolution models.

Definition 4.1. The geometric data $[\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}] \approx [\tilde{\theta}(\cdot,\cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot,\cdot),\ {}^{\theta}\widetilde{\mathcal{D}} = \overline{\boldsymbol{\nabla}} + \widetilde{\mathcal{Z}}]$ for a complete Riemannian metric $\overline{\mathbf{g}}$ on a smooth $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and corresponding $\mathcal{K}^{\mathbf{E}}\mathbf{E}$ define a gradient almost Kähler–Ricci d-algebroid soliton if there exists a smooth potential function on $\tilde{\kappa}(x^i, y^C)$ such that

$$\widetilde{\mathbf{R}}_{\ \overline{\beta}\overline{\gamma}} + \widetilde{\mathcal{D}}_{\overline{\beta}}\widetilde{\mathcal{D}}_{\overline{\gamma}}\tilde{\kappa} = \lambda\tilde{\mathbf{g}}_{\ \overline{\beta}\overline{\gamma}}. \tag{37}$$

Using the almost symplectic form (18), these equations can be written equivalently in the form

$${}^{\theta}\widetilde{\mathbf{R}}_{\ [\overline{\beta}\overline{\gamma}]} + {}^{\theta}\widetilde{\mathcal{D}}_{[\overline{\beta}}\ {}^{\theta}\widetilde{\mathcal{D}}_{\overline{\gamma}]}\tilde{\kappa} = \lambda\tilde{\theta}_{\ \overline{\beta}\overline{\gamma}}.$$

There are three types of such Ricci solitons determined by $\lambda = \text{const}$: steady ones, for $\lambda = 0$; shrinking, for $\lambda > 0$; and expanding, for $\lambda < 0$.

The above classification is important because shrinking solutions for the Riemannian Levi-Civita solitons helps us to understand the asymptotic behaviour of ancient solutions of Ricci flows (see, for instance, Proposition 11.2 in [4] and/or Theorem 6.2.1 in [7]). In general, complete gradient shrinking Ricci solitons describe possible Type I singularity models in the Ricci flow theory. If $\tilde{\kappa} = \text{const}$, the Eq. (37) transform into distorted Einstein equations, but for Ricci solitonic configurations.

Proposition 4.1. *Let* $(\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}; \tilde{\kappa})$ *be a complete shrinking soliton on* $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ *and/or* $\mathcal{K}^{\mathbf{E}}\mathbf{E}$. *Using nonholonomic frame deformations, we can construct a redefined potential function* $\widehat{\kappa}(x^i, y^C)$, *for* $\overline{\mathbf{g}} \sim \tilde{\mathbf{g}}$, *when* (37) *are equivalent to*

$$\widehat{\mathbf{R}}_{\ \overline{\beta}\overline{\gamma}} + \widehat{\mathcal{D}}_{\overline{\beta}}\widehat{\mathcal{D}}_{\overline{\gamma}}\widehat{\kappa} = \lambda\overline{\mathbf{g}}_{\ \overline{\beta}\overline{\gamma}}. \tag{38}$$

Proof. Using Conclusion 2.1 and contracting indices in (37), we obtain that ${}^{s}\widetilde{\mathbf{R}} + |\widetilde{\mathcal{D}}\tilde{\kappa}|^2 - \tilde{\kappa} = \text{const}$. Distortion relations of type $\widetilde{\mathcal{D}}_{\overline{\alpha}} = \widehat{\mathcal{D}}_{\overline{\alpha}} + \widehat{\mathcal{Z}}_{\overline{\alpha}}$ allow us to compute ${}^{s}\widehat{\mathbf{R}} + {}^{s}\widehat{\mathbf{Z}} + |(\widehat{\mathcal{D}} + \widehat{\mathcal{Z}})\tilde{\kappa}|^2 - \tilde{\kappa} = \text{const}$, which can be rewritten as ${}^{s}\widehat{\mathbf{R}} + |\widehat{\mathcal{D}}\widehat{\kappa}|^2 - \widehat{\kappa} = \text{const}$ for certain nonlinear transform $\tilde{\kappa} \to \widehat{\kappa}$. In general, the systems (38) and (37) have different solutions. Nevertheless, conditions of type $\widehat{\mathcal{Z}} = 0$ and/or $\widetilde{\mathcal{Z}} = 0$ result in the Levi-Civita configurations and equivalent classes of solutions. □

4.2. Generalized Einstein Equations Encoding Lie d-Algebroids

We can construct very general classes of off-diagonal solutions of (38) if we impose the condition that in some N-adapted frames

$$\widehat{\mathcal{D}}_{\overline{\gamma}}\widehat{\kappa} = \mathbf{e}_{\overline{\gamma}}\widehat{\kappa} = \kappa_{\overline{\gamma}} = \text{const}, \qquad (39)$$
$$\text{i.e. } \delta_a\widehat{\kappa} = \mathcal{X}_a\widehat{\kappa} - \mathcal{N}_a^C\kappa_C = 0 \quad \text{and} \quad \mathcal{V}_A\widehat{\kappa} = \kappa_A.$$

The information from potential functions $\widehat{\kappa}$ is encoded into the data for N-connection structure with coefficients $\mathcal{N}_a^C$.

For simplicity, we shall consider in this section nonholonomic distributions on a nonholonomic $\mathbf{E} = \mathbf{P}$ with $2+2$ splitting when $a, b, \ldots = 1, 2; i', j', \ldots = 1, 2$ and $A, B, \ldots = 3, 4$. The local coordinates are parameterized in the form $u^\mu = (x^i, y^a) = (x^1, x^2, y^3, y^4)$. We study nonholonomic deformations of a d-metric $\mathring{\mathbf{g}}$ on $\mathbf{E}$ into a target metric $\overline{\mathbf{g}}$ (A.3) on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, $\mathring{\mathbf{g}} \to \overline{\mathbf{g}}$, which results in solutions of the Ricci solitonic equations (38) and (39). The prime metric is parameterized as

$$\mathring{\mathbf{g}} = \mathring{g}_\alpha(u)\mathring{\mathbf{e}}^\alpha \otimes \mathring{\mathbf{e}}^\beta = \mathring{g}_i(x)\mathrm{d}x^i \otimes \mathrm{d}x^i + \mathring{h}_a(x, y)\mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a,$$
$$\text{for } \mathring{\mathbf{e}}^\alpha = (\mathrm{d}x^i, \mathbf{e}^a = \mathrm{d}y^a + \mathring{N}_i^a(u)\mathrm{d}x^i), \qquad (40)$$
$$\mathring{\mathbf{e}}_\alpha = (\mathring{\mathbf{e}}_i = \partial/\partial y^a - \mathring{N}_i^b(u)\partial/\partial y^b,\ e_a = \partial/\partial y^a).$$

For physical applications, we can consider that the coefficients of such metrics are with two Killing vector symmetries and that in certain systems of coordinates it can be diagonalized.[4] In general, we can consider arbitrary (semi) Riemannian metrics. The target Lie algebroid d-metrics are chosen

$$\overline{\mathbf{g}} = \mathbf{g}_{\overline{\alpha}\overline{\beta}}\mathbf{e}^{\overline{\beta}} \otimes \mathbf{e}^{\overline{\beta}} = \mathbf{g}_a\ \mathcal{X}^a \otimes \mathcal{X}^a + \mathbf{g}_A\delta^A \otimes \delta^A \qquad (41)$$
$$= \eta_a(x^k)\mathring{g}_a\mathcal{X}^a \otimes \mathcal{X}^a + \eta_A(x^k, y^3)\mathring{h}_A\delta^A \otimes \delta^A,$$

where we shall construct exact solutions with Killing symmetry on $\partial/\partial y^4$ (non-Killing configurations need a more advanced geometric techniques). Let us denote by $h^* := \partial_3 h$ and $\mathcal{N}_a^3 = w_a(x^k, y^3)$, $\mathcal{N}_a^4 = n_a(x^k, y^3)$.

Proposition 4.2. *The nontrivial components of the Ricci soliton d-algebroid equations* (38) *and* (39)*, with respect to N-adapted bases* (8), (9) *and for*

[4] This includes the bulk of physically important exact solutions of Einstein equations.

coordinate transforms when $\partial_a \to \mathcal{X}_a$ and $\mathcal{V}_A = \partial_A$ for a metric (41), are

$$-\widehat{\mathbf{R}}^1_1 = -\widehat{\mathbf{R}}^2_2 = \frac{1}{2g_1g_2}\left[\mathcal{X}_1(\mathcal{X}_1 g_2) - \frac{\mathcal{X}_1 g_1\ \mathcal{X}_1 g_2}{2g_1} - \frac{(\mathcal{X}_1 g_2)^2}{2g_2} + \mathcal{X}_2(\mathcal{X}_2 g_1) - \frac{\mathcal{X}_2 g_1\ \mathcal{X}_2 g_2}{2g_2} - \frac{(\mathcal{X}_2 g_1)^2}{2g_1}\right] = \lambda, \quad (42)$$

$$-\widehat{\mathbf{R}}^3_3 = -\widehat{\mathbf{R}}^4_4 = \frac{1}{2h_3h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}\right] = \lambda, \quad (43)$$

$$\widehat{\mathbf{R}}_{3a} = \frac{w_a}{2h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}\right] + \frac{h_4^*}{4h_4}\left(\frac{\mathcal{X}_a h_3}{h_3} + \frac{\mathcal{X}_a h_4}{h_4}\right) - \frac{\mathcal{X}_a h_4^*}{2h_4} = 0, \quad (44)$$

$$\widehat{\mathbf{R}}_{4a} = \frac{h_4}{2h_3} n_a^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_a^*}{2h_3} = 0; \quad (45)$$

for the equations for the potential function

$$\mathcal{X}_a\widehat{\kappa} - w_a\kappa_3 - n_a\kappa_4 = 0 \quad \textit{and} \quad \mathcal{V}_A\widehat{\kappa} = \kappa_A,$$

when the torsionless (Levi-Civita, LC) conditions $\widehat{\mathcal{Z}} = 0$ transform into

$$w_a^* = (\mathcal{X}_a - w_a\partial_3)\ln\sqrt{|h_3|}, (\mathcal{X}_a - w_a\partial_3)\ln\sqrt{|h_4|} = 0, \quad (46)$$
$$\mathcal{X}_b w_a = \mathcal{X}_a w_b, n_a^* = 0, \partial_a n_b = \partial_b n_a.$$

Proof. It follows from straightforward computations of the Ricci d-tensor on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ using formulas (A.5) and (A.2). The local frames are redefined in the form $\partial_a = e^{a'}_{\ a}\mathcal{X}_{a'}$ to include Lie algebroid anchor structure functions and commutation relations of type (5) and (10). For nonholonomic $2+2+2+\cdots+$ decompositions, this can be performed by local frame/coordinate transforms. Details of calculus on $T\mathbf{V}$ are provided in [17] and references therein. We can work similarly both on $T\mathbf{V}$ and $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ but with different N-adapted nonholonomic frames and N-elongated partial derivatives and differentials. The system of Eqs. (42)–(46) possesses an important decoupling property. For instance, the Eq. (42) is a 2-d version of Laplace/d'Alambert equation (it depends on the signature of the h-metric) with prescribed local source λ. Such equations can be integrated in general form and even the algebroid structure functions $\rho^i_a(x^k)$ are not trivial. The Eq. (43) is the same both on $T\mathbf{V}$ and $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ and contains partial derivatives only on ∂_3 and can be also in similar form. □

4.3. Generating Off-Diagonal Solutions

We can integrate the algebroid Ricci soliton equations (38) and (39) for a nontrivial source $\lambda, g_a = \epsilon_a e^{\psi(x^k)}$, $\epsilon_a = \pm 1$ and $h_a^* \neq 0$.

Theorem 4.1. *The system* (42)–(45) *decoupled in N-adapted form,*

$$\epsilon_1 \mathcal{X}_1(\mathcal{X}_1\psi) + \epsilon_2 \mathcal{X}_2(\mathcal{X}_2\psi) = 2\lambda \tag{47}$$
$$\phi^* h_4^* = 2h_3 h_4 \lambda \tag{48}$$
$$\beta w_a - \alpha_a = 0, \tag{49}$$
$$n_a^{**} + \gamma n_a^* = 0, \tag{50}$$
$$\textit{for } \alpha_a = h_4^* \mathcal{X}_a \phi, \beta = h_4^* \phi^*, \gamma = \left(\ln |h_4|^{3/2}/|h_3|\right)^*, \tag{51}$$
$$\textit{where } \phi = \ln |h_4^*/\sqrt{|h_3 h_4|}|, \tag{52}$$

is considered as a generating function.

Proof. It follows from explicit computations for d-metrics (41) with Killing symmetry on ∂_4. It is convenient to use the value $\Phi := e^\phi$. □

Corollary 4.1. *The above systems of nonlinear partial differential equations, PDE, can be integrated in very general forms.*

Proof. We should follow such a procedure:

1. The (47) is just a 2-d Laplace/d'Alambert equation which can be solved for any given λ.
2. For $h_A := \epsilon_A z_A^2(x^k, y^3)$, $\epsilon_A = \pm 1$ (we do not consider summation on repeating indices in this formula), the system of two Eqs. (48) and (52) can be written as $\phi^* z_4^* = \epsilon_3 z_4 (z_3)^2 \lambda$ and $e^\phi z_3 = 2\epsilon_4 z_4^*$. Multiplying both equations for nonzero z_4^*, ϕ^*, z_A and introducing the result instead of the first equation, this system transforms into $\Phi^* = 2\epsilon_3\epsilon_4 z_3 z_4 \lambda$ and $\Phi z_3 = 2\epsilon_4 z_4^*$. Introducing z_3 from the second equation into the first one, we obtain $[(z_4)^2]^* = \epsilon_3 [\Phi^2]^*/4\lambda$. We can integrate on y^3,
$$h_4 = \epsilon_4 (z_4)^2 = {}^0h_4(x^k) + \frac{\epsilon_3\epsilon_4}{4\lambda}\Phi^2, \tag{53}$$
for an integration function ${}^0h_4(x^k)$. From the first equation in the above system, we compute
$$h_3 = \epsilon_3 (z_3)^2 = \frac{\phi^*}{\lambda}\frac{z_4^* z_4}{z_4 z_4} = \frac{1}{2\lambda}(\ln|\Phi|)^*(\ln|h_4|)^*. \tag{54}$$
Redefining the coordinates and Φ and introducing $\epsilon_3\epsilon_4$ in λ, we express the solutions in functional form, $h_3[\Phi] = (\Phi^*)^2/\lambda\Phi^2,\ h_4[\Phi] = \Phi^2/4\lambda$.
3. To find w_a we have to solve certain algebraic equations which can be obtained if we introduce the coefficients (51) in (49),
$$w_a = \mathcal{X}_a \phi/\phi^* = \mathcal{X}_a \Phi/\Phi^*. \tag{55}$$
4. Integrating two times on y^3 in (50), we find
$$n_b = {}_1n_b + {}_2n_b \int dy^3\ h_3/(\sqrt{|h_4|})^3, \tag{56}$$
where ${}_1n_b(x^i), {}_2n_b(x^i)$ are integration functions.

5. The nonholonomic constraints for the LC-conditions (46) can be solved in explicit form for certain classes of integration functions ${}_1n_b$ and ${}_2n_b$. We can find explicit solutions if ${}_2n_b = 0$ and ${}_1n_b = \mathcal{X}_b n$ with a function $n = n(x^k)$. We get $(\mathcal{X}_a - w_a\partial_3)\Phi \equiv 0$ for any $\Phi(x^k, y^3)$ if w_a is defined by (55). For any functional $H(\Phi)$, we obtain $(\mathcal{X}_a - w_a\partial_3)H = \frac{\partial H}{\partial \Phi}(\mathcal{X}_a - w_a\partial_3)\Phi = 0$. It is possible to solve the equations $(\mathcal{X}_a - w_a\partial_3)h_4 = 0$ for $h_4 = H(|\tilde{\Phi}(\Phi)|)$. This way we solve the second system of equations in (46) when $(\mathcal{X}_a - w_a\partial_3)\ln\sqrt{|h_4|} \sim (\mathcal{X}_a - w_a\partial_3)h_4$. We can consider a subclass of generating functions $\Phi = \check{\Phi}$ for which $(\mathcal{X}_a\check{\Phi})^* = \mathcal{X}_a(\check{\Phi}^*)$. Then, we can compute for the left part of the second equation in (46), $(\mathcal{X}_a - w_a\partial_3)\ln\sqrt{|h_4|} = 0$. The first system of equations in (46) can be solved explicitly for any w_a determined by formulas (55), and $h_3[\tilde{\Phi}]$ and $h_4[\tilde{\Phi}, \tilde{\Phi}^*]$. Let us consider $\tilde{\Phi} = \tilde{\Phi}(\ln\sqrt{|h_3|})$ for a functional dependence $h_3[\tilde{\Phi}[\check{\Phi}]]$. This allows us to obtain the formulas $w_a = \mathcal{X}_a|\tilde{\Phi}|/|\tilde{\Phi}|^* = \mathcal{X}_a|\ln\sqrt{|h_3|}|/|\ln\sqrt{|h_3|}|^*$. Taking derivative ∂_3 on both sides of this equation, we get $w_a^* = \frac{(\mathcal{X}_a|\ln\sqrt{|h_3|}|)^*}{|\ln\sqrt{|h_3|}|^*} - w_a\frac{|\ln\sqrt{|h_3|}|^{**}}{|\ln\sqrt{|h_3|}|^*}$. The condition $w_a^* = (\mathcal{X}_a - w_a\partial_3)\ln\sqrt{|h_3|}$ is necessary for the zero torsion conditions. It is satisfied for $\Phi = \check{\Phi}$. We can choose $w_a = \check{w}_a = \mathcal{X}_a\check{\Phi}/\check{\Phi}^* = \mathcal{X}_a\widetilde{A}$, with a nontrivial function $\widetilde{A}(x^k, y^3)$ depending functionally on generating function $\check{\Phi}$ to solve the equations $\mathcal{X}_a w_b = \mathcal{X}_b w_a$ from the second line in (46).

□

Conclusion 4.1. *The class of off-diagonal metrics of type* (41) *with coefficients computed following the method outlined in the above Proof are determined by quadratic elements of type* $\mathrm{d}s^2 =$

$$e^{\psi(x^k)}[\epsilon_1(\mathcal{X}^1)^2 + \epsilon_2(\mathcal{X}^2)^2] + \epsilon_3\frac{(\check{\Phi}^*)^2}{\lambda\check{\Phi}^2}[\mathcal{V}^3 + (\mathcal{X}_a\widetilde{A}[\check{\Phi}])\mathcal{X}^a]^2$$
$$+\epsilon_4\frac{\check{\Phi}^2}{4|\lambda|}[\mathcal{V}^4 + (\mathcal{X}_a n)\mathcal{X}^a]^2. \quad (57)$$

In general, on prolongation of Lie d-algebroids, the solutions defining Ricci solitons can be with nontrivial torsion.

Remark 4.1. For arbitrary ϕ and related Φ, or $\widetilde{\Phi}$, we can generate off-diagonal solutions of (42)–(45) with nonholonomically induced torsion,

$$\mathrm{d}s^2 = e^{\psi(x^k)}[\epsilon_1(\mathcal{X}^1)^2 + \epsilon_2(\mathcal{X}^2)^2]$$
$$+\epsilon_3(z_3)^2\left[\mathcal{V}^3 + \frac{\mathcal{X}_a\Phi}{\Phi^*}\mathcal{X}^a\right]^2$$
$$+\epsilon_4(z_4)^2\left[\mathcal{V}^4 + \left({}_1n_a + {}_2n_a\int \mathrm{d}y^3\frac{(z_3)^2}{(z_4)^3}\right)\mathcal{X}^a\right]^2, \quad (58)$$

where the values $z_3(x^k, y^3)$ and $z_4(x^k, y^3)$ are defined by formulas (54) and (53). In N-adapted frames, the ansatz for such solutions defines a nontrivial distorting tensor $\widehat{\mathbf{Z}} = \{\widehat{\mathbf{Z}}^\alpha{}_{\beta\gamma}\}$; see (A.6).

Taking data $\mathring{g}_a(x^k)$ and $\mathring{h}_A(x^k)$ for a prime metric (40) to define, for instance, a black hole solution for Einstein–de Sitter spaces, and reparameterizing the metric (57) in the form (41), we can study nonholonomic deformations of black hole metrics into Lie algebroid solitonic configurations. In explicit form, such examples of algebroid black holes are proved in [19] and reference therein.

4.4. On Lie Algebroid and Almost Kähler–Finsler Ricci Solitons

We show how a Finsler geometry model can be nonholonomically deformed into a Ricci soliton d-algebroid configuration. Let $\mathbf{E} = \mathbf{TM}$ for a tangent bundle TM on the base space M be a real C^∞ manifold of dimension $\dim M = n = 2$.

Definition 4.2. A Finsler fundamental, or generating, function (metric) is a function $F: TM \to [0, \infty)$ for which (1) $F(x, y)$ is C^∞ on $\widetilde{TM} := TM \backslash \{0\}$, where $\{0\}$ is the set of zero sections of TM on M; (2) $F(x, \beta y) = \beta F(x, y)$, for any $\beta > 0$, i.e. it is a positive 1-homogeneous function on the fibers of TM; (3) for any $y \in \widetilde{T_x M}$, the Hessian ${}^v\tilde{g}_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is considered as a "vertical" (v) metric on typical fiber, i.e. it is nondegenerate and positive definite, $\det |{}^v\tilde{g}_{ij}| \neq 0$.

For the conditions of Theorem 2.1, we take $\mathcal{L} = L = F^2$ and construct the geometric data $(\tilde{\mathbf{g}}, \tilde{\mathbf{N}})$.

Theorem 4.2. *Any Finsler–Cartan geometry* (*equivalently modelled as an almost Kä hler–Finsler space*) *with nonholonomic splitting* $2+2$ *can be encoded as a canonical Ricci d-algebroid soliton with metric of type* (58) *and a respective almost Kähler d-algebroid soliton; see Definition* 4.1.

Proof. It follows from explicit computations for d-metrics (41) with Killing symmetry on ∂_4. We consider for the class of metrics (41) that up to frame/coordinate transforms the prime configuration is determined by a total bundle metric $\mathring{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'} = e^{\overline{\alpha}}_{\ \overline{\alpha}'} e^{\overline{\beta}}_{\ \overline{\beta}'} \tilde{\mathbf{g}}_{\overline{\alpha}\overline{\beta}}$. The target metric $\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'}$ (41) defines generic off-diagonal solutions for prolongation Lie d-algebroids with canonical d-connections if $\overline{\mathbf{g}}_{\overline{\alpha}'\overline{\beta}'}$ is of type (58). Such solutions of Ricci soliton d-algebroid equations (38) define nonholonomic transforms $(\mathring{\mathbf{g}} \sim \tilde{\mathbf{g}}, \mathcal{L} = F^2, \widetilde{\mathcal{N}}, \widetilde{\mathcal{D}}; \tilde{\kappa}) \to (\overline{\mathbf{g}} \sim \widehat{\mathbf{g}}, \widehat{\mathcal{N}}, \widehat{\mathcal{D}}; \widehat{\kappa})$. We reencode a Finsler–Cartan geometry into the canonical data for a Ricci soliton solution on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$, for $\mathbf{E} = \mathbf{TM}$ Via additional d-connection distortions ${}^\theta\widetilde{\mathcal{D}} = \widehat{\mathcal{D}} + \mathcal{Z}$, completely defined by a Ricci soliton d-algebroid solution (58), we redefine the geometric constructions on $\mathcal{K}^{\mathbf{TM}}\mathbf{TM}$. For fundamental geometric values, $(\overline{\mathbf{g}} \sim \widehat{\mathbf{g}}, \widehat{\mathcal{N}}, \widehat{\mathcal{D}}; \widehat{\kappa}) \approx [\tilde{\theta}(\cdot, \cdot) := \tilde{\mathbf{g}}(\widetilde{\mathcal{J}}\cdot, \cdot),\ {}^\theta\widetilde{\mathcal{D}} = \widehat{\mathcal{D}} + \mathcal{Z}]$.

The constructions for this theorem can be extended for nonholonomic splitting of any finite dimension $2 + 2 \cdots + 2$. □

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Appendix A. Formulas in Coefficient Forms

In this section, we summarize some important local constructions and coefficient formulas which are necessary for formulating Ricci evolution equations and deriving exact solutions on Lie algebroid models.

A.1. Torsions and Curvatures on $\mathcal{T}^{\mathbf{E}}\mathbf{P}$

The N-adapted components $\mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = (\mathbf{L}^a_{bf}, \mathbf{L}^A_{Bf}\ ; \mathbf{B}^a_{bC}, \mathbf{B}^A_{BC})$ of a d-connection Definition 2.5 and corresponding covariant operator $\mathcal{D}_{\overline{\alpha}} = (\mathbf{e}_{\overline{\alpha}} \rfloor \mathcal{D})$, where $\rfloor$ is the interior product, are computed following equations $\mathbf{\Gamma}^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}} = (\mathcal{D}_{\overline{\alpha}} \mathbf{e}_{\overline{\beta}}) \rfloor \mathbf{e}^{\overline{\gamma}}$. The h- and v-covariant derivatives are defined, respectively, $h\mathcal{D} = \{\mathcal{D}_\gamma = (\mathbf{L}^a_{bf}, \mathbf{L}^A_{Bf}\ )\}$ and $v\mathcal{D} = \{\mathcal{D}_C = (\mathbf{B}^a_{bC}, \mathbf{B}^A_{BC})\}$, where

$$\mathbf{L}^a_{bf} \coloneqq (\mathcal{D}_f \delta_b) \rfloor \mathcal{X}^a, \mathbf{L}^A_{Bf} \coloneqq (\mathcal{D}_f \mathcal{V}_B) \rfloor \delta^A, \mathbf{B}^a_{bC} \coloneqq (\mathcal{D}_C \delta_b) \rfloor \mathcal{X}^a, \mathbf{B}^A_{BC} \coloneqq (\mathcal{D}_C \mathcal{V}_B) \rfloor \delta^A$$

are computed for N-adapted bases (8) and (9).

Using rules of absolute differentiation (5) for N-adapted bases $\mathbf{e}_{\overline{\alpha}} := \{\delta_\alpha, \mathcal{V}_A\}$ and $\mathbf{e}^{\overline{\beta}} := \{\mathcal{X}^\alpha, \delta^B\}$ and the d-connection 1-form $\mathbf{\Gamma}^{\overline{\gamma}}_{\ \overline{\alpha}} := \mathbf{\Gamma}^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}} \mathbf{e}^{\overline{\beta}}$, we can compute the torsion and curvature 2-forms of $\mathcal{D}$ on $\mathcal{T}^E\mathbf{P}$. For instance, let us consider such a calculus for the d-torsion 2-form. We take some sections $\overline{x}, \overline{y}, \overline{z}$ of $\mathcal{T}^E\mathbf{P}$ parameterized in the form, for example, $\overline{z} = z^{\overline{\alpha}} \mathbf{e}_{\overline{\alpha}} = z^a \delta_a + z^A \mathcal{V}_A$. Following an N-adapted differential form calculus, we prove some important formulas for the d-torsion and d-curvature.

Theorem A.1. *For a d-connection $\mathcal{D}$, we can compute*

- (a) *the torsion $\mathcal{T}^{\overline{\alpha}} := \mathcal{D}\mathbf{e}^{\overline{\alpha}} = d\mathbf{e}^{\overline{\alpha}} + \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}} \wedge \mathbf{e}^{\overline{\beta}}$; the h–v coefficients $\mathcal{T}^{\overline{\alpha}} = \{\mathbf{T}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}\} = \{\mathbf{T}^a_{\ bf}, \mathbf{T}^a_{\ bA}, \mathbf{T}^A_{\ bf}, \mathbf{T}^A_{\ Ba}, \mathbf{T}^A_{\ BC}\}$ with N-adapted coefficients*

$$\mathbf{T}^a_{\ bf} = \mathbf{L}^a_{\ bf} - \mathbf{L}^a_{\ fb} + C^a_{\ bf},\ \mathbf{T}^a_{\ bA} = -\mathbf{T}^a_{\ Ab} = \mathbf{B}^a_{\ bA},\ \mathbf{T}^A_{\ ba} = \Omega^A_{\ ba},$$
$$\mathbf{T}^A_{\ Ba} = \frac{\partial \mathcal{N}^A_a}{\partial u^B} - \mathbf{L}^A_{\ Ba},\ \mathbf{T}^A_{\ BC} = \mathbf{B}^A_{\ BC} - \mathbf{B}^A_{\ CB}. \tag{A.1}$$

- (b) *The curvature $\mathcal{R}^{\overline{\alpha}}_{\ \overline{\beta}} := \mathcal{D}\mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}} = d\mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}} - \mathbf{\Gamma}^{\overline{\gamma}}_{\ \overline{\beta}} \wedge \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\gamma}} = \mathbf{R}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}} \mathbf{e}^{\overline{\gamma}} \wedge \mathbf{e}^{\overline{\delta}}$, where $\mathbf{R}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}} = \mathbf{e}_{\overline{\delta}} \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} - \mathbf{e}_{\overline{\gamma}}\ \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\delta}} + \mathbf{\Gamma}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\gamma}}\ \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\delta}} - \mathbf{\Gamma}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\delta}}\ \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\varphi}\gamma} + \mathbf{\Gamma}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\varphi}} W^{\overline{\varphi}}_{\ \overline{\gamma}\overline{\delta}}$ with*

N-adapted coefficients

$$\begin{aligned}
&\mathcal{R}^{\overline{\alpha}}_{\ \overline{\beta}} = \{\mathbf{R}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}}\} = \{\mathbf{R}^{\alpha}_{\ \varepsilon\beta\gamma}, \mathbf{R}^{A}_{\ B\beta\gamma}, \mathbf{R}^{\alpha}_{\ \varepsilon\beta A}, \mathbf{R}^{C}_{\ B\beta A}, \mathbf{R}^{\alpha}_{\ \beta BA}, \mathbf{R}^{C}_{\ BEA}\}, \quad \textit{for}\\
&\mathbf{R}^{a}_{\ ebf} = \delta_f \mathbf{L}^{a}_{\ eb} - \delta_b \mathbf{L}^{a}_{\ eb} + \mathbf{L}^{d}_{\ eb}\mathbf{L}^{a}_{\ df} - \mathbf{L}^{d}_{\ ef}\mathbf{L}^{a}_{\ db} + \mathbf{L}^{a}_{\ ed} C^{d}_{bf} - \mathbf{B}^{a}_{\ eA}\Omega^{A}_{\ fb},\\
&\mathbf{R}^{A}_{\ Bbf} = \delta_f \mathbf{L}^{A}_{\ Bb} - \delta_b \mathbf{L}^{A}_{Bf} + \mathbf{L}^{C}_{Bb}\mathbf{L}^{A}_{\ Cf} - \mathbf{L}^{C}_{Bf}\mathbf{L}^{A}_{\ Cb} + \mathbf{L}^{A}_{Bd} C^{d}_{bf} - \mathbf{B}^{A}_{\ BC}\Omega^{C}_{fb},\\
&\mathbf{R}^{a}_{\ ebA} = \mathcal{V}_A \mathbf{L}^{a}_{\ eb} - \mathcal{D}_b \mathbf{B}^{a}_{eA} + \mathbf{B}^{a}_{\ eB}\mathbf{T}^{B}_{\ bA},\\
&\mathbf{R}^{C}_{\ BfA} = \mathcal{V}_A \mathbf{L}^{C}_{\ Bf} - \mathcal{D}_\gamma \mathbf{B}^{C}_{\ BA} + \mathbf{B}^{C}_{\ BD}\mathbf{T}^{D}_{\ \gamma A}, \qquad\qquad (A.2)\\
&\mathbf{R}^{a}_{\ bBA} = \mathcal{V}_A \mathbf{B}^{a}_{\ bB} - \mathcal{V}_B \mathbf{B}^{a}_{bC} + \mathbf{B}^{d}_{\ bB}\mathbf{B}^{a}_{\ dC} - \mathbf{B}^{d}_{\ bC}\mathbf{B}^{a}_{\ dB},\\
&\mathbf{R}^{A}_{\ ECB} = \mathcal{V}_E \mathbf{B}^{A}_{\ BC} - \mathcal{V}_C \mathbf{B}^{A}_{\ BE} + \mathbf{B}^{F}_{\ BC}\mathbf{B}^{A}_{\ FE} - \mathbf{B}^{F}_{\ BE}\mathbf{B}^{A}_{\ FC}.
\end{aligned}$$

The formulas (A.1) and (A.2) can be used for $\mathcal{D} = \widehat{\mathcal{D}}$, or $\mathcal{D} = \tilde{\mathcal{D}}$ [in the last case, see respective formulas (A.16) and (A.18)].

A.2. N-Adapted Coefficients for the Canonical d-Connection

A d-metric structure $\overline{\mathbf{g}} = \{\mathbf{g}_{\overline{\alpha}\overline{\beta}}\}$ on $\mathcal{T}^E\mathbf{P}$ is defined by

$$\overline{\mathbf{g}} = \mathbf{g}_{\overline{\alpha}\overline{\beta}}\mathbf{e}^{\overline{\beta}} \otimes \mathbf{e}^{\overline{\beta}} = \mathbf{g}_{ab}\ \mathcal{X}^a \otimes \mathcal{X}^b + \mathbf{g}_{AB}\ \delta^A \otimes \delta^B. \qquad (A.3)$$

It can be represented in generic off-diagonal form,[5] $\mathbf{g} = g_{\overline{\alpha}\overline{\beta}} \mathrm{d}z^{\overline{\alpha}} \otimes \mathrm{d}z^{\overline{\beta}}$ where

$$g_{\overline{\alpha}\overline{\beta}} = \begin{bmatrix} \mathbf{g}_{ab} + \mathcal{N}^A_a\ \mathcal{N}^B_b\ \ \mathbf{g}_{AB} & \mathcal{N}^A_b\ \ \mathbf{g}_{AC} \\ \mathcal{N}^E_a\ \mathbf{g}_{ED} & \mathbf{g}_{DC} \end{bmatrix}. \qquad (A.4)$$

Considering dual frame and/or frame transforms, $\mathbf{e}^{\overline{\beta}} \to \mathrm{d}z^{\overline{\beta}'}$ and/or $\partial_{\overline{\alpha}'} \to \mathbf{e}_{\overline{\alpha}}^{\ \overline{\alpha}'}\partial_{\overline{\alpha}'}$, with $\mathbf{e}_{\overline{\alpha}}^{\ \overline{\alpha}'} = \begin{bmatrix} \mathbf{e}_a^{\ a'} & \mathcal{N}^B_a\ \mathbf{e}_B^{\ A'} \\ 0 & \mathbf{e}_A^{\ A'} \end{bmatrix}$, we obtain quadratic relations between coefficients $g_{\overline{\alpha}\overline{\beta}} = e_{\overline{\alpha}}^{\ \overline{\alpha}'} e_{\overline{\beta}}^{\ \overline{\beta}'} \eta_{\overline{\alpha}'\overline{\beta}'}$, for $\eta_{\overline{\alpha}'\overline{\beta}'} = \mathrm{diag}[\pm 1, \ldots \pm 1]$ fixing a local signature for the metric field on $\mathcal{T}^E\mathbf{P}$.

Now, we can provide a proof of Theorem 2.3. Let us consider $\widehat{\mathbf{\Gamma}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = (\widehat{\mathbf{L}}^{\alpha}_{\beta\gamma}, \widehat{\mathbf{L}}^{A}_{B\gamma}; \widehat{\mathbf{B}}^{\alpha}_{\beta C}, \widehat{\mathbf{B}}^{A}_{BC})$, where

$$\begin{aligned}
\widehat{L}^a_{bf} &= \frac{1}{2}\mathbf{g}^{ae}\left(\delta_f \mathbf{g}_{be} + \delta_b \mathbf{g}_{fe} - \delta_e \mathbf{g}_{bf}\right) + \frac{1}{2}\mathbf{g}^{ae}\left(\mathbf{g}_{bd}C^{d}_{\ fe} + \mathbf{g}_{fd}C^{d}_{\ eb} - \mathbf{g}_{ed}C^{d}_{\ bf}\right),\\
\widehat{L}^A_{Bf} &= \mathcal{V}_B(\mathcal{N}^A_f) + \frac{1}{2}\mathbf{g}^{AC}\left(\delta_f \mathbf{g}_{BC} - \mathbf{g}_{DC}\mathcal{V}_B(\mathcal{N}^D_f) - \mathbf{g}_{DB}\mathcal{V}_C(\mathcal{N}^D_f)\right), \qquad (A.5)\\
\widehat{B}^{\alpha}_{\beta C} &= \frac{1}{2}\mathbf{g}^{\alpha\tau}\mathcal{V}_C \mathbf{g}_{\beta\tau},\ \ \widehat{B}^{A}_{BC} = \frac{1}{2}\mathbf{g}^{AD}\left(\mathcal{V}_C \mathbf{g}_{BD} + \mathcal{V}_B \mathbf{g}_{CD} - \mathcal{V}_D \mathbf{g}_{BC}\right).
\end{aligned}$$

Using such values as N-adapted coefficients for a canonical d-connection $\widehat{\mathcal{D}}$, we can check that $\widehat{\mathcal{D}}\overline{\mathbf{g}} = 0$ and that the h- and v-torsions (A.1) are computed $\widehat{T}^{a}_{\ bf} = C^{a}_{\ bf}$ and $\widehat{T}^{A}_{\ BC} = 0$. There are nontrivial N-adapted coefficients of torsion of $\widehat{\mathcal{D}}$, i.e. $\widehat{T}^{a}_{\ bf}$, $\widehat{T}^{a}_{\ bA}, \widehat{T}^{A}_{\ ba}$ and $\widehat{T}^{A}_{\ Ba}$, which can be computed by introducing the coefficients (A.5) into formulas (A.1).

---

[5] I.e. such metrics cannot be diagonalized by coordinate transforms.

Let us show how we can compute the N-adapted coefficients of the distorting relation $\widehat{\mathcal{D}} = \overline{\nabla} + \widehat{\mathcal{Z}}$ in Remark 2.1. Having a d-metric structure $\overline{\mathbf{g}}$ on $\mathcal{T}^E\mathbf{P}$ we can always construct a metric-compatible Levi-Civita connection $\overline{\nabla}$ which is completely defined by the zero torsion condition, ${}^{\nabla}\mathcal{T}^{\overline{\alpha}} = \{K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}\} = 0$. Parameterizing the N-adapted coefficients in the form $K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = (\overline{L}^{\alpha}_{\beta\gamma}, \overline{L}^{A}_{B\gamma}\,; \overline{B}^{\alpha}_{\beta C}, \overline{B}^{A}_{BC})$, we can verify via straightforward computations with respect to (8) and (9) that the conditions of the mentioned Remark satisfied by distortion relation

$$K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = \widehat{\mathbf{\Gamma}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} + \widehat{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}, \tag{A.6}$$

where the N-adapted coefficients of the distortion d-tensor $\widehat{\mathcal{Z}} = \{\widehat{\mathbf{Z}}^{\gamma}_{\ \alpha\beta}\}$ are

$$\begin{aligned}
&\widehat{\mathbf{Z}}^{a}_{bf} = 0, \quad \widehat{\mathbf{Z}}^{A}_{bf} = -\widehat{\mathbf{B}}^{a}_{bB}\mathbf{g}_{af}\mathbf{g}^{AB} - \frac{1}{2}\Omega^{A}_{bf}, \quad \widehat{\mathbf{Z}}^{a}_{Bf} = \frac{1}{2}\Omega^{C}_{af}\mathbf{g}_{CB}\mathbf{g}^{ba} - \Xi^{ad}_{bf}\ \widehat{\mathbf{B}}^{b}_{dB},\\
&\widehat{\mathbf{Z}}^{A}_{Bf} = {}^{+}\Xi^{AB}_{CD}\ \widehat{\mathbf{T}}^{C}_{\gamma B}, \widehat{\mathbf{Z}}^{A}_{BC} = 0, \quad \widehat{\mathbf{Z}}^{a}_{fB} = \frac{1}{2}\Omega^{A}_{bf}\mathbf{g}_{CB}\mathbf{g}^{ba} + \Xi^{ad}_{bf}\ \widehat{\mathbf{B}}^{b}_{dB},\\
&\widehat{\mathbf{Z}}^{A}_{bB} = -\ {}^{-}\Xi^{AD}_{CB}\ \widehat{\mathbf{T}}^{C}_{bD}, \quad \widehat{\mathbf{Z}}^{a}_{AB} = -\frac{\mathbf{g}^{ab}}{2}\left[\widehat{\mathbf{T}}^{C}_{bA}\mathbf{g}_{CB} + \widehat{\mathbf{T}}^{C}_{bB}\mathbf{g}_{CA}\right],
\end{aligned}$$

for $\Xi^{ad}_{bf} = \frac{1}{2}(\delta^{a}_{b}\delta^{d}_{f} - g_{\beta\gamma}g^{\alpha\tau})$ and ${}^{\pm}\Xi^{AB}_{CD} = \frac{1}{2}(\delta^{A}_{C}\delta^{B}_{D} \pm g_{CD}g^{AB})$.

Introducing $K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = \widehat{\mathbf{\Gamma}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ (A.5) into formulas (A.2), (24) and (25), we compute respectively the coefficients of curvature, $\widehat{\mathbf{R}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}}$, Ricci tensor, $\widehat{\mathbf{R}}_{\overline{\alpha}\overline{\beta}}$, and scalar curvature, ${}^{s}\widehat{\mathbf{R}}$. The distortions $K = \widehat{\mathbf{\Gamma}} + \widehat{\mathbf{Z}}$ (A.6) allows us to compute the distorting tensors ($\widehat{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}}, \widehat{\mathbf{Z}}_{\overline{\alpha}\overline{\beta}}$ and ${}^{s}\widehat{\mathbf{Z}}$) resulting in similar values for the (pseudo) Riemannian geometry on $\mathcal{T}^E\mathbf{P}$ determined by $(\overline{\mathbf{g}}, K)$, i.e. to define $R^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\delta}}, R_{\ \overline{\beta}\overline{\gamma}}$ and ${}^{s}R$.

A.3. Lie Algebroid Mechanics and Kern–Matsumoto Models

Let us briefly outline some basic constructions [19] when the canonical N- and d-connections and d-metric on $\mathcal{T}^E\mathbf{E}$, for $\mathbf{P} = \mathbf{E}$ can be generated from a regular Lagrangian $\mathcal{L}$ as a solution of the corresponding Euler–Lagrange equations. The approach was developed in geometric mechanics with regular Lagrangians on prolongations of Lie algebroids on bundle maps; see [27,29,30] and references therein (the first models on mechanics on algebroids were elaborated in [31,32]).

For a generating function $\mathcal{L}(x^i, y^a) \in C^{\infty}(\mathbf{E})$ (or Lagrangian $L(x^i, y^a)$ if $\mathbf{E} = \mathbf{TM}$), we can compute $d^E\mathcal{L} = \rho^i_a(\partial_i\mathcal{L})\mathcal{X}^a + (\partial_A\mathcal{L})\mathcal{V}^A$. A vertical endomorphism $S : \mathcal{T}^{\mathbf{E}}\mathbf{E} \to \mathcal{T}^{\mathbf{E}}\mathbf{E}$ is constructed by $S(a, b, v) = \xi^V(a, b) = (a, 0, b^V_a)$. We consider b^V_a as the vector tangent to the curve $a + \tau b$, the curve parameter $\tau = 0$. The vertical lift is a map $\xi^V : \tau^*\mathbf{E} \to \mathcal{T}^{\mathbf{E}}\mathbf{E}$ and the Liouville dilaton vector field $\triangle(a) = \xi^V(a, a) = (a, 0, b^V_a)$. This allows us to construct a model of Lie algebroid mechanics for $\mathcal{L}$ which can be geometrized on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ in terms of three geometric objects,

$$\begin{aligned}&\text{the Cartan 1-section: } \theta_{\mathcal{L}} := S^*(d\mathcal{L}) \in \mathrm{Sec}((\mathcal{T}^{\mathbf{E}}\mathbf{E})^*);\\ &\text{the Cartan 2-section: } \omega_{\mathcal{L}} := -d\theta_{\mathcal{L}} \in \mathrm{Sec}(\wedge^2(\mathcal{T}^{\mathbf{E}}\mathbf{E})^*);\\ &\text{the Lagrangian energy: } E_{\mathcal{L}} := \mathcal{L}i_{\triangle}\mathcal{L} - \mathcal{L} \in C^\infty(\mathbf{E}),\end{aligned} \tag{A.7}$$

where the Lie derivative $\mathcal{L}i_{\triangle}$ is considered in the last formula. The dynamical equations for $\mathcal{L}$ are geometrized

$$i_{SX}\omega_{\mathcal{L}} = -S^*(i_X\omega_{\mathcal{L}}) \quad \text{and} \quad i_{\triangle}\omega_{\mathcal{L}} = -S^*(d\mathbf{E}_{\mathcal{L}}), \quad \forall X \in \mathrm{Sec}(\mathcal{T}^{\mathbf{E}}\mathbf{E}). \tag{A.8}$$

The geometric objects (A.7) and equations (A.8) can are known for various applications in coefficient forms. Using local coordinates $(x^i, y^a) \in \mathbf{E}$ and choosing a basis $\{\mathcal{X}_a, \mathcal{V}_A\} \in \mathrm{Sec}(\mathcal{T}^{\mathbf{E}}\mathbf{E})$, for all a, we have

$$S\mathcal{X}_a = \mathcal{V}_a, \quad S\mathcal{V}_a = 0, \quad \triangle = y^a\mathcal{V}_a, \; E_{\mathcal{L}} = y^a\partial\mathcal{L}/\partial y^a - \mathcal{L}, \tag{A.9}$$
$$\omega_{\mathcal{L}} = \frac{\partial^2\mathcal{L}}{\partial y^a\partial y^b}\mathcal{X}^a \wedge \mathcal{V}^b + \frac{1}{2}\left(\rho^i_b\frac{\partial^2\mathcal{L}}{\partial x^i\partial y^a} - \rho^i_a\frac{\partial^2\mathcal{L}}{\partial x^i\partial y^b} + C^f_{ab}\frac{\partial\mathcal{L}}{\partial y^f}\right)\mathcal{X}^a \wedge \mathcal{X}^b,$$

for Lie algebroid structure functions (ρ^i_a, C^f_{ab}). As a vertical endomorphism (equivalently, tangent structure) can be used, the operator $S := \mathcal{X}^a \otimes \mathcal{V}_a$. A regular system is characterized by a non-degenerate Hessian

$$\tilde{g}_{ab} := \frac{\partial^2\mathcal{L}}{\partial y^a\partial y^b}, \quad |\tilde{g}_{ab}| = \det|\tilde{g}_{ab}| \neq 0. \tag{A.10}$$

A Euler–Lagrange section associated with $\mathcal{L}$ is given by any $\Gamma_{\mathcal{L}} = y^a\mathcal{X}_a + \varphi^a\mathcal{V}_a \in \mathrm{Sec}(\mathcal{T}^{\mathbf{E}}\mathbf{E})$, when functions $\varphi^a(x^i, y^b)$ solve this system of linear equations $\varphi^b\frac{\partial^2\mathcal{L}}{\partial y^b\partial y^a} + y^b(\rho^i_b\frac{\partial^2\mathcal{L}}{\partial x^i\partial y^a} + C^f_{ab}\frac{\partial\mathcal{L}}{\partial y^f}) - \rho^i_a\frac{\partial\mathcal{L}}{\partial x^i} = 0$. The semi-spray vector

$$\varphi^e = \tilde{g}^{eb}\left(\rho^i_b\frac{\partial\mathcal{L}}{\partial x^i} - \rho^i_a\frac{\partial^2\mathcal{L}}{\partial x^i\partial y^b}y^a - C^f_{ba}\frac{\partial\mathcal{L}}{\partial y^f}y^a\right) \tag{A.11}$$

can be found in explicit forms for regular configurations when $\tilde{g}^{ab}$ is inverse to $\tilde{g}_{ab}$. The section $\Gamma_{\mathcal{L}}$ transforms into a spray which states that the functions φ^b are homogenous of degree 2 on y^b if the condition $[\triangle, \Gamma_{\mathcal{L}}]_{\mathbf{E}} = \Gamma_{\mathcal{L}}$ is satisfied. The solutions of the Euler–Lagrange equations for $\mathcal{L}$,

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \rho^i_a y^a \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial y^a}\right) + y^bC^f_{ab}\frac{\partial\mathcal{L}}{\partial y^f} - \rho^i_a\frac{\partial\mathcal{L}}{\partial x^i} = 0, \tag{A.12}$$

are parameterized by curves $c(\tau) = (x^i(\tau), y^a(\tau)) \in \mathbf{E}$.

A.4. The Torsion and Curvature of the Normal d-Connection

Let us consider a 1-form associated to the normal d-connection $\widetilde{\mathcal{D}} = (\widetilde{\mathcal{D}}_a, \widetilde{\mathcal{D}}_A)$, see ${}^n\mathcal{D}$ (22) and $\widetilde{\mathbf{\Gamma}}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} = (\widehat{\mathbf{L}}^a_{bf}, \widehat{\mathbf{B}}^a_{bc})$ (23), $\widetilde{\mathbf{\Gamma}}^a_b := \widehat{\mathbf{L}}^a_{bf}\mathcal{X}^f + \widehat{\mathbf{B}}^a_{bc}\delta^c$, where $\widetilde{\mathbf{e}}_{\overline{\alpha}} = (\widetilde{\mathbf{e}}_a = \delta_a, \mathcal{V}_A)$ and $\widetilde{\mathbf{e}}^{\overline{\beta}} := \{\mathcal{X}^a, \delta^b = \mathcal{V}^b + \widetilde{\mathcal{N}}^b_f\mathcal{V}^f\}$ are taken as in (13). We can prove that the Cartan structure equations are satisfied,

$$d\mathcal{X}^a - \mathcal{X}^b \wedge \widetilde{\mathbf{\Gamma}}^a_b = -h\widetilde{\mathcal{T}}^a, \quad d\delta^c - \delta^b \wedge \widetilde{\mathbf{\Gamma}}^c_b = -v\widetilde{\mathcal{T}}^c, \tag{A.13}$$
$$\text{and} \quad d\widetilde{\mathbf{\Gamma}}^a_b - \widetilde{\mathbf{\Gamma}}^c_b \wedge \widetilde{\mathbf{\Gamma}}^a_c = -\widetilde{\mathcal{R}}^a{}_b. \tag{A.14}$$

The h- and v-components of the torsion 2-form $\widetilde{\mathcal{T}}^{\alpha} = \left(h\widetilde{\mathcal{T}}^{a}, v\widetilde{\mathcal{T}}^{a} \right) = \widetilde{\mathbf{T}}^{a}{}_{cb}\delta^{c} \wedge \delta^{b}$ from (A.13). The N-adapted coefficients are computed

$$h\widetilde{\mathcal{T}}^{a} = \widetilde{\mathbf{B}}^{a}_{bc}\mathcal{X}^{b} \wedge \delta^{c}, v\widetilde{\mathcal{T}}^{a} = \frac{1}{2}\tilde{\Omega}^{a}_{bc}\mathcal{X}^{b} \wedge \mathcal{X}^{c} + (\mathcal{V}_{c}\widetilde{\mathcal{N}}^{a}_{b} - \widetilde{\mathbf{L}}^{a}_{bc})\mathcal{X}^{b} \wedge \delta^{c}, \tag{A.15}$$

where $\tilde{\Omega}^{a}_{bc}$ are coefficients of the canonical N-connection curvature (11) of the canonical N-connection, $\mathcal{N}^{b}_{f} \rightarrow \widetilde{\mathcal{N}}^{b}_{f}$ (12). In explicit form, the N-adapted coefficients of d-torsion (A.1) of $\widetilde{\mathcal{D}}$ are

$$\widetilde{\mathbf{T}}^{a}{}_{cb} = 0, \widetilde{\mathbf{T}}^{a}{}_{cB} = \widetilde{\mathbf{B}}^{a}_{cB}, \widetilde{\mathbf{T}}^{A}{}_{cb} = \tilde{\Omega}^{A}_{bc}, \widetilde{\mathbf{T}}^{A}{}_{cB} = \mathcal{V}_{B}\widetilde{\mathcal{N}}^{A}_{c} - \widetilde{\mathbf{L}}^{A}_{cB}, \widetilde{\mathbf{T}}^{A}{}_{CB} = 0, \tag{A.16}$$

where indices $A, B, C, \ldots$ are transformed into, respectively, $a, b, c, \ldots$ at the end (to keep the convention on h- and v-indices). We note that we have chosen such nonholonomic distributions when the Y-fields from the Theorem 2.6 are stated in such way that the formulas (A.16) on $\mathcal{T}^{\mathbf{E}}\mathbf{E}$ are similar to those on **TM**.

Using a differential form calculus, we can also compute the curvature 2-form (A.14),

$$\widetilde{\mathcal{R}}^{e}{}_{f} = \widetilde{\boldsymbol{R}}^{e}{}_{fab}\delta^{a} \wedge \delta^{b} = \frac{1}{2}\widetilde{\mathcal{R}}^{e}{}_{fab}\mathcal{X}^{a} \wedge \mathcal{X}^{b} + \widetilde{\mathcal{P}}^{e}{}_{faB}\mathcal{X}^{a} \wedge \delta^{B} + \frac{1}{2}\widetilde{\mathcal{S}}^{e}{}_{fAB}\delta^{A} \wedge \delta^{B}, \tag{A.17}$$

where the nontrivial N-adapted coefficients are

$$\begin{aligned} \widetilde{\mathcal{R}}^{e}{}_{fab} &= \delta_{b}\widetilde{\mathbf{L}}^{e}_{fa} - \delta_{a}\widetilde{\mathbf{L}}^{e}_{fb} + \widetilde{\mathbf{L}}^{d}_{fa}\widetilde{\mathbf{L}}^{e}_{db} - \widetilde{\mathbf{L}}^{d}_{fb}\widetilde{\mathbf{L}}^{e}_{da} - \widetilde{\mathbf{B}}^{e}_{fA}\tilde{\Omega}^{A}_{ba}, \\ \widetilde{\mathcal{P}}^{a}{}_{bfB} &= \mathcal{V}_{B}\widehat{L}^{a}_{bf} - \widetilde{\mathcal{D}}_{f}\widetilde{\mathbf{B}}^{a}_{bB}, \ \widetilde{\mathcal{S}}^{e}{}_{fAB} = \mathcal{V}_{B}\widetilde{\mathbf{B}}^{e}_{fA} - \mathcal{V}_{A}\widetilde{\mathbf{B}}^{e}_{fB} + \widetilde{\mathbf{B}}^{d}_{fA}\widetilde{\mathbf{B}}^{e}_{dB} \\ &\quad -\widetilde{\mathbf{B}}^{d}_{fB}\widetilde{\mathbf{B}}^{e}_{dA}. \end{aligned} \tag{A.18}$$

The distortion relations for the Ricci tensor are computed as

$$\begin{aligned} R_{\overline{\alpha}\overline{\beta}} &= \widetilde{\mathbf{R}}_{\overline{\alpha}\overline{\beta}} + \widetilde{\mathbf{Z}}_{\overline{\alpha}\overline{\beta}}, \\ R_{\overline{\beta}\overline{\gamma}} &= R^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\alpha}} = \mathbf{e}_{\overline{\delta}}K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} - \mathbf{e}_{\overline{\gamma}}K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\delta}} + K^{\overline{\varphi}}_{\ \overline{\beta}\overline{\gamma}}K^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\delta}} - K^{\overline{\varphi}}_{\ \overline{\beta}\overline{\delta}}K^{\overline{\alpha}}_{\ \overline{\varphi}\gamma} + K^{\overline{\alpha}}_{\ \overline{\beta}\overline{\varphi}}W^{\overline{\varphi}}_{\ \overline{\gamma}\overline{\delta}}, \\ \widetilde{\mathbf{Z}}_{\ \overline{\beta}\overline{\gamma}} &= \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}\overline{\alpha}} = \mathbf{e}_{\overline{\alpha}}\widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} - \mathbf{e}_{\overline{\gamma}}\ \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\alpha}} + \widetilde{\mathbf{Z}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\gamma}}\ \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\alpha}} - \widetilde{\mathbf{Z}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\alpha}}\ \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\gamma}} \\ &\quad + \widetilde{\mathbf{\Gamma}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\gamma}}\ \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\alpha}} - \widetilde{\mathbf{\Gamma}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\alpha}}\ \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\gamma}} + \widetilde{\mathbf{Z}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\gamma}}\widetilde{\mathbf{\Gamma}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\alpha}} - \widetilde{\mathbf{Z}}^{\overline{\varphi}}_{\ \overline{\beta}\overline{\alpha}}\ \widetilde{\mathbf{\Gamma}}^{\overline{\alpha}}_{\ \overline{\varphi}\overline{\gamma}} + \widetilde{\mathbf{Z}}^{\overline{\alpha}}_{\ \overline{\beta}\overline{\varphi}}W^{\overline{\varphi}}_{\ \overline{\gamma}\overline{\alpha}}. \end{aligned} \tag{A.19}$$

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