Large Scale Geometry of Box Spaces

THÈSE

présentée à la Faculté des Science pour l'obtention du grade de docteur ès Sciences en Mathématiques

par

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Soutenue le 23 may 2018

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Titre:

"Large Scale Geometry of Box Spaces"

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Neuchâtel, le 20 septembre 2018

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This document is the result of my PhD during which I investigated box spaces.

We investigate coarse equivalence for full box spaces of free groups. Both for abelian free groups and non-abelian free group. In both cases we find a restriction on the number of generators.
We also construct an example of a box spaces of a free group that do not coarsely embed into a Hilbert space, but do not contain coarsely nor weakly embedded expanders, such example did not yet exist.
We also prove a rigidity result for finitely presented group. We show that the most of the filtration can be recovered by the coarse equivalence class of the box space.
We also construct an example of a box space of a free group that embeds into a Hilbert space, but where the index of the subgroups in the filtration grows only slowly.
Finally we show that Box spaces of virtually nilpotent groups have finite asymptotic dimension.

This research was funded by grant 200021_163417 of the Swiss National Science Foundation.

The following papers are part of this research:

[Del17] Thiebout Delabie,
Full box spaces of free groups.

[DK16] Thiebout Delabie and Ana Khukhro,
Box spaces of the free group that neither contain expanders nor embed into a Hilbert space.

[DK18] Thiebout Delabie and Ana Khukhro,
Coarse fundamental groups and box spaces.

[DT18] Thiebout Delabie and Matthew Tointon,
The asymptotic dimension of box spaces of virtually nilpotent groups.

Keywords: Box spaces, coarse geometry, embeddability and expanders.

Contents

Acknowledgements

Heartfelt thanks go to my supervisor Alain Valette. Merci de m'avoir accepté comme doctorant et pour tout le travail que tu as fournis, c'est facile de sous-estimer l'importance de ce que tu as fait. Another heartfelt thanks goes to my co-supervisor Ana Khukhro. At first I came to you because your office was conveniently placed across the corridor, but I came back because of the useful advice, the interesting ideas and the amusing discussions. Next I want to thank the other members of the jury, Martin Finn-Sell and Romain Tessera, for their excellent remarks. I also want to thank Paul Jolissaint and David Frenkel, c'était un plaisir d'être votre assistant.

During the past three years I had many great moments with my colleagues.
In particular I would like to thank Sanaz Pooya, it was a pleasure sharing an office with you. I also want to thank my compatriot Tom Kaiser, ik heb sterk genoten van de vele, interessante discussies en de treinritten waren veel gezelliger met z'n twee. Next I like to thank Matthew Tointon, it was a pleasure working with you. I also want to thank the other group theorists: Laura Ciobanu, her postdoc Dominik Gruber and PhD-student Valentin Mercier.
Next I want to thank the rest of math department, in particular the picnics at the lake and the movie nights were great fun. Specifically I want to thank Carsten Haug, merci pour les croisières sur le lac. I also want to thank Lucas Dahinden, thank you for the amusing walk on the Chaumont. Next I want to thank Luc Pétiard, Aïssa Ndiaye and Katie Gittins, votre bureau était clairement le plus dynamique. De plus merci à Luc pour les matchs de ping-pong stimulants. I also want to thank Mireille Tissot-Daguette, Edouard Strickler and Carl-Erik Gauthier, merci pour l'organisation des activités. I should not forget to thank Tobias Hurth, Joé Brendel and Radu Slobodeanu. Merci pour ajouter à l'ambiance.
Next I want to thank the students I taught over the years. I also want to thank Rachel and Loris Paratte, merci pour le logement. Finally I want to thank my family, in particular my siblings, Lisa, Mathijs and Annemoon, and my parents, Diderik and Myriam. Bedankt om mij te steunen doorheen deze drie jaar.

Chapter 1

Preliminaries

1.1 Groups

In this section we will look at groups. A group consists of a set G and an operation that is a well-defined map $G^2 \to G$. Every group has a neutral element, which is denoted by either e or 1. There are many examples of groups with wildly different properties. In this section we introduce some of the most well-known properties of groups. For more details we refer to [Rob12], a book Robinson.

1.1.1 Subgroups

A subgroup H of a group G denoted by $H < G$ is a subset of G that is a group itself according to the restriction of the operation. A coset of H is a set $gH = \{gh : h \in H\}$ for $g \in G$. These cosets form a partition of the group. The set of these cosets is denoted by G/H. There is a natural operation on the set of cosets, that is $g_1H \cdot g_2H = g_1g_2H$. However this does not necessarily form a group. It is a group if the subgroup is conjugacy invariant. These subgroups are called normal, equivalently a subgroup H of G is normal if $ghg^{-1} \in H$ for every $g \in G$ and $h \in H$ and is denote by $H \lhd G$. This property can be strengthened. A subgroup is called a characteristic subgroup if H is invariant under every automorphism of G and is denoted by $H \lhd_{\text{char}} G$.

If H is a normal subgroup we can consider the quotient group G/H. There is a natural map $G \to G/H$ called the quotient map and a natural action of G on G/H with $g \cdot (xH) = (gx)H$.

An example of a characteristic subgroup of G is the commutator group $[G, G]$, which is the group generated by elements $[g, h] = ghg^{-1}h^{-1}$.
We remark that $N \lhd H \lhd G$ does not imply $N \lhd G$, for example $G = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}_4$ where $\alpha(x, y) = (-y, x)$, $H = \mathbb{Z}^2$ and $N = \{(x, 0) \in H\} = \mathbb{Z}$. Here G as a set is $\mathbb{Z}^2 \times \mathbb{Z}_4$ with operation $(v, k) \cdot (w, \ell) = (v\,\alpha(w), k + \ell)$. So $((0, 0), 1) \cdot ((x, 0), 0) \cdot ((0, 0), -1) = (\alpha(x, 0), 0) = ((0, x), 0)$.
However for we do have that $N \lhd_{\text{char}} H \lhd_{\text{char}} G$ implies $N \lhd_{\text{char}} G$ and that $N \lhd_{\text{char}} H \lhd G$ implies $N \lhd G$.

Of every subgroup $H < G$ we can consider the index, that is the number of cosets denoted by $[G : H]$.
Note that every finite index subgroup contains a finite index normal subgroup. Indeed, for $H < G$ take $N = \bigcap_{g \in G} gHg^{-1}$. Clearly $N \lhd G$ and $N < H$, so it suffices to show that N is a finite index subgroup of G. For g_1 and g_2 in the same coset of H there exists an $h \in H$ such that $g_1 = g_2h$. So $g_1Hg_1^{-1} = g_2hHh^{-1}g_2^{-1} = g_2Hg_2^{-1}$. So N is equal to the intersection of $[G : H]$ conjugates of H, therefore N is of finite index in G.
If G is finitely generated, then H contains a finite index characteristic subgroup of G.

A finite index subgroup of a finitely generated group is also finitely generated. In fact the Nielsen-Schreier rank formula says that $\text{rk}(H) - 1 \leq [G : H](\text{rk}(G) - 1)$ with equality if G and H are free groups (see section 1.1.2).
One technique we will use to construct finite index subgroups is to consider the group generated by the squares, denoted as $\Gamma(G)$. For a finitely generated group G we have that $\Gamma(G)$ is of finite index in G. Note $\Gamma(G)$ is the normal subgroup of G such that $G/\Gamma(G) = \mathbb{Z}_2^n$ with n the biggest possible value.
Similarly we take $\Gamma_m(G)$ such that $G/\Gamma_m(G) = \mathbb{Z}_m^n$ with n the biggest possible value. Here G is generated by m^{th}-powers of elements in G and commutators.

1.1.2 Presentations

Let S be a set. The free group F_S is the group consisting of words with letters in S. The group operation is the concatenation of words and words can be reduced by removing subwords of the form ss^{-1} and $s^{-1}s$ with

$s \in S$.
Groups can be presented by a set of generators S and a set of relators R, denoted by $G = \langle S, R \rangle$. Here G is defined as F_S/N where N is the normal subgroup generated by the elements in R, meaning that N is the smallest normal subgroup of F_S such that $R \subset N$.

Note that every group has a presentation. We can take $S = G$ and $R = \{ghk^{-1} : g, h, k \in G, k = gh\}$. Then $G = \langle S, R \rangle$. If G is finite, then both S and R are finite. Groups with a presentation such that S and R are finite are called finitely presented. Such groups are not necessarily finite for example $\mathbb{Z}^2$ has a presentation with $S = \{a, b\}$ and R only containing $[a, b]$.
At the same time not all finitely generated groups are finitely presented, for example the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. Here $G \wr H$ is equal to $\bigoplus_H G \rtimes H$ where H acts by shifts on $\bigoplus_H G$. This group can be represented by a man lighting lamps $\mathbb{Z}_2$ on a street $\mathbb{Z}$.

1.1.3 Nilpotency and solvability

Two elements of a group commute if $gh = hg$ for g and h in G. The groups of which the elements commute are called abelian, here $[G, G] = \{e\}$. These groups are well understood, however not all groups are abelian.
A first generalization is being nilpotent. Consider $G_1 = [G, G]$ and $G_{n+1} = [G, G_n]$. A group is nilpotent if there exists an n such that G_n is trivial. The smallest such n is called the step of a nilpotent group.
For a nilpotent group G with step n consider the map $\varphi \colon G_{n-2} \to G_{n-1}$.

For any property P we can define a new property called poly-P. A group G is poly-P if there exists a sequence $G = G_0 \rhd G_1 \rhd \ldots \rhd G_n = \{e\}$ such that G_{k-1}/G_k is P for every k between 1 and n.
One particular such property is polyabelian, also known as solvable. Every nilpotent group is solvable, because $G_n/G_{n+1} = G_n/[G, G_n]$ is abelian. An other such property is polycyclic. A cyclic group is a group that can be generated by a single element. All cyclic groups are abelian, so every polycyclic group is solvable. We also have that every finitely generated nilpotent group is polycyclic.
Remark that every polycyclic group is finitely generated. Also remark that a group G is solvable if and only if the n^{th} derived group $G^{(n)}$ is trivial for some n. The derived group G' is equal to $[G, G]$.
Note that not all polycyclic groups are nilpotent, for example $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where $\alpha(1, 0) = (0, 1)$ and $\alpha(0, 1) = (1, 1)$. Here every $G_n = \mathbb{Z}^2$ for $n \geq 1$. Also note that not all finitely generated solvable groups are polycyclic, for example the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}$ defined above, where α is the shift operator.

1.1.4 Residual and virtual properties

We can also define new properties using already existing properties. For a property P we say that a group G is virtually P if there exists a subgroup $H < G$ of finite index such that H is P.
Note that some properties are preserved by finite extentions, in those case we do not have a new property. However for properties like nilpotent we do get a new property called virtually nilpotency.

For a property P we say that a group G is residually P, if for every $g \in G \setminus \{e\}$ there exists a quotient G/N of G such that $g \notin N$ and G/N is P.
An example of such a property is being residually finite. We will often assume groups to be finitely generated and residually finite, because we want to take a sequence of finite index normal subgroups N_n such that they are nested ($N_{n+1} \lhd N_n$) and their intersection is trivial ($\bigcap N_n = \{e\}$). Such a sequence is called a filtration.
A group with such a filtration is residually finite by definition. In fact any finitely generated residually finite group G has such a sequence. Indeed, a finitely generated group is countable, therefore $G = \{e, g_1, g_2, \ldots\}$. As G is residually finite there exist normal subgroups $H_n \lhd G$ such that $g_n \notin H_n$ and G/H_n is finite. Now for $N_n = \bigcap_{i=1}^n H_i$ we have that G/N_n is finite. As $g_i \notin N_n$ for every $i \leq n$ we have that $\bigcap N_n = \{e\}$.

1.2 Graphs and box spaces

In this section we define graphs. Colloquially graphs are representations of a network. They contain dots (called vertices) and connections (called edges). We also introduce Cayley graphs, which are graphs that are constructed using a group.

1.2.1 Graphs

A graph consists of a set of vertices and set of edges that connect some pairs of these vertices. We will be working with simple graphs, i.e. undirected graphs without loops and without multiple edges. Considering these properties we can define a graph as follows.

Definition 1.2.1. *A graph $\mathcal{G}$ is a pair (V, E) consisting of a set V of vertices and a set $E \subset V^{[2]} = \{\{x, y\} \mid x, y \in V, x \neq y\}$ of edges.*

Now we will look at some properties for graphs. A graph $\mathcal{G} = (V, E)$ is called finite if both V and E are finite sets. Note that if V is finite, then E is also finite. Now consider a vertex x of the graph. We call the number of edges that contain x the degree of x. A graph is called k-regular if the degree of every vertex is k. For these regular graphs, the degree k of every vertex is also called the degree of the graph.

A path of length $n \in \mathbb{N}$ between the vertices x and y of a graph is a sequence of vertices $x = x_0, x_1, \ldots, x_n = y$ where $\{x_{i-1}, x_i\}$ is an edge for every i in $\{1, \ldots, n\}$. (Not to be confused with r-paths as defined in section 1.6.2.) Now a graph is called connected if there exists a path between every two vertices. Connected graphs have a natural metric: the distance between two vertices is defined as the shortest path between them.

A cycle is a path in a graph that starts and end in the same point, does not have any backtracks and is not of length 0. The girth of a graph is the length of the shortest cycle. Note that for any graph $\mathcal{G}$ we have that $\mathrm{girth}(\mathcal{G}) \leq 2\,\mathrm{diam}(\mathcal{G}) + 1$.

The boundary ∂F of a subset $F \subset V$ of a graph $\mathcal{G} = (V, E)$ is equal to the set of edges $\{v, w\} \in E$ such that $v \in F$ and $w \notin F$ (or vice versa).

Finally note that if two graphs are coarsely equivalent, then they are quasi-isometric. As both graphs are quasi-geodesic spaces, due to the observation made in section 1.4.1, we can assume ρ_- and ρ_+ to be linear.

1.2.2 Cayley graphs

When we are working with graphs, we will mainly be interested in Cayley graphs, which are graphs that represent the structure of finitely generated groups.

Definition 1.2.2. *Let G be a group and let S be a finite generating subset of G not containing 1. Then the Cayley graph $\mathrm{Cay}(G, S)$ is the graph with G as the set of vertices and $\{(g, gs) : g \in G, s \in S\}$ as the set of edges.*

Note that if $\{g, gs\}$ is an edge of a Cayley graph, then $\{gs^{-1}, (gs^{-1})s\} = \{g, gs^{-1}\}$ is also an edge. Therefore replacing S with $S \cup S^{-1}$ does not make a difference, so we can assume that S is symmetric.
Also note that Cayley graphs are connected regular graphs.

Proposition 1.2.3. *Let G be a finitely generated group and let $S \subset G \setminus \{1\}$ be a finite generating set. Then the Cayley graph $\mathrm{Cay}(G, S)$ is a connected $|S \cup S^{-1}|$-regular graph.*

Proof. Let g be an element of G. Then the set of edges containing g is given by $\{\{g, gs\}|s \in S\} \cup \{\{gs^{-1}, g\}|s \in S\}$. This set equals $\{\{g, gs\}|s \in S \cup S^{-1}\}$, so the cardinality of this set equals $|S \cup S^{-1}|$. So the degree of g equals $|S \cup S^{-1}|$.
Since S is a generating set we have that every vertex of $\mathrm{Cay}(G, S)$ is connected to $1 \in G$, therefore $\mathrm{Cay}(G, S)$ is connected. □

As $\mathrm{Cay}(G, S)$ is constructed using a group G there exists a natural isometric group action of G on $\mathrm{Cay}(G, S)$. This action is the left regular action $\lambda \colon G \to \mathrm{Iso}(\mathrm{Cay}(G, S)) \colon g \mapsto \lambda_g$ also known as the action by left multiplication as $\lambda_g(h) = gh$ and $\lambda_g(\{h, hs\}) = \{gh, ghs\}$ for every $h \in G$ and $s \in S$.

Every edge of a Cayley graph $\mathrm{Cay}(G, S)$ corresponds to an element of the generating set S. Similarly the paths in $\mathrm{Cay}(G, S)$ that start in the neutral element e have a one-to-one corresponds with the words with letters in S. Note that the words corresponding to loops in $\mathrm{Cay}(G, S)$ are the relators of $G = \langle S|R \rangle$.

1.3 Representations of groups

In this section we introduce representations of groups these are maps from the group to the linear operators on a vector space. We also introduce Banach spaces, certain normed vector spaces and representations on these Banach spaces. Then we define characters of representation and how they can be used to study the representations. Finally we define properties of groups related to representations called amenability, the Haagerup property and property (T). For more information on representations and property (T) we refer to [BHV08].

1.3.1 Banach spaces

A vector space $(\mathbb{F}, V, +)$ over a field $\mathbb{F}$ is an abelian group $(V, +)$ with a scalar multiplication, i.e. every vector in V can be multiplied with a scalar in $\mathbb{F}$.
We will only consider vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

A normed vector space is a vector space with a norm $\|\cdot\|$, the norm represents the length of the vector. Normed vector spaces have a natural metric $d(v,w) = \|v-w\|$ for every v and w in the vector space.
A Banach space is a normed vector space such that the norm induced metric is complete: i.e. every Cauchy sequence converges. A Hilbert space is a Banach space with an inner product $\langle\cdot,\cdot\rangle$ such that $\|v\|^2 = \langle v,v\rangle$.
Example of Banach spaces are the ℓ^p spaces. For $p \geq 1$ define the p-norm $\|\cdot\|_p$ such that $\|(x_n)_n\|_p^p = \sum_{i=0}^{\infty} |x_n|^p$ for any sequence $(x_n)_n$. The space of sequences (either in $\mathbb{R}$ or $\mathbb{C}$) with finite p-norm is called ℓ^p and is a Banach space. For $p = 2$ we have a Hilbert space with inner product $\langle (x_n)_n, (y_n)_n\rangle = \sum_{i=0}^{\infty} x_n\overline{y_n}$.

1.3.2 Representations

A representation of a group is a map from the group to the operators on a vector space.

Definition 1.3.1. *A representation of a group G on a vector space V is a map $\pi\colon G \to L(V)\colon g \mapsto \pi_g$ such that $\pi_1 = \mathrm{Id}_V$ and $\pi_{gh} = \pi_g \circ \pi_h$ for every $g,h \in G$.*

Note that $L(V)$ is the space of linear operators on V.
The degree of a representations is the dimension of the vector space V.
We will be working exclusively with unitary representations on a Hilbert space. In fact we will often omit the word unitary. Unitary refers to unitary operators, which are linear operators that are isometric.

Definition 1.3.2. *A unitary representation of a group G on a Hilbert space $\mathcal{H}$ is a map $\pi\colon G \to B(\mathcal{H})\colon g \mapsto \pi_g$ such that $\pi_1 = \mathrm{Id}_{\mathcal{H}}$, $\pi_{gh} = \pi_g \circ \pi_h$ for every $g,h \in G$ and π_g is unitary for every $g \in G$.*

Note that representations also exist for topological groups, these are groups with a topology such that the multiplication and inversion maps are continuous. For those topological groups we consider strongly continuous representations. These are representations for which the map $g \mapsto \pi_g\xi$ is continuous for every ξ.
We will only consider discrete groups, for these groups all representations are strongly continuous. In that sense the groups we use could be considered as a discrete groups even though the topology never gets brought up.

An important aspect of representation theory is the concept of invariant vectors. An invariant vector of a representation of a group G on a Hilbert space $\mathcal{H}$ is a vector $\xi \in \mathcal{H}$ such that $\pi_g(\xi) = \xi$ for every $g \in G$. These vectors form a subspace of $\mathcal{H}$, which will be denoted by $\mathcal{H}^{\pi(G)}$.

Every group G has an trivial representation where every element of G gets mapped to the identity operator. It also has the left regular representation $\lambda\colon G \to B(\ell^2(G))$ with $\lambda_g\delta_h = \delta_{g^{-1}h}$ for every $g,h \in G$.
It is also possible to do the following constructions:
We can lift a representation π of a quotient G/N. Here $\tilde{\pi}_g$ is equal to π_{gN}.
We can add two representations π and ρ together $\pi\oplus\rho\colon G \to B(\mathcal{H}_\pi \oplus \mathcal{H}_\rho)$ with $(\pi\oplus\rho)_g(\xi,\eta) = (\pi_g(\xi), \rho_g(\eta))$.
It is possible to decompose representations according to this addition. Representations that can not be decomposed are called irreducible representations. Equivalently representations are irreducible if they only have trivial G-invariant subspaces.
Finally we can create new representations by taking the tensor product $\pi\otimes\rho\colon G \to B(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ with $(\pi\otimes\rho)_g(\xi\otimes\eta) = \pi_g(\xi)\otimes\rho_g(\eta)$.

1.3.3 Character theory

The character χ_π of a unitary representation π of G on a finite dimensional Hilbert space is $\chi_\pi\colon G \to \mathbb{C}$ such that $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$.
Note that for any two unitary representations π and ρ of a group G and any $g \in G$ we have that $\chi_{\pi\oplus\rho}(g) = \chi_\pi(g) + \chi_\rho(g)$ and $\chi_{\pi\otimes\rho}(g) = \chi_\pi(g)\cdot\chi_\rho(g)$.

As χ_π is an element of $\ell^2(G)$ we can consider the inner product $\langle\chi_\pi,\chi_\rho\rangle = \dfrac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_\rho(g)}$.

Character theory is useful because of the orthogonality properties:

- For every two irreducible unitary representations π and ρ we have that $\langle\chi_\pi,\chi_\rho\rangle$ is equal to 1 if π and ρ are isomorphic and 0 otherwise.
- For any $g,h \in G$ we have that the sum over all irreducible unitary representations π of $\chi_\pi(g)\overline{\chi_\pi(h)}$ is equal to the size of the centralizer of g if g and h are conjugates or 0 otherwise.

As all characters of unitary representations are the linear combination of characters of irreducible unitary representations and the characters of non-isomorphic irreducible unitary representations are orthogonal we

have a one to one correspondence between characters and isomorphism classes of unitary representations.
We also have that

$$|G| = \sum_{\pi \text{ irred.}} |\chi_\pi(e)|^2 = \sum_{\pi \text{ irred.}} |\chi_\pi|^2 = \sum_{\pi \text{ irred.}} \deg(\pi)^2.$$

1.3.4 Amenability and the Haagerup property

Amenability is a group property that is a generalization of both finiteness and being abelian. A group G is amenable if it has a left invariant mean, that is there exists a linear map $\varphi\colon \ell^\infty(G) \to \mathbb{R}$ that is left invariant, non-negative and has norm 1. Here $\ell^\infty(G)$ is the space containing all bounded maps $G \to \mathbb{R}$. This map is non-negative if every map $G \to \mathbb{R}^+$ is mapped to a non-negative value. For non-negative maps $\ell^\infty(G) \to \mathbb{R}$ norm 1 means that the image of the constant 1 map is mapped to 1. Left invariant means that for every $g \in G$ and every $f \in \ell^\infty(G)$ we have $\varphi(g \cdot f) = \varphi(f)$ where $g \cdot f(x) = f(g^{-1}x)$ for every $x \in G$.

Remark that finite groups have a natural mean, that is the average: $\varphi\colon \ell^\infty(G) \to \mathbb{R}\colon f \mapsto \dfrac{1}{|G|}\displaystyle\sum_{g\in G} f(g)$.
Also remark that for any normal subgroup N is G we have that if N is amenable and G/N is amenable, then G is amenable as well. Indeed, if there exist two left invariant means φ_N and $\varphi_{G/N}$, then we can define a map $\varphi\colon \ell^\infty(G) \to \ell^\infty(G/N)$ such that $\varphi(f)(gN) = \varphi_N(f|_{gN})$. Then $\varphi_{G/N} \circ \varphi$ is a left invariant mean of G.
Here we say that G is an extension of N by G/N. So amenability is preserved by amenable extensions.

Amenability has many alternative definitions. We will use the Følner condition, shown in [Fol55].
A finitely generated group G is amenable if for every finite subset S and every $\varepsilon > 0$ there exists a finite subset $F \subset G$ such that $|F \triangle gF| < \varepsilon|F|$ for every $g \in S$. This is equivalent with having finite sets in the Cayley graph $\mathrm{Cay}(G, S)$ with small boundary.
Indeed, if $|F \triangle gF| < \varepsilon|F|$ for every $g \in S$, then $|\partial F| = \sum_{s\in S} |F \setminus sF| \leq \sum_{s\in S} |F \triangle sF| \leq |S|\varepsilon|F|$. In the other direction we remark that small boundary is a quasi-isometric invariant, so due to Proposition 1.4.8 we have for every finite generating set S there exists a finite subset $F \subset \mathrm{Cay}(G, S)$ such that $|\partial F| < \varepsilon|F|$. So for every $s \in S$ we have that $|F \triangle gF| \leq |\partial F| < \varepsilon|F|$. Therefore G satisfies the Følner condition.

A last characterization of amenable uses representation theory. A group is amenable if and only if the left regular representation λ has almost invariant vectors. Indeed, for any generating set S and any $\varepsilon > 0$ there exists a finite set F in G such that $|F \triangle gF| < \varepsilon|F|$ due to the Følner condition. Now for every $g \in S$ we have $\|\lambda_g \chi_F - \chi_F\| = |gF \Delta F| < \varepsilon|F| = \varepsilon\|\chi_F\|$, so λ indeed has almost invariant vectors.

With the FøIner condition it is easy to show that $\mathbb{Z}$ is amenable. In $\mathrm{Cay}(\mathbb{Z}, \{1\})$ we can take $F = [1, 2n] \cap \mathbb{Z}$, then $|\partial F| = 2$ while $|F| = 2n$, so $|\partial F|/|F|$ can be arbitrary small.
As amenability is preserved by amenable extentions we have that all abelian groups and even all solvable groups are amenable.

Remark that class of amenable groups is closed under directed unions. This means that for every sequence $G_1 < G_2 < \ldots$ of amenable groups we have that $\bigcup_{n=1}^\infty G_n$ is amenable.
We can consider the smallest class of groups that contains abelian and finite groups and is closed under extentions and directed unions. Groups in this class are called elementary amenable groups. These are not all amenable group for example the Grigorchuk group is amenable, but not elementary amenable.

An other important property is the Haagerup property also known as a-T-menability. A group has the Haagerup property if it is either finite or has a C_0 unitary representation with almost invariant vectors. A representation π on $\mathcal{H}$ is C_0 if for every $\xi, \nu \in \mathcal{H}$ we have that $\lim_{g\to\infty}\langle \pi_g(\xi), \nu\rangle = 0$.
Note that the left regular representation of an infinite group is C_0, so every amenable group has the Haagerup property.

1.3.5 Property (T) and property (τ)

Another property related to the notion of almost invariant vectors is that of property (T). This is a property of locally compact groups. However we will restrict to discrete groups.
To define property (T) we need the notion of almost invariant vectors. A representation $\pi\colon G \to B(\mathcal{H})$ of the group G has almost invariant vectors if for every finite $S \subset G$ and every $\varepsilon > 0$ there is a $\xi \in X$ such that $\sup\limits_{g\in S} \|\pi_g(\xi) - \xi\| < \varepsilon\|\xi\|$.

Definition 1.3.3. *A discrete group G has property* (T) *if every unitary representation π of G having almost invariant vectors has a non-zero invariant vector.*

This property has many different characterisations. The name property (T) comes from the characterisation that the trivial representation is isolated for the Fell topology. For a proof of this equivalence and other details we refer to [BHV08].
Another characterisation is the existence of a Kazhdan pair. A group G has a property (T) if there exists a finite generating set S and a Kazhdan constant $C > 0$ such that for every unitary representation π of G on $\mathcal{H}$ without non-trivial invariant vectors and any $\xi \in \mathcal{H}$ there exists an $s \in S$ such that $\|\pi_s(\xi) - \xi\| \geq C\|\xi\|$. This is proved in Proposition 2.1 of [BHV08].
Property (T) is often used as an obstruction to amenability. If a discrete group is both amenable and has property (T), then it is finite. This is proved in Theorem 1.1.6 of [BHV08].

Let G be a group with an infinite subset Y. The pair (G, Y) has relative Property (T) if for every representation π of G and every almost invariant sequence of unit vectors ξ_n we have $\sup_{y \in Y} \|\pi_y \xi_n - \xi_n\| \to 0$ as $n \to \infty$.
Property (τ) is a weaker version of property (T). The group G has property (τ) relative to a sequence of finite index normal subgroups N_n if it has a Kazhdan pair relative to the representations of G without invariant vectors that are trivial on one of the subgroups N_n.

Definition 1.3.4. *A group G with a sequence of finite index normal subgroups N_n has property (τ), if there exists a constant $C > 0$ and a finite generating set S such that for every unitary representation π of G on $\mathcal{H}$ that is trivial on one of the subgroups N_n and without any non-trivial invariant vectors we have that for every $\xi \in \mathcal{H}$ there exists an $s \in S$ such that $\|\pi_s(\xi) - \xi\| \geq C\|\xi\|$.*

Note that every group with property (T) has property (τ) for any sequence of finite index normal subgroups.

1.4 Large scale geometry

In this section we will be looking at metric spaces. A metric space (X, d) consists of a set X and a metric $d \colon X \times X \to \mathbb{R}^+$, which determines a distance between every two point in X.
We want to study the large scale structure of metric spaces. First we will define bi-Lipschitz, quasi-isometric and coarse maps, these kinds of maps partially preserve the distance, in all three cases sets that are bounded get mapped to something bounded, i.e. they preserve the large scale structure. For these maps there exists a related equivalence relation. Then we will define some properties that are invariant for these equivalence relations.
For more information on large scale geometry we refer to [NY12].

1.4.1 Quasi-isometries and coarse equivalence

In this section we will take a look at maps that at least partially preserve distance and we will define equivalences that partitions the family of metric spaces in classes with a similar metric structure.

Consider two metric spaces (X, d_X) and (Y, d_Y) and a map $f \colon X \to Y$. We will define a variety of conditions that partially preserve the distance.
The strongest preservation of distance is being isometric. We say that f is isometric or an isometric embedding if $d_X(x, x') = d_Y(f(x), f(x'))$ for every x and x' in X. If f is also a bijection, then X and Y are isometric and f is a isometry.
The first weakening that we consider is called bi-Lipschitz, f is bi-Lipschitz if there exists a constant $C > 1$ such that for every $x, x' \in X$ we have that $\frac{1}{C} d_X(x, x') \leq d_Y(f(x), f(x')) \leq C d_X(x, x')$. This property is called bi-Lipschitz because the map f and its inverse $f^{-1} \colon \mathrm{Im}(f) \to X$ are both C-Lipschitz for some constant C. If f is also a bijection, then X and Y are bi-Lipschitz equivalent.
The next weakening we consider is quasi-isometric, this is a large scale version of bi-Lipschitz.

Definition 1.4.1. *A map $f \colon (X, d_X) \to (Y, d_Y)$ is quasi-isometric or a quasi-isometric embedding if there exists two constants B and C such that for every x and x' in X we have that $\frac{1}{C} d_X(x, x') - B \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + B$. If on top of that the image of f is C-dense for some constant C, i.e. for every element in $y \in Y$ there exists an element z in the image of f such that $d_Y(y, z) \leq C$, then f is a quasi-isometry and X and Y are quasi-isometric.*

To say that X and Y are quasi-isometric we write $X \cong_{\mathrm{QI}} Y$.
The weakest preservation to distance is being coarse.

Definition 1.4.2. *A map $f\colon (X, d_X) \to (Y, d_Y)$ is coarse or a coarse embedding if there exists two functions $\rho_\pm$ such that $\rho_\pm(n) \to +\infty$ as $n \to +\infty$ for every x and x' in X we have that $\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))$. If on top of that the image of f is C-dense for some constant C, then f is a coarse equivalence and X and Y are coarsely equivalent.*

If X is a geodesic space, then we can take ρ_+ to be a linear function. Indeed, for every x and x' in X with $d_X(x, x') - 1 \leq n \in \mathbb{N}$ we can take $x_1, \dots, x_n$ such that $d_X(x, x_1), d_X(x_1, x_2), \dots, d_X(x_n, x') \leq 1$. Then

$$d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_1)) + d_Y(f(x_1), f(x_2)) + \ldots + d_Y(f(x_n), f(x')) \leq (n+1)\rho_+(1),$$

so we can conclude that $d_Y(f(x), f(x')) \leq \rho_+(1)(d_X(x, x') + 1)$.
Similarly if X is a quasi-geodesic bounded geometry and Y is a Hilbert space, then we may suppose that $\rho_+(x) = x$ for every $x \in \mathbb{R}^+$.

An alternative definition of coarse is the preservation of boundedness.

Proposition 1.4.3. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f\colon X \to Y$ is a coarse embedding if and only if*

$$d_X(x_n, y_n) \to +\infty \iff d_Y(f(x_n), f(y_n)) \to +\infty$$

for any two sequences $(x_n)_n$ and $(y_n)_n$ in X.

Note that the equivalences we defined are indeed equivalence relations. However in the quasi-isometric and coarse case, it is not straightforward the relation is symmetric. Let $f\colon X \to Y$ be the map realizing the quasi-isometry (or coarse equivalence), then for every $x, x' \in X$ we have that $\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))$ where $\rho_\pm$ is linear (or tends to $+\infty$ respectively) and the image of f is C-dense. So for every $y \in Y$ we can take $x_y \in X$ such that $d_Y(f(x_y), y) \leq C$. Then the map $f'\colon Y \to X\colon y \mapsto x_y$ can be used to show that Y is quasi-isometric with X (or that Y is coarsely equivalent with X respectively).

1.4.2 Metrized disjoint unions

Given a sequence of bounded metric spaces $(X_n)_n$ we want to consider it as a single metric space. As a set we can take the disjoint union $\bigsqcup X_n$. Unfortunately there is no natural way to define the metric on this disjoint union. Therefore we take any possible metric such that the distance between two points of the same component is the distance within that metric space and the distance between elements of different components only depends on those components and for every R the pairs of components such that the distance between their elements is less than R is finite.

Every such metric is called a metrization of the disjoint union $\bigsqcup X_n$. Fortunately all these metrizations are coarsely equivalent.

Proposition 1.4.4. *Let X_n be a sequence of bounded metric spaces and let d_1 and d_2 be two metrizations of the disjoint union, then $(\bigsqcup X_n, d_1)$ is coarsely equivalent to $(\bigsqcup X_n, d_2)$.*

Proof. Let $f\colon (\bigsqcup X_n, d_1) \to (\bigsqcup X_n, d_2)$ be the identity map. Due to Proposition 1.4.3 it suffices to show that if $d_1(x_n, y_n) \not\to +\infty$, then $d_2(x_n, y_n) \not\to +\infty$.

Assuming $d_1(x_n, y_n)$ does not go to infinity, we know there exists a subsequence n_i and a constant C such that $d_1(x_{n_i}, y_{n_i}) \leq C$ for every i. The number of pairs of components where the elements of one are at a distance of at most C to the elements of the other one is finite, for each of these pairs we can take the distance between the elements of these pair for d_2. Let D be the maximum of these distances over all such pairs.
Now if x_{n_i} and y_{n_i} are in the same component, then $d_2(x_{n_i}, y_{n_i}) \leq C$. If x_{n_i} and y_{n_i} are in different components, then $d_2(x_{n_i}, y_{n_i}) \leq D$. So $d_2(x_{n_i}, y_{n_i}) \leq \max(C, D)$ for every i and therefore $d_2(x_n, y_n)$ does not go to infinity either. $\square$

As all these metrization are coarsely equivalent we can take the metrized disjoint union to be that coarse equivalence class containing all the different metrization. If the diameter of X_n goes to infinity as $n \to \infty$, then we can represent that coarse equivalence class with the metrized disjoint union for which the distances between elements of X_n and X_m is equal to $\operatorname{diam}(X_n) + \operatorname{diam}(X_m)$ for every $n \neq m$.

We will mainly be working with sequences of graphs with bounded degree, these metric spaces are bounded geometries, meaning that the for every R the size of the balls of radius R is bounded. For sequences of graphs we have that the metrized disjoint union is of bounded geometry as well.

Proposition 1.4.5. *Let X_n be a sequence of finite graphs with bounded degree. Then the metrized disjoint union $\bigsqcup X_n$ is of bounded geometry.*

Proof. Let K be the upper bound on the degree and take $R > 0$. Let C be the number of pairs of components with elements at a distance at most R. Now let $x \in \bigsqcup X_n$ and let i be such that $x \in X_i$. Note that K must be at least 2, so the number of elements in X_i at a distance of at most R is at most $1 + K\frac{(K-1)^R-1}{K-2}$ if $K \geq 3$ and at most $2R + 1$ if $K = 2$, so in both cases it is less than $K^{R+1} + 1$. For other elements take $y \in X_j \neq X_i$ with $d(x, y) \leq R$. Then $\operatorname{diam}(X_j) \leq 2R$, so $|X_j| \leq K^{2R+1} + 1$. So $|B[x, R]| \leq (K^{R+1} + 1) + C(K^{2R+1} + 1)$, which is independent of x, therefore $\bigsqcup X_n$ is of bounded geometry. □

1.4.3 Box Spaces

Given a residually finite, finitely generated group G, we say that a sequence of nested finite index normal subgroups of the group is a filtration if this sequence of subgroups has trivial intersection. Given such a filtration $\{N_i\}$ of G and fixing a generating set of G, we can consider each finite quotient G/N_i with the Cayley graph metric induced by image the generating set of G.

Definition 1.4.6. *Let G be a finitely generated group with generating set S. The box space $\square_{N_i} G$ of a group G with respect to a filtration $\{N_i\}$ is the metrized disjoint union of the Cayley graphs $\operatorname{Cay}(G/N_i, \bar{S})$, where $\bar{S}$ is the image of S under the quotient map $G \to G/N_i$.*

Note that such a filtration only exists if G is residually finite. There also exist some similar constructions. The most notable example is the full box space.

Definition 1.4.7. *Let G be a finitely generated group with generating set S. Then $\square_f G$, the full box space of G, is the metrized disjoint union of $\operatorname{Cay}(G/N, \bar{S})$ over all normal subgroup N of G and where $\bar{S}$ is the image of S under the quotient map $G \to G/N$.*

There exist more variations on this construction. One such variation weakens the definition of filtration to Farber sequence, here the subgroups are not necessarily normal, so $\operatorname{Cay}(G/N, \bar{S})$ is replaced by a Schreier graph.
As a variation on the full box space it is possible to take the metrized disjoint union over all subgroup or all characteristic subgroup.

Note that two Cayley graphs of the same group are quasi-isometric.

Proposition 1.4.8. *Let G be a finitely generated group and let S and T be two finite generating groups. Then $\operatorname{Cay}(G, S)$ and $\operatorname{Cay}(G, T)$ are quasi-isometric.*

Proof. Take $\varphi\colon \operatorname{Cay}(G, S) \to \operatorname{Cay}(G, T)$ to be the identity on G and take $D_s = \operatorname{diam}_{\operatorname{Cay}(G,T)}(S)$ and $D_t = \operatorname{diam}_{\operatorname{Cay}(G,S)}(T)$. Then for every $g \in G$, $s \in S$ and $t \in T$ we have that $d_{\operatorname{Cay}(G,T)}(g, gs) \leq D_s$ and $d_{\operatorname{Cay}(G,S)}(g, gt) \leq D_t$. So for every g and h in G we have that $\frac{1}{D_s} d_{\operatorname{Cay}(G,T)}(g, h) \leq d_{\operatorname{Cay}(G,S)}(g, h) \leq D_t d_{\operatorname{Cay}(G,T)}(g, h)$. □

As two Cayley graphs of the same group are quasi-isometric we know that the coarse equivalence class of a box space is independent of the generating set for any of these constructions.

The following standard result says that the components of a box space locally 'look like' the group.

Proposition 1.4.9. *Let G be a residually finite, finitely generated group and let $(N_n)_n$ be a filtration of G. Then there exists an increasing sequence $(i_n)_n$ such that for every $k \in \mathbb{N}$ the balls of radius k of G are isometric to the balls of radius k of G/N_i, where $i \geq i_n$.*

Proof. For a given k and large enough i we have $B_G(e, 2k) \cap N_i = \{1\}$, which in turn implies that $B_G(e, k)$ is isometric to $B_{G/N_n}(e, k)$. □

1.4.4 Asymptotic dimension

Asymptotic dimension is a coarse version of the topological dimension and was defined by Gromov [Gro93]. There exist many equivalent definitions of asymptotic dimension, see theorem 19 of [BD08]. In every one of these definition we define when the asymptotic dimension is smaller that an natural number n. Then the asymptotic dimension is the smallest such n.

A first definition of asymptotic dimension uses the notion of R-multiplicity. Let X be a metric space and let $\mathcal{U}$ be a covering of X, a family of subsets of X such that every element of X is contains in one subset in

$\mathcal{U}$. For a number $R > 0$ the R-multiplicity of $\mathcal{U}$ at $x \in X$ is equal to the number of sets $U \in \mathcal{U}$ containing an element $u \in U$ such that $d(x, u) \leq R$. The R-multiplicity of $\mathcal{U}$ is the supremum of the R-multiplicity at every point.
A covering is uniformly bounded if there exists an $S > 0$ such that $\operatorname{diam}(U) \leq S$ for every $U \in \mathcal{U}$.

Definition 1.4.10. *The metric space X has asymptotic dimension smaller than n, denoted as* $\operatorname{asdim}(X) \leq n$, *if for every $R > 0$ there exists a uniformly bounded covering with R-multiplicity at most $n + 1$.*

Note that some metric spaces do not satisfy $\operatorname{asdim}(X) \leq n$ for any n. In this case we say that the asymptotic dimension of X is infinite.

A second definition of asymptotic dimension is very similar. The asymptotic dimension of X is smaller than n if for every $R > 0$ there exist families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ such that the union of these families is a uniformly bounded covering of X and every U and V in the same family are R-disjoint, for every $u \in U$ and $v \in V$ we have that $d(u, v) > R$.

The last definition of asymptotic dimension considers coarse maps from X to a simplicial complex. A simplicial complex is a set composed of simplices, i.e. sets $\sigma_n = \{(x_0, \ldots, x_n) \in (\mathbb{R}^+)^{n+1} : x_0 + \ldots + x_n = 1\}$.
The asymptotic dimension of X is smaller than n, if for every $\varepsilon > 0$ there exists a coarse map $X \to K$ where K is an n dimensional simplicial complex and the upper control function ρ_+ is such that $\rho^+(x) = \varepsilon x$.

Note that asymptotic dimension is a coarse invariant. Indeed, suppose $\operatorname{asdim} X = n$ and there exists a coarse map $\varphi \colon X \to Y$. Let $\rho_\pm$ be the control functions of φ and let C be such that the image of φ is C-dense.
Let $R > 0$, then there exists a covering $\mathcal{U}$ of X with $\rho_+^{-1}(R + C)$-multiplicity is at most $n + 1$ and let $\mathcal{V}$ be the C-neighbourhood of $\varphi(\mathcal{U})$. The $(R + C)$-multiplicity of $\varphi(\mathcal{U})$ is at most $n + 1$ and therefore the R-multiplicity of $\mathcal{V}$ is at most $n + 1$ as well. As the image of φ is C-dense and $\varphi(\mathcal{U})$ is a covering of that image, we have that $\mathcal{V}$ is a covering of Y. So we can conclude that $\operatorname{asdim} Y = n$.

Some examples of metric spaces with finite asymptotic dimension are the Cayley graphs of polycyclic groups (see [BD06] and [DS06]), $\mathbb{R}^n$ with the standard metric, as it is coarsely equivalent to any Cayley graph of $\mathbb{Z}^n$ and the Cayley graphs of hyperbolic groups (see [Gro93]). An example of a metric space with infinite asymptotic dimension is $\mathbb{Z} \wr \mathbb{Z}$, as it contains a subspace that is coarsely equivalent to $\mathbb{Z}^n$ for every n.

1.4.5 Property A

Property A is a generalization of amenability similar to the Følner condition.

Definition 1.4.11. *A metric space X has property A if for every $R, \varepsilon > 0$ there exists a constant C and there exist finite sets $A_x \subset X \times \mathbb{N}$ for every $x \in X$ such that $A_x \subset B(x, C) \times \mathbb{N}$ and for every $x, y \in X$ with $d(x, y) \leq R$ we have that $\frac{A_x \Delta A_y}{A_x \cap A_y} \leq \varepsilon$.*

A space with finite asymptotic dimension has property A. Indeed, let $\operatorname{asdim} X = n - 1$ and let $R, \varepsilon > 0$. There exists a uniformly bounded covering $\mathcal{U}$ of X with $\frac{R}{n\varepsilon}$-multiplicity at most n. If S is the uniform bound on $\mathcal{U}$, then we take $C = S + \frac{R}{n\varepsilon}$.
For every set $U \in \mathcal{U}$ we can fix $x_U \in U$. For every $x \in X$ we take A_x to be the set containing the elements (x_U, k) with $U \in \mathcal{U}$ and $k \in \mathbb{N}$ with $d(x, U) + k \leq \frac{R}{n\varepsilon}$.
We know that (x_U, k) can only be in A_x if $d(x, U) \leq \frac{R}{n\varepsilon}$ and x_U is at most at a distance S from the closest element to x. So $d(x, x_U) \leq S + \frac{R}{n\varepsilon} = C$ and therefore $A_x \subset B(x, C) \times \mathbb{N}$.
Now there exists at most n sets in $\mathcal{U}$ that intersect the ball $B\left(x, \frac{R}{n\varepsilon}\right)$, so for any y with $d(x, y) \leq R$ we have that $|A_x \Delta A_y| \leq |A_x| \leq n \cdot \frac{R}{n\varepsilon} = \frac{R}{\varepsilon}$.

However not every metric space with property A has finite asymptotic dimension for example any Cayley graph of $\mathbb{Z} \wr \mathbb{Z}$.

It is also known that spaces with property A can be coarsely embedded into a Hilbert space, this is shown in Theorem 2.7 of [Yu00]. A metric space X embeds into a Hilbert space if there exists a Hilbert space $\mathcal{H}$ and a coarse map $\varphi \colon X \to \mathcal{H}$.

1.5 Expanders

In this section we give several definitions of what it means for a graph (or family of graphs) to be an expander family. We also compare their coarse structure to that of a Hilbert space. For an extended introduction to

expanders we refer to [HLW06].

1.5.1 Expanders

An expander is a property for sequence of finite graphs $\mathcal{G}_n$ such that $|\mathcal{G}_n| \to \infty$. It has several equivalent definitions.
The first definition uses the Cheeger constant. Recall that the boundary of a subset F of a finite graph $\mathcal{G}$ is $\partial F = \big\{\{x,y\} \mid x \in F, y \notin F\big\}$. Now the Cheeger constant, also called the edge expansion ratio of $\mathcal{G}$, is defined as

$$h(\mathcal{G}) = \min_{F \subset V} \left\{ \frac{|\partial F|}{|F|} \,\middle|\, 2|F| \le |V| \right\}.$$

A sequence of finite graphs $(\mathcal{G}_n)_n$ is an expander, if $|\mathcal{G}_n| \to \infty$ and there exists a constant $\varepsilon > 0$ such that $\varepsilon \le h(\mathcal{G}_n)$ for every n.

An alternative definition is the existence of a spectral gap, a gap in the eigenvalues of the Laplacian: The Laplacian of a regular graph $\mathcal{G}$ is an operator on $\ell^2(\mathcal{G})$, where $\ell^2(\mathcal{G})$ is the set of functions from the vertex set of $\mathcal{G}$ to $\mathbb{C}$ with the 2-norm. The Laplacian is defined by $\Delta(f)(x) = \sum\limits_{y\sim x} f(x) - f(y)$ for every $f \in \ell^2(\mathcal{G})$ and $x \in \mathcal{G}$.

If f is constant, then $\Delta(f) = 0$. Therefore 0 is an eigenvalue of Δ. All eigenvalues of Δ lie in $[0, 2k]$, where k is the degree of $\mathcal{G}$ in every vertex. Indeed, for f an eigenvector for eigenvalue λ we have that $\langle f, \Delta(f)\rangle = \lambda \|f\|^2$ and at the same time we have the following:

$$\begin{aligned}
\langle f, \Delta(f)\rangle &= \sum_{x\in\mathcal{G}} f(x) \sum_{y\sim x} f(x) - f(y) \\
&= \sum_{x\in\mathcal{G}} \sum_{y\sim x} f(x)(f(x) - f(y)) \\
&\le \sum_{x\in\mathcal{G}} \sum_{y\sim x} f(x)(f(x) + f(x)) \\
&= \sum_{x\in\mathcal{G}} \sum_{y\sim x} 2f(x)^2 \\
&= 2k\|f\|^2
\end{aligned}$$

and

$$\begin{aligned}
0 &\le \sum_{x\in\mathcal{G}} \sum_{y\sim x} (f(x) - f(y))^2 \\
&= \sum_{x\in\mathcal{G}} \sum_{y\sim x} f(x)(f(x) - f(y)) - \sum_{x\in\mathcal{G}} \sum_{y\sim x} f(y)(f(x) - f(y)) \\
&= 2\sum_{x\in\mathcal{G}} \sum_{y\sim x} f(x)(f(x) - f(y)) \\
&= 2\langle f, \Delta(f)\rangle.
\end{aligned}$$

Now we can take $\lambda_1(\mathcal{G})$ to be the smallest eigenvalue of an eigenvector that is not a constant function. The Cheeger inequality states that $2h(\mathcal{G}) \ge \lambda_1(\mathcal{G}) \ge \frac{h(\mathcal{G})^2}{2}$

A last definition of being an expander is the existence of a certain Poincaré inequality. Specifically we want that $\frac{1}{|\mathcal{G}_n|^2} \sum_{x,y\in\mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2$ is uniformly bounded over all n and all 1-Lipschitz maps $\varphi\colon \mathcal{G}_n \to \ell^2$.

Theorem 1.5.1. *Let $(\mathcal{G}_n)_n$ be a sequence of k-regular Cayley graphs. This sequence is an expander if one of the following equivalent statements is true:*

1. *There exists a $c > 0$ such that $h(\mathcal{G}_n) \ge c$ for every n.*
2. *There exists an $\varepsilon > 0$ such that $\lambda_1(\mathcal{G}_n) \ge \varepsilon$.*
3. *There exists a C such that for every n and every 1-Lipschitz map $\varphi\colon \mathcal{G}_n \to \ell^2$ we have*

$$\sum_{x,y\in\mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \le C|\mathcal{G}_n|^2.$$

Proof. The equivalence $1 \Leftrightarrow 2$ is due to the Cheeger-Buser inequality.
The proof of $2 \Rightarrow 3$ is based on Proposition 5.7.2 of [NY12].
Set $C = \frac{k}{\varepsilon}$. Now for any n we can take $v_0 = 1_{\mathcal{G}_n}, v_1, \dots, v_{|\mathcal{G}_n|-1}$ to be the eigenvectors of the Laplacian Δ_n on $\mathcal{G}_n$.
Let $f \colon \mathcal{G}_n \to \mathbb{R}$ such that $\sum\limits_{x \in \mathcal{G}_n} f(x) = 0$, we can write $f = a_1 v_1 + \ldots + a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1}$. Using that $\langle v_i, v_j \rangle = 0$ if $i \neq j$ we can make the following computations:

$$\begin{aligned}
\sum_{d(x,y)=1} |f(x) - f(y)|^2 &= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) - f(y)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} f(x)(f(x) - f(y)) + f(x)(f(x) - f(y)) \\
&= \sum_{d(x,y)=1} 2f(x)(f(x) - f(y)) \\
&= \sum_{x \in \mathcal{G}_n} 2f(x)(\Delta_n(f)(x)) \\
&= 2\langle f, \Delta_n(f) \rangle \\
&= 2 \left\langle a_1 v_1 + \ldots + a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1}, a_1 \lambda_1 v_1 + \ldots + a_{|\mathcal{G}_n|-1} \lambda_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1} \right\rangle \\
&= 2 \left(\lambda_1 \|a_1 v_1\|^2 + \ldots + \lambda_{|\mathcal{G}_n|-1} \left\| a_{|\mathcal{G}_n|-1} v_{|\mathcal{G}_n|-1} \right\|^2 \right) \\
&\geq 2\lambda_1 \|f\|^2 \\
&= 2\lambda_1(\mathcal{G}_n) \sum_{x \in \mathcal{G}_n} |f(x)|^2.
\end{aligned}$$

Now let $\varphi \colon \mathcal{G}_n \to \ell^2$ be a 1-Lipschitz map. Without loss of generality we may assume that $\sum\limits_{x \in \mathcal{G}_n} \varphi(x) = 0$. We can decompose φ according to an orthonormal basis. Using this decomposition we find that $2\lambda_1(\mathcal{G}_n) \sum\limits_{x \in \mathcal{G}_n} \|\varphi(x)\|^2 \leq \sum\limits_{d(x,y)=1} \|\varphi(x) - \varphi(y)\|^2 \leq \sum\limits_{d(x,y)=1} 1 \leq k|\mathcal{G}_n|$.
Now we can bound $\sum\limits_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2$ as follows:

$$\begin{aligned}
\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 &= \sum_{x,y \in \mathcal{G}_n} \|\varphi(x)\|^2 + \|\varphi(y)\|^2 - 2\langle \varphi(x), \varphi(y) \rangle \\
&= \sum_{x \in \mathcal{G}_n} 2|\mathcal{G}_n| \|\varphi(x)\|^2 - 2 \left\langle \sum_{x \in \mathcal{G}_n} \varphi(x), \sum_{y \in \mathcal{G}_n} \varphi(y) \right\rangle \\
&\leq \frac{k|\mathcal{G}_n|}{\lambda_1(\mathcal{G}_n)} |\mathcal{G}_n| \\
&\leq C|\mathcal{G}_n|^2.
\end{aligned}$$

This proves that $2 \Rightarrow 3$.
Now we only have to prove that $3 \Rightarrow 2$. Set $\varepsilon = \frac{1}{C}$ and suppose that $\lambda_1(\mathcal{G}_n) < \varepsilon$ for some n. Let f be the eigenvector v_1 for this n. Set $B = \sum\limits_{d(x,y)=1} |f(x) - f(y)|^2$. Now we can take $\varphi \colon \mathcal{G}_n \to \ell^2(\mathcal{G}_n)$ with $\varphi(x) \colon \mathcal{G}_n \to \mathbb{R} \colon y \to \frac{1}{\sqrt{B}} f(y^{-1}x)$. Note that $\mathcal{G}_n$ is a Cayley graph, therefore $y^{-1}x$ is well-defined. Now φ is 1-Lipschitz because for every $x, y \in \mathcal{G}_n$ with $d(x,y) = 1$ we have

$$\|\varphi(x) - \varphi(y)\|^2 \leq \frac{1}{B} \sum_{z \in \mathcal{G}_n} |f(z^{-1}x) - f(z^{-1}y)|^2 \leq \frac{1}{B} \sum_{d(x',y')=1} |f(x') - f(y')|^2 = 1.$$

Showing that $2\varepsilon \sum\limits_{x \in \mathcal{G}_n} |f(x)|^2 > B$ would show that φ does not satisfy $\sum\limits_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \leq C|\mathcal{G}_n|^2$, because

of the following argument:

$$\sum_{x,y\in\mathcal{G}_n} \|\varphi(x)-\varphi(y)\|^2 = \sum_{x,y\in\mathcal{G}_n} \|\varphi(x)\|^2 + \|\varphi(y)\|^2 - 2\langle\varphi(x),\varphi(y)\rangle$$
$$= \sum_{x\in\mathcal{G}_n} 2|\mathcal{G}_n|\,\|\varphi(x)\|^2 - 2\left\langle \sum_{x\in\mathcal{G}_n}\varphi(x), \sum_{y\in\mathcal{G}_n}\varphi(y)\right\rangle.$$

But we have the following computation

$$\left(\sum_{x\in\mathcal{G}_n}\varphi(x)\right)(z) = \sum_{x\in\mathcal{G}_n}\frac{1}{\sqrt{B}}f(z^{-1}x) = \sum_{y\in\mathcal{G}_n}\frac{1}{\sqrt{B}}f(y) = \frac{1}{\sqrt{B}}\langle f, 1_{\mathcal{G}_n}\rangle = 0$$

so that we have

$$\left\langle \sum_{x\in\mathcal{G}_n}\varphi(x), \sum_{y\in\mathcal{G}_n}\varphi(y)\right\rangle = 0$$

and thus

$$\begin{aligned}\sum_{x,y\in\mathcal{G}_n}\|\varphi(x)-\varphi(y)\|^2 &= \sum_{x\in\mathcal{G}_n} 2|\mathcal{G}_n|\,\|\varphi(x)\|^2\\ &= \frac{2}{B}|\mathcal{G}_n|\sum_{x,y\in\mathcal{G}_n}|f(y^{-1}x)|^2\\ &= \frac{2}{B}|\mathcal{G}_n|^2\sum_{y\in\mathcal{G}_n}|f(y)|^2\\ &\geq C|\mathcal{G}_n|^2.\end{aligned}$$

To show that $2\varepsilon\sum_{x\in\mathcal{G}_n}|f(x)|^2 > B$ we make the following computations:

$$\begin{aligned} B &= \sum_{d(x,y)=1}|f(x)-f(y)|^2\\ &= \sum_{d(x,y)=1} f(x)(f(x)-f(y)) - f(y)(f(x)-f(y))\\ &= \sum_{d(x,y)=1} f(x)(f(x)-f(y)) + f(x)(f(x)-f(y))\\ &= \sum_{d(x,y)=1} 2f(x)(f(x)-f(y))\\ &= 2\sum_{x\in\mathcal{G}_n} f(x)(\Delta_n(f)(x))\\ &= 2\sum_{x\in\mathcal{G}_n}\lambda_1(\mathcal{G}_n)f(x)f(x)\\ &= 2\lambda_1(\mathcal{G}_n)\sum_{x\in\mathcal{G}_n}|f(x)|^2\\ &< 2\varepsilon\sum_{x\in\mathcal{G}_n}|f(x)|^2.\end{aligned}$$

This concludes the proof. □

1.5.2 Weakly embedded expanders

An a priori weaker notion of coarse embedding is that of a weak embedding. It was used in [Gro03] to construct a group which does not admit a coarse embedding into a Hilbert space.

Definition 1.5.2. *Given a sequence of finite metric spaces $(X_n)_{n\in\mathbb{N}}$, and a metric space Y, a sequence of maps $f_n : X_n \to Y$ is a weak embedding if there is $C>0$ such that each f_n is C-Lipschitz, and for all $r>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in X_n}\frac{|f_n^{-1}(B_Y(f_n(x),r))|}{|X_n|} = 0,$$

where $B_Y(y,r)$ denotes the ball of radius r about $y \in Y$.

When the target space Y is of bounded geometry (i.e. the cardinality of balls is uniformly bounded by some constant depending only on the radius), then the above condition is equivalent to

$$\lim_{n\to\infty} \sup_{x \in X_n} \frac{|f_n^{-1}(f_n(x))|}{|X_n|} = 0.$$

We remark that a coarse embedding of a metrized sequence of finite graphs of bounded geometry into a space Y implies a weak embedding of the sequence of graphs into Y.
We also remark that expanders do not weakly embed into a Hilbert space. This follows from Theorem 1.5.1 with a similar argument as in section 1.5.4.

1.5.3 Generalized expanders

It is well known that expanders do not coarsely embed into a Hilbert, however these are not the only such metric spaces. In order to include all non-embeddable metric spaces we generalise expanders.

Definition 1.5.3. *A sequence of bounded metric spaces X_n is a generalised expander if there exists a sequence $r_n > 0$, a sequence of probability measures μ_n on $X_n \times X_n$ and a constant C such that the following conditions are met:*

- *The sequence r_n tends to infinity as $n \to \infty$.*
- *We have that $\mu_n(D) = 0$ for $D = \{(x,y) : d(x,y) < r_n\}$.*
- *For every $\varphi\colon X_n \to \ell^2$ that is 1-Lipschitz we have*

$$\sum_{x,y \in X_n} \|\varphi(x) - \varphi(y)\|^2 \mu_n(x,y) \le C.$$

This is a generalization of being an expander. Indeed, for any expander $\mathcal{G}_n$ the set $\mathcal{G}_n \times \mathcal{G}_n$ can be partitioned into two equal subsets of pairs that are close and pairs that are far. Let Y_n be the subset with pairs that are far and let r_n be equal to minimum of $d(x,y)$ over all (x,y) in Y_n. As every expander is a bounded geometry and $|\mathcal{G}_n|$ tends to infinity, we have that $r_n \to \infty$. Let μ_n be the uniform measure on Y_n. By definition $D = \mathcal{G}_n \setminus Y_n$, so $\mu_n(D) = \mu_n(\mathcal{G}_n \setminus Y_n) = 1 - 1 = 0$.
Finally there exists a constant C such that for every 1-Lipschitz map $\varphi\colon X_n \to \ell^2$ we have

$$\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \le C|\mathcal{G}_n|^2.$$

So we can conclude that

$$\begin{aligned}
\sum_{x,y \in \mathcal{G}_n} \|\varphi(x) - \varphi(y)\|^2 \mu_n(x,y) &= \sum_{(x,y) \in Y_n} \|\varphi(x) - \varphi(y)\|^2 \frac{1}{|Y_n|} \\
&= \sum_{(x,y) \in Y_n} \frac{2}{|\mathcal{G}_n|\,|\mathcal{G}_n|} \|\varphi(x) - \varphi(y)\|^2 \\
&\le 2C.
\end{aligned}$$

Many examples of generalized expander that are not expanders can be found in [AT15]. In that paper Arzhantseva and Tessera define expanders relative to subsets. The Cayley graphs of a sequence of groups G_n with generating sets S_n is an expander relative to the subsets Y_n of G_n, if the following conditions are satisfied:

- the cardinality of S_n is bounded,
- the sets Y_n are unbounded in G_n, and
- $\displaystyle\sum_{x \in G_n} \sum_{y \in Y_n} \|\varphi(x) - \varphi(xy)\|^2 \le C|G_n|\,|Y_n|.$

Due to proposition 3 of [AT15] we know that if (G,Y) has relative property (T) and N_n is a filtration of G, then $\square_{N_n} G$ is a relative expander.

1.5.4 Non embeddability in Hilbert spaces

As mentioned before expanders do not coarsely embed into a Hilbert space. In fact generalized expander do not embed either. Indeed, if there exists a coarse embedding φ form a generalized expander $\mathcal{G}_n$ to a Hilbert space $\mathcal{H}$, then

$$C \geq \sum_{x,y\in X_n} \|\varphi(x)-\varphi(y)\|^2\mu_n(x,y) \geq \sum_{x,y\in X_n} \rho_-(r_n)^2\mu_n(x,y) = \rho_-(r_n)^2.$$

As r_n tends to infinity and $\lim\limits_{t\to+\infty} \rho_-(t)=\infty$ we get a contradiction.

In fact a metric space embeds into a Hilbert space if and only if it contains a generalized expander. See Theorem 5.7.3 of [NY12].

1.6 Algebraic topology

In this section we give an introduce some objects from algebraic topology. First we introduce fundamental groups, this group provides information about the paths in a given topological space. Then we look at a coarse version of this construction called the coarse fundamental group. We also introduce simplicial homology and finally we introduce covering spaces. For an introduction to topology and algebraic topology we refer to [Mun00] and [Hat02].

1.6.1 Fundamental groups

A path in a topological space X is a continuous map $p\colon [0,1]\to X$ and a loop is a path p such that $p(0)=p(1)$. Two paths p and q with the same begin point and end point are homotopic if there exists a continuous map $f\colon [0,1]^2\to X$ such that $f(0,x)=p(0)=q(0)$, $f(1,x)=p(1)=q(1)$, $f(x,0)=p(x)$ and $f(x,1)=q(x)$. A loop p is nulhomotopic if it is homotopic to the trivial loop $q\colon [0,1]\to X : x\mapsto p(0)$.
Two paths p and q can be composed such that

$$\begin{aligned} (p*q)(x) &= p\left(\frac{x}{2}\right) && \text{if } x\in[0,\tfrac{1}{2}] \text{ and}\\ (p*q)(x) &= q\left(\frac{1+x}{2}\right) && \text{if } x\in[\tfrac{1}{2},1]. \end{aligned}$$

This defines a group action on the homotopy classes of loops p in X with $p(0)$ fixed. This group is called the fundamental group $\pi_1(X)$ of X.
If X is connected, then $\pi_1(X)$ is independent of the base point.

Graphs are not connected not even connected graphs. However given a connected graph we can add a path between every two adjacent vertices. Then the metric space we find is connected. So we can consider the fundamental group of a connected graph to be the fundamental group of that metric space.
Note that the fundamental group of a graph is always a free group. In fact for a group G with representation $\langle S|R\rangle$ we have that $\pi_1(\mathrm{Cay}(G,S))=\langle R\rangle$.

1.6.2 Coarse fundamental groups

In the classical fundamental group, two paths are homotopic if we can deform one path into the other in a continuous way. The coarse version first defined for simplicial complexes in [BKLW01] and then in full generality in [BCW14], (see also [BBdLL06], [BL05]). In this coarse setting, we take quasi-paths and we will make these deformations in discrete steps. This will depend on a constant $r>0$. An r-path p in a metric space X is a map $p\colon \{0,\dots,n\}\to X$ with $n\in\mathbb{N}$ and $d(p(i-1),p(i))\leq r$ for every $i\in\{1,\dots,n\}$. Remark that paths on a graph, as defined in section 1.2.1, are 1-paths. The deformation of the paths will also depend on the constant r, and if such a deformation exists, we will call the two paths r-close.
We say that two paths p and q in a graph $\mathcal{G}$ are r-close if one of the two following cases is satisfied:

(a) For every $i\leq \min(\ell(p),\ell(q))$ we have that $p(i)=q(i)$ and for bigger i we either have $p(i)=p(\ell(q))$ or $q(i)=q(\ell(p))$, depending on which path is defined at i.

(b) We have that $\ell(p)=\ell(q)$ and for every $0\leq i\leq \ell(p)$ we have $d(p(i),q(i))\leq r$.

Now we define a coarse version of homotopy by combining these deformations.

Definition 1.6.1. *Let $\mathcal{G}$ be a graph, let $r > 0$ be a constant and let p and q be two r-paths in $\mathcal{G}$. We say that p and q are r-homotopic if there exists a sequence $p_0 = p$, $p_1, \ldots, p_n = q$ such that p_i is r-close to p_{i-1} for every $i \in \{1, 2, \ldots, n\}$.*

Note that r-homotopy is an equivalence relation, so we can now define the fundamental group up to r-homotopy as the group of r-homotopy equivalence classes of r-loops rooted in a basepoint, with the group operation corresponding to concatenation of loops.

Definition 1.6.2. *The fundamental group up to r-homotopy $\pi_{1,r}(\mathcal{G}, x)$ is defined to be the group of equivalence classes of r-loops rooted in a basepoint x with the operation $*\colon \pi_{1,r}(\mathcal{G}, x)^2 \to \pi_{1,r}(\mathcal{G}, x)\colon ([p], [q]) \mapsto [p * q]$ with*

$$p * q\colon \{0, 1, \ldots, \ell(p) + \ell(q)\} \to \mathcal{G}\colon \begin{cases} i & \mapsto & p(i) & \text{if } 0 \leq i \leq \ell(p), \\ i & \mapsto & q(i - \ell(p)) & \text{if } \ell(p) + 1 \leq i \leq \ell(p) + \ell(q). \end{cases}$$

As we will focus on the case where $\mathcal{G}$ is a Cayley graph $\mathrm{Cay}(G, S)$, we will write $\pi_{1,r}(\mathcal{G}) = \pi_{1,r}(\mathcal{G}, e)$, since the basepoint will always be taken to be the identity element.

1.6.3 Homology

In this section we will define simplicial homology.
Fix a ring R. The simplex σ_n is the space $\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + \ldots + x_n = 1\}$. For a topological space X consider the free R-module $C_n(X)$ generated by the image of continuous maps $\sigma_n \to X$, consider for every i between 0 and n the map $\partial_n^i\colon C_n(X) \to C_{n-1}(X)$ such that for any $f\colon \sigma_n \to X$ we have

$$\partial_n^i(f)(x_0, x_1, \ldots, x_{n-1}) = f(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

and finally consider the boundary map $\partial_n\colon C_n(X) \to C_{n-1}(X)$ such that for any $f\colon \sigma_n \to X$ we have

$$\partial_n(f) = \sum_{i=0}^{n} (-1)^i \partial_n^i(f).$$

Note that for every $f\colon \sigma_{n+1} \to X$ and every i and j such that $0 \leq i \leq j \leq n$ we have that $\partial_n^j \circ \partial_{n+1}^i(f) = \partial_n^i \circ \partial_{n+1}^{j+1}(f)$. So we have

$$\begin{aligned}
\partial_n \circ \partial_{n+1}(f) &= \sum_{i=0}^{n} (-1)^i \partial_n^i \left(\sum_{j=0}^{n+1} (-1)^j \partial_{n+1}^j(f) \right) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \partial_n^i \circ \partial_{n+1}^j(f) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j} \partial_n^i \circ \partial_{n+1}^j(f) + \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j} \partial_n^i \circ \partial_{n+1}^j(f) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j} \partial_n^j \circ \partial_{n+1}^{i+1}(f) + \sum_{j=1}^{n+1} \sum_{i=0}^{j-1} (-1)^{i+j} \partial_n^i \circ \partial_{n+1}^j(f) \\
&= 0.
\end{aligned}$$

As $\partial_n \circ \partial_{n+1}$ is the zero map. The following sequence is called a chain complex:

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0 \longrightarrow 0\ .$$

Now the simplicial homology groups of this chain complex are $H_n(X, R) = \dfrac{\mathrm{Im}(\partial_{n+1})}{\ker(\partial_n)}$. Elements of $\mathrm{Im}(\partial_{n+1})$ are the boundaries and elements of $\ker(\partial_n)$ are called cycles, not to by confused with cycles in a graph.

Remark that $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$. Also remark that if X is connected, then $H_0(X, R)$ is equal to R.

1.6.4 Covering spaces

A topological space Y is a covering space of a topological space X if there exists a map $P\colon Y \to X$ such that for every $x \in X$ there exists a neighbourhood U of x such that $P^{-1}(U)$ is the union of disjoint sets that are homeomorphic to U. Note that if X and Y are path connected, then there is a natural inclusion of $\pi_1(Y)$ into $\pi_1(X)$.

A specific covering of a path connected space X we use is the m-homology cover, where m is an integer greater than 1. This is the covering space Y such that $\pi_1(Y)$ is equal to the kernel of the quotient map $\pi_1(X) \to H_1(X, \mathbb{Z}_m)$.

Given a finite graph X, one can construct a covering graph $\widetilde{X}$ of X such that $\widetilde{X}$ is the cover corresponding to the the quotient $\pi(X) \to \bigoplus^r \mathbb{Z}_m$ of highest rank r possible. Indeed, since $\pi(X)$ is a free group, the rank r is simply the rank of this free group.

Note that as a graph, the cover can be viewed in the following way. First, choose a maximal spanning tree T of the graph X. Construct the Cayley graph of $\bigoplus^r \mathbb{Z}_m$ with respect to the image of the free generating set of $\pi(X)$. Note that the free generating set of $\pi(X)$ is in bijection with its image in $\bigoplus^r \mathbb{Z}_m$, and also in bijection with the edges of X not contained in the maximal tree T. Let κ be the bijection between the edges not in T and this generating set of $\bigoplus^r \mathbb{Z}_m$.

Now, replace by a copy of T each of the vertices of the Cayley graph of $\bigoplus^r \mathbb{Z}_m$ with respect to the image of the free generating set of $\pi(X)$, where the different copies of T are connected according to how the vertices in the Cayley graph $\bigoplus^r \mathbb{Z}_m$ are connected, via the correspondence between the edges not in T and the generating set of $\bigoplus^r \mathbb{Z}_m$ (that is, if two vertices v and w in X are connected by an edge e which is not in T, then given such a vertex $\widetilde{v}$ in one of the copies of T corresponding to a vertex a of $\bigoplus^r \mathbb{Z}_m$, we connect it via an edge to a vertex $\widetilde{w}$ in the copy of T corresponding to the element $a\kappa(e)$ of $\bigoplus^r \mathbb{Z}_m$).

The covering space $\widetilde{X}$ obtained in this way is called the *m-homology cover of X*.

Chapter 2

Overview

2.1 History

2.1.1 Gromov's polynomial growth theorem

One of the most important results in geometric group theory is Gromov's polynomial growth theorem. In a sparse graph $\mathcal{G}$, i.e. a graph with bounded degree, we can consider the balls of a certain radius r. We say that $B(x, r)$ is such a ball centred at $x \in \mathcal{G}$. If $\mathcal{G}$ is a Cayley graph then the size of these balls is independent of the point x, so we can define $\beta(r) = |B(x, r)|$, which is called the growth function of $\mathcal{G}$.

The growth function of the Cayley graph of a finitely generated infinite group can be polynomial, exponential or in between, which is called intermediate growth. This growth type is a quasi-isometry invariant, so for any group this growth type is independent of the finite generating set. For a group with polynomial growth even the degree of the polynomial is independent of the finite generating set.
Due to the Følner condition non-amenable groups always have exponential growth. However there also exist amenable groups with exponential growth for example $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where α corresponds to the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

In [Mil68] and [Wol68], Milnor and Wolf show that a finitely generated solvable group has polynomial growth if and only if it is virtually nilpotent. They also show that finitely generated solvable groups, who do not have polynomial growth, have exponential growth.
In [Gro81] Gromov shows that any finitely generated groups has polynomial growth if and only if it is virtually nilpotent. This theorem gives a one to one correspondence between a group theoretical property and geometric property of the Cayley graph. There exists many other such results, for example the different characterizations of amenability: the existence of a left invariant mean is a group analytic property, while the existence of Følner set (sets with an arbitrary small boundary) is a geometric property.

2.1.2 Margulis: property (T) and expanders

The existence of expanders was shown in 1967 by Kolmogorov and Barzdin (see [Bar93]). However they did not provide a construction. The first explicit construction came 6 years later by Margulis in [Mar73]. He considers a sequence of increasing quotient of a group G and shows if G has property (T) then the sequence of their Cayley graphs is an expander sequence. In fact this result can be strengthened: If a group G with a sequence of finite index normal subgroups N_n has property (τ) and $[G : N_n] \to \infty$, then $(\mathrm{Cay}(G/N_n, \bar{S}))$ is an expander for any finite generating set S of G with $\bar{S}$ the projection of S in G/N_n.
Indeed, for any n we have the representation $\pi\colon G/N_n \to \ell_0^2(G/N_n)$, which is the restriction of the left regular representation to the functions f that sum to zero, i.e. $\sum_{g \in G} f(g) = 0$. For the eigenvalue $\lambda_1(\mathrm{Cay}(G/N_n, \bar{S}))$ we have an eigenvector ξ in $\ell_0^2(G/N_n)$. As π does not have any invariant vectors there exists a $g \in \bar{S}$ such that $\|\pi_g(\xi) - \xi\| \geq C\|\xi\|$ where $C > 0$ is independent of n.

Recall that $\langle \Delta(\xi), \xi \rangle = \lambda_1(\mathrm{Cay}(G/N_n, \bar{S}))\|\xi\|^2$. Now we have that

$$\begin{aligned}
\langle \Delta(\xi), \xi \rangle &= \sum_{x \in G/N_n} \left\langle \sum_{s \in \bar{S}} \xi(x) - \pi_s(\xi)(x), \xi(x) \right\rangle \\
&= \sum_{s \in \bar{S}} \sum_{x \in G/N_n} \langle \xi(x) - \pi_s(\xi)(x), \xi(x) \rangle \\
&\geq \langle \xi(x) - \pi_g(\xi)(x), \xi(x) \rangle && \pi \text{ is unitary: } \langle \xi(x) - \pi_s(\xi)(x), \xi(x) \rangle \geq 0 \\
&= \frac{1}{2} \|\xi(x) - \pi_g(\xi)(x)\|^2 && \pi \text{ is unitary: } \|\pi_g(\xi)\| = \|\xi\| \\
&\geq \frac{C^2}{2} \|\xi\|^2,
\end{aligned}$$

so $\lambda_1(\mathrm{Cay}(G/N_n, \bar{S})) \geq \frac{C^2}{2}$.

We can therefore conclude that the sequence $\mathrm{Cay}(\mathrm{SL}(3, \mathbb{Z}/n\mathbb{Z}), \bar{S})$ is an expander for any generating set S of $\mathrm{SL}(3, \mathbb{Z})$.

Remark that this result states the box space of a group with property (T) to be an expander.

Note that this result by Margulis can not be reversed, there exist box spaces of groups without property (T) that are expanders, for example the sequence $\mathrm{Cay}(\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}), \bar{S})$ with any generating set S of $\mathrm{SL}(2, \mathbb{Z})$, this follows from Selberg's theorem, see theorem 4.4.2 of [Lub10]. However consider a group G with a filtration N_n. If the box space $\square_{N_n} G$ is an expander, then G has property (τ) with respect to N_n.
Indeed, if G does not have property (τ) with respect to N_n, then for any generating set S of G and for every $\varepsilon > 0$ there exists a representation of G that is trivial on N_n for some fixed n such that $\sup_{g \in S} \|\pi_g(\xi_\varepsilon) - \xi_\varepsilon\| < \varepsilon\|\xi\|$. Now this representation induces a representation π of G/N_n. This is a representation of a finite group, so it can be decomposed into a direct sum of irreducible representations and these irreducible representation are also contained in the decomposition of $\rho\colon G/N_n \to \ell_0^2(G/N_n)$, which is again the restriction of the left regular representation to the functions that sum to zero.
So ξ_ε can be decomposed into $\bigoplus_i \xi_{\varepsilon,i}$ where all $\xi_{\varepsilon,i}$ are vectors in $\ell_0^2(G/N_n)$. So

$$\begin{aligned}
\langle \Delta(\xi_\varepsilon), \xi_\varepsilon \rangle &= \sum_i \langle \Delta(\xi_{\varepsilon,i}), \xi_{\varepsilon,i} \rangle \\
&= \sum_i \sum_{x \in G/N_n} \left\langle \sum_{s \in \bar{S}} \xi_{\varepsilon,i}(x) - \pi_s(\xi_{\varepsilon,i})(x), \xi_{\varepsilon,i}(x) \right\rangle \\
&= \sum_i \sum_{x \in G/N_n} \sum_{s \in \bar{S}} \frac{1}{2} \|\xi_{\varepsilon,i}(x) - \pi_s(\xi_{\varepsilon,i})(x)\|^2 && \pi \text{ is unitary: } \|\pi_s(\xi)\| = \|\xi\| \\
&\leq \sum_i \sum_{x \in G/N_n} \frac{|\bar{S}|}{2} \varepsilon^2 \|\xi_{\varepsilon,i}(x)\|^2 \\
&= \frac{|\bar{S}|}{2} \varepsilon^2 \|\xi_\varepsilon\|^2
\end{aligned}$$

As $\lambda_1(G/N_n)$ is the smallest eigenvalue of ρ and $\xi_\varepsilon \neq 0$ we have that $\lambda_1(G/N_n) \leq \frac{|\bar{S}|}{2}\varepsilon^2$. So for every $\varepsilon > 0$ there exists an n such that $\lambda_1(G/N_n) \leq \frac{|\bar{S}|}{2}\varepsilon^2$, therefore $(\mathrm{Cay}(G/N_n, \bar{S}))_n$ is not an expander.

2.1.3 Amenability and property A

An other result that links a property of the group to a property of its box space is given by Guentner. Proposition 11.39 of [Roe03] shows that for a residually finite group G with a filtration N_n we have that G is amenable if and only if its box space $\square_{N_n} G$ has property A.

2.1.4 Rigidity of box spaces

Finally there also exist some rigidity results considering box spaces a first one is given by Khukhro and Valette in [KV15]. First they show that following lemma:

Lemma 2.1.1 (Khukhro-Valette, Lemma 1). *Let* $X = \bigsqcup_{k>0}^{+\infty} X_k$ *and* $Y = \bigsqcup_{k>0}^{+\infty} Y_k$ *be coarse disjoint unions of graphs such that the diameter tends to infinity as* k *tends to infinity and let* $\Phi\colon X \to Y$ *be a coarse equivalence between these metric spaces. Then there exists a constant* A *and an almost permutation* ϕ *between the components of* X *and the components of* Y *such that* $\Phi|_{X_i}$ *is an* (A, A)*-quasi-isometry between* X_i *and* $\phi(X_i)$*.*

Note that an almost permutation between sets A and B is a bijection between a co-finite subset of A and a co-finite subset of B. Also note that the lemma was only stated for coarse disjoint unions of graphs with strictly increasing diameter, however the proof only uses that the diameter tends to infinity.

We can therefore conclude that the if two box space $\Box G$ and $\Box H$ are coarsely equivalent, then the balls are quasi-isometric. By taking a limit in the space of marked group we find that G and H must be quasi-isometric.

Theorem 2.1.2 (Khukhro-Valette, Theorem 7). *Let* G *and* H *be residually finite groups, with filtrations* $N_n \lhd G$ *and* $M_n \lhd H$*. If* $\Box_{N_n} G$ *is coarsely equivalent to* $\Box_{M_n} H$*, then* G *is quasi-isometric to* H*.*

This theorem implies that the partition of the class of box spaces due to coarse equivalence is finer than the partition caused by the quasi-isometry of the underlying group. Therefore every quasi-isometric property of a group corresponds to a coarse property of the box space. So amenability is not the only property of the group that can be detected by the coarse equivalence class of a box space, it is also possible to determine the growth type, the asymptotic dimension, the number of ends and many more.

An other rigidity result is given by Das in [Das15].

Theorem 2.1.3 (Das, Theorem 1.1). *Given two finitely generated groups* G *and* H *with respective filtrations* N_i *and* M_i*,* $\Box_{(N_i)} G \simeq_{CE} \Box_{(M_i)} H$ *implies that* G *and* H *are uniformly measure equivalent.*

An application of these results is being able distinguish box spaces up to coarse equivalence, which one can for example use as in [KV15] to show that there exist uncountably many expanders with geometric property (T) of Willett and Yu ([WY12]).

2.2 Summary

2.2.1 Full box spaces

Chapter 3 is based on [Del17], there we investigate full box spaces. In particular what can be said about a residually finite group G if the full box space is coarsely equivalent to the full box space $\mathbb{Z}^n$ for some n. We know that it has to be virtually $\mathbb{Z}^n$.
Indeed, due to Theorem 2.1.2 we know that G is quasi-isometric to a quotient of $\mathbb{Z}^n$, so G is virtually $\mathbb{Z}^m$ with $m \leq n$. Due to Theorem 2.1.2 we also know that $\mathbb{Z}^n$ is quasi-isometric to a quotient of G, so $m \geq n$, therefore G is virtually $\mathbb{Z}^n$.

In that chapter we show that G can not be 2-generated if $n > 2$. We also show that the full box space of free groups F_d and F_k are not coarsely equivalent whenever $d \geq 8k + 10$.

2.2.2 A box space of the free group

Chapter 4 is based on [DK16], joint work with Khukhro. In that chapter we investigate box spaces of the free group. In [LPS88] Lubotzky, Phillips and Sarnak construct a box space of the free group that is an expander. In [AGŠ12] Arzhantseva, Guentner and Špakula construct a box space of the free group that embeds into a Hilbert space. This technique is generalized in [Khu14].
In that chapter we construct box spaces of a free group that do not coarsely embed into a Hilbert space, but do not contain coarsely nor weakly embedded expanders. We do this by considering two sequences of subgroups of the free group: the sequence provided by Lubotzky, Phillips and Sarnak which forms an expander, and another created using the techniques from [Khu14] which gives rise to a box space that can be coarsely embedded into a Hilbert space.
We then take certain intersections of these subgroups, and prove that the corresponding box space contains generalized expanders. We show that there are no weakly embedded expanders in the box space corresponding to our chosen sequence by proving that a box space that covers another box space of the same group that is coarsely embeddable into a Hilbert space cannot contain weakly embedded expanders.

2.2.3 Coarse fundamental groups and box spaces

Chapter 5 is based on [DK18], joint work with Khukhro. In that chapter we prove the following strong rigidity theorem for box spaces of finitely presented groups.

Theorem 2.2.1. *Let G and H be finitely presented groups with respective filtrations N_i and M_i such that $\Box_{(N_i)}G \simeq_{CE} \Box_{(M_i)}H$. Then there exists an almost permutation with bounded displacement f of $\mathbb{N}$ such that $N_i \cong M_{f(i)}$ for every i in the domain of f.*

Here, an almost permutation is defined as in Lemma 2.1.1.

Definition 2.2.2. *An almost permutation f of $\mathbb{N}$ is a bijection between two cofinite subsets of $\mathbb{N}$. We say that f has bounded displacement if there exist A and N such that $|f(n) - n| \leq A$ for all $n \geq N$.*

We achieve this rigidity result by using the coarse fundamental group of [BCW14]. The idea of using the coarse fundamental group in this context comes from the following intuitive idea: if $G = \langle S|R\rangle$ is a finitely presented group, and N is a normal subgroup of G such that non-trivial elements in N are "sufficiently long" with respect to the generating set of G, then one can use the coarse fundamental group at a scale which lies between the length of the longest element in R and the shortest element in N, so that only loops coming from N are detected in $\mathrm{Cay}(G/N)$.
This idea works well for eventually detecting the normal subgroups used to construct box spaces, since the condition that a filtration (N_i) must be nested and have trivial intersection implies that the length of non-trivial elements in the N_i tends to infinity as i tends to infinity.

We give applications of this theorem in various contexts, in particular, we prove the following results.

Theorem 2.2.3. *There exist two box spaces $\Box_{N_i}G \not\simeq_{CE} \Box_{M_i}G$ of the same group G such that $G/N_i \twoheadrightarrow G/M_i$ with $[M_i : N_i]$ bounded.*

Theorem 2.2.4. *There exist infinitely many coarse equivalence classes of box spaces of the free group F_3 that contain Ramanujan expanders.*

2.2.4 A slowly growing box space of a free group that embeds into a Hilbert space

Chapter 6 is based on upcoming research, we give an example of a box space $\Box_{N_i}G$ of a non amenable group G that embeds into a Hilbert space such that the size of the components grows as slowly as possible, i.e. we have that $[N_i : N_{i-1}] = 2$ for all i in $\mathbb{N}$.
This resolves a question posed by Damian Sawicki.

We show this result by generalizing the techniques used in [AGŠ12] and [Khu14]. We consider the sequence from [AGŠ12] which is defined inductively by $N_0 = F_2$ and $N_{i+1} = \Gamma(N_i)$. Then we define subgroups in between and show that the entire box space still embeds with techniques similar to those of [Khu14].

2.2.5 Box spaces of virtually nilpotent groups and asymptotic dimension

Chapter 7 is based on [DT18], joint work with Tointon. In that chapter we look at asymptotic dimension.

It is known that a virtually polycyclic group with a Cayley-graph metric has finite asymptotic dimension. Indeed, given a virtually polycyclic group G we write $h(G)$ for the *Hirsch length* of G, which is to say the number of infinite factors in a normal polycyclic series of a finite-index polycyclic subgroup of G. Dranishnikov and Smith [DS06, Theorem 3.5] show that if G is residually finite and virtually polycyclic then

$$\operatorname{asdim} G = h(G). \tag{2.1}$$

It is not unreasonable to expect that box spaces of virtually polycyclic groups should also have finite asymptotic dimension, and indeed there have been some results in this direction. For example, Szabó, Wu and Zacharias [SWZ14] show that every finitely generated virtually nilpotent group has some box space with finite asymptotic dimension. Finn-Sell and Wu [FSW15] show moreover that for certain box spaces of virtually polycyclic groups the asymptotic dimension of the box space is, like that of the group itself, equal to the Hirsch length of the group. These results are not ideal in their current form, as they rely on certain subgroup inclusions inducing coarse embeddings of the corresponding box spaces, a fact that doesn't hold in general due to [DK18, Theorem 4.9].

The main purpose of that chapter is to clarify the situation and strengthen these results in the case of virtually nilpotent groups. Indeed, we show that if G is a finitely generated virtually nilpotent group then in fact every box space of G has asymptotic dimension equal to the Hirsch length of G, as follows.

Theorem 2.2.5. *Let G be a finitely generated residually finite virtually nilpotent group and let $(N_n)_n$ be a filtration of G. Then* $\operatorname{asdim} \square_{(N_n)} G = h(G)$.

2.2.6 Large girth and asymptotic dimension

In appendix A is part of unpublished research. we show that a metrized disjoint union of a graph sequence of large girth does not have asymptotic dimension 2.

Theorem 2.2.6. *Let X_n be a sequence of finite, connected graphs with large girth.*
Then $\operatorname{asdim} \left(\coprod_n X_n \right) \neq 2$.

We decided not to publish this result due to the publication of [Yam17]. Theorem 1.3 of that paper is a stronger result than Theorem 2.2.6.

Chapter 3

Full box spaces of free and free abelian groups

This chapter is based on [Del17]. Here we investigate some full box spaces and coarse equivalences between them. We do this in two parts. In part one we compare the full box spaces of free groups on different numbers of generators. In particular the full box space of a free group F_k is not coarsely equivalent to the full box space of a free group F_d, if $d \geq 8k + 10$. In part two we compare $\Box_f \mathbb{Z}^n$ to the full box spaces of 2-generated groups. In particular we prove that the full box space of $\mathbb{Z}^n$ is not coarsely equivalent to the full box space of any 2-generated group, if $n \geq 3$.

In both cases we use Lemma 2.1.1, which states that if two groups have coarsely equivalent full box spaces, then there exists an almost bijection ϕ between the normal subgroups of both groups such that the quotient by N and $\phi(N)$ are (A, A)-quasi-isometric for some A independent of N.
Then we will use some counting argument on the normal subgroups. In the first part we will count the normal subgroups with a low index and in the second part we will count normal subgroups with a quotient with small diameter.

3.1 The full box spaces of the free groups

In this section we will prove that the full box spaces of free groups are different, at least if the amount of generators is sufficiently different. This suggests that the full box spaces of all free groups are different.
In the proof we make use of normal subgroup growth, for further reading on (normal) subgroup growth we refer to [LS12]. Here we only need the number of normal subgroups of a given index, denoted by: $a_n^{\triangleleft}(G)$, where n is the given index.

Theorem 3.1.1. *Let $2 \leq k \leq d$ with $2(k+1) < \frac{(d-1)^2}{4d}$. Then $\Box_f F_d$ is not coarsely equivalent to $\Box_f F_k$.*

Proof. Suppose that the full box spaces of the free groups F_d and F_k are coarsely equivalent where $2(k+1) < \frac{(d-1)^2}{4d}$, i.e. there is a coarse equivalence Φ between $\Box_f F_d$ and $\Box_f F_k$. Due to Lemma 2.1.1 there exists an almost permutation ϕ between the components of $\Box_f F_d$ and the components of $\Box_f F_k$. As there is some C' such that $\operatorname{Im}\Phi$ is C'-dense, components of order less than some n must be mapped to a component of order less than $n \cdot |B[0, C']|$, where $B[0, C']$ is the closed ball of radius C'. Now set $C = |B[0, C']|$ and set D equal to the number of components that are not in the domain of ϕ.
So $|\{N \triangleleft F_d \mid \#(F_d/N) \leq n\}| - D$ is not greater than $|\{N \triangleleft F_k \mid \#(F_k/N) \leq Cn\}|$ for any n. Note that

$$\{N \triangleleft F_d \mid \#(F_d/N) \leq n\} = \{N \triangleleft F_d \mid [F_d : N] \leq n\} = \sum_{i=1}^{n} a_i^{\triangleleft}(F_d),$$

so we find the following inequality:

$$\sum_{i=1}^{Cn} a_i^{\triangleleft}(F_k) + D \geq \sum_{i=1}^{n} a_i^{\triangleleft}(F_d) \geq a_n^{\triangleleft}(F_d)$$

It suffices to find an n for which this is not the case.
Let n be a power of 2, $n = 2^m$. Then $a_n^{\triangleleft}(F_d) \geq 2^{cm^2}$ if $c < \frac{(d-1)^2}{4d}$ due to Theorem 3.7 of [LS12]. As $\frac{(d-1)^2}{4d} > 2(k+1)$ we can take $c = 2(k+1) + 2\delta$, where $\delta > 0$. Due to Theorem 2.6 and Lemma 2.5 of

[LS12], $a_i^{\triangleleft}(F_k) \leq i^k i^{2(k+1)\log_2(i)}$ for every $i \in \mathbb{N}$. By combining these two bounds we can make the following computation:

$$\begin{aligned}
2^{cm^2} - D &\leq a_n^{\triangleleft}(F_d) - D \\
&\leq \sum_{i=1}^{Cn} a_i^{\triangleleft}(F_k) \\
&\leq \sum_{i=1}^{Cn} i^k i^{2(k+1)\log_2(i)} \\
&\leq \sum_{i=1}^{Cn} (Cn)^k (Cn)^{2(k+1)\log_2(Cn)} \\
&= C^{k+1} n^{k+1} C^{2(k+1)(\log_2(n)+\log_2(C))} n^{2(k+1)(\log_2(n)+\log_2(C))} \\
&= 2^{(k+1)\log_2(C)} 2^{m(k+1)} 2^{2(k+1)(m+\log_2(C))\log_2(C)} 2^{2m(k+1)(m+\log_2(C))} \\
&= 2^{(k+1)\log_2(C)+m(k+1)+2(k+1)(m+\log_2(C))\log_2(C)+2m(k+1)(m+\log_2(C))} \\
&= 2^{2(k+1)m^2+m(k+1)+4m(k+1)\log_2(C)+2(k+1)\log_2(C)^2+(k+1)\log_2(C)} \\
&= 2^{(k+1)(2m^2+m+4m\log_2(C)+2\log_2(C)^2+\log_2(C))}
\end{aligned}$$

Now we can take $m \gg 0$ such that $2m^2 + m + 4m\log_2(C) + 2\log_2(C)^2 + \log_2(C) \leq (2 + \frac{\delta}{k+1})m^2$ and $D < 2^{cm^2} - 2^{(2(k+1)+\delta)m^2}$. But then we find the following contradiction.

$$\begin{aligned}
2^{cm^2} - D &\leq 2^{(k+1)(2m^2+m+4m\log_2(C)+2\log_2(C)^2+\log_2(C))} \\
&\leq 2^{(k+1)(2+\frac{\delta}{k+1})m^2} \\
&= 2^{(2(k+1)+\delta)m^2} \\
&< 2^{cm^2} - D
\end{aligned}$$

This proves that $\Box_f F_d$ is not coarsely equivalent to $\Box_f F_k$ for $2(k+1) < \frac{(d-1)^2}{4d}$. $\Box$

To use Theorem 3.1.1 we only need to find appropriate values for k and d. The condition $2(k+1) < \frac{(d-1)^2}{4d}$ is satisfied if and only if d is not smaller than $8k + 10$. For example $\Box_f F_2$ is not coarsely equivalent to $\Box_f F_{26}$.

3.2 The full box spaces of $\mathbb{Z}^n$

In this section we will prove that the full box space of $\mathbb{Z}^n$ is not coarsely equivalent to the full box space of a 2-generated group for every $n \geq 3$.To do so we will compare the growth in k of $\#\{$quotients with diameter $\leq k\}$, which we will call the diameter growth of the components of these full box spaces. Note that the term diameter growth is often used to compare the growth of the diameter with that of the index, however that is not how we will use it.

Theorem 3.2.1. *Let $n \geq 3$ and let H be a 2-generated group. Then $\Box_f H$ is not coarsely equivalent to $\Box_f \mathbb{Z}^n$.*

Once we know the diameter growth we can compare the full box spaces using the following result:

Proposition 3.2.2. *Let G and H be two groups, with $\Box_f G$ coarsely equivalent to $\Box_f H$ and let $a \in \mathbb{N}$. If $\#\{N \triangleleft G \mid \operatorname{diam}(G/N) \leq k\} = \mathcal{O}(k^a)$, then $\#\{N \triangleleft H \mid \operatorname{diam}(H/N) \leq k\} = \mathcal{O}(k^a)$.*

Proof. As there exists a coarse equivalence $\Phi\colon \Box_f G \to \Box_f H$ we can use Lemma 2.1.1 to find an almost permutation ϕ between the components of $\Box_f G$ and the components of $\Box_f H$ such that $\Phi|_{G/N}$ is an (A, A)-quasi-isometry between G/N and $\phi(G/N)$, if G/N lies in the domain of ϕ. Therefore $\operatorname{diam}(\phi(G/N)) \leq A \operatorname{diam}(G/N) + A$. We can take a constant C such that $\#\{N \triangleleft G \mid \operatorname{diam}(G/N) \leq k\} \leq Ck^a$ for every k. Now ϕ is an almost permutation, so we can define $D = |(\operatorname{Im}\phi)^c|$. Then we can bound $\#\{N \triangleleft H \mid \operatorname{diam}(H/N) \leq k\}$ as follows:

$$\begin{aligned}
\#\{N \triangleleft H \mid \operatorname{diam}(H/N) \leq k\} &\leq \#\{N \triangleleft H \mid \operatorname{diam}(H/N) \leq k, H/N \in \operatorname{Im}(\phi)\} + D \\
&\leq \#\{N \triangleleft G \mid \operatorname{diam}(\phi(G/N)) \leq k, G/N \in \operatorname{dom}(\phi)\} + D \\
&\leq \#\{N \triangleleft G \mid \operatorname{diam}(G/N) \leq Ak + A^2, G/N \in \operatorname{dom}(\phi)\} + D \\
&\leq \#\{N \triangleleft G \mid \operatorname{diam}(G/N) \leq Ak + A^2\} + D \\
&\leq C(Ak + A^2)^a + D.
\end{aligned}$$

So $\#\{N \lhd H \mid \operatorname{diam}(H/N) \leq k\} = \mathcal{O}(k^a)$. $\square$

Now we want to calculate the diameter growth of $\square_f \mathbb{Z}^n$.

Proposition 3.2.3. *For every $n \in \mathbb{N}$ we have a lower bound*

$$\#\{N \lhd \mathbb{Z}^n \mid \operatorname{diam}(\mathbb{Z}^n/N) \leq k\} = \Omega\left(k^{n^2}\right).$$

Note that $f(x) = \Omega(g(x))$ if there exists a $C > 0$ such that $f(x) \geq Cg(x)$ for every x.

Proof. Fix a k and consider the subgroups of $\mathbb{Z}^n$ generated by $x_1, \dots, x_n$ with $\frac{k}{2n} < x_{ii} \leq \frac{k}{n}$ and $|x_{ij}| \leq \frac{k}{2n^2}$ for every $i \neq j$, where $x_i = (x_{i1}, \dots, x_{in})$. The number of possibilities for $x_1, \dots, x_n$ is $\left(\frac{k}{2n}\right)^n \left(\frac{2k}{2n^2} + 1\right)^{n(n-1)}$. This is more than $\frac{1}{(2n)^n} \frac{1}{n^{2n(n-1)}} k^{n^2}$. So it suffices to show that all these subgroups N are different and the diameter of $\mathbb{Z}/N$ is not greater than k.
To show that these subgroups are different take $N = N'$ where N is generated by $x_1, \dots, x_n$ and N' is generated by $x_1', \dots, x_n'$. For every $i \leq n$ we can take $x_i' = a_1 x_1 + \ldots + a_n x_n$ with $a_1, \dots, a_n \in \mathbb{Z}$, since $N = N'$. Now take $j \neq i$ such that a_j is maximal. By projecting on the j^{th}-component we get the following:

$$\begin{aligned}
\frac{k}{2n^2} &\geq |a_1 x_{1j} + \ldots + a_n x_{nj}| \\
&\geq |a_j| x_{jj} - \sum_{k \neq j} |a_k x_{kj}| \\
&> \frac{k}{2n}|a_j| - \sum_{k \neq j} \frac{k}{2n^2}|a_k| \\
&\geq \frac{k}{2n}|a_j| - (n-1)\frac{k}{2n^2}|a_j| \\
&= \frac{k}{2n^2}|a_j|.
\end{aligned}$$

We can conclude that $a_j = 0$, therefore only a_i can be different from 0, which has to be equal to 1, because $\frac{k}{2n} < x_{ii}, x_{ii}' \leq \frac{k}{n}$, so $x_i' = x_i$. This is true for every i, so N and N' are generated by the same vectors $x_1, \dots, x_n$.
To prove that $\operatorname{diam}(\mathbb{Z}^n/N) \leq k$ suppose there is such a subgroup N for which $\operatorname{diam}(\mathbb{Z}^n/N) > k$. So there is an element in $\mathbb{Z}^n/N$ such that for every representing vector $y = (y_1, \dots, y_n)$ in $\mathbb{Z}^n$ we have $\sum_{i=1}^n |y_i| > k$. Let y be the representing vector for which $\|y\|$ is minimal and let i be such that $|y_i|$ is maximal. Without loss of generality we may assume y_i to be positive. Now as $\sum_{i=1}^n |y_i| > k$, we find that $y_i > \frac{k}{n} \geq x_{ii} > 0$ and we get the following:

$$\begin{aligned}
\|y - x_i\| &= \sum_{j=1}^n |y_j - x_{ij}| \\
&\leq y_i - x_{ii} + \sum_{j \neq i} (|y_j| + |x_{ij}|) \\
&< y_i - \frac{k}{2n} + \sum_{j \neq i} \left(|y_j| + \frac{k}{2n^2}\right) \\
&= \|y\| - \frac{k}{2n^2}.
\end{aligned}$$

Now $y - x_i$ is a smaller representing vector of the same element as y, which is a contradiction.
So for all these subgroups N we have $\operatorname{diam}(\mathbb{Z}^n/N) \leq k$, which proves that $\#\{N \lhd \mathbb{Z}^n \mid \operatorname{diam}(\mathbb{Z}^n/N) \leq k\} = \Omega\left(k^{n^2}\right)$. $\square$

In the proof of Theorem 3.2.1 we will show that H must be quasi-isometric to $\mathbb{Z}^n$, therefore it will suffice to know the diameter growth of 2-generated virtually $\mathbb{Z}^n$, as virtually $\mathbb{Z}^n$ groups are the only ones quasi-isometric to $\mathbb{Z}^n$. Consequently H must be $\mathbb{Z}^n$-by-finite, because one can turn the finite index subgroup $\mathbb{Z}^n$ into a finite index normal subgroup by taking the intersection of all conjugates, which is again $\mathbb{Z}^n$ as it is a finite index subgroup in $\mathbb{Z}^n$. In order to calculate this growth we will first restrict the normal subgroups of H to the finite index normal subgroup $\mathbb{Z}^n \lhd H$. To better understand these normal subgroups of $\mathbb{Z}^n$ we define minimal generating sets.

Definition 3.2.4. *A minimal generating set of $N \lhd \mathbb{Z}^n$ is the subset $\{x_1, \ldots, x_n\}$ of N where x_1 is the smallest vector in N (for the euclidean norm) and x_i is the smallest vector in $N \setminus \langle x_1, \ldots, x_{i-1} \rangle$ such that $N \cap \operatorname{span}(x_1, \ldots, x_i) = \langle x_1, \ldots, x_i \rangle$.*

A minimal generating set is a generating set of N, because it is linearly independent and therefore $N = N \cap \operatorname{span}(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n \rangle$.
Note that such a generating set always exists. Also note that a subset of a minimal generating set is a minimal generating set of what it generates. This notion will be important to control the diameter of $\mathbb{Z}^n/(N \cap \mathbb{Z}^n)$.

Lemma 3.2.5. *For every $n \in \mathbb{N}$ there exists a constant $D_n \in \mathbb{N}$ such that for every subgroup N of $\mathbb{Z}^n$ and every minimal generating set $\{x_1, \ldots, x_n\}$ we have $\|a_1x_1 + \ldots + a_nx_n\| \geq \frac{1}{D_n} \max_i \{\|a_ix_i\|\}$ for every $a_1, \ldots, a_n \in \mathbb{R}$.*

As we will do in the proof of Lemma 3.2.5, we define D_n recursively with $D_1 = 1$ and $D_n = D_{n-1}^2(4n^2D_{n-1}^3)^n$.
If the minimal generating set we choose happens to be orthogonal, this lemma would be obvious. The main idea behind the proof is to show that minimal generating sets are sufficiently similar to being orthogonal. In the proof we will assume that Lemma 3.2.5 is true up to some value n. We will use this to prove an intermediate result (Lemma 3.2.6 for $m = n$) and then we will use that to show that Lemma 3.2.5 is true for $n + 1$.

Lemma 3.2.6. *Let $\{x_1, \ldots, x_{m+1}\}$ be a minimal generating set and let p be the orthogonal projection on $\operatorname{span}(x_1, \ldots, x_m)$, so we can write $p(x_{m+1}) = a_1x_1 + \ldots + a_mx_m$. Suppose Lemma 3.2.5 is satisfied for all $n \leq m$. Then $|a_m| \leq \frac{m}{2}D_m$ and $|a_i| \leq \frac{m}{2}D_m^2$ for all $i < m$.*

For this lemma we will also assume that $D_{i+1} \geq \frac{i}{2}D_i^2 + 1$ for every $i \geq 1$, which will be the case in the proof of Lemma 3.2.5.

Proof. We proceed by contradiction. Let $\{x_1, \ldots, x_{m+1}\}$ be a minimal generating set with the smallest m such that it does not satisfy Lemma 3.2.6. Then we find

$$\|p(x_{m+1})\| = \|a_1x_1 + \ldots + a_mx_m\| \geq \frac{1}{D_m} \max_i \|a_ix_i\| \geq \frac{|a_m|}{D_m} \|x_m\|.$$

However as $\{x_1, \ldots, x_{m+1}\}$ is a minimal generating set we have that for every $b_1, \ldots, b_m$ in $\mathbb{Z}$ $\|x_{m+1}\| \leq \|x_{m+1} - b_1x_1 - \ldots - b_mx_m\|$, we even have $\|p(x_{m+1})\| \leq \|p(x_{m+1}) - b_1x_1 - \ldots - b_mx_m\|$, because the projections of both vectors onto $\operatorname{span}(x_1, \ldots, x_m)^\perp$ are equal. If we take b_i such that $|b_i - a_i| \leq \frac{1}{2}$, then we find the following inequality:

$$\|p(x_{m+1})\| \leq \|p(x_{m+1}) - b_1x_1 - \ldots - b_mx_m\| \leq \|(a_1 - b_1)x_1\| + \ \ldots\ + \|(a_m - b_m)x_m\| \leq \frac{m}{2} \|x_m\|.$$

Combining these inequalities we conclude that $|a_m| \leq \frac{m}{2}D_m$. As we assume this minimal generating set does not satisfy the lemma there must be an a_i such that $|a_i| > \frac{m}{2}D_m^2$, let l be the largest such i.
Now let p_i be the orthogonal projection onto $\operatorname{span}(x_1, \ldots, x_i)$. We will use these projections to bound the corresponding $|a_i|$. We already have $p(x_{m+1}) = a_1x_1 + \ldots + a_mx_m$. Now we take something similar for the projections p_i:

$$\begin{aligned}
p_{m-1}(a_mx_m) &= a_{m-1,m}x_{m-1} + \ldots + a_{1,m}x_1 \\
p_{m-2}((a_{m-1} + a_{m-1,m})x_{m-1}) &= a_{m-2,m-1}x_{m-2} + \ldots + a_{1,m-1}x_1 \\
&\ \vdots \\
p_l((a_{l+1} + a_{l+1,m} + \ldots + a_{l+1,l+2})x_{l+1}) &= a_{l,l+1}x_l + \ldots + a_{1,l+1}x_1
\end{aligned}$$

Let m' be such that $l \leq m' < m$. As before we have

$$\begin{aligned}
\|p_{m'}(x_{m+1})\| &= \|a_1x_1 + \ldots + a_{m'}x_{m'} + a_{1,m}x_1 + \ldots + a_{m',m}x_{m'} + \ldots + a_{m',m'+1}x_{m'}\| \\
&\geq \frac{1}{D_{m'}} \|(a_{m'} + a_{m',m} + \ldots + a_{m',m'+1})x_{m'}\| \geq \frac{|a_{m'} + a_{m',m} + \ldots + a_{m',m'+1}|}{D_{m'}} \|x_{m'}\|.
\end{aligned}$$

As before we can take $b_1, \ldots, b_m$ in $\mathbb{Z}$ such that $|b_i - a_i - a_{i,m} - \ldots - a_{i,m'+1}| \leq \frac{1}{2}$ for every i. Now we find

$$\|p_{m'}(x_{m+1})\| \leq \|p_{m'}(x_{m+1}) - b_1x_1 - \ldots - b_{m'}x_{m'}\| \leq \frac{1}{2} \|x_1\| + \ \ldots\ + \frac{1}{2} \|x_{m'}\| \leq \frac{m'}{2} \|x_{m'}\|.$$

So $\frac{m'}{2}D_{m'} \geq |a_{m'} + a_{m',m} + \ldots + a_{m',m'+1}|$. As m is assumed to be the smallest value for which this lemma is not true, we have that when $p_{m'}(x_{m'+1})$ is written as a linear combination of $x_1, \ldots, x_{m'}$, where the coefficient of x_m is not greater than $\frac{m'}{2}D_{m'}$ and the other coefficients are not greater than $\frac{m'}{2}D^2_{m'}$. Now as

$$p_{m'}((a_{m'+1} + a_{m'+1,m} + \ldots + a_{m'+1,m'+2})x_{m'+1}) = a_{m',m'+1}x_{m'} + \ldots + a_{1,m'+1}x_1$$

we have $|a_{m',m'+1}| \leq \frac{m'}{2}D_{m'}|a_{m'+1} + a_{m'+1,m} + \ldots + a_{m'+1,m'+2}| \leq \frac{m'}{2}D_{m'}\frac{m'+1}{2}D_{m'+1}$ and $|a_{i,m'+1}| \leq \frac{m'}{2}D^2_{m'}|a_{m'+1} + a_{m'+1,m} + \ldots + a_{m'+1,m'+2}| \leq \frac{m'}{2}D^2_{m'}\frac{m'+1}{2}D_{m'+1}$ for $i < m'$.
Now we had $\frac{l}{2}D_l \geq |a_l + a_{l,m} + \ldots + a_{l,m'+1}|$, so using the fact that $D_{i+1} \geq \frac{i}{2}D_i^2 + 1$ and $iD_i \leq (i+1)D_{i+1}$ for every $i \geq 1$, we can make the following computation.

$$\begin{aligned}
|a_l| &\leq |a_{l,m}| + \ldots + |a_{l,l+1}| + \frac{l}{2}D_l \\
&\leq \frac{m-1}{2}D^2_{m-1}\frac{m}{2}D_m + \ldots + \frac{l+1}{2}D^2_{l+1}\frac{l+2}{2}D_{l+2} + \frac{l}{2}D_l\frac{l+1}{2}D_{l+1} + \frac{l}{2}D_l \\
&\leq \frac{m-1}{2}D^2_{m-1}\frac{m}{2}D_m + \ldots + \frac{l+1}{2}D^2_{l+1}\frac{l+2}{2}D_{l+2} + \frac{l+1}{2}D_{l+1}\left(\frac{l}{2}D_l + 1\right) \\
&\leq \frac{m-1}{2}D^2_{m-1}\frac{m}{2}D_m + \ldots + \frac{l+1}{2}D^2_{l+1}\frac{l+2}{2}D_{l+2} + \frac{l+1}{2}D^2_{l+1} \\
&\leq \frac{m-1}{2}D^2_{m-1}\frac{m}{2}D_m + \ldots + \frac{l+1}{2}D^2_{l+1}\frac{l+2}{2}D_{l+2} + \frac{l+2}{2}D_{l+2} \\
&\vdots \\
&\leq \frac{m-1}{2}D^2_{m-1}\frac{m}{2}D_m + \frac{m}{2}D_m \\
&\leq \left(\frac{m-1}{2}D^2_{m-1} + 1\right)\frac{m}{2}D_m \\
&\leq \frac{m}{2}D^2_m
\end{aligned}$$

But we assumed $|a_l| > \frac{m}{2}D^2_m$, and so we have a contradiction, which proves this lemma. □

Now we can use this result to prove Lemma 3.2.5.

Proof of Lemma 3.2.5. We define D_n recursively with $D_1 = 1$ and $D_n = D^2_{n-1}\left(4n^2D^3_{n-1}\right)^n$. For every subgroup $N \lhd \mathbb{Z}^n$ we can take a minimal generating set $x_1, \ldots, x_n$.
Let n be the smallest value for which the lemma is not true, i.e. there exist a_i such that $\frac{1}{D_n}\max_i\{\|a_ix_i\|\} > \|a_1x_1 + \ldots + a_nx_n\|$. As the lemma is obvious for $n = 1$, we may assume that $n \geq 2$.
First we observe that $\|a_ix_i\|$ must be similar for all i, that is $\min_i\{\|a_ix_i\|\} > \frac{1}{2D_{n-1}}\max_i\{\|a_ix_i\|\}$. We can see this by combining the reverse triangular inequality with the fact that a subset of a minimal generating set is a minimal generating set of what it generates: $\frac{1}{D_n}\max_i\{\|a_ix_i\|\} > \|a_1x_1 + \ldots + a_nx_n\| \geq \frac{1}{D_{n-1}}\max_i\{\|a_ix_i\|\} - \min_i\{\|a_ix_i\|\}$.
So we get the desired result that $\min_i\{\|a_ix_i\|\} > \left(\frac{1}{D_{n-1}} - \frac{1}{D_n}\right)\max_i\{\|a_ix_i\|\} \geq \frac{1}{2D_{n-1}}\max_i\{\|a_ix_i\|\}$.

To continue we would prefer for x_n to be orthogonal to $\operatorname{span}(x_1, \ldots, x_{n-1})$. However a partial result will suffice. We will show that the angle between x_n and the span of $x_1, \ldots, x_{n-1}$ can not be arbitrarily small, which will prove the lemma. So let p be the orthogonal projection onto $\operatorname{span}(x_1, \ldots, x_{n-1})$.

Now distinguish two cases according to whether or not $nD^2_{n-1}|a_n|$ is greater or smaller than $\max_i\{|a_i|\}$.

Suppose $\max_i\{|a_i|\} > nD^2_{n-1}|a_n|$. As such we can write $p(x_n)$ as the linear combination $a'_1x_1 + \ldots + a'_{n-1}x_{n-1}$. Due to Lemma 3.2.6 we know that $|a'_i| \leq \frac{n}{2}D^2_{n-1}$ for every i. Now we can take k such that $|a_k|$ is maximized. By combining $|a'_k| \leq \frac{n}{2}D^2_{n-1}$ with $\max_i\{|a_i|\} = |a_k| > nD^2_{n-1}|a_n|$ we find that $|a_k + a'_ka_n| \geq |a_k| - \frac{nD^2_{n-1}}{2}|a_n| \geq \frac{1}{2}|a_k|$.

This admits the following computation:

$$\begin{aligned}
\frac{1}{D_n}\max_i\{\|a_ix_i\|\} &\geq \|a_1x_1+\ldots+a_nx_n\| \\
&\geq \|p(a_1x_1+\ldots+a_{n-1}x_{n-1}+a_nx_n)\| \\
&\geq \|a_1x_1+\ldots+a_{n-1}x_{n-1}+a_np(x_n)\| \\
&\geq \left\|(a_1+a_1'a_n)x_1+\ldots+(a_{n-1}+a_{n-1}'a_n)x_{n-1}\right\| \\
&\geq \frac{1}{D_{n-1}}\max_i\{\|(a_i+a_i'a_n)x_i\|\} \\
&\geq \frac{1}{2D_{n-1}}\|a_kx_k\| \\
&\geq \frac{1}{2D_{n-1}}\min_i\{\|a_ix_i\|\} \\
&\geq \frac{1}{4D_{n-1}^2}\max_i\{\|a_ix_i\|\}.
\end{aligned}$$

Now $n \geq 2$, so $D_n = 2D_{n-1}\left(2n^2D_{n-1}\right)^n > 4D_{n-1}^2$, which contradicts the earlier computations.

Up to this point we essentially only used that x_n can not be shortened by adding a linear combination $\lambda_1x_1+\ldots+\lambda_{n-1}x_{n-1}$ with $\lambda_1,\ldots,\lambda_{n-1} \in \mathbb{Z}$. However if $\max_i\{|a_i|\} \leq nD_{n-1}^2|a_n|$ this will not be possible. For example for every $\varepsilon > 0$ we have $(2,0,0,0,0)$, $(0,2,0,0,0)$, $(0,0,2,0,0)$, $(0,0,0,2,0)$, $(1,1,1,1,\varepsilon)$, but the group generated by these vectors contains $(0,0,0,0,2\varepsilon)$. In the continuation of this proof we will look for a vector like $(0,0,0,0,2\varepsilon)$, more precisely a short vector that is almost orthogonal to $x_1,\ldots,x_{n-1}$.

As $\max_i\{|a_i|\} \leq nD_{n-1}^2|a_n|$, we have $nD_{n-1}^2\|a_nx_n\| \geq \max_i\|a_ix_i\|$, as x_n is the biggest vector in the basis $\{x_1,\ldots,x_n\}$. Let e be a unit vector perpendicular to $\operatorname{span}(x_1,\ldots,x_{n-1})$. Then we have $\frac{nD_{n-1}^2}{D_n}\|a_nx_n\| \geq \|a_1x_1+\ldots+a_nx_n\| \geq |a_nx_n\cdot e|$, so $\|x_n\| \geq \frac{D_n}{nD_{n-1}^2}|x_n\cdot e|$.
Now for every $m \in \mathbb{N}$ we can take $p(mx_n) = b_1x_1+\ldots+b_{n-1}x_{n-1}+c_1x_1+\ldots+c_{n-1}x_{n-1}$ with $b_i \in \mathbb{Z}$ and $|c_i| \leq \frac{1}{2}$ for every i. What we are looking for is an m such that $c_1,\ldots,c_{n-1}$ are close to zero. In that case $p(mx_n - b_1x_1 - \ldots - b_{n-1}x_{n-1})$ is small.
To make this precise: for every $i < n$ there exists a $k_i \in \mathbb{N}$ such that $c_i \in \left[\frac{k_i}{4n^2D_{n-1}^3}, \frac{k_i+1}{4n^2D_{n-1}^3}\right]$, with k_i between $-2n^2D_{n-1}^3$ and $2n^2D_{n-1}^3-1$. Now due to the pigeonhole principle there will be an $m, m' \leq (4n^2D_{n-1}^3)^{n-1}$ with $k_i = k_i'$ for every i. Now $(m-m')x_n$ will be the vector we are looking for, because $c_i' - c_i \in \left[\frac{-1}{4n^2D_{n-1}^3}, \frac{1}{4n^2D_{n-1}^3}\right]$.
As x_1 is the smallest vector in N we can make the following computation:

$$\begin{aligned}
\|x_1\|^2 &\leq \left\|(b_1-b_1')x_1+\ldots+(b_{n-1}-b_{n-1}')x_{n-1}+(m'-m)x_n\right\|^2 \\
&= \left\|(b_1-b_1')x_1+\ldots+(b_{n-1}-b_{n-1}')x_{n-1}+p(m'x_n)-p(mx_n)\right\|^2+|m'-m|^2|x_n\cdot e|^2 \\
&\leq \left(\sum_{i=1}^{n-1}\|(c_i'-c_i)x_i\|\right)^2+(4n^2D_{n-1}^3)^{2n-2}\frac{n^2D_{n-1}^4}{D_n^2}\|x_n\|^2 \\
&\leq \left(\sum_{i=1}^{n-1}\frac{\|x_i\|}{4n^2D_{n-1}^3}\right)^2+\left(\frac{(4n^2D_{n-1}^3)^n}{4nD_{n-1}D_n}\right)^2\|x_n\|^2 \\
&\leq \left(\frac{(n-1)\|x_n\|}{4n^2D_{n-1}^3}\right)^2+\frac{1}{8n^2D_{n-1}^6}\|x_n\|^2 \\
&< \frac{1}{4n^2D_{n-1}^6}\|x_n\|^2
\end{aligned}$$

However, this contradicts the earlier results that $\max_i|a_i| \leq 2nD_{n-1}^2$ and $\min_i\|a_ix_i\| \geq \|a_nx_n\|$

$$\|x_1\| = \frac{1}{|a_1|}\|a_1x_1\| \geq \frac{1}{nD_{n-1}^2|a_n|}\min_i\|a_ix_i\| \geq \frac{1}{2nD_{n-1}^3}\frac{1}{|a_n|}\|a_nx_n\| \geq \frac{1}{2nD_{n-1}^3}\|x_n\|.$$

So for every $a_1,\ldots,a_n \in \mathbb{R}$ we have $\|a_1x_1+\ldots+a_nx_n\| \geq \frac{1}{D_n}\max_i\{\|a_ix_i\|\}$. □

As mentioned earlier this lemma will help us to control the diameter of $\mathbb{Z}^n/(N\cap\mathbb{Z}^n)$. However we need to control the diameter of H/N. We will show that H must be $\mathbb{Z}^n$-by-finite and then we will consider $\mathbb{Z}^n/(N\cap\mathbb{Z}^n)$. While it is not necessarily true that $\operatorname{diam}(H/N)\geq \operatorname{diam}(\mathbb{Z}^n/(N\cap\mathbb{Z}^n))$, it is true up to a constant.

Lemma 3.2.7. *Let G and H be two groups such that H is G-by-finite. Then there exists a constant C such that $C\operatorname{diam}(H/N)\geq \operatorname{diam}(G/(N\cap G))$ for every $N\lhd H$.*

Proof. Due to Proposition 2 of [Khu12], there exists a C' such that $\operatorname{diam}_G(G/(N\cap G))\leq C'\operatorname{diam}_H(G/(N\cap G))$ for every $N\in H$. So it suffices to show that there exists a C such that $\operatorname{diam}_H(G/(N\cap G))\leq C\operatorname{diam}(H/N)$. As H is G-by-finite we can take $F=H/G$ finite and set $C=3|F|$.
We can take $g\in G$ such that $|g|_H=|gN|_{H/N}=\operatorname{diam}_H(G/(N\cap G))$. Now take a path between 1 and g and take $1=b_0,b_1,\ldots,b_{|F|}=g$ on this path with $d_H(b_i,b_{i+1})\geq\left\lceil\frac{|g|_H}{|F|}\right\rceil$. Then for every i there exists an $n_i\in N$ such that $d_H(b_i,n_i)\leq\operatorname{diam}(H/N)$. As we have $|F|+1$ elements n_i, there will be two indices $i<j$ such that n_i and n_j lie in the same coset of G. So there exists an $x\in G\cap N$ such that $n_j=xn_i$. Now we can make the following computation:

$$\begin{aligned}
|g|_H &\leq d_H(x,g)\\
&\leq d_H(x,xb_i)+d(xb_i,xn_i)+d(n_j,b_j)+d(b_j,g)\\
&\leq d_H(1,b_i)+d(b_j,g)+d(b_i,n_i)+d(n_j,b_j)\\
&\leq |g|_H-d_H(b_i,b_j)+d(b_i,n_i)+d(n_j,b_j)\\
&\leq |g|_H-\left\lceil\frac{|g|_H}{|F|}\right\rceil+2\operatorname{diam}(H/N)
\end{aligned}$$

So $\dfrac{|g|_H}{|F|}\leq 2\operatorname{diam}(H/N)+1\leq 3\operatorname{diam}(H/N)$, which proves the lemma. □

Finally we need to show that as the 2-generated group H contains an element with a non-trivial conjugation action on $\mathbb{Z}^n$.

Lemma 3.2.8. *Let $n\in\mathbb{N}$ and let H be an $(n-1)$-generated group that is $\mathbb{Z}^n$-by-finite. Then there exists an element $h\in H$ such that the conjugation action of h on $\mathbb{Z}^n$ is neither* Id, *nor* $-$Id.

Proof. Let K be the subgroup of H for which the conjugation action on $\mathbb{Z}^n$ is trivial. Suppose that the conjugation action of every element is either Id or $-$Id. Then $[H:K]$ is either 1 or 2. Due to Schur's Theorem $[K,K]$ is finite, because K is a finite central extention of $\mathbb{Z}^n$. So $\mathbb{Z}^n<H/[K,K]$ and therefore we may assume without loss of generality that K is abelian. If $[H:K]=1$, then $H=K$ is abelian of rank at least n, so H is not $(n-1)$-generated.
If $[H:K]=2$ and K is an abelian subgroup of H of index 2. Let g be in H, but not in K. Now remark that for $x,y\in K$ we have that $[x,y]=e$, $[gx,y]=gxyx^{-1}g^{-1}y^{-1}=y^{-2}$ and $[gx,gy]=e$. So $[H,H]=(2\mathbb{Z})^n$ and therefore $\mathbb{Z}_2^n<H/[H,H]$. So we can conclude that H is not $(n-1)$-generated. □

Now we can calculate the amount of intersections $N\cap\mathbb{Z}^n$ we can have such that $\operatorname{diam}(H/N)\leq k$.

Lemma 3.2.9. *Let H be 2-generated and $\mathbb{Z}^n$-by-finite with $n\geq 3$. Then $\#\{N\cap\mathbb{Z}^n\mid N\lhd H,\operatorname{diam}(H/N)\leq k\}=\mathcal{O}\left(k^{n^2-1}\right)$.*

Proof. Due to Lemma 3.2.7 it suffices to show that $\#\{N\cap\mathbb{Z}^n\mid N\lhd H,\operatorname{diam}(\mathbb{Z}^n/(N\cap\mathbb{Z}^n))\leq k\}=\mathcal{O}\left(k^{n^2-1}\right)$. So take $N\lhd H$ such that $\operatorname{diam}(\mathbb{Z}^n/(N\cap\mathbb{Z}^n))\leq k$. Then we can take $N\cap\mathbb{Z}^n$ generated by $\{x_1,\ldots,x_n\}$ as in Lemma 3.2.5 and without loss of generality we can assume $\|x_1\|\geq\ldots\geq\|x_n\|$. Now for every vector $x\in\mathbb{R}^n$ we have $d(x,\mathbb{Z}^n)\leq\frac{\sqrt{n}}{2}$, in particular we have $d\left(\frac{x_1}{2},\mathbb{Z}^n\right)\leq\frac{\sqrt{n}}{2}$. So we can make the following computation:

$$\begin{aligned}
k+\frac{\sqrt{n}}{2} &\geq \operatorname{diam}(\mathbb{Z}^n/(N\cap\mathbb{Z}^n))+\frac{\sqrt{n}}{2}\\
&\geq d\left(\frac{x_1}{2}+N,0+N\right)\\
&= \inf_{a_1,\ldots,a_n\in\mathbb{Z}}\left(\left\|\left(\frac{1}{2}+a_1\right)x_1+a_2x_2+\ldots+a_nx_n\right\|\right)\\
&\geq \frac{1}{D_n}\inf_{a_1}\left(\left|\frac{1}{2}+a_1\right|\,\|x_1\|\right) && \text{by Lemma 3.2.5}\\
&= \frac{1}{2D_n}\|x_1\|.
\end{aligned}$$

We can conclude that $2D_nk + D_n\sqrt{n} \geq \|x_1\| \geq \ldots \geq \|x_n\|$. So for any i we have that x_i lies within $[-D_n(2k+\sqrt{n}), D_n(2k+\sqrt{n})]^n$.
Due to Lemma 3.2.8 we can choose $h \in H$ such that $\alpha_h \in \operatorname{Aut}(\mathbb{Z}^n)$ is different from $\pm\operatorname{Id}$, with $\alpha_h(x) = hxh^{-1}$. Note that α_h is of finite order and note that $N \cap \mathbb{Z}^n$ is α_h-independent. So there exist a_i such that $\alpha_h(x_n) = a_1x_1 + \ldots + a_nx_n$. Note that α_h is an bounded operator on $\mathbb{R}^n$, which allows the following computation:

$$\begin{aligned} \|\alpha_h\| \|x_n\| &\geq \|a_1x_1 + \ldots + a_nx_n\| \\ &\geq \frac{1}{D_n} \max_i\{\|a_ix_i\|\} \\ &\geq \frac{1}{D_n} \max_i\{|a_i|\} \|x_n\| . \end{aligned}$$

So $D_n \|\alpha_h\| \geq \max_i\{|a_i|\}$.
Now we still have to count the different possibilities for N. There are fewer of these than the different possibilities for $x_1, \ldots, x_n$, as different subgroups have different generators. Note that every possibility of $x_1, \ldots, x_n$ admits values of $a_1, \ldots, a_n$ associated to α_h.

Now we will show that for any given a sequence $a_1, \ldots, a_n$, the number of $x_1, \ldots, x_n$ satisfying earlier conditions is bounded by $(4D_nk+2D_n\sqrt{n}+1)^{n^2-1}$. As the number of possibilities for any a_i is bounded by $2D_n\|\alpha_h\|$, the total number of possibilities for $x_1, \ldots, x_n$ is bounded by $(2D_n\|\alpha_h\|)^n(4D_nk+2D_n\sqrt{n}+1)^{n^2-1} = \mathcal{O}(k^{n^2-1})$. These earlier conditions are $D_n\|\alpha_h\| \geq \max_i\{|a_i|\}$, $2D_nk+D_n\sqrt{n} \geq \|x_1\| \geq \ldots \geq \|x_n\|$ and $\alpha_h(x_n) = a_1x_1+\ldots+a_nx_n$.
If there is an $i < n$ such that $a_i \neq 0$, then x_i can be deduced from all other x_j. So the number of possibilities of $x_1, \ldots, x_n$ is bounded by $(4D_nk + 2\sqrt{n}D_n + 1)^{(n-1)n}$.
If for every $i < n$ we have $a_i = 0$, then $a_n = \pm 1$, because otherwise α_h is not an automorphism. Since $\alpha_h \neq \pm\operatorname{Id}$ we know that $\{x \in \mathbb{R}^n \mid \alpha_h(x) = a_nx\}$ is not the entirety of $\mathbb{R}^n$. Therefore it is at most an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, which reduces the possibilities for x_n to at most $(4D_nk + 2D_n\sqrt{n}+1)^{n-1}$, while the possibilities of other $x_1, \ldots, x_{n-1}$ is bounded by $(4D_nk + 2D_n\sqrt{n}+1)^{(n-1)n}$. Therefore the total number of possibilities in this case is also bounded by $(4D_nk + 2D_n\sqrt{n}+1)^{n^2-1}$.

In conclusion we have that for any fixed sequence $a_1, \ldots, a_n$ the number of possibilities of $x_1, \ldots, x_n$ is bounded by $(4D_nk + 2D_n\sqrt{n}+1)^{n^2-1}$. So the total number of possibilities for $x_1, \ldots, x_n$ is bounded by $(D_n\|\alpha_h\|)^n(4D_nk + 2D_n\sqrt{n}+1)^{n^2-1}$. Therefore the possibilities of $N \cap \mathbb{Z}^n$ is bounded by that same number, which means $\#\{N \cap \mathbb{Z}^n \mid N \lhd H, \operatorname{diam}(H/N) \leq k\} = \mathcal{O}\left(k^{n^2-1}\right)$. □

Now every intersection $\mathbb{Z}^n \cap N$ can be realized by multiple normal subgroups $N \lhd H$. However this amount is bounded. We give the following improved version of our original proposition, due to Alain Valette.

Proposition 3.2.10 (A.Valette). *let H be a finite group with a normal abelian subgroup A generated by $n \geq 1$ elements, and with index $d = [H : A]$. Let $S(H, A)$ be the set of normal subgroups $N \lhd H$ such that $N \cap A = \{1\}$. Then $|S(H, A)|$ is bounded above by a function only depending on n and d.*

Proof. Indeed, let $\pi : H \to H/A$ be the quotient map. For $N_1 \in S(H, A)$, since $\pi|_{N_1}$ is injective, there are (very crudely) at most 2^d possibilities for $\pi(N_1)$.
Now we estimate how many $N_2 \in S(H, A)$ are such that $\pi(N_1) = \pi(N_2)$. The subgroup N_1A is isomorphic to the direct product $N_1 \times A$, we write its elements as pairs (n_1, a). Now since $N_2A = N_1A$ we may view N_2 as the graph of a map $\alpha : N_1 \to A$ (we identify π on $N_1 \times A$ with the projection on the first factor). So we write $N_2 = \{(g, \alpha(g)) : g \in N_1\}$ and $N_1 \to N_2 : g \mapsto (g, \alpha(g))$ is an isomorphism.
Fixing $g \in N_1$, we estimate the number of possibilities for $\alpha(g)$. Since $g^d = 1$, we must have $\alpha(g)^d = 1$ in A. So we must bound d-torsion in A.
By the theory of elementary divisors, there exist integers $f_1, \ldots, f_k$, with $f_i|f_{i+1}$, such that $A \simeq \bigoplus_{i=1}^k \mathbb{Z}/f_i\mathbb{Z}$. We have $k \leq n$ as A is n-generated. Now there are at most d elements of d-torsion in a cyclic group (by uniqueness of subgroups). So there are at most d^k elements of d-torsion in A. So the number of possibilities for $\alpha(g)$ is at most $d^k \leq d^n$.
Therefore the number of possibilities for N_2 is at most $(d^n)^{|N_2|} \leq d^{nd}$. Finally we have $|S(H, A)| \leq 2^d \cdot d^{nd}$. □

Corollary 3.2.11. *Let H be $\mathbb{Z}^n$-by-finite for some $n \geq 3$. Then there exists a $C > 0$ such that for every $\mathcal{N} \lhd \mathbb{Z}^n$ of finite index the set $\#\{N \lhd H \mid N \cap \mathbb{Z}^n = \mathcal{N}\} \leq C$.*

This is an easy consequence of Proposition 3.2.10 as $\#\{N \lhd H \mid N \cap \mathbb{Z}^n = \mathcal{N}\} = |S(H/\mathcal{N}, \mathbb{Z}^n/\mathcal{N})|$.
Now combining Lemma 3.2.9 and Corollary 3.2.11 we can control the diameter growth of H, which suffices to prove Theorem 3.2.1.

In the proof of Theorem 3.2.1 we will use a generalized version of Theorem 7 of [KV15]. We will essentially find two coarsely equivalent sequences of groups that each converge to a group in the space of marked groups. Now by combining Lemma 2.1.1 and Proposition 3 in [KV15] we find that these two groups are quasi-isometric.

Proof of Theorem 3.2.1. Suppose there is a coarse equivalence Φ between $\Box_f H$ and $\Box_f \mathbb{Z}^n$, with H 2-generated. We may assume that H is residually finite, because if H is not residually finite, i.e. $\bigcap_{N \lhd H} N \neq \{1\}$, then $\Box_f H = \Box_f H / \bigcap_{N \lhd H} N$ and $H / \bigcap_{N \lhd H} N$ is residually finite. Note that H is still 2-generated.
Now due to Lemma 2.1.1 there is an almost permutation ϕ between the components of $\Box_f H$ and the components of $\Box_f \mathbb{Z}^n$, where $\Phi|_X$ is a quasi-isometry between X and $\phi(X)$ for every component X of $\Box_f H$ in the domain of ϕ. Since H is residually finite, there is a box space $\Box_{(N_k)} H$ contained in $\Box_f H$. Via ϕ this corresponds to a subspace $\coprod_k \mathbb{Z}^n / M_k$ of $\Box_f \mathbb{Z}^n$. Now this sequence $(\mathbb{Z}^n/M_k)_k$ has a subsequence that is constant on bigger and bigger balls, i.e. there exists a sequence k_r such that $k_r \to \infty$ as $r \to \infty$ and for every $k, k' \geq k_r$ in this subsequence we have $M_k \cap B[1,r] = M_{k'} \cap B[1,r]$. Now due to a generalized version of Theorem 7 of [KV15] H is quasi-isometric a quotient of $\mathbb{Z}^n$, because the intersection of the subsequence M_k converges to a normal subgroup of $\mathbb{Z}^n$. So H is virtually $\mathbb{Z}^m$ with $m \leq n$, due to the quasi-isometric rigidity of $\mathbb{Z}^m$.
Due to Lemma 3.2.9 we have $\#\{N \cap \mathbb{Z}^n \mid N \lhd H, \operatorname{diam}(H/N) \leq k\} = \mathcal{O}\left(k^{m^2-1}\right)$ and due to Corollary 3.2.11 we have $\#\{N \lhd H \mid \operatorname{diam}(H/N) \leq k\} = \mathcal{O}\left(k^{m^2-1}\right)$. However due to Proposition 3.2.2 we have that $\#\{N \lhd \mathbb{Z}^n \mid \operatorname{diam}(\mathbb{Z}^n/N) \leq k\} = \mathcal{O}\left(k^{m^2-1}\right)$, but as $m \leq n$ this is in contradiction with Proposition 3.2.3. $\Box$

Chapter 4

Box spaces of the free group that neither contain expanders nor embed into a Hilbert space

As stated in section 1.5.2, expander do not embed into a Hilbert space. For a long time, the presence of weakly embedded expanders was in fact the only known obstruction to a bounded geometry metric space coarsely embedding into a Hilbert space. Note that if one does not impose the condition of bounded geometry, then ℓ^p with $p > 2$ is a space which does not contain expanders, yet does not admit an embedding into a Hilbert space ([JR06]).

An important step towards answering the question of whether expanders are indeed the only possible obstruction was the paper of Tessera [Tes09], in which he was able to give a characterization of spaces which do not embed coarsely into a Hilbert space in terms of *generalized expanders*, which satisfy corresponding Poincaré inequalities relative to a measure.

In the groundbreaking article [AT15], Arzhantseva and Tessera gave examples of sequences of finite Cayley graphs of uniformly bounded degree which do not contain weakly embedded expanders but do not embed coarsely into a Hilbert space. Their examples make use of *relative expanders*, which are a specific case of generalized expanders. One of their examples is a box space of $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$, a group with relative property (T): this box space does not embed into a Hilbert space because the parent group does not have the Haagerup property, and it does not contain expanders thanks to a proposition (Proposition 2, [AT15]) which shows that expanders cannot be embedded into a sequence of group extensions where the sequence of quotients and the sequence of normal subgroups which make up the extension both embed coarsely into a Hilbert space. They also give constructions of box spaces of wreath products, including an example which admits a fibred coarse embedding into a Hilbert space (i.e. it is a box space of a group with the Haagerup property, see [CWW13] for the proof of this equivalence). All of these examples are constructed using sequences of finite groups which do embed into a Hilbert space, and the non-embeddability of the resulting spaces is encoded in the action of one subgroup on another.

The following problem ([AT15], section 8: Open Problems) remained open: does there exist a sequence of finite graphs with bounded degree and girth (i.e. the length of the smallest cycle) tending to infinity that does not coarsely embed into a Hilbert space but does not contain a weakly embedded expander? The original motivation for this question of Arzhantseva and Tessera was the possibility to use such a sequence for the construction of a group with these properties (although the presence of such a sequence would not guarantee that the group constructed would not contain expanders elsewhere). Arzhantseva and Tessera have since constructed such a group without the use of such a sequence of graphs ([AT15]), and the question about the existence of such a large girth sequence remained unanswered. A natural way to construct such a sequence would be to use a box space of a non-abelian free group. This requires a different method to the one used in [AT15], since there are no obvious "building blocks" which can be used to construct the sequence (as with the semidirect products of embeddable groups in [AT15]).

This chapter is based on [DK16], joint work with Khukhro. There we answer this question by showing that there exists a filtration of the free group F_3 such that the corresponding box space does not coarsely embed into a Hilbert space, but does not admit a weakly embedded expander sequence.

4.1 Overview

In this chapter we prove the following theorem.

Theorem 4.1.1. *There exists a filtration of the free group F_3 such that the corresponding box space does not coarsely embed into a Hilbert space, but does not admit a weakly embedded expander sequence.*

The overall structure of the proof is as follows. We construct a sequence of subgroups $\{N_i\}$ of F_3 which gives rise to expanders (section 4.3.1), and consider the sequence of homology covers of the quotients $\{F_3/N_i\}$; this gives rise to another sequence of subgroups $\Gamma_q(N_i) < N_i$ of F_3 (section 4.3.2) such that the corresponding quotients of F_3 coarsely embed into a Hilbert space. Recall that $\Gamma_m(G)$ is the subgroup of G generated by m^{th}-powers of elements in G and commutators. We then consider the quotients of F_3 by intersections of these sequences of subgroups, as in the following diagram, where the arrows represent quotient maps.

$$\begin{tikzcd}
 & & & F_3/\Gamma_q(N_3)\cdots \arrow[d] \\
 & & F_3/\Gamma_q(N_2) \arrow[d] & F_3/(N_3\cap\Gamma_q(N_2))\cdots \arrow[l] \arrow[d] \\
 & F_3/\Gamma_q(N_1) \arrow[d] & F_3/(N_2\cap\Gamma_q(N_1)) \arrow[l] \arrow[d] & F_3/(N_3\cap\Gamma_q(N_1))\cdots \arrow[l] \arrow[d] \\
\{1\} & F_3/N_1 \arrow[l] & F_3/N_2 \arrow[l] & F_3/N_3\cdots \arrow[l]
\end{tikzcd}$$

In section 4.4, we choose a subsequence of quotients $\{F_3/(N_{n_i}\cap\Gamma_q(N_{k_i}))\}$ which lie on some path that moves sufficiently slowly away the horizontal (expander) sequence in this "triangle" of intersections.

We do this so that for such a quotient $F_3/(N_{n_i}\cap\Gamma_q(N_{k_i}))$, we can control the eigenvalues corresponding to those eigenvectors of the Laplacian which are not coming from lifts of eigenvectors of the Laplacian on the quotient $F_3/(N_{n_i-1}\cap\Gamma_q(N_{k_i}))$ which is horizontally to the left of $F_3/(N_{n_i}\cap\Gamma_q(N_{k_i}))$ (we do this using representation theory in section 4.3.3). This ensures, via the results on generalized expanders of section 4.2.1 that the chosen sequence will not coarsely embed into a Hilbert space.

On the other hand, each of the quotients $F_3/(N_{n_i}\cap\Gamma_q(N_{k_i}))$ surjects onto $F_3/\Gamma_q(N_{k_i})$, and we prove in section 4.2.2 that such a sequence then cannot contain weakly embedded expanders.

4.2 Expanders and embeddability into Hilbert spaces

4.2.1 Expanders and generalized expanders

Let $X=(E,V)$ be a finite, k-regular graph, and number the vertices of X, $V=\{v_1,v_2,...,v_n\}$. The *adjacency matrix* of X is the matrix A indexed by pairs of vertices $v_i,v_j\in V$ such that A_{ij} is equal to the number of edges connecting v_i to v_j. We will restrict ourselves to considering simple graphs, and so for us, this number will always be equal to either 0 or 1.

The *Laplacian* is defined as the matrix $\Delta := k\,\mathrm{Id}-A$, which can be viewed as an operator $\ell^2(V)\to\ell^2(V)$. If $|V|=n$, then Δ is an $n\times n$ symmetric matrix and thus, counting multiplicities, has n real eigenvalues,

$$\lambda_0=0\le\lambda_1\le\cdots\le\lambda_{n-1}.$$

Note that the corresponding eigenvectors are orthogonal. The first non-trivial eigenvalue λ_1 is linked to connectivity properties of the graph X, namely via the *Cheeger constant* $h(X):=\inf|\partial F|/|F|$, where the infimum is taken over all subsets F of X satisfying $0<|F|\le|X|/2$. The well-known *Cheeger-Buser inequality* links the first non-trivial eigenvalue of the Laplacian with the Cheeger constant: $\frac{\lambda_1}{2}\le h(X)\le\sqrt{2k\lambda_1}$.

While any finite connected graph X has a non-zero Cheeger constant, it is rather difficult to construct a sequence of k-regular graphs of growing size such that their Cheeger constants are bounded uniformly away from zero. Given a sequence of k-regular graphs $\{X_n\}$ with $|X_n|\to\infty$, we say that $\{X_n\}$ is an *expander sequence* if there exists an $\varepsilon>0$ such that $h(X_n)>\varepsilon$ for all n. In Section 1.5.1 we give three characterizations of expanders.

Theorem 4.2.1 (Theorem 1.5.1)**.** *Let $(\mathcal{G}_n)_n$ be a sequence of k-regular Cayley graphs. This sequence is an expander if one of the following equivalent statements is true:*

1. *There exists a $c>0$ such that $h(\mathcal{G}_n)\ge c$ for every n.*

2. *There exists an $\varepsilon>0$ such that $\lambda_1(\mathcal{G}_n)\ge\varepsilon$.*

3. There exists a C such that for every n and every 1-Lipschitz map $\varphi\colon \mathcal{G}_n \to \ell^2$ we have

$$\sum_{x,y\in\mathcal{G}_n} \|\varphi(x)-\varphi(y)\|^2 \leq C|\mathcal{G}_n|^2.$$

The following definition of Tessera [Tes09] was introduced in order to characterize the failure to embed into a Hilbert space.

Definition 4.2.2. *Let $(\mathcal{G}_n)_n$ be a sequence of graphs. This sequence is said to be a generalized expander if there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$, a sequence of probability measures μ_n on $\mathcal{G}_n \times \mathcal{G}_n$ and a constant $C > 0$ such that for every 1-Lipschitz map $\varphi\colon (\mathcal{G}_n)_n \to \ell^2$ we have the following condition:*

$$\sum_{x,y\in\mathcal{G}_n} \|\varphi(x)-\varphi(y)\|^2 \mu_n(x,y) \leq C.$$

In particular, expanders in the usual sense (as above) are generalized expanders. It is proved in [Tes09] that a metric space does not embed coarsely into a Hilbert space if and only if it contains a coarsely embedded sequence of expanders. In [AT15], Arzhantseva and Tessera define the notions of expansion relative to subgroups, partitions, and measures, to differentiate between different cases of generalized expansion, and give examples of box spaces which do not embed into a Hilbert space and do not contain coarsely (and even weakly) embedded expanders. We now give a natural way to find generalized expanders, which coincides with the special case of expansion relative to subgroups.

Proposition 4.2.3. *Let r_n be a sequence such that $r_n \to \infty$ as $n \to \infty$. Let G_n be a sequence of finite k-generated groups with their corresponding Cayley graphs, and let H_n be a sequence of quotient groups of G_n with the induced metrics such that the kernel N_n of $G_n \to H_n$ is non-trivial, but $B_{G_n}(e, r_n) \cap N_n = \{e\}$.*

If there exists a constant $\varepsilon > 0$ such that for every eigenvector of the Laplacian Δ_n on G_n that is not the lift of an eigenvector of the Laplacian of H_n, the corresponding eigenvalue is bigger than ε. Then the Cayley graphs of G_n form a generalized expander.

Proof. Take $D = |G_n|(|N_n| - 1)$ and take μ such that $\mu(x, y)$ is equal to $\frac{1}{D}$ if $x^{-1}y$ lies in $N_n \setminus \{e\}$ and 0 otherwise. Take $C = \frac{2k}{\varepsilon}$.

Now for any n we can take M_{N_n} to be the averaging operator, i.e. $M_{N_n}(f)(x) = \frac{1}{|N_n|}\sum_{g\in N_n} f(gx)$. The space generated by the lifts of eigenvectors of the Laplacian of H_n is equal to the image of M_{N_n}, so the space generated by all other eigenvectors is the image of $\mathrm{Id} - M_{N_n}$. These eigenvectors correspond to eigenvalues bigger than ε, so $\Delta_n(1 - M_N) \geq \varepsilon(1 - M_N)$, where Δ_n denotes the Laplacian on G_n. Note that as N_n is a normal subgroup we have for every $s \in S$, every $f \in \ell(G_n)$ and every $x \in G_n$ that $(\lambda_s \circ M_{N_n})(f)(x) = M_{N_n}(f)(s^{-1}x) = \dfrac{1}{|N_n|}\sum_{g\in N_n} f(gs^{-1}x) = \dfrac{1}{|N_n|}\sum_{h\in N_n} f(s^{-1}hx) = (M_{N_n} \circ \lambda_s)(f)(x)$. So as $\Delta_n = k\,\mathrm{Id} - \sum_{s\in S}\lambda_s$ we have that M_{N_n} commutes with Δ_n.

As M_{N_n} is an orthogonal projection and the Laplacian Δ_n is a positive self-adjoint operator, we can conclude that $\Delta_n \geq (1 - M_N)\Delta_n = \Delta_n(1 - M_N) \geq \varepsilon(1 - M_N)$.

So we can make the following computation:

$$\begin{aligned}
\sum_{d(x,y)=1} |f(x)-f(y)|^2 &= \sum_{d(x,y)=1} f(x)(f(x)-f(y)) - f(y)(f(x)-f(y)) \\
&= \sum_{d(x,y)=1} 2f(x)(f(x)-f(y)) \\
&= \sum_{x\in G_n} 2f(x)(\Delta_n(f)(x)) \\
&= 2\langle f, \Delta_n(f)\rangle \\
&\geq 2\varepsilon\langle f, (\mathrm{Id} - M_{N_n})f\rangle \\
&= 2\varepsilon \sum_{x\in G_n} f(x)\left(f(x) - \frac{1}{|N_n|}\sum_{z\in N_n} f(xz)\right)
\end{aligned}$$

Now let $\varphi\colon \mathcal{G}_n \to \ell^2$ be a 1-Lipschitz map. We can decompose φ according to an orthonormal basis. Using this decomposition we find the following:

$$2\varepsilon \sum_{x\in G_n}\left\langle \varphi(x), \varphi(x) - \frac{1}{|N_n|}\sum_{z\in N_n}\varphi(xz)\right\rangle \leq \sum_{d(x,y)=1}\|\varphi(x)-\varphi(y)\|^2 \leq \sum_{d(x,y)=1} 1 \leq k|G_n|.$$

Now we can bound $\sum_{x,y\in G_n} \|\varphi(x)-\varphi(y)\|^2$ as follows:

$$\begin{aligned}
\sum_{x,y\in G_n} \|\varphi(x)-\varphi(y)\|^2\mu(x,y) &= 2\sum_{x,y\in G_n} \left(\|\varphi(x)\|^2 - \langle\varphi(x),\varphi(y)\rangle\right)\mu(x,y) \\
&= \frac{2}{D}\sum_{x\in G_n}\sum_{z\in N_n\setminus\{e\}} \left(\|\varphi(x)\|^2 - \langle\varphi(x),\varphi(xz)\rangle\right) \\
&= \frac{2}{D}\sum_{x\in G_n}\sum_{z\in N_n} \left(\|\varphi(x)\|^2 - \langle\varphi(x),\varphi(xz)\rangle\right) \\
&\leq \frac{2}{D}\sum_{x\in G_n} \left(|N_n|\|\varphi(x)\|^2 - \sum_{z\in N_n}\langle\varphi(x),\varphi(xz)\rangle\right) \\
&\leq \frac{2|N_n|}{D}\sum_{x\in G_n} \left(\|\varphi(x)\|^2 - \left\langle\varphi(x), \frac{1}{|N_n|}\sum_{z\in N_n}\varphi(xz)\right\rangle\right) \\
&\leq \frac{2|N_n|}{D}\cdot\frac{k|G_n|}{2\varepsilon} \\
&\leq \frac{2k}{\varepsilon} \\
&= C.
\end{aligned}$$

Note that the second-to-last inequality follows from the inequality $\frac{|N_n|\cdot|G_n|}{D} \leq \frac{|N_n|}{|N_n|-1} \leq 2$. Therefore we can conclude that G_n is a generalized expander. $\square$

4.2.2 Expanders and finite covers

Proposition 2 of [AT15] states that given a sequence of short exact sequences of finite groups $\{N_n \to G_n \to Q_n\}_n$ such that the quotient groups $\{Q_n\}$ and the subgroups $\{N_n\}$ coarsely embed into a Hilbert space (with respect to the induced metrics from $\{G_n\}$), the sequence $\{G_n\}$ cannot contain weakly embedded expanders.

We now show that the assumption on the subgroups is satisfied if the sequences $\{Q_n\}$ and $\{G_n\}$ both approximate the same group, i.e. if they are both box spaces of the same infinite group.

Proposition 4.2.4. *Let G be a finitely generated, residually finite group with a filtration $\{N_i\}$, and let $\{M_i\}$ be another sequence of finite index normal subgroups of G such that $N_i > M_i$ for all i. If $\square_{(N_i)}G$ coarsely embeds into a Hilbert space, then $\square_{(M_i)}G$ does not contain weakly (and thus coarsely) embedded expanders.*

Proof. Consider the sequence of short exact sequences

$$\{N_i/M_i \to G/M_i \to G/N_i\}_i,$$

where G/M_i and G/N_i are considered with the metric induced by the restriction of the respective box space metrics, and N_i/M_i is considered with the metric induced by viewing N_i/M_i as a subspace of G/M_i.

Since both $\square_{(M_i)}G$ and $\square_{(N_i)}G$ are box spaces of G and G/N_i is a quotient of G/M_i, for all R there is some $m(R)$ such that for all $i \geq m(R)$, the balls of radius R in G/M_i and G/N_i are isometric to balls of radius R in G; moreover, the quotient map $\pi_i : G/M_i \to G/N_i$ is an isometry when restricted to a ball of radius R. This means that the ball of radius R in G/M_i does not contain any non-trivial element of N_i/M_i, and so we see that the N_i/M_i are *sparse* with respect to the subspace metric, i.e. there exists a sequence $r_i \to \infty$ such that any two points of N_i/M_i are at distance at least r_i from each other.

We can deduce from this that the sequence (N_i/M_i) coarsely embeds into a Hilbert space: indeed, consider the embedding of each N_i/M_i into $\ell^2(N_i/M_i)$ defined by sending each element x of N_i/M_i to $r_i\chi_x$, that is, the characteristic function of x in $\ell^2(N_i/M_i)$ scaled by r_i.

Thus, since we also assume that the box space $\square_{(N_i)}G$ coarsely embeds into a Hilbert space, this sequence of short exact sequences satisfies the assumptions of Proposition 2 of [AT15], and so the box space $\square_{(M_i)}G$ does not contain a weakly embedded expander. $\square$

4.3 Subgroups of the free group

To construct box spaces of the free group which do not admit a coarse embedding into a Hilbert space without containing weakly embedded expanders, we will use two sequences of subgroups of the free group: one which

gives rise to a box space which is an expander, and one which does admit a coarse embedding into a Hilbert space. We then use information about these two sequences to prove that the box space obtained using certain intersections of these subgroups has the desired properties. In the following two subsections, we will describe the two sequences of subgroups.

4.3.1 Constructing subgroups of F_3

In this section we will define a sequence of nested finite index normal subgroups N_n of the free group F_3. We will rely heavily on the machinery described in [Lub10] to construct a sequence of Ramanujan graphs, and will frequently refer to relevant results and proofs in [Lub10].

We fix the prime $p = 5$, noting that $p \equiv 1 \bmod 4$, and an odd prime $q \neq 5$ such that -1 is a quadratic residue modulo q and 5 is a quadratic residue modulo $2q$. Such a prime exists, for example $q = 29$ (in fact, there exist infinitely many such primes, see for example the proof of Theorem 5.2.6).

Consider $\mathbb{H}(\mathbb{Z})$, the integer quaternions, with the equivalence relation $a \sim b$ if there exists $m, n \in \mathbb{N}$ such that $5^n a = \pm 5^m b$. Note that the equivalence relation $\sim$ is compatible with multiplication in $\mathbb{H}(\mathbb{Z})$. Recall that the norm N on $\mathbb{H}(\mathbb{Z})$ is defined by $N(\alpha) = \alpha\bar{\alpha}$, where $\bar{\alpha}$ is the quaternion conjugate to α. Abusing the notation, we will also write α for the equivalence class of α with respect to $\sim$. Note that for elements $\alpha \in \mathbb{H}(\mathbb{Z})/\sim$ with $N(\alpha) = 5^m$ for some $m \in \mathbb{Z}$, we have $\alpha^{-1} = \bar{\alpha}$.

Proposition 4.3.1. *The subgroup $\Lambda(2)$ of $\mathbb{H}(\mathbb{Z})/\sim$ generated multiplicatively by the set $S_5 := \{1 + 2i, 1 + 2j, 1 + 2k\}$ is the free group F_3 on the set S_5.*

Proof. This is precisely Corollary 2.1.11 of [Lub10]. □

An equivalent way to see this free group is as in section 7.4 of [Lub10]. Consider Γ to be the group $\mathbb{H}(\mathbb{Z}[\frac{1}{p}])^\times / Z(\mathbb{H}(\mathbb{Z}[\frac{1}{p}])^\times)$, where Z denotes the center. Following the notation of [Lub10][1], we can define a sequence of subgroups of Γ by $\Gamma(N) := \ker(\Gamma \to \mathbb{H}(\mathbb{Z}[\frac{1}{p}]/N\mathbb{Z}[\frac{1}{p}])^\times / Z(\mathbb{H}(\mathbb{Z}[\frac{1}{p}]/N\mathbb{Z}[\frac{1}{p}])^\times))$. The subgroup $\Gamma(2)$ is generated by the image of the set $S_5^{\pm} := \{1 \pm 2i, 1 \pm 2j, 1 \pm 2k\}$ and is exactly the free group $\Lambda(2)$ above.

The following theorem of [Lub10] tells us that we can construct quotients of the free group which are expanders. Recall that a *Ramanujan graph* is a k-regular graph such that all of its eigenvalues apart from 0 and possibly $2k$ lie in the interval $[k - 2\sqrt{k-1}, k + 2\sqrt{k-1}]$ (thus a family of Ramanujan graphs achieves the best possible spectral gap).

Theorem 4.3.2. *[Theorem 7.4.3, [Lub10]] Let $p \equiv 1 \bmod 4$ be a prime, and let $N = 2M$ be an integer such that $(M, 2p) = 1$. Assume that there is $\varepsilon \in \mathbb{Z}$ such that $\varepsilon^2 \equiv -1 \bmod M$. Consider the set $S_p^{\pm}$ of the $p+1$ solutions $x_0 + x_1 i + x_2 j + x_3 k$ of $x_0^2 + x_1^2 + x_2^2 + x_3^2 = p$ (where $x_0 > 0$ is odd, and x_1, x_2, x_3 are even). Associate to each element $x_0 + x_1 i + x_2 j + x_3 k$ of S_p the matrix $\begin{bmatrix} x_0 + x_1\varepsilon & x_2 + x_3\varepsilon \\ -x_2 + x_3\varepsilon & x_0 - x_1\varepsilon \end{bmatrix} \bmod M$ in $\mathrm{PGL}_2(M)$. Then the image of the group generated by $S_p^{\pm}$ under this map is the quotient $\Gamma(2)/\Gamma(N)$, which is isomorphic to $\mathrm{PSL}_2(M)$ if p is a quadratic residue modulo N, and the Cayley graph of $\Gamma(2)/\Gamma(N)$ with respect to the image of $S_p^{\pm}$ is a non-bipartite Ramanujan graph.*

We will apply this theorem to a particular sequence of our free group $\Gamma(2)$ (to which we will from now on refer to simply as F_3), namely the sequence of subgroups $N_n := \Gamma(2q^n)$.

We have chosen q such that -1 is a quadratic residue modulo q, and now we show that it is also a quadratic residue modulo q^n for any n.

Proposition 4.3.3. *Let q be an odd prime. For every $u \in \mathbb{Z}$ and every $n \in \mathbb{N}$, if u is a quadratic residue modulo q and u is not zero modulo q, then u is a quadratic residue modulo q^n.*

Proof. By induction we may assume that there exists a number b such that $b^2 \equiv u \bmod q^{n-1}$. So there exists a number c such that $b^2 = u + cq^{n-1}$. Now take $a = b - tcq^{n-1}$, where t is the inverse of $2b$ modulo q ($2b$ is invertible modulo q since u is not zero modulo q). Now $a^2 = b^2 - 2btcq^{n-1} + t^2c^2q^{2n-2} \equiv u + cq^{n-1} - cq^{n-1} = u \bmod q^n$. □

We note that this implies that there exists a q-adic integer ε such that $\varepsilon^2 = -1$. We can thus use this ε to define the map in Theorem 4.3.2 so that the maps are compatible for M equal to different powers of q.

Similarly, since we chose q so that 5 will be a quadratic residue modulo $2q$, it is also a quadratic residue modulo $2q^n$ for all n.

Proposition 4.3.4. *Let q be an odd prime. For every $u \in \mathbb{Z}$ and every $n \in \mathbb{N}$, if u is a quadratic residue modulo $2q$ and u is not zero modulo q, then u is a quadratic residue modulo $2q^n$.*

[1] Note that we use this notation in this subsection only to make it easier for the reader to refer to results in [Lub10]; the notation Γ is redefined and used differently in the next subsection.

Proof. By induction, we can assume that there is a number b such that $b^2 \equiv u \bmod 2q^{n-1}$. So there exists a number c such that $b^2 = u + 2cq^{n-1}$. Now take $a = b - tcq^{n-1}$, where t is the inverse of b modulo q (b is invertible modulo q since u is not zero modulo q). Now $a^2 = b^2 - 2btcq^{n-1} + t^2c^2q^{2n-2} \equiv u + 2cq^{n-1} - 2cq^{n-1} = u \bmod 2q^n$. □

Thus, the assumptions of Theorem 4.3.2 are satisfied, and we obtain the following.

Corollary 4.3.5. *For any $n \in \mathbb{N}$ we have that F_3/N_n is isomorphic to* $\mathrm{PSL}_2(q^n)$.

We will now investigate the properties of certain intermediate quotients N_k/N_n. We first need the following lemma.

Lemma 4.3.6. *Let $k, n \in \mathbb{N}$ with $0 < k \leq n \leq 2k$. Then the kernel $\ker(\mathrm{PSL}_2(q^n) \to \mathrm{PSL}_2(q^k))$ of reduction modulo q^k is isomorphic to $\mathbb{Z}^3_{q^{n-k}}$.*

Proof. We have that $\ker(\mathrm{PSL}_2(q^n) \to \mathrm{PSL}_2(q^k))$ is equal to $\{B \in \mathrm{PSL}_2(q^n) \mid B \equiv I_2 \bmod q^k\}$, where I_2 denotes the 2-by-2 identity matrix. This is precisely the set of matrices of the form $\begin{bmatrix} 1+aq^k & bq^k \\ cq^k & 1-aq^k \end{bmatrix}$ with a, b and c in $\mathbb{Z}_{q^{n-k}}$. For every two such matrices, we find

$$\begin{bmatrix} 1+aq^k & bq^k \\ cq^k & 1-aq^k \end{bmatrix} \begin{bmatrix} 1+a'q^k & b'q^k \\ c'q^k & 1-a'q^k \end{bmatrix} \equiv \begin{bmatrix} 1+aq^k+a'q^k & bq^k+b'q^k \\ cq^k+c'q^k & 1-aq^k-a'q^k \end{bmatrix} \bmod q^n.$$

Thus we have that $\ker(\mathrm{PSL}_2(q^n) \to \mathrm{PSL}_2(q^k))$ is isomorphic to $\mathbb{Z}^3_{q^{n-k}}$. □

Corollary 4.3.7. *Let $k, n \in \mathbb{N}$ with $0 < k \leq n \leq 2k$. Then N_k/N_n is isomorphic to $\mathbb{Z}^3_{q^{n-k}}$.*

Proof. The map in Theorem 4.3.2 which provides the isomorphism between $\Gamma(2)/\Gamma(2q^n)$ and $\mathrm{PSL}_2(q^n)$ commutes with the map of reduction modulo q^k, and thus we have that $N_k/N_n \cong \Gamma(2q^k)/\Gamma(2q^n) \cong \ker(\mathrm{PSL}_2(q^n) \to \mathrm{PSL}_2(q^k)) \cong \mathbb{Z}^3_{q^{n-k}}$. □

Remark 4.3.8. *A fact that will be of direct use to us later is that, since every element of N_{n-1}/N_n can be viewed as a matrix of the form $\begin{bmatrix} 1+cq^{n-1} & dq^{n-1} \\ fq^{n-1} & 1-cq^{n-1} \end{bmatrix}$ with c, d and f in $\mathbb{Z}_q$, a generating set of N_{n-1}/N_n can be given by the matrices*

$$\begin{bmatrix} 1+q^{n-1} & 0 \\ 0 & 1-q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix}.$$

We note that for any $k < n-1$, the matrix $\begin{bmatrix} 1+q^{n-1} & 0 \\ 0 & 1-q^{n-1} \end{bmatrix}$ is equivalent to the matrix

$$\begin{bmatrix} q^{2n-2}+q^{n-1}+1 & -q^{n+k} \\ q^{2n-k-3} & -q^{n-1}+1 \end{bmatrix},$$

which is the commutator of the matrices $\begin{bmatrix} 1 & q^{k+1} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ q^{n-k-2} & 1 \end{bmatrix}$.

4.3.2 Homology covers

Recall the construction of the *m-homology cover* given in section **??**. Given a finite graph X, one can construct a covering graph $\widetilde{X}$ of X such that $\widetilde{X}$ is the cover corresponding to the the quotient $\pi(X) \to \bigoplus^r \mathbb{Z}_m$ of highest rank r possible. Indeed, since $\pi(X)$ is a free group, the rank r is simply the rank of this free group.

The situation we are interested in is as follows: we have a sequence of quotients of the free group, $\{F_3/N_i\}$, metrized using the Cayley graph metric coming from the free generating set of F_3, and we consider the sequence of their q-homology covers, with q as in the previous subsection. By covering space theory (for details, see for example [Khu14]), the q-homology covers of the F_3/N_i are also quotients of F_3, by the subgroups

$$\Gamma_q(N_i) := N_i^q[N_i, N_i]$$

where N_i^q denotes the subgroup $\langle g^q : g \in N_i \rangle$ of F_3 generated by all the qth powers of elements of N_i. Since $\Gamma_q(N_i) < N_i$, we have that $\cap_i N_i = \{1\}$ implies that $\cap_i \Gamma_q(N_i) = \{1\}$ and so we can consider the box space $\square_{\Gamma_q(N_i)} F_3$.

The box space of a free group corresponding to a 2-homology cover was first considered by Arzhantseva, Guentner and Špakula in [AGŠ12], who proved that such a box space coarsely embeds into a Hilbert space, as one can construct a wall structure on it using the covering space structure.

In [Khu14], this was generalised as follows: given any $m \geq 2$ and any sequence $\{X_i\}$ of 2-connected finite graphs where the number of maximal spanning trees in X_i not containing a given edge does not depend on the edge, the sequence of $\mathbb{Z}_m$-homology covers of the X_i coarsely embeds into a Hilbert space (uniformly with respect to i) if $\operatorname{girth}(X_i) \to \infty$. Note that this holds even if the sequence $\{X_i\}$ does not embed coarsely into a Hilbert space.

In particular, we have that the box space $\square_{\Gamma_q(N_i)} F_3$ corresponding to the q-homology covers of any box space $\square_{N_i} F_3$ of the free group embeds coarsely into a Hilbert space, even if the box space $\square_{N_i} F_3$ is an expander sequence.

We now restrict ourselves to the following setting: the sequence $\{N_i\}$ is as defined in the previous subsection, and we consider the sequence of subgroups $\{\Gamma_q(N_i)\}$ corresponding to the q-homology covers. We have the following relation between the sequences, which we will need in the subsequent sections.

Proposition 4.3.9. *Let $k, n \in \mathbb{N}$ with $0 < k < n$. Then $N_n \Gamma_q(N_k) = N_{k+1}$.*

Proof. We will prove this proposition by induction on $n - k$. For $n = k + 1$ we clearly have that $N_{k+1} < N_{k+1}\Gamma_q(N_k)$. So it suffices to show that $\Gamma_q(N_k) = N_k^q[N_k, N_k] < N_{k+1}$. We will in fact show that $N_k^q < N_{k+1}$ and $[N_k, N_k] < N_{k+1}$.

To see that $N_k^q < N_{k+1}$, take an element $x \in N_k < F_3$. Up to the equivalence relation $\sim$, we can assume that x has the form $x = 1 + aq^k + bq^k i + cq^k j + dq^k k$. Then we can make the following computation:

$$\begin{aligned} x^q &= \left(1 + aq^k + bq^k i + cq^k j + dq^k k\right)^q \\ &= 1 + q^{k+1}(a + bi + cj + dk) + \frac{q(q-1)}{2} q^{2k}(a + bi + cj + dk)^2 + \dots \\ &\equiv 1 \qquad \qquad \operatorname{mod} q^{k+1} \end{aligned}$$

and so we have that $x^q \in N_{k+1}$ and thus $N_k^q < N_{k+1}$.

We also have that $[N_k, N_k] < N_{k+1}$, since the quotient N_k/N_{k+1} is abelian by Corollary 4.3.7. Therefore we have $\Gamma_q(N_k) \subset N_{k+1}$, and this proves the proposition for $n = k + 1$.

Now by induction we may assume that $N_{n-1}\Gamma_q(N_k) = N_{k+1}$. As $N_n < N_{n-1}$ we have that $N_n\Gamma_q(N_k) < N_{k+1}$.

We have that $N_{n-1}\Gamma_q(N_k) > N_n\Gamma_q(N_k)$. It suffices now to show that $N_{n-1}\Gamma_q(N_k) < N_n\Gamma_q(N_k)$, or equivalently that $N_{n-1}\Gamma_q(N_k)/N_n\Gamma_q(N_k)$ is trivial. Due to the second isomorphism theorem we have that

$$\frac{N_{n-1}\Gamma_q(N_k)}{N_n\Gamma_q(N_k)} = \frac{N_{n-1}N_n\Gamma_q(N_k)}{N_n\Gamma_q(N_k)} \cong \frac{N_{n-1}}{N_{n-1} \cap N_n\Gamma_q(N_k)}.$$

Now this is a quotient of N_{n-1}/N_n since $N_{n-1} \cap N_n\Gamma_q(N_k) > N_n$, and is therefore isomorphic to a quotient of $\mathbb{Z}_q^3$ as a consequence of Corollary 4.3.7. So it suffices to take a generating set of N_{n-1}/N_n and show that the elements of this generating set lie in $N_n\Gamma_q(N_k)$ modulo N_n. This will ensure that the quotient $N_{n-1}/N_{n-1} \cap N_n\Gamma_q(N_k)$ of N_{n-1}/N_n is trivial.

In fact it suffices to show that the generating elements lie in $\Gamma_q(N_{n-2})$ modulo the subgroup N_n, since $\Gamma_q(N_{n-2}) < N_n\Gamma_q(N_k)$.

Due to Corollary 4.3.5 we can view N_{n-1}/N_n as a subgroup of $\mathrm{PSL}_2(q^n)$. As in Remark 4.3.8, an example of a generating set of N_{n-1}/N_n is

$$\left\{ \begin{bmatrix} q^{n-1}+1 & 0 \\ 0 & -q^{n-1}+1 \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix} \right\}.$$

Now modulo q^n,

$$\begin{aligned} \begin{bmatrix} q^{n-1}+1 & 0 \\ 0 & -q^{n-1}+1 \end{bmatrix} &\equiv \begin{bmatrix} q^{n-2}+1 & 0 \\ 0 & -q^{n-2}+1 \end{bmatrix}^q \\ \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix} &\equiv \begin{bmatrix} 1 & q^{n-2} \\ 0 & 1 \end{bmatrix}^q \\ \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 0 \\ q^{n-2} & 1 \end{bmatrix}^q. \end{aligned}$$

Thus all elements of N_{n-1}/N_n lie in $N_n\Gamma_q(N_k)/N_n$ so $N_{n-1}/(N_{n-1} \cap N_n\Gamma_q(N_k))$ is trivial and therefore $N_n\Gamma_q(N_k) = N_{n-1}\Gamma_q(N_k) = N_{k+1}$. $\square$

4.3.3 Representation theory

The aim of this section is to study representations of the quotients $F_3/(N_n \cap \Gamma_q(N_k))$ for certain values of n and k.

All representations of $F_3/(N_{n-1} \cap \Gamma_q(N_k))$ can be lifted to representations of $F_3/(N_n \cap \Gamma_q(N_k))$. In this section we want to show that the dimensions of the representations of $F_3/(N_n \cap \Gamma_q(N_k))$ which are not such lifts grow like q^n for k fixed. [2]

For $k, n \in \mathbb{N}$ with $0 < 2k \leq n$ define $B_{k,n}$ as follows:

$$B_{k,n} := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in N_k/N_n \mid a \in \mathbb{Z}_{q^n}^\times, b \in \mathbb{Z}_{q^n} \right\}.$$

Another way of stating this condition on $a \in \mathbb{Z}_{q^n}^\times$ and $b \in \mathbb{Z}_{q^n}$ is that $a \equiv 1 \bmod q^k$ and $b \equiv 0 \bmod q^k$. Note that $B_{k,n}$ is a subgroup of N_k/N_n. In fact, for every such choice of a and b, we have that $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ is an element of N_k/N_n and we thus see that $B_{k,n}$ has order $(q^{n-k})^2 = q^{2n-2k}$.

Lemma 4.3.10. *Let $k, n, l \in \mathbb{N}$ with $0 < 2k \leq 2k + l \leq n$. Then every irreducible representation π of $B_{k,n}$ for which $\pi\left(\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1 \end{bmatrix}\right) = \mathrm{Id}$ and $\pi\left(\begin{bmatrix} 1 & q^{n-l-1} \\ 0 & 1 \end{bmatrix}\right) \neq \mathrm{Id}$ has dimension q^{n-2k-l}.*

Proof. If $n = 2k$, then $l = 0$ and due to Corollary 4.3.7 we know that $B_{k,n}$ is abelian. In this case all irreducible representations of $B_{k,n}$ have dimension 1, which satisfies this proposition.

For other values of k and n, we will now consider the irreducible representations of $B_{k,n}$.

Take $\omega = e^{\frac{2\pi i}{q^{n-k}}}$. As $k \geq 1$ we have that $1 + q^k$ is of order q^{n-k} in $\mathbb{Z}_{q^n}^\times$ and therefore generates $\{\alpha \equiv 1 \bmod q^k \mid \alpha \in \mathbb{Z}_{q^n}^\times\}$. Now for every $j \in \{0, 1, \ldots, q^k - 1\}$ define $\rho_j \colon B_{k,n} \to \mathbb{C}$ by

$$\begin{bmatrix} (1+q^k)^\beta & b \\ 0 & (1+q^k)^{-\beta} \end{bmatrix} \mapsto \omega^{\beta j}.$$

For every such j with $j \not\equiv 0 \bmod q$ set V_j to be the finite-dimensional Hilbert space with $\{\xi_x \mid x \equiv j \bmod q^k, x \in \mathbb{Z}_{q^{n-k}}\}$ as orthogonal basis, where ξ_x denotes the sequence indexed by elements of $\mathbb{Z}_{q^{n-k}}$ which takes the value 1 at $x \in \mathbb{Z}_{q^{n-k}}$ and 0 elsewhere. Let π_j be the representation of $B_{k,n}$ on V_j such that

$$\pi_j\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right)\xi_x = e^{\frac{2\pi i abx}{q^n}}\xi_{a^2x}.$$

Now we can calculate the characters of these representations:

$$\begin{aligned}
\chi_{\rho_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) &= \rho_j\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) \\
\chi_{\pi_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) &= \sum_{x \equiv j \bmod q^k} \left\langle \xi_x, \pi_j\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right)\xi_x \right\rangle \\
&= \sum_{x \equiv j \bmod q^k} \left\langle \xi_x, e^{\frac{2\pi i abx}{q^n}}\xi_{a^2x} \right\rangle \\
&= \sum_{x \equiv j \bmod q^k} e^{\frac{2\pi i abx}{q^n}} \langle \xi_x, \xi_{a^2x} \rangle
\end{aligned}$$

Note that if $a \equiv 1 \bmod q^k$ and $a^2 \equiv 1 \bmod q^{n-k}$, then $a \equiv 1 \bmod q^{n-k}$. Thus, if $a \not\equiv 1 \bmod q^{n-k}$, then for every $x \in \mathbb{Z}_{q^{n-k}}$ we have $\langle \xi_x, \xi_{a^2x} \rangle = 0$, so $\chi_{\pi_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) = 0$. If $b \not\equiv 0 \bmod q^{n-k}$, then $\sum_{x \equiv j \bmod q^k} e^{\frac{2\pi i abx}{q^n}} = 0$, so $\chi_{\pi_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\right) = 0$. If $a \equiv 1 \bmod q^{n-k}$ and $b \equiv 0 \bmod q^{n-k}$, then $a^2x \equiv x \bmod q^{n-k}$ and $\sum_{x \equiv j \bmod q^k} e^{\frac{2\pi i abx}{q^n}} = q^{n-2k}e^{\frac{2\pi i jb}{q^n}}$. Now for every $j, j' \in \{0, \ldots, q^k - 1\}$ with $j \not\equiv 0 \bmod q$ we can compute $\langle \chi_{\pi_j \otimes \rho_{j'}}, \chi_{\pi_j \otimes \rho_{j'}} \rangle$ using the

[2] Alain Valette pointed out to us that a proof of this can also be given using the Mackey machine.

fact that $|\chi_{\rho_j}(g)| = 1$ for every $g \in B_{k,n}$:

$$\begin{aligned}
\langle \chi_{\pi_j\otimes\rho_{j'}}, \chi_{\pi_j\otimes\rho_{j'}}\rangle &= \frac{1}{|B_{k,n}|}\sum_{a\equiv 1, b\equiv 0 \bmod q^k}\left|\chi_{\rho_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}\right)\chi_{\pi_j}\left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}\right)\right|^2 \\
&= \frac{1}{q^{2n-2k}}\sum_{a\equiv 1, b\equiv 0 \bmod q^{n-k}}\left|q^{n-2k}e^{\frac{2\pi i j b}{q^n}}\right|^2 \\
&= \frac{1}{q^{2n-2k}}q^{2k}q^{2n-4k} \\
&= 1.
\end{aligned}$$

Varying j and j', we find $q^{2k} - q^{2k-1}$ irreducible representations of dimension q^{n-2k}. Note that all of these representations are different.

For every irreducible representation π of $B_{k,n-1}$, we can lift this to an irreducible representation $\widetilde{\pi}$ of $B_{k,n}$. We can now consider the (also irreducible and pairwise distinct) representations $\widetilde{\pi}\otimes\rho_j$, for $j \in \{0, 1, \ldots, q-1\}$, π running through irreducible representations of $B_{k,n-1}$. For these representations we have that $\widetilde{\pi}\otimes\rho_j\left(\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1\end{bmatrix}\right) = \mathrm{Id}$, since the matrix $\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1\end{bmatrix}$ lies in N_{n-1} and thus is trivial in $B_{k,n-1}$.

Now we can check if we have found all irreducible representations of $B_{k,n}$:

$$\begin{aligned}
\sum_{\pi \text{ rep. of} B_{k,n}} |\chi_\pi(I_2)|^2 &= \sum_{j=0}^{q^k-1}\sum_{j'\not\equiv 0 \bmod q}\left|\chi_{\pi_j\otimes\rho_{j'}}(I_2)\right|^2 + \sum_{j=0}^{q-1}\sum_{\pi \text{ rep. of } B_{k,n-1}}\left|\chi_{\widetilde{\pi}\otimes\rho_j}(I_2)\right|^2 \\
&= \sum_{j=0}^{q^k-1}\sum_{j'\not\equiv 0 \bmod q} q^{2n-4k} + \sum_{j=0}^{q-1}\sum_{\pi \text{ rep. of } B_{k,n-1}}|\chi_{\widetilde{\pi}}(I_2)|^2 \\
&= (q^{2k} - q^{2k-1})q^{2n-4k} + \sum_{j=0}^{q-1} q^{2n-2k-2} \\
&= q^{2n-2k} - q^{2n-2k-1} + q^{2n-2k-1} \\
&= |B_{k,n}|.
\end{aligned}$$

Thus we have found all the irreducible representations of $B_{k,n}$.

By induction we may assume that the proposition is true for $B_{k,n-1}$. If $l = 0$, then all the irreducible representations of $B_{k,n}$ where the image of $\begin{bmatrix} 1 & q^{n-1} \\ 0 & 1\end{bmatrix}$ is not the identity have dimension q^{n-2k}, as they are necessarily those representations not arising as $\widetilde{\pi}\otimes\rho_j$ with π an irreducible representation of $B_{k,n-1}$, i.e. they are those representations of the form $\pi_j\otimes\rho_{j'}$ constructed above.

If $l > 0$, then all irreducible representations where the image of $\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1\end{bmatrix}$ is the identity, but the image of $\begin{bmatrix} 1 & q^{n-l-1} \\ 0 & 1\end{bmatrix}$ is not, are of the form $\widetilde{\pi}\otimes\rho_j$ where $\widetilde{\pi}$ is the lift of an irreducible representation π of $B_{k,n-1}$. This is because if we consider the other representations, which are of the form $\pi_j\otimes\rho_{j'}$, considering where the vector ξ_1 is mapped by $\pi_j\left(\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1\end{bmatrix}\right)$, we see that the image of $\begin{bmatrix} 1 & q^{n-l} \\ 0 & 1\end{bmatrix}$ cannot be equal to the identity.

Now due to the induction hypothesis we have that the dimension of π is $q^{(n-1)-2k-(l-1)} = q^{n-2k-l}$. Now the representation $\widetilde{\pi}\otimes\rho_j$ has the same dimension, which completes the proof of the theorem. □

Proposition 4.3.11. *Let $k, n \in \mathbb{N}$ be such that $3k \leq n-1$, then every representation of $F_3/(N_n \cap \Gamma_q(N_k))$ that is not the lift of a representation of $F_3/(N_{n-1}\cap\Gamma_q(N_k))$ has dimension at least q^{n-3k-3}.*

Proof. First note that $\Gamma_q(N_k)/(N_n\cap\Gamma_q(N_k))$ is isomorphic to N_{k+1}/N_n:

$$\begin{aligned}
\Gamma_q(N_k)/(N_n\cap\Gamma_q(N_k)) &\cong (N_n\Gamma_q(N_k))/N_n \\
&\cong N_{k+1}/N_n.
\end{aligned}$$

We have used the second isomorphism theorem and Proposition 4.3.9. Let us call this isomorphism Ψ,

$$\Psi : \Gamma_q(N_k)/(N_n\cap\Gamma_q(N_k)) \to N_{k+1}/N_n.$$

We can thus view N_{k+1}/N_n as a subgroup of $F_3/(N_n \cap \Gamma_q(N_k))$, via Ψ.

Let π be a representation of $F_3/(N_n \cap \Gamma_q(N_k))$ that is not the lift of a representation of $F_3/(N_{n-1} \cap \Gamma_q(N_k))$. This means that π is non-trivial on the kernel of the map

$$F_3/(N_n \cap \Gamma_q(N_k)) \to F_3/(N_{n-1} \cap \Gamma_q(N_k)).$$

This kernel is equal to $(N_{n-1} \cap \Gamma_q(N_k))/(N_n \cap \Gamma_q(N_k))$. Considering this kernel, we see that it is in fact isomorphic to N_{n-1}/N_n:

$$\begin{aligned}(N_{n-1} \cap \Gamma_q(N_k))/(N_n \cap \Gamma_q(N_k)) &\cong (N_{n-1} \cap \Gamma_q(N_k))/(N_n \cap (N_{n-1} \cap \Gamma_q(N_k))) \\ &\cong ((N_{n-1} \cap \Gamma_q(N_k))N_n)/N_n \\ &\cong (N_{n-1}N_n \cap \Gamma_q(N_k)N_n)/N_n \\ &\cong (N_{n-1} \cap N_{k+1})/N_n \\ &\cong N_{n-1}/N_n.\end{aligned}$$

Here, we have used the fact that the N_i are nested, the second isomorphism theorem, Proposition 4.3.9, and that n is sufficiently larger than k. Let us call this isomorphism Φ,

$$\Phi : (N_{n-1} \cap \Gamma_q(N_k))/(N_n \cap \Gamma_q(N_k)) \to N_{n-1}/N_n.$$

Now the isomorphisms Ψ and Φ are compatible, in the sense that Φ is just a restriction of Ψ. This means that when we restrict the representation π to N_{k+1}/N_n (viewed as a as a subgroup of $F_3/(N_n \cap \Gamma_q(N_k))$, via Ψ), this restriction is non-trivial on N_{n-1}/N_n as π is not a lift. This implies that at least one of the following elements of N_{n-1}/N_n has an image under π that is not the identity:

$$\begin{bmatrix} 1+q^{n-1} & 0 \\ 0 & 1-q^{n-1} \end{bmatrix}, \begin{bmatrix} 1 & q^{n-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q^{n-1} & 1 \end{bmatrix}.$$

This is because, as in Remark 4.3.8, these matrices generate N_{n-1}/N_n. The matrix $\begin{bmatrix} 1+q^{n-1} & 0 \\ 0 & 1-q^{n-1} \end{bmatrix}$ is equivalent to $\begin{bmatrix} q^{2n-2}+q^{n-1}+1 & -q^{n+k} \\ q^{2n-k-3} & -q^{n-1}+1 \end{bmatrix}$, which is the commutator of $\begin{bmatrix} 1 & q^{k+1} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ q^{n-k-2} & 1 \end{bmatrix}$, as we have seen in Remark 4.3.8, and so the images of both of these must be non-trivial, if the image of their commutator is non-trivial. The transpose-inverse map is an automorphism, and thus, we may assume without loss of generality that $\pi\left(\begin{bmatrix} 1 & q^{n-k-2} \\ 0 & 1 \end{bmatrix}\right) \neq \mathrm{Id}$ (since one of the other two generators have non-trivial images, this also implies that this matrix has a non-trivial image).

Let B be the subgroup corresponding to upper triangular matrices of N_{k+1}/N_n under the isomorphism Ψ between $\Gamma_q(N_k)/(N_n \cap \Gamma_q(N_k))$ and N_{k+1}/N_n. Due to Lemma 4.3.10 we know that $\pi|_B$ contains a representation of dimension at least q^{n-3k-3} (considering $B_{k+1,n}$ and $l = k+1$). Thus we can conclude that π has dimension at least q^{n-3k-3}. □

4.4 Box spaces of the free group

In this section we will prove that there exist box spaces of the free group F_3 that do not embed into a Hilbert space, but do not contain weakly embedded expanders either. To do so we will use the following diagram, made up of quotients of the free group F_3 by intersections of the subgroups N_i with the subgroups $\Gamma_q(N_k)$ coming from the q-homology covers of the F_3/N_k (see Figure 1).

Note that the quotients F_3/N_i appearing along the bottom row are expanders by Corollary 4.3.5 and the result of Lubotzky (Theorem 4.3.2).

Set $f_{n,k}(m) = \#\{g \in N_n \cap \Gamma_q(N_k) : |g| \leq m\}$ and set $A_{n,k} = [F_3 : N_n \cap \Gamma_q(N_k)]$.

Lemma 4.4.1. *If $a^2 \equiv b^2 \bmod q^n$ and $q \nmid b$, then $a \equiv \pm b \bmod q^n$.*

Proof. We will prove this lemma by induction. For $n = 2$ the lemma follows from Exercise 1 in section 4.3 of [DSV03]. For bigger n, we have that $a^2 \equiv b^2 \bmod q^n$ implies $a^2 \equiv b^2 \bmod q^{n-1}$, so by induction we have that $a \equiv \pm b \bmod q^{n-1}$. Therefore there exists a $c \in \mathbb{Z}_q$ such that $a \equiv cq^{n-1} \pm b$ modulo q^n.

Now it suffices to show that $c \equiv 0 \bmod q$. We have that $b^2 \equiv a^2 \equiv b^2 \pm 2cbq^{n-1} \bmod q^n$, so $q \mid 2cb$. As q is prime, either $q \mid c$ or $q \mid 2b$. As $q \nmid b$, we have that $q \mid c$ and therefore $a \equiv \pm b \bmod q^n$. □

Lemma 4.4.2. *For any $k, n, m \in \mathbb{N}$ with m even, we have $f_{n,k}(m) = \mathcal{O}\left(\frac{5^{\frac{13}{12}m}}{q^{3n}} + \frac{5^{\frac{7}{12}m}}{q^n}\right)$.*

$$\begin{tikzcd}
 & & & & F_3/\Gamma_q(N_4)\cdots \arrow[d] \\
 & & & F_3/\Gamma_q(N_3) \arrow[d] & F_3/(N_4\cap\Gamma_q(N_3))\cdots \arrow[l] \arrow[d] \\
 & & F_3/\Gamma_q(N_2) \arrow[d] & F_3/(N_3\cap\Gamma_q(N_2)) \arrow[l] \arrow[d] & F_3/(N_4\cap\Gamma_q(N_2))\cdots \arrow[l] \arrow[d] \\
 & F_3/\Gamma_q(N_1) \arrow[d] & F_3/(N_2\cap\Gamma_q(N_1)) \arrow[l] \arrow[d] & F_3/(N_3\cap\Gamma_q(N_1)) \arrow[l] \arrow[d] & F_3/(N_4\cap\Gamma_q(N_1))\cdots \arrow[l] \arrow[d] \\
\{1\} & F_3/N_1 \arrow[l] & F_3/N_2 \arrow[l] & F_3/N_3 \arrow[l] & F_3/N_4\cdots \arrow[l]
\end{tikzcd}$$

Figure 4.1: The magic triangle

Proof. Clearly it suffices to prove the theorem for $k=0$. We have that

$$f_{n,0}(m)=\#\left\{\alpha\in\mathbb{H}(\mathbb{Z})\mid[\alpha]\in N_n, N(\alpha)=5^m\right\}=\#\left\{a+q^n(bi+cj+dk)\mid a^2+q^{2n}(b^2+c^2+d^2)=5^m\right\}.$$

Now $a^2\equiv 5^m \bmod q^{2n}$, so due to Lemma 4.4.1 we have $a\equiv\pm 5^{\frac{m}{2}} \bmod q^{2n}$. This leaves at most $\frac{4\cdot 5^{\frac{m}{2}}}{q^{2n}}+2$ possibilities for a.

Now due to [DSV03] we know that for any fixed $\varepsilon>0$ we have that $\#\{(a,b,c)\mid a^2+b^2+c^2=x\}=\mathcal{O}\left(x^{\frac{1}{2}+\varepsilon}\right)$. So we find a bound for $f_{n,0}(m)$:

$$\begin{aligned}
f_{n,0}(m) &\leq \sum_a \mathcal{O}\left(\left(\frac{5^m-a^2}{q^{2n}}\right)^{\frac{1}{2}+\varepsilon}\right)\\
&\leq \left(\frac{4\cdot 5^{\frac{m}{2}}}{q^{2n}}+2\right)\mathcal{O}\left(\left(\frac{5^m}{q^{2n}}\right)^{\frac{1}{2}+\varepsilon}\right)\\
&\leq \mathcal{O}\left(\left(\frac{4\cdot 5^{\frac{m}{2}}}{q^{2n}}+2\right)\left(\frac{5^{m(\frac{1}{2}+\varepsilon)}}{q^n}\right)\right)\\
&= \mathcal{O}\left(\frac{5^{m(1+\varepsilon)}}{q^{3n}}+\frac{5^{m(\frac{1}{2}+\varepsilon)}}{q^n}\right).
\end{aligned}$$

Now for $\varepsilon=\frac{1}{12}$ we find $f_{n,k}(m)=\mathcal{O}\left(\frac{5^{\frac{13}{12}m}}{q^{3n}}+\frac{5^{\frac{7}{12}m}}{q^n}\right)$.

□

Theorem 4.4.3. *There exists $N>0$ such that for every $k,n\in\mathbb{N}$ with $n\geq N$, $18<18(k+1)\leq n$ and $A_{n,k}\leq q^{\frac{19}{6}n}$, we have that every eigenvalue λ of the adjacency operator A of $F_3/(N_n\cap\Gamma_q(N_k))$ such that some corresponding eigenvector is not the lift of an eigenvector of the adjacency operator of $F_3/(N_{n-1}\cap\Gamma_q(N_k))$ satisfies $\lambda\leq 5^{\frac{71}{72}}+5^{\frac{1}{72}}<6$.*

Proof. Without loss of generality we may assume that $\lambda\geq 2\sqrt{5}$. Take θ_j such that $\mu_j=2\sqrt{5}\cos(\theta_j)$, where μ_j are the eigenvalues of the adjacency operator A. Due to the results of section 4.4 of [DSV03] we have

$$f_{n,k}(m)\geq\frac{1}{A_{n,k}}5^{\frac{m}{2}}\sum_{j=0}^{A_{n,k}-1}\frac{\sin(m+1)\theta_j}{\sin\theta_j}.$$

Take $\psi_j=i\theta_j$. If $|\mu_j|\leq 2\sqrt{5}$, then θ_j is real and $\left|\frac{\sin(m+1)\theta_j}{\sin\theta_j}\right|\leq(m+1)$, and if $|\mu_j|\geq 2\sqrt{5}$, then ψ_j is real and $\frac{\sin(m+1)\theta_j}{\sin\theta_j}=\frac{\sinh(m+1)\psi_j}{\sinh\psi_j}\geq 0$. So we find the following inequality for any l, and in particular for $\mu_l=\lambda$:

$$\frac{A_{n,k}}{5^{\frac{m}{2}}}f_{n,k}(m)\geq\sum_{j=0}^{A_{n,k}-1}\frac{\sin(m+1)\theta_j}{\sin\theta_j}\geq M(\lambda)\frac{\sinh(m+1)\psi_l}{\sinh\psi_l}-(m+1)A_{n,k},$$

where $M(\lambda)$ denotes the multiplicity of the eigenvalue λ. When we take m to be the biggest even integer such that $5^{\frac{m}{2}} \leq q^{3n}$, we can use Lemma 4.4.2 and the fact that we chose $A_{n,k} \leq q^{\frac{19}{6}n}$ to obtain the following:

$$\begin{aligned}
(m+1)A_{n,k} + \frac{A_{n,k}}{5^{\frac{m}{2}}} f_{n,k}(m) &\leq q^{\frac{19}{6}n}\left(m+1+\mathcal{O}\left(\frac{5^{\frac{7}{12}m}}{q^{3n}} + \frac{5^{\frac{1}{12}m}}{q^n}\right)\right) \\
&\leq q^{\frac{19}{6}n}\left(6n\log_5(q)+1+\mathcal{O}\left(\frac{q^{\frac{7}{2}n}}{q^{3n}} + \frac{q^{\frac{1}{2}n}}{q^n}\right)\right) \\
&\leq q^{\frac{19}{6}n}\left(6n\log_5(q)+1+\mathcal{O}\left(q^{\frac{1}{2}n} + q^{\frac{-1}{2}n}\right)\right) \\
&= \mathcal{O}\left(q^{\frac{22}{6}n}\right).
\end{aligned}$$

Let V_λ be the eigenspace of A corresponding to λ on $F_3/N_n \cap \Gamma_q(N_k)$; since some eigenvector is not a lift from $F_3/N_{n-1} \cap \Gamma_q(N_k)$, the representation of $F_3/N_n \cap \Gamma_q(N_k)$ on V_λ is not a lift from a representation of $F_3/N_{n-1} \cap \Gamma_q(N_k)$. Since the eigenspace V_λ is a representation space of the group $F_3/N_n \cap \Gamma_q(N_k)$ (see for example [DSV03]), we thus have $M(\lambda) \geq q^{n-3k-3}$ due to Proposition 4.3.11.

We also have

$$\frac{\sinh(m+1)\psi_l}{\sinh\psi_l} \geq \frac{e^{(m+1)|\psi_l|}}{e^{|\psi_l|}} > e^{(6n\log_5(q)-2)|\psi_l|} = \frac{q^{\frac{6n}{\log(5)}|\psi_l|}}{e^{-2|\psi_l|}}.$$

We assumed $\lambda \geq 2\sqrt{5}$, so $\psi_l \geq 0$. As $e^{2\psi_l}$ is bounded by $e^{\sqrt{5}}$ we have the following:

$$q^{n-3k-3+\frac{6n}{\log(5)}\psi_l} \leq e^{\sqrt{5}} M(\lambda)\frac{\sinh(m+1)\psi_l}{\sinh\psi_l} = \mathcal{O}\left(q^{\frac{22}{6}n}\right)$$

So for big n we find $n-3k-3+\frac{6n}{\log(5)}\psi_l \leq \frac{45}{12}n$. As $18(k+1) \leq n$ we see that $n - \frac{n}{6} + \frac{6n}{\log(5)}\psi_l \leq \frac{45}{12}n$. So $\frac{6}{\log(5)}\psi_l \leq \frac{35}{12}$ and therefore $\psi_l \leq \frac{35}{72}\log(5)$. Now we can compute λ as follows:

$$\lambda = 2\sqrt{5}\cos(\theta_l) = 2\sqrt{5}\cosh(\psi_l) \leq \sqrt{5}\left(5^{\frac{35}{72}} + 5^{\frac{-35}{72}}\right) = 5^{\frac{71}{72}} + 5^{\frac{1}{72}} < 6.$$

This proves the theorem. □

Corollary 4.4.4. *Let k_i and n_i be non-decreasing sequences in $\mathbb{N}$ with n_i increasing, $18 < 18(k_i+1) \leq n_i$ and $A_{n_i,k_i} \leq q^{\frac{19}{6}n_i}$. Then $\Box_{N_{n_i}\cap\Gamma_q(N_{k_i})}F_3$ does not embed into a Hilbert space.*

Proof. We want to apply Proposition 4.2.3, so we need to check that all the hypotheses hold.

Due to Theorem 4.4.3 we know there exists an $N > 0$ such that for all $n_i \geq N$, for eigenvalues λ of the adjacency operator A of $F_3/(N_{n_i} \cap \Gamma_q(N_{k_i}))$ such that the corresponding eigenvector is not the lift of an eigenvector of the adjacency operator of $F_3/(N_{n_i-1} \cap \Gamma_q(N_{k_i}))$, we have that $\lambda \leq 5^{\frac{35}{36}} + 5^{\frac{1}{36}}$.

Since the Laplacian Δ is in this case equal to $6\,\mathrm{Id} - A$ we have that every non-trivial eigenvalue of the Laplacian is greater than $6 - 5^{\frac{35}{36}} - 5^{\frac{1}{36}}$. The quotients $F_3/(N_{n_i-1} \cap \Gamma_q(N_{k_i}))$ and $F_3/(N_{n_i} \cap \Gamma_q(N_{k_i}))$ look like F_3 (and thus like each other) on bigger and bigger balls, so there exists a sequence r_i such that $r_i \to \infty$ as $i \to \infty$ with

$$B(e, r_i) \cap \Big(N_{n_i-1} \cap \Gamma_q(N_{k_i}))/(N_{n_i} \cap \Gamma_q(N_{k_i})\Big) = \{e\},$$

where $B(e, r_i)$ denotes the ball of radius r_i about the identity in $F_3/(N_{n_i} \cap \Gamma_q(N_{k_i}))$. But on the other hand, due to the isomorphism Φ given as part of the proof of Proposition 4.3.11, and Corollary 4.3.7, we have $N_{n_i-1} \cap \Gamma_q(N_{k_i}) \neq N_{n_i} \cap \Gamma_q(N_{k_i})$, since $(N_{n_i-1} \cap \Gamma_q(N_{k_i}))/(N_{n_i} \cap \Gamma_q(N_{k_i})) \cong N_{n_i-1}/N_{n_i} \cong \mathbb{Z}_q^3$.
Now Proposition 4.2.3 can be applied to the subsequence of $F_3/(N_{n_i} \cap \Gamma_q(N_{k_i}))$ with $n_i \geq N$. So $\Box_{N_{n_i}\cap\Gamma_q(N_{k_i})}F_3$ contains a generalized expander and therefore does not embed into a Hilbert space, by the characterization of Tessera [Tes09]. □

This chapters Main Theorem now follows from the following result.

Theorem 4.4.5. *There exist increasing sequences k_i and n_i in $\mathbb{N}$ such that $18 < 18(k_i+1) \leq n_i$ and $A_{n_i,k_i} \leq q^{\frac{19}{6}n_i}$, and for such n_i, k_i, the box space $\Box_{N_{n_i}\cap\Gamma_q(N_{k_i})}F_3$ does not embed into a Hilbert space, but does not contain weakly embedded expanders.*

Proof. Let us first check that such a sequence (n_i, k_i) exists. We have, using the information we have obtained in sections 4.3.1 and 4.3.2 about the sizes of quotients in Figure 1,

$$\begin{aligned}
A_{n_i,k_i} &= |F_3/(N_{n_i} \cap \Gamma_q(N_{k_i}))| \\
&\leq |F_3/N_{n_i}| \cdot |F_3/\Gamma_q(N_{k_i})| \\
&\leq |F_3/N_{n_i}| \cdot |F_3/N_{k_i}| \cdot |N_{k_i}/\Gamma_q(N_{k_i})| \\
&\leq |\operatorname{PSL}_2(q^{n_i})| \cdot |\operatorname{PSL}_2(q^{k_i})| \cdot |\mathbb{Z}_q^{2|\operatorname{PSL}_2(q^{k_i})|+1}| \\
&\leq q^4 \cdot q^{3(n_i-1)} \cdot q^4 \cdot q^{3(k_i-1)} \cdot q^{2(q^4 q^{3(k_i-1)})+1} \\
&= q^{3n_i+3k_i+3+2q^{3k_i+1}}.
\end{aligned}$$

This means that we need $3k_i + 3 + 2q^{3k_i+1}$ to be less than or equal to $\frac{1}{6}n_i$ in order to satisfy $A_{n_i,k_i} \leq q^{\frac{19}{6}n_i}$. Now it is clear that for a sequence of large enough n_i we can take a sequence of k_i which will simultaneously satisfy this and the condition $18 < 18(k_i + 1) \leq n_i$. By taking subsequences if necessary, we can ensure that the sequences n_i and k_i are increasing.

Corollary 4.4.4 gives us the first part of the statement. For the second part of the statement, we can now apply Proposition 4.2.4 to the box space $\square_{N_{n_i} \cap \Gamma_q(N_{k_i})} F_3$ and the box space $\square_{\Gamma_q(N_{k_i})} F_3$, which is embeddable into Hilbert space by the main result in [Khu14] (described in Section 4.3.2) as it is a sequence of q-homology covers of the graphs F_3/N_{k_i} which satisfy the necessary conditions. $\square$

Chapter 5

Coarse fundamental groups and box spaces

Given a free group $G = F_S$ on a set S, and the Cayley graph of a quotient G/N of this free group with respect to the image of S, one can recover the subgroup N by computing the fundamental group of this Cayley graph, which will be isomorphic to the free group N. If the group G is not free, one cannot use the fundamental group in this way.

This chapter is based on [DK18], joint work with Khukhro. Here we use a coarse version of the fundamental group, first defined in [BCW14], to recover normal subgroups used to construct quotients in the context of box spaces of finitely presented groups. Specifically we prove Theorem 2.2.1 which states that if two box spaces of finitely presented groups are coarsely equivalent, then there exists a almost bijection between the normal subgroups in the filtrations such that the corresponding subgroups are isomorphic.

5.1 Coarse homotopy

In this section we give some properties of paths with respect to r-homotopy, and then use these to prove our main results.

5.1.1 r-homotopic paths

We will now prove some elementary propositions about 1-paths, and then show that for a given r-path, there is a 1-path which is r-homotopic to it. We do this because it will be useful to consider 1-paths in the proof of the main theorem. We remark that a 1-path is also an r-path for $r \geq 1$.

The first proposition shows that we can remove any backtracking (including staying at the same vertex) of a 1-path. Note that given a 1-path in the Cayley graph, there is a corresponding word in $S \cup \{e\}$, which is the word read along the labelled edges of the path (with an occurrence of e denoting staying at a vertex).

Proposition 5.1.1. *Let $G = \langle S \rangle$ be a group and let $r \geq 1$. Let p be a 1-path in $\mathrm{Cay}(G)$, let w be the word in elements of the set $S \cup \{e\}$ corresponding to p and let q be the path corresponding to the reduced version of w. Then q is r-homotopic to p.*

Proof. Suppose that w is a word that is not in its reduced form, then it can be written as $w = u_1 s s^{-1} u_2$ or $w = u_1 e u_2$, where u_1 and u_2 are words in the elements of $S \cup \{e\}$ and $s \in S$. By induction it suffices to show that the 1-path corresponding to $u_1 u_2$ is r-homotopic to p, because we can reduce w by removing these ss^{-1} and e until w is in its reduced form.

In the case that $w = u_1 e u_2$ we see that p is r-homotopic to the 1-path corresponding to the word $u_1 u_2 e$, call this path $\tilde{q}$. Then removing $\tilde{q}(\ell(\tilde{q}))$, we recover the 1-path corresponding to $u_1 u_2$, and thus we see that the 1-path corresponding to w is r-homotopic to the 1-path corresponding to $u_1 u_2$.

In the case that $w = u_1 s s^{-1} u_2$, we see that p is r-homotopic to the 1-path corresponding to the word $u_1 e^2 u_2$, call this path q' (in fact p and q' are r-close, because $\ell(p) = \ell(u_1) + 2 + \ell(u_2) = \ell(q')$ and $d(p(i), q'(i)) \leq r$). Then, by inductively using the other case, we show that p is r-homotopic to the 1-path corresponding to $u_1 u_2$. □

The second proposition shows that r-homotopies can pass holes of "size" $2r$.

Proposition 5.1.2. *Let G be a group and let $r \geq 1$. Let u, v and w be three words in the elements S such that uw and v correspond to e in G and $\ell(v) \leq 2r$. Then the 1-paths corresponding to uw and uvw are r-homotopic.*

In fact it is possible to show that the proposition still holds for holes of size $4r$ (i.e. $\ell(v) \leq 4r$), but we restrict ourselves to $\ell(v) \leq 2r$ to avoid unnecessarily complicating the proof.

Proof. As $\ell(v) \leq 2r$ the 1-path corresponding to uvw is r-homotopic to the one corresponding to $ue^{\ell(v)}w$ and due to Proposition 5.1.1 they are r-homotopic to the 1-path corresponding to uw. □

Finally, we prove a proposition which will allow us to work with 1-paths in the next section.

Proposition 5.1.3. *Let $\mathcal{G}$ be a graph, let $r \geq 1$ and p be an r-path in $\mathcal{G}$, then there exists a 1-path q in $\mathcal{G}$ which is r-homotopic to p.*

Proof. If $\ell(p) = 0$, then p is already a 1-path.

If $\ell(p) \geq 1$, then by induction we can assume that $d(p(i), p(i+1)) \leq 1$ for $i \geq 1$. As $\mathcal{G}$ is a graph we can take a geodesic 1-path g between $p(0)$ and $p(1)$. Then take

$$q\colon \{0, \ldots, \ell(p) + \ell(g)\} \to \mathcal{G}\colon \begin{cases} i \mapsto g(i) & \text{if } 0 \leq i \leq \ell(g) \\ i \mapsto p(i - \ell(g)) & \text{if } \ell(g) + 1 \leq i \leq \ell(p) + \ell(g). \end{cases}$$

Now q is a 1-path and since $\ell(g) \leq r$, it is r-close to

$$\tilde{q}\colon \{0, \ldots, \ell(p) + \ell(g)\} \to \mathcal{G}\colon \begin{cases} i \mapsto p(i) & \text{if } 0 \leq i \leq \ell(p) \\ i \mapsto p(\ell(p)) & \text{if } \ell(p) + 1 \leq i \leq \ell(p) + \ell(g), \end{cases}$$

which is r-close to p. Thus q is r-homotopic to p. □

5.1.2 Box spaces of finitely presented groups

In this section we will prove the main result, that box spaces of finitely presented groups eventually detect the normal subgroups used to construct them, even when we look up to coarse equivalence, i.e. given the coarse equivalence class of a box space, one can deduce the sequence of normal subgroups (from some index onwards).

In order to show this, we first prove that coarse fundamental groups of a Cayley graph of a quotient can detect the normal subgroup used to construct the quotient.

Given a presentation $G = \langle S|R \rangle$, and a normal subgroup $N \lhd G$, we denote by $|-|_{F_S}$ the length of relators in R viewed as a subset of the free group F_S on the set S with its natural metric, and by $|-|_G$ the length of elements of G in the Cayley graph $\mathrm{Cay}(G, S)$.

Lemma 5.1.4. *Let G be a finitely presented group with $G = \langle S|R \rangle$, let k be equal to $\max\{|g|_{F_S} : g \in R\}$, let $N \lhd G$ with $2k < n = \inf\{|g|_G : g \in N \setminus \{e\}\}$, and let r be a constant such that $2k \leq 4r < n$. Then $A_{1,r}(\mathrm{Cay}(G/N, \bar{S}), e)$ is isomorphic to N.*

Proof. Define $\Phi\colon N \to A_{1,r}(\mathrm{Cay}(G/N), e)$ as follows: For $g \in N$ write g as a word in the elements of S. This corresponds to a 1-loop in $\mathrm{Cay}(G/N)$ based at e. We take $\Phi(g)$ equal to this loop (note that it is in particular an r-loop).

To show that Φ is uniquely defined, suppose that $g \in N$ can be written in two ways as a word in the elements of S, say v and w. Due to Proposition 5.1.1 we can assume these words are reduced as elements of the free group F_S on S. As v and w realize the same element in G we can write $w^{-1}v = a$ where a is in the normal subgroup of F_S generated by the elements of R, so we can write $a = a_1 a_2 \ldots a_m$ with each a_i the conjugate of an element in R.

As being r-homotopic is an equivalence relation, it suffices to show that the paths corresponding to the words w and $whbh^{-1}$ are r-homotopic for every $b \in R$, every $h \in F_S$ and every word in w in F_S representing a loop in G/N.

Since $\ell(b) \leq k \leq 2r$, we have that $whbh^{-1}$ and whh^{-1} correspond to r-homotopic loops due to Proposition 5.1.2, and then by Proposition 5.1.1 we have that these loops are r-homotopic to the one corresponding to w.

Thus Φ is well-defined. We now need to show that Φ is an isomorphism.

It is clearly a homomorphism, because of the correspondence between words in F_S and paths in $\mathrm{Cay}(G/N)$.

To show that it is injective, it suffices to show that $\Phi(g)$ is not null-r-homotopic if $g \in N \setminus \{e\}$. Therefore we suppose there exists a element g that does correspond to a null-r-homotopic loop. This means there exists a sequence $p_0, p_1, \ldots, p_n = \Phi(g)$, where p_0 is the trivial loop and p_i is r-close to p_{i+1} for every i.

Now as $2r < n$, there is a unique way of lifting an r-path in $\mathrm{Cay}(G/N)$ to an r-path in $\mathrm{Cay}(G)$, and so we can take $\tilde{p}_i$ to be the lift of p_i for every i. We know that $\tilde{p_0}(\ell(p_0)) = e$ and $\tilde{p}_n(\ell(p_n)) = g$. Now take i to be the biggest value such that $\tilde{p}_i(\ell(p_i)) = e$. Then $\tilde{p}_{i+1}(\ell(p_{i+1})) \neq e$ (since $\tilde{p}_n(\ell(p_n)) = g$ and $g \neq e$). However $\tilde{p}_{i+1}(\ell(p_{i+1})) \in N$, since p_{i+1} is an r-loop in $\mathrm{Cay}(G/N)$, so $d(\tilde{p}_i(\ell(p_i)), \tilde{p}_{i+1}(\ell(p_{i+1}))) \geq n > r$.

We know that p_i is r-close to p_{i+1} and there are two ways, (a) and (b), in which this can happen, according to the definition. If we were in the case of (a), then p_i would be equal to p_{i+1} on the interval where both of them are defined and constantly e on the rest, and so then we would have that $e = \tilde{p}_i(\ell(p_i)) = \tilde{p}_{i+1}(\ell(p_{i+1}))$. But we had assumed that i is the biggest value for which $\tilde{p}_i(\ell(p_i)) = e$, so the paths must be r-close as in (b), i.e. p_i and p_{i+1} are of the same length and $d(p_i(x), p_{i+1}(x)) \leq r$ for every $0 \leq x \leq \ell(p_i)$.

Now $d(\tilde{p}_i(\ell(p_i)), \tilde{p}_{i+1}(\ell(p_i))) \geq n$ while $d(\tilde{p}_i(0), \tilde{p}_{i+1}(0)) = 0$. Therefore there exists a j between 0 and $\ell(p_i)$ such that $d(\tilde{p}_i(j), \tilde{p}_{i+1}(j)) \leq r$ and $d(\tilde{p}_i(j+1), \tilde{p}_{i+1}(j+1)) > r$. However, since $d(p_i(j+1), p_{i+1}(j+1)) \leq r$, there exists a $h \in N \setminus \{e\}$ such that $d(\tilde{p}_i(j+1), \tilde{p}_{i+1}(j+1)h) \leq r$.

We also note that $n \leq d(\tilde{p}_{i+1}(j+1), \tilde{p}_{i+1}(j+1)h)$, since $d(\tilde{p}_{i+1}(j+1), \tilde{p}_{i+1}(j+1)h) = |h|_G$, and h is a non-trivial element of N, so that $|h|_G > n$.

Now we can make the following computation:

$$\begin{aligned}
n &\leq d(\tilde{p}_{i+1}(j+1), \tilde{p}_{i+1}(j+1)h) \\
&\leq d(\tilde{p}_{i+1}(j+1), \tilde{p}_{i+1}(j)) + d(\tilde{p}_{i+1}(j), \tilde{p}_i(j)) \\
&\quad + d(\tilde{p}_i(j), \tilde{p}_i(j+1)) + d(\tilde{p}_i(j+1), \tilde{p}_{i+1}(j+1)h) \\
&\leq r + r + r + r \\
&\leq 4r < n.
\end{aligned}$$

As this is impossible, we find that $\Phi(g)$ can not be null-r-homotopic and therefore Φ is injective.

To show that Φ is surjective, take an r-loop in $\mathrm{Cay}(G/N)$. Due to Proposition 5.1.3 this loop is r-homotopic to a 1-loop p. This loop corresponds to a word w in $F(S)$ and this word gets mapped to an element g in G via the map $F_S \to G$. As p is a loop, $g \in N$. Now by definition $\Phi(g) = p$.

So we can conclude that Φ is an isomorphism. $\square$

We need the following lemma which tells us how $A_{1,r}$ behaves under quasi-isometries of the underlying graph. Given two graphs $\mathcal{G}$ and $\mathcal{H}$, recall that $\phi : \mathcal{G} \to \mathcal{H}$ is a quasi-isometry with constant $C > 0$ if for all $x, y \in \mathcal{G}$,

$$d_{\mathcal{H}}(\phi(x), \phi(y)) \leq C d_{\mathcal{G}}(x, y) + C$$

and there exists a quasi-inverse map $\phi' : \mathcal{H} \to \mathcal{G}$ which satisfies $d_{\mathcal{G}}(\phi'(x), \phi'(y)) \leq C d_{\mathcal{H}}(x, y) + C$ for all $x, y \in \mathcal{H}$, $d_{\mathcal{G}}(x, \phi'(\phi(x))) \leq C$ for all $x \in \mathcal{G}$ and $d_{\mathcal{H}}(y, \phi(\phi'(y))) \leq C$ for all $y \in \mathcal{H}$.

Lemma 5.1.5. *Let $C > 0$ be a constant, let $\mathcal{G} \simeq_{QI} \mathcal{H}$ be two Cayley graphs that are quasi-isometric with constant C and let $r \geq 2C$. Then there exists a homomorphism $\Psi \colon A_{1,r}(\mathcal{G}) \to A_{1,Cr+C}(\mathcal{H})$ that is surjective.*

It will be very useful to assume that both the quasi-isometry and its quasi-inverse map the neutral element of one group to the neutral element of the other group. It is always possible to do this, as Cayley graphs have a natural isometric action of the group. The composition of the quasi-isometry and its quasi-inverse may then not be C-close to the identity map, but it will be $2C$-close.

Proof. Let $\phi \colon \mathcal{G} \to \mathcal{H}$ be a quasi-isometry with constant C. To construct the map $\Psi \colon A_{1,r}(\mathcal{G}) \to A_{1,Cr+C}(\mathcal{H})$, given a path p in $\mathcal{G}$, we define $\Psi(p)$ by $\Psi(p)(i) = \phi(p(i))$ for every $0 \leq i \leq \ell(p)$. As we have assumed that $\phi(e) = e$, we have that $\Psi(p)$ is a $(Cr + C)$-loop based at e.

In order to show that Ψ is well-defined, we also have to show that when p and q are r-loops in $\mathcal{G}$ that are r-close, then $\Psi(p)$ and $\Psi(q)$ are $(Cr + C)$-close. We can check this for the two cases, (a) and (b), of being r-close.

In case (a), $p(i) = q(i)$ for every $0 \leq i \leq \ell(p)$ and $q(i) = e$ for every $i \geq \ell(p)$, and so $\Psi(p)(i) = \Psi(q)(i)$ for every $0 \leq i \leq \ell(p)$ and $\Psi(q)(i) = e$ for every $i \geq \ell(p)$.

In case (b), $\ell(p) = \ell(q)$ and $d(p(i), q(i)) \leq r$, and so $d(\Psi(p)(i), \Psi(q)(i)) = d(\phi(p(i)), \phi(q(i))) \leq Cr + C$. So $\Psi(p)$ and $\Psi(q)$ are $(Cr + C)$-close, whenever p and q are r-close. Therefore Ψ is well-defined.

Now we will show that Ψ is surjective. To do so, take a $(Cr + C)$-loop q in $\mathcal{H}$. Due to Proposition 5.1.3 q is $(Cr + C)$-homotopic to a 1-loop $\tilde{q}$.

As ϕ is a quasi-isometry, there exists a quasi-inverse ϕ'. Now we can define an r-loop p in $\mathcal{G}$ such that $p(i) = \phi'(\tilde{q}(i))$. This is an r-loop since $d(p(i), p(i+1)) = d(\phi'(\tilde{q}(i)), \phi'(\tilde{q}(i+1))) \leq 2C \leq r$ (as $\tilde{q}$ is a 1-loop and so $d(\tilde{q}(i), \tilde{q}(i+1)) \leq 1$).

It suffices to show that $\Psi(p)$ is $(Cr + C)$-homotopic to $\tilde{q}$. By definition we have that $\Psi(p)(i) = \phi(p(i)) = \phi(\phi'(\tilde{q}(i)))$ for every $0 \leq i \leq \ell(p)$, so $d(\Psi(p)(i), \tilde{q}(i)) \leq 2C \leq Cr + C$. So $\Psi(p)$ is $(Cr + C)$-homotopic to q. Therefore Ψ is surjective. $\square$

Now we are ready to prove the main result, that coarsely equivalent box spaces must be constructed using essentially the same sequence of normal subgroups.

Proof Theorem 2.2.1. As $\square_{(N_i)}G \simeq_{\mathrm{CE}} \square_{(M_i)}H$, we know due to Lemma 1 of [KV15] that there exists an almost permutation of $\mathbb{N}$ with bounded displacement and a constant C such that G/N_i is quasi-isometric, with constant C, to $H/M_{f(i)}$ for every i in the domain of f.

Now take finite presentations of $G = \langle S, R\rangle$ and $H = \langle S', R'\rangle$, where S and S' are the generating sets used for the construction of the box spaces. Also take $k = \max\left(\{|g| : g \in R \cup R'\} \cup \{2C\}\right)$ and take I such that $\inf\{|g| : g \in N_i \setminus \{e\}\} > Ck + C$ and $\inf\{|g| : g \in M_{f(i)} \setminus \{e\}\} > Ck + C$ for every $i \geq I$. Such an I exists, because N_i and M_i are filtrations and f is an almost permutation of $\mathbb{N}$ with bounded displacement.

By combining Lemma 5.1.4 and Lemma 5.1.5 we find for any $i \geq I$ that

$$N_i \cong A_{1,k}\left(\mathrm{Cay}(G/N_i)\right) \twoheadrightarrow A_{1,Ck+C}\left(\mathrm{Cay}(H/M_{f(i)})\right) \cong M_{f(i)}.$$

Similarly we find for any $i \geq I$ that

$$M_{f(i)} \cong A_{1,k}\left(\mathrm{Cay}(H/M_{f(i)})\right) \twoheadrightarrow A_{1,Ck+C}\left(\mathrm{Cay}(G/N_i)\right) \cong N_i.$$

Combining these maps provides a surjective homomorphism $N_i \twoheadrightarrow N_i$. Now $N_i \lhd G$ is residually finite, therefore it is Hopfian. This means that this surjective homomorphism is also injective. Therefore $N_i \cong M_{f(i)}$, when $i \geq I$. Now we can remove all $i < I$ from the domain of f. Then f is still an almost permutation of $\mathbb{N}$ with bounded displacement and $N_i \cong M_{f(i)}$ for every i in the domain of f. $\square$

5.2 Applications

In this section we look at some applications of Theorem 2.2.1.

5.2.1 Bounded covers

A first application of Theorem 2.2.1, is that there exist two box spaces $\square_{N_i}G \not\simeq_{\mathrm{CE}} \square_{M_i}G$ of the same group G such that $G/N_i \twoheadrightarrow G/M_i$ with $[M_i : N_i]$ bounded. This is surprising, because for two groups $G \twoheadrightarrow H$ with finite kernel, we have that G and H are quasi-isometric.

An example of such box spaces can be found for the free group. Consider F_2 as a subgroup of $\mathrm{SL}(2, \mathbb{Z})$, where the generating set is taken to be

$$\left\{\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\right\}.$$

Then take N_i to be the kernel of $F_2 \to \mathrm{SL}(2, \mathbb{Z}/4^i\mathbb{Z})$ and take M_i to be the kernel of $F_2 \to \mathrm{SL}(2, \mathbb{Z}/2^{2i+1}\mathbb{Z})$. Now for $i \geq 2$ we have $[F_2 : N_i] = 2^{6i-4}$, while $[F_2 : M_i] = 2^{6i-1}$. As they are subgroups of F_2, N_i and M_i are free groups for every i, and we can calculate the rank for any $N \lhd F_2$ using the Nielsen-Schreier rank formula, $\mathrm{rk}(N) = [F_2 : N] + 1$.

If we assume that $\square_{N_i}F_2 \simeq_{\mathrm{CE}} \square_{M_i}F_2$, then due to Theorem 2.2.1 there exist some i and j in $\mathbb{N}$ such that $N_i \cong M_j$. So $2^{6i-4} + 1 = \mathrm{rk}(N_i) = \mathrm{rk}(M_j) = 2^{6j-1} + 1$. So $6i - 4 = 6j - 1$ which is impossible if i and j are both in $\mathbb{N}$.

5.2.2 Rigidity of box spaces

Another consequence of Theorem 2.2.1 is that for finitely presented groups we can strengthen Theorem 7 of [KV15], which states that two box spaces that are coarsely equivalent stem from quasi-isometric groups.

Theorem 5.2.1. *Let G and H be finitely presented groups, then there exist two filtrations N_i and M_i of G and H respectively such that $\square_{(N_i)}G$ and $\square_{(M_i)}H$ are coarsely equivalent if and only if G is commensurable to H via a normal subgroup.*

Note that two groups are commensurable via a normal subgroup if there exist finite index normal subgroups of each of the groups such that these subgroups are isomorphic.

One of the implications of this theorem follows immediately from Theorem 2.2.1. For the other direction, we first recall the following well-known result.

Proposition 5.2.2. *If a group G is finitely generated and residually finite, then it has a filtration consisting of characteristic subgroups.*

Proof. As G is residually finite, it has a filtration N_i. Now take $M_i = \cap_\varphi \varphi(N_i)$, where the intersection is taken over all automorphisms φ in $\mathrm{Aut}(G)$. The subgroups M_i are characteristic, since for any $\psi \in \mathrm{Aut}(G)$, $\psi(M_i) = \psi\left(\cap_\varphi \varphi(N_i)\right) = \cap_\varphi (\psi\varphi)(N_i) = M_i$.

Now we need to show that M_i is a filtration of G, in other words we need to show that M_i is a sequence of nested finite index subgroups of G such that their intersection is trivial. As the intersection of all N_i is trivial,

we find the same for M_i, since $M_i < N_i$. To show that M_i is nested, we use that $N_{i+1} < N_i$, because this implies that we have $\varphi(N_{i+1}) < \varphi(N_i)$, so $M_{i+1} < M_i$. Now we only need to show that M_i is of finite index in G for every i. For every automorphism φ of G we have $[G : N_i] = [G : \varphi(N_i)]$. Thus, there are finitely many possibilities for $\varphi(N_i)$, as G is finitely generated and the index of $\varphi(N_i)$ in G is independent of φ. Now M_i is the intersection of finitely many finite index subgroups, and therefore it is also of finite index. $\square$

We can use this proposition to prove Theorem 5.2.1.

Proof Theorem 5.2.1. If $\square_{(N_i)}G$ is coarsely equivalent to $\square_{(M_i)}H$, then due to Theorem 2.2.1 we have that for a large enough i there exists a j such that N_i is isomorphic to M_j. As both N_i and M_j are normal subgroups, this proves one direction of the theorem.

Conversely, if G and H are commensurable via normal subgroups, they both have finite index normal subgroups that are isomorphic to one another. In fact we may say that they have a common finite index normal subgroup K. As G is residually finite, we have that K is also residually finite. Now due to Proposition 5.2.2 we can take a filtration N_i of K consisting of characteristic subgroups. Now as K is a finite index normal subgroup of G and H, we have that N_i is a filtration of G and of H, and the corresponding box spaces are coarsely equivalent. $\square$

We remark that this result is in some sense optimal, since by Proposition 10 of [KV15], there exist non-commensurable groups which admit isometric box spaces. The groups used are the wreath products $\mathbb{Z}/4\mathbb{Z} \wr \mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$, which are “easy” examples of groups which are not fnitely presented.

This result also allows us to answer the question posed at the end of [Das15], which asks whether residually finite groups that are uniformly measure equivalent necessarily admit coarsely equivalent box spaces. The answer can be seen to be negative, by taking two residually finite, finitely presented, uniformly measure equivalent groups which are not commensurable (for example, cocompact lattices in $SL(2, \mathbb{C})$ are a source of such examples).

5.2.3 Rank gradient and first ℓ^2 Betti number

From Theorem 2.2.1 we can deduce that non-zero rank gradient of the group is a coarse invariant for box spaces of finitely presented groups.

For a finitely generated, residually finite group G, the rank gradient $\mathrm{RG}(G, (N_i))$ of G with respect to a filtration (N_i) is defined by

$$\mathrm{RG}(G, (N_i)) := \lim_{i\to\infty} \frac{\mathrm{rk}(N_i) - 1}{[G : N_i]},$$

where $\mathrm{rk}(N)$ denotes the rank of N, i.e. the minimal cardinality of a generating set. This notion was first introduced by Lackenby in [Lac05], and is connected in interesting ways to analytic properties of G, see for example [AJZN11].

Theorem 5.2.3. *Given two finitely presented, residually finite groups G and H and respective filtrations (N_i) and (M_i), if $\square_{(N_i)}G$ is coarsely equivalent to $\square_{(M_i)}H$ then $\mathrm{RG}(G, (N_i)) > 0$ if and only if $\mathrm{RG}(H, (M_i)) > 0$.*

Proof. We begin with the remark that “ $\mathrm{rk} - 1$ ” is submultiplicative, that is, if $H < G$, then

$$\mathrm{rk}(H) - 1 \leq (\mathrm{rk}(G) - 1)[G : H].$$

Thus the limit in the definition is an infimum, and for all i,

$$\frac{\mathrm{rk}(N_i) - 1}{[G : N_i]} \leq \frac{\mathrm{rk}(N_{i-1}) - 1}{[G : N_{i-1}]} \text{ and } \frac{\mathrm{rk}(M_i) - 1}{[H : M_i]} \leq \frac{\mathrm{rk}(M_{i-1}) - 1}{[H : M_{i-1}]}.$$

By Theorem 2.2.1 and Lemma 1 of [KV15], there is an almost permutation $f : \mathbb{N} \to \mathbb{N}$ with bounded displacement such that $N_i \cong M_{f(i)}$ for every i in the domain of f, and such that there is a constant $C > 0$ with G/N_i and $H/M_{f(i)}$ C-quasi-isometric for all i in the domain of f. Thus we have

$$\inf_i \frac{\mathrm{rk}(N_i) - 1}{[G : N_i]} = \inf_i \frac{\mathrm{rk}(M_{f(i)}) - 1}{[G : N_i]} \geq \inf_i \frac{\mathrm{rk}(M_i) - 1}{|B_G(C^2)| \cdot [H : M_i]},$$

where $|B_G(C^2)|$ denotes the cardinality of a ball of radius C^2 in G. So we have that $\mathrm{RG}(H, (M_i)) > 0$ implies that $\mathrm{RG}(G, (N_i)) > 0$, and the theorem follows. $\square$

As another corollary of Theorem 2.2.1, we have that having the first ℓ^2 Betti number equal to zero is a coarse invariant of box spaces of finitely presented groups. For the definition and properties of the first ℓ^2 Betti number, we refer the reader to [Lüc02]. By the Lück Approximation Theorem ([Lüc94]), we have that for a

finitely presented group G, the first ℓ^2 Betti number $\beta_1^{(2)}(G)$ can be approximated using a filtration (N_i) of G as follows:

$$\beta_1^{(2)}(G) = \lim_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]},$$

where $b_1(N)$ denotes the classical first Betti number of N.

Thus, by the same argument as above, we have

$$\lim_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]} = \lim_{i\to\infty} \frac{b_1(M_{f(i)})}{[G:N_i]} \geq \lim_{i\to\infty} \frac{b_1(M_i)}{|B_G(C^2)|\cdot[H:M_i]},$$

under the same hypotheses as above, and so we obtain the following theorem.

Theorem 5.2.4. *Given two finitely presented, residually finite groups G and H and respective filtrations (N_i) and (M_i), if $\square_{(N_i)}G$ is coarsely equivalent to $\square_{(M_i)}H$ then $\beta_1^{(2)}(G) > 0$ if and only if $\beta_1^{(2)}(H) > 0$.*

5.2.4 Box spaces of free groups

The main theorem also allows us to easily distinguish box spaces of the free group.

Theorem 5.2.5. *For each $n \geq 2$, there exist infinitely many coarse equivalence classes of box spaces of the free group F_n that coarsely embed into a Hilbert space.*

Proof. Given $m \geq 2$, consider the m-homology filtration (N_i), defined inductively by $N_1 := F_n$, $N_{i+1} = N_i^m[N_i, N_i]$. Since for each j, $N_j/N_{j+1} \cong \mathbb{Z}_m^{\mathrm{rk}(N_j)}$, using the Nielsen-Schreier rank formula we can deduce that

$$\mathrm{rk}(N_i) = m^{\sum \mathrm{rk}(N_j)}(n-1)+1,$$

where the sum in the exponent runs from $j=1$ to $j=i-1$. By consideration of these ranks for coprime m and k, we see that the corresponding m-homology and k-homology filtrations (N_i) and (M_i) will satisfy $N_i \ncong M_j$ for all i,j sufficiently large. Thus, considering m-homology filtrations for various pairwise coprime m gives rise to box spaces of F_n which are not coarsely equivalent by Theorem 2.2.1. Such box spaces are coarsely embeddable into a Hilbert space by the main result of [Khu14]. ☐

We now prove Theorem 2.2.4 from the summary. Note that since being a Ramanujan expander is not preserved by coarse equivalences, the relevant question becomes how many coarse equivalence classes of box spaces there are such that each equivalence class contains at least one box space which is a Ramanujan expander.

Theorem 5.2.6. *There exist infinitely many coarse equivalence classes of box spaces of the free group F_3 that contain Ramanujan expanders.*

Proof. We begin by fixing an odd prime q such that -1 is a quadratic residue modulo q, and 5 is a quadratic residue modulo $2q$. By Theorem 7.4.3 of [Lub10], there exists a filtration (N_i) of F_3 with the property that the quotients $F_3/Ni \cong \mathrm{PSL}_2(q^i)$ form a Ramanujan expander sequence, and such that $N_i/N_{i+1} \cong \mathbb{Z}_q^3$ (see [DK16]). Using this, and the Nielsen-Schreier rank formula, we obtain

$$\mathrm{rk}(N_i) = 2q^{3i}+1.$$

By taking such expanders corresponding to different primes q satisfying the above conditions, consideration of the ranks mean that we obtain box spaces of F_3 which are not coarsely equivalent by Theorem 2.2.1. ☐

See also [Hum17], where a continuum of regular equivalence classes of expanders with large girth (i.e. the length of the smallest loop tending to infinity) is constructed.

Theorem 5.2.7. *Given $n \geq 3$, there exists a box space of the free group F_n such that no box space of F_m with $m-1$ coprime to $n-1$ is coarsely equivalent to it.*

Proof. Consider the $(n-1)$-homology filtration of F_n, (N_i). If (M_i) is a filtration of F_m, then by Theorem 2.2.1, if $\square_{(N_i)}F_n \simeq_{CE} \square_{(M_i)}F_m$ then for some i,j, we must have

$$(n-1)^a + 1 = [F_m : M_j](m-1)+1$$

with $a \in \mathbb{N}$ (by rank considerations). But this is impossible due to the assumptions on m. ☐

Chapter 6

A slowly growing box space of the free group that embeds into a Hilbert space

In this chapter we create a box space of the free group F_2 with a filtration N_n such that $\square_{N_i} G$ embeds into a Hilbert space and $[N_{i+1} : N_i] = 2$ for i in $\mathbb{N}$.
The filtration is defined as follows: Let a and b be the two generators of F_2. Then the normal subgroups N_i are defined inductively: we take $N_0 = F_2$, $N_1 = \langle a^2, ab, ab^{-1} \rangle$ and $N_2 = \Gamma(N_0)$. For every $n > 2$ there exists a unique i such that $n > i + 2^i + 1$, but $n \leq i + 2 + 2^{i+1}$. Then we take $N_n = [N_i, N_{n-2^i-1}]\Gamma(N_{i+1})$.
Recall that $\Gamma(G)$ is that subgroup of G generated by the squares of elements in G.

6.1 Overview

In order to show that $\square_{(N_n)} F_2$ is a linearly growing box space that embeds into a Hilbert space, we need to show two propositions: We need to show that $[N_n : N_{n+1}]$ is bounded and that $\square_{(N_n)} F_2$ embeds coarsely into a Hilbert space.

In Proposition 6.2.4 we show that $[N_n : N_{n+1}]$ is equal to 2. At the same time we show that the map $\varphi_n \colon \dfrac{N_{n+1}}{\Gamma(N_n)} \to \dfrac{[N_n, N_{n+1}]\Gamma(N_{n+1})}{\Gamma(N_{n+1})} \colon x \mapsto [g, x]$ is an isomorphism for every n. We also show that $\Gamma(N_n) = N_{n+2^n+1}$.
Unfortunately the proofs of these three propositions are nested. In Proposition 6.2.1, Lemma 6.2.2 and Proposition 6.2.3 we will assume that the index is 2. Then in Proposition 6.2.4 will show that the index is indeed 2, such that we have all three results.

To show that $\square_{(N_n)} F_2$ embeds coarsely into a Hilbert space we use the same technique as is used in [AGŠ12] and [Khu14]. First we define a new metric on $\square_{(N_n)} F_2$ that clearly embeds into ℓ^1. As ℓ^1 embeds into a Hilbert space it suffices to show that the new metric is coarsely equivalent to the old one.
The new metric d_n on $\square_{(N_n)} F_2$ which is used in Theorem 6.3.9 is the linear combination of pseudo-metrics defined using a maximal spanning tree of F_2/N_k with $k < n$. These pseudo-metrics are defined by lifting the chosen maximal spanning tree via the quotient map $F_2/N_n \to F_2/N_k$ and contracting the different copies of the tree, this graph which is the contraction of the Cayley graph of F_2/N_n defines a pseudo-distance between elements of F_2/N_n.
Note that a maximal spanning tree can be shifted via the left multiplication action on F_2/N_k, when we average over these translation we find a pseudo-metric that is translation invariant as shown in Proposition 6.3.2.
In order to construct the metric such that it is coarsely equivalent to the old metric we need to coarsely differenciate between any two elements of F_2/N_n which are coarsely different in the old metric. For any n we can take i to be the biggest integer such that $N_n < \Gamma(N_i)$. Then for any two elements x and y in F_2/N_n consider $d(f(x), f(y))$ where $f \colon F_2/N_n \to F_2/\Gamma(N_i)$ is the quotient map. This pseudo-distance coarsely differentiates between a lot of point, in fact now we only need to differentiate between x and y with $x^{-1}y$ in $\Gamma(N_i)$.
So now we need some other pseudo-metric that differentiates between these elements. If $N_n = \Gamma(N_i)$, then there are no elements to differentiate. Due to Proposition 3.11 of [AGŠ12] we already have a coarse bound for this embedding.
Otherwise we take $d_n(x, y) = \frac{1}{2}(d(f(x), f(y)) + \tilde{d}_n(x, y))$, where $\tilde{d}_n(x, y)$ is a pseudo-metric that coarsely differentiates between elements x and y of F_2/N_n with $x^{-1}y \in \Gamma(N_i)$.
For $N_{n-1} = \Gamma(N_i)$ the pseudo-metric $\tilde{d}_n$ only needs to provide a small adjustment provided in Proposition 6.3.5.
For other cases we use Proposition 6.3.4 to inductively differentiate between all elements x and y in F_2/N_n with

$x^{-1}y \in \Gamma(N_i)$.
Finally in Theorem 6.3.9 we we combine these pseudo-metrics to conclude that d_n is coarsely equivalent to the old metric and therefore $\square_{(N_n)}F_2$ embeds coarsely into a Hilbert space.

6.2 Algebraic properties of the sequence

In this section we will prove some algebraic properties of the sequence $(F_2/N_n)_n$. We want to show that the index of N_n in F_2 indeed grows linearly and we want to define a generating set which will be used in Section 6.3.

We want to show that for every i we have $[N_i : N_{i+1}] = 2$. This will be shown by induction, but first we need some other observations.

Proposition 6.2.1. *For any free group N and any $M \lhd N$ with $[N : M] = 2$ we have that $[N, \Gamma(N)] < \Gamma(M)$ and $[N, N] = [N, M]$.*

Proof. Let x and y be in N. To show that $[N, \Gamma(N)] < \Gamma(M)$ it suffices to show that $[x, y^2] \in \Gamma(M)$. Note that $[M, M] < \Gamma(M)$ and $\Gamma(N) < M$, so if $x \in M$, then $[x, y^2] \in \Gamma(M)$. If however $x \in N \setminus M$ and $y \in M$, then $[x, y^2] = (xyx^{-1})^2y^2 \in \Gamma(M)$. Finally if x and y lie in $N \setminus M$, then $[x, y^2] = xy(yx^{-1})^2xy \in \Gamma(M)$. So $[N, \Gamma(N)] < \Gamma(M)$.
To show that $[N, N] = [N, M]$ it suffices to show that $[x, y] \in [N, M]$. If either x or y in M, then we have that $[x, y] \in [N, M]$. So we may suppose that x and y in $N \setminus M$. As $[N : M] = 2$ there exists a $z \in M$ such that $y = xz$. Therefore $[x, y] = x^2\, z\, x^{-2}\, xz^{-1}x^{-1} = [x^2, z][z, x] \in [N, M]$. □

Lemma 6.2.2. *For any free group N and any $M \lhd N$ with $[N : M] = 2$ we have $[\Gamma(N) : [N, M]\Gamma(M)] = 2$.*

Proof. First consider the quotient map $\rho\colon N \to N/[N, N]$. Then $\rho(\Gamma(M)) = \Gamma(\rho(M))$ as both groups are generated by squares of elements in $\rho(M)$. So $[N, N]\Gamma(M)/[N, N] = \Gamma(M/[N, N])$, because $[N, N] < M$.
Due to Proposition 6.2.1 we have that

$$N/([N, M]\Gamma(M)) = N/([N, N]\Gamma(M)) = \frac{N/[N, N]}{[N, N]\Gamma(M)/[N, N]} = \frac{N/[N, N]}{\Gamma(M/[N, N])}.$$

Now $N/[N, N] = \mathbb{Z}^{\mathrm{rk}(N)}$ and $M/[N, N]$ is an index 2 subgroup of $N/[N, N]$ and $\Gamma(M/[N, N]) = \{(2x_1, \ldots, 2x_{\mathrm{rk}(N)}) : (x_1, \ldots, x_{\mathrm{rk}(N)}) \in M/[N, N]\}$. So we can conclude that $[N : [N, M]\Gamma(M)] = [N/[N, N] : \Gamma(M/[N, N])] = 2 \cdot 2^{\mathrm{rk}(N)} = 2[N : \Gamma(N)]$ and therefore $[\Gamma(N) : [N, M]\Gamma(M)] = 2$.

□

Proposition 6.2.3. *For any free group N, any $M \lhd N$ with $[N : M] = 2$ and any $g \in N \setminus M$ we have that the map $\varphi\colon \dfrac{M}{\Gamma(N)} \to \dfrac{[N, M]\Gamma(M)}{\Gamma(M)} : x \mapsto [g, x]$ is an isomorphism.*

Proof. Note that the map $\tilde{\varphi}\colon \frac{N}{\Gamma(M)} \to \frac{[N,N]\Gamma(M)}{\Gamma(M)} : x \mapsto [g, x]$ is a homomorphism. In fact this is true for every 2-step nilpotent group and due to Proposition 6.2.1 we have that $[N, [N, N]] < [N, \Gamma(N)] < \Gamma(M)$ and therefore $N/\Gamma(M)$ is a 2-step nilpotent group. Due to Proposition 6.2.1 we have that $\Gamma(N)/\Gamma(M)$ lies in the kernel of $\tilde{\varphi}$ and $\frac{[N,N]\Gamma(M)}{\Gamma(M)} = \frac{[N,M]\Gamma(M)}{\Gamma(M)}$. Therefore φ is a homomorphism as well.
Let $[h, x]\Gamma(M)$ be an element of $\frac{[N,M]\Gamma(M)}{\Gamma(M)}$ with $h \in N \setminus M$ and $x \in M$. We have that $\varphi(x\Gamma(N)) = [h, x]\Gamma(M)$, because $[h, x] = g\, g^{-1}h\, x\, h^{-1}g\, g^{-1}\, x^{-1} = g[g^{-1}h, x]g^{-1}[g, x] \in [g, x]\Gamma(M)$. Therefore φ is surjective, so it suffices to show that $[M : \Gamma(N)]$ is equal to $[[N, M]\Gamma(M) : \Gamma(M)]$.
Due to the Nielsen-Schreier formula we have $[M : \Gamma(N)] = \frac{1}{2}[N : \Gamma(N)] = \frac{1}{2}2^{\mathrm{rk}(()N)} = 2^{[F_2:N]}$. On the other hand we have due to Lemma 6.2.2 that

$$\begin{aligned}
[[N, M]\Gamma(M) : \Gamma(M)] = \frac{1}{2}[\Gamma(N) : \Gamma(M)] = \frac{1}{2}\frac{[N : \Gamma(M)]}{[N : \Gamma(N)]} = \frac{[N : M]\,[M : \Gamma(M)]}{2\,[N : \Gamma(N)]} &= \frac{2^{\mathrm{rk}(()M)}}{2^{\mathrm{rk}(()N)}} \\
&= 2^{[F_2:M]-[F_2:N]} = 2^{[F_2:N]}.
\end{aligned}$$

So we can conclude that φ is an isomorphism. □

Using these results we can compute the index of each N_n.

Proposition 6.2.4. *For every n we have $[N_n : N_{n+1}] = 2$ and $[F_2 : N_n] = 2^n$.*

Proof. For $n = 0$ we have that N_1 is the subgroup of words with even length, because it contains every word of length 2: a^2, ab, ab^{-1}, a^{-2}, $a^{-1}b = a^{-2}ab$, $ab^{-1} = a^{-2}ab^{-1}$, $b^2 = (ab^{-1})^{-1}ab$, $ba = (ab^{-1})^{-1}a^2$, $ba^{-1} = (ab^{-1})^{-1}$, $b^{-2} = (ab)^{-1}ab^{-1}$, $b^{-1}a = (ab)^{-1}a^2$ and $b^{-1}a^{-1} = (ab)^{-1}$, so $[N_0 : N_1] = 2$.
For $n = 1$ we have that $N_2 = \Gamma(N_0)$, so $[N_0 : N_2] = [N_0 : \Gamma(N_0)] = 2^{\mathrm{rk}(()N_0)} = 4$. Therefore $[N_1 : N_2] = \frac{[N_0:N_2]}{[N_0:N_1]} = \frac{4}{2} = 2$.
So it suffices to prove the theorem for $n \geq 2$. Take i the biggest integer such that $n \geq i+2^i+1$. As the proposition is true for $n \leq 1$ we may assume by induction that for every $n \leq i + 2^i$ we have that $[N_n : N_{n+1}] = 2$.
For $n = i + 2^i + 1$ we have that $[N_n : N_{n+1}] = [\Gamma(N_i) : [N_i, N_{i+1}]\Gamma(N_{i+1})] = 2$ due to Lemma 6.2.2. So we only need to show that $[N_n : N_{n+1}] = 2$ for every n with $i + 2^i + 2 \leq n \leq i + 1 + 2^{i+1}$ or equivalently that $[N_{i+2^i+2} : N_{n+1}] = 2^{n-i-2^i-1}$.
Fix $g \in N_i \setminus N_{i+1}$, then due to Proposition 6.2.3 we have that $\varphi\colon \dfrac{N_{i+1}}{\Gamma(N_i)} \to \dfrac{[N_i, N_{i+1}]\Gamma(N_{i+1})}{\Gamma(N_{i+1})} \colon x \mapsto [g, x]$ is an isomorphism, since $[N_i : N_{i+1}] = 2$.
Now $\varphi(N_{n-2^i}) = [N_i, N_{n-2^i}]\Gamma(N_{i+1}) = N_{n+1}$, so as φ is an isomorphism we have that $[N_{i+2^i+2} : N_{n+1}] = [N_{i+1} : N_{n-2^i}] = 2^{n-i-2^i-1}$.
So we can conclude that $[N_n : N_{n+1}] = 2$ for every n and therefore $[F_2 : N_n] = 2^n$. $\square$

Corollary 6.2.5. *For every i we have that $N_{i+2^i+1} = \Gamma(N_i)$.*

Proof. For $i = 0$ we have $N_2 = \Gamma(N_0)$, so by induction we may assume that $N_{i+2^{i-1}} = \Gamma(N_{i-1})$. Therefore $N_{i+2^i+1} = [N_{i-1}, \Gamma(N_{i-1})]\Gamma(N_i)$. Now by combining Proposition 6.2.1 and Proposition 6.2.4 we have that $[N_{i-1}, \Gamma(N_{i-1})] < \Gamma(N_i)$, so $N_{i+2^i+1} = \Gamma(N_i)$. $\square$

Now we want to define a generating set of N_n for each n. We define this generating set using a maximal spanning tree T_n on F_2/N_n.

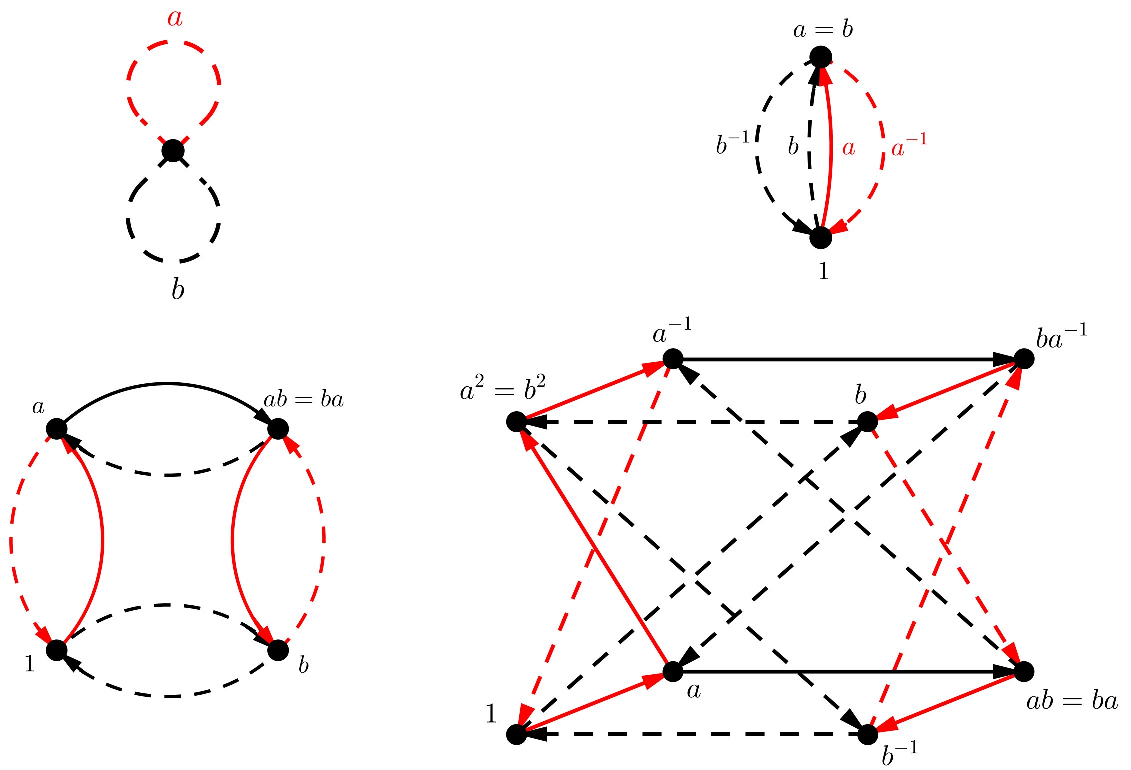


Figure 6.1: The spanning trees T_n for $n \in \{0, 1, 2, 3\}$ and for $g_0 = a$, $g_1 = ab$, $g_2 = a^2$. The edges in red are due to the generator $a \in F_2$ and the black edges are due to the generator $b \in F_2$. Edges of F_2/N_n that are not contained in T_n are represented by a dotted line.

These trees are defined inductively: T_0 is trivial and for any other T_n we can choose a generator g_n of N_n, which is not in N_{n+1}, to construct T_{n+1}, an example in given in Fig. 6.1. As we lift T_n from F_2/N_n to F_2/N_{n+1} we get two copies of T_n and the element g_n which corresponds to a loop in F_2/N_n gets lifted to a path from the neutral element in F_2/N_{n+1} to the other element in N_n/N_{n+1}. Note that this element in unique, because $[N_n : N_{n+1}] = 2$ as shown in Proposition 6.2.4.
As g_n is a generator we have that its corresponding path crosses only one edge that is not in the lift of T_n, that

edge connects the two copies of T_n in F_2/N_{n+1}. Adding this edge gives a maximal spanning tree of F_2/N_{n+1}, we take T_{n+1} to be that spanning tree.

Remark 6.2.6. *Remark that the generating set of N_{n+1} is the set containing g_n^2, g_nh and g_nh^{-1} for h a generator of N_n that is not in N_{n+1} and h and $g_nhg_n^{-1}$ for h a generator of N_n that is in N_{n+1}. An example for $n = 1$ and $n = 2$ is given in Fig. 6.3. To prove that the generators of N_{n+1} are of that form, consider the two copies of T_n in $\mathrm{Cay}(F_2/N_{n+1})$. When we contract these two copies of T_n, we find $\mathrm{Cay}(N_n/N_{n+1}, S)$ where S is the generating set of N_n constructed by T_n. The tree T_{n+1} gets contracted to the graph with two vertices and one edge, this edge corresponds to the generator g_n. Now Fig. 6.2 shows why we have this classification.*
We say that a generator of N_{n+1} lift-corresponds to a generator h of N_n if that generator is equal to either g_nh, g_nh^{-1}, h or $g_nhg_n^{-1}$.

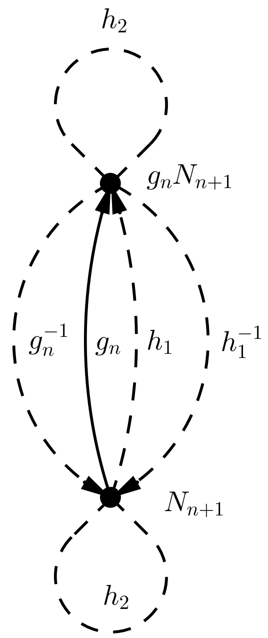


Figure 6.2: The Cayley graph of N_n/N_{n+1} as a contraction of $\mathrm{Cay}(F_2/N_{n+1})$ using T_n. In other words the two vertices represent the two copies of T_n in $\mathrm{Cay}(F_2/N_{n+1})$, every edge represents an edge that is not in a copy of T_n and every such edge corresponds to a generator of N_n. The tree T_{n+1} consists of the two copies of T_n connected by the edge corresponding to a fixed generator g_n, drawn with a full edge. Other generators of N_n that are not contained in N_{n+1} are represented by h_1, while the generators of N_n that are contained in N_{n+1} are represented by h_2.
The generators of N_{n+1} are loops in $\mathrm{Cay}(N_n/N_{n+1})$ that only contain one dotted line. On this picture you pass g_n and then return by either g_n^{-1}, h_1 or h_1^{-1}, giving the generators g_n^2, $g_nh_1^{-1}$ and g_nh_1 respectively. You can also cross h_2 or pass g_n, cross h_2 and return via g_n giving the generators h_2 and $g_nh_2g_n^{-1}$.

Now we need to fix a generator g_n of N_n that is not in N_{n+1}, for $n \leq 1$ we take $g_0 = a$ and $g_1 = ab$ where a and b are the generators of F_2. For $n \geq 2$ we could take g_n to be an arbitrary generator of N_n that is not in N_{n+1}. However we want to classify the generators of N_n. In order to have a good classification we need to take specific generators of N_n. For any such $n \geq 2$ we can take i to be the biggest integer such that $i + 2^i + 1 \leq n$. If $n = i + 2^i + 1$, then we take $g_n = g_i^2$. If $n > i + 2^i + 1$ the choice of g_n is more complicated, we want $g_n \equiv [g_i, g_{n-2^i-1}]$ modulo $\Gamma(N_{i+1})$. Unfortunately it is not obvious that such a generator exists, so for now we will only assume that g_n is a generator of N_n that is not in N_{n+1}. In Corollary 6.2.11 we will show that g_n can be taken such that $g_n \equiv [g_i, g_{n-2^i-1}]$.

Proposition 6.2.7. *For every i we can classify the generators of $N_{i+2^i+1} = \Gamma(N_i)$ as follows:*

- *For every generator $h \neq g_i$ of N_i which is not in N_{i+1}, there are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$.*
- *For every generator h of N_i which is in N_{i+1}, there are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$.*
- *There are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to g_i^2 modulo $\Gamma(N_{i+1})$.*

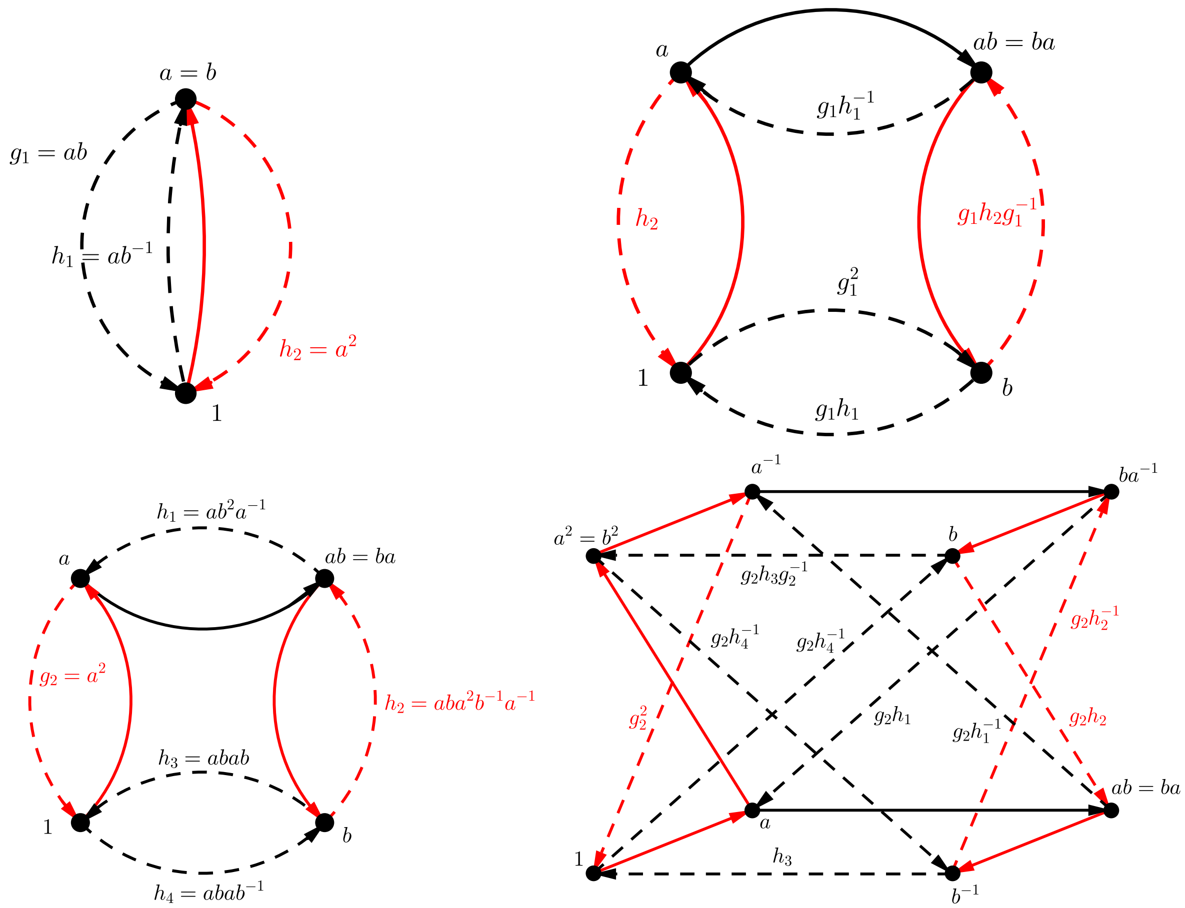


Figure 6.3: Every generator of N_n corresponds to an edge in $\mathrm{Cay}(F_2/N_n)$ that is not in T_n. These graphs show the relation between the generators of N_n on the left and those of N_{n+1} on the right for n equal to 1 or 2 and for $g_0 = a$, $g_1 = ab$, $g_2 = a^2$. The edges in red are due to the generator $a \in F_2$ and the black edges are due to the generator $b \in F_2$. Edges of F_2/N_n that are not contained in T_n are represented by a dotted line.

- *Every other generator of $\Gamma(N_i)$ lies in $\Gamma(N_{i+1})$.*

Lemma 6.2.8. *Let i and j be such that $i < j \leq i + 2^i + 1$. Then for every generator h of N_i that is not in $\langle g_i, g_{i+1}, \ldots, g_{j-1}\rangle\Gamma(N_i)$ there are two sets of 2^{j-i-1} generators of N_j with the following properties:*

- *generators from the same set are equal modulo $\Gamma(N_{i+1})$,*
- *generators from any of the two sets are equal to h modulo $\langle g_i, g_{i+1}, \ldots, g_{j-1}\rangle\Gamma(N_i)$, and*
- *generators from the two different sets differ by $h_1'^{-1}h_2'$ modulo $\Gamma(N_{i+1})$, where h_1' and h_2' are the two generators of N_{i+1} lift-corresponding to h.*

Proof. First we show that $\langle g_i, g_{i+1}, \ldots, g_{j-1}\rangle\Gamma(N_i)$ is a normal subgroup of N_i: for every $k \in N_i$ and $k' \in \langle g_i, g_{i+1}, \ldots, g_{j-1}\rangle$ we have that $kk'\Gamma(N_i)k^{-1} = k'[k'^{-1}, k]\Gamma(N_i) = k'\Gamma(N_i)$, because $[k'^{-1}, k] \in \Gamma(N_i) \lhd N_i$.

Now we consider the case where $j = i + 1$. Let h be a generator of N_i that is not in $\langle g_i\rangle\Gamma(N_i)$. Due to Remark 6.2.6 we can take the sets as follows:
If $h \in N_{i+1}$, then we have two set $\{h\}$ and $\{g_ihg_i^{-1}\}$ of 2^{j-i-1} generators.
If $h \in N_i \setminus N_{i+1}$, then we have two set $\{g_ih\}$ and $\{g_ih^{-1}\}$ of 2^{j-i-1} generators.
In both cases these sets satisfy all three properties.

By induction we can assume to sets of generators for N_{j-1}. Now let h_1' and h_2' be the generators of N_{i+1} lift-corresponding to h. Then there are two sets of 2^{j-i-2} generators of N_{j-1}, which are equal to each other modulo $\Gamma(N_{i+1})$, which are equal to h_1' or h_2' respectively modulo $\langle g_{i+1}, \ldots, g_{j-2}\rangle\Gamma(N_i)$ and generators from the two different sets differ by $h_1'^{-1}h_2'$ modulo $\Gamma(N_{i+1})$.
Let k be such a generator of N_{j-1}. If $k \in N_j$, then k and $g_{j-1}kg_{j-1}^{-1}$ are generators of N_j and they are equal to k modulo $\Gamma(N_{i+1})$. If $k \in N_{j-1} \setminus N_j$, then $g_{j-1}k$ and $g_{j-1}k^{-1}$ are generators of N_j and they are equal to $g_{j-1}k$ modulo $\Gamma(N_{i+1})$.

All generators are modulo $\Gamma(N_{i+1})$ either equal or differ by $h_1'^{-1}h_2'$, so we have that whether or not $k \in N_j$ depends only on h, because due to Remark 6.2.6 we have that $h_1'^{-1}h_2'$ is equal to $(g_ih)^{-1}g_ih^{-1} = h^{-1}$ or $h^{-1}g_ihg_i^{-1} = [h^{-1}, g_i]$, in either case we have $h_1'^{-1}h_2' \in \Gamma(N_i) < N_j$. So we have that all elements of the two sets of 2^{j-i-1} generators are equal modulo $\Gamma(N_{i+1})$, they are all equal to h_1 or h_2 modulo $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$ and two such generators of different sets still differ by $h_1'^{-1}h_2'$ modulo $\Gamma(N_{i+1})$. □

Lemma 6.2.9. *Let i and j be such that $i < j \leq i + 2^i + 1$. Then there are $j - i$ generators of N_i that are in $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$.*

Proof. Note that we need to prove the lemma for $2^i + 1$ values of j, so it suffices to show that there is always one generator of N_i that is in $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$, but not in $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$.
Due to Remark 6.2.6 we know that for every generator h' of N_j there exists a generator of N_{j-1} that is equivalent to h' modulo $\langle g_{j-1}\rangle\Gamma(N_i)$. So by induction there exists a generator of N_i that is equivalent to h' modulo $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$ for every j. Therefore there exists a generator h of N_i that is equivalent to g_{j-1} modulo $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$.
Now it suffices to show that g_{j-1} is not in $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$, because then h is in $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$, but not in $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$.

For any such j we have that $N_j \lneq \langle g_{j-1}\rangle N_j < N_{j-1}$, because $g_{j-1} \in N_{j-1} \setminus N_j$. Due to Proposition 6.2.4 we have that $[N_{j-1} : N_j] = 2$, so $N_{j-1} = \langle g_{j-1}\rangle N_j$. Now due to Corollary 6.2.5 we have that $\langle g_i, \dots, g_{i+2^i}\rangle\Gamma(N_i) = \langle g_i, \dots, g_{i+2^i}\rangle N_{i+2^i+1} = N_i$.
Now suppose that $g_{j-1} \in \langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$. Then $N_i = \langle g_i, \dots, g_{j-2}, g_j, \dots, g_{i+2^i}\rangle\Gamma(N_i)$, so $N_i/\Gamma(N_i) = \mathbb{Z}_2^{\mathrm{rk}(()N_i)}$ is generated by 2^i elements, but to the Nielsen-Schreier formula and Proposition 6.2.4 we have that $\mathrm{rk}(()N_i) = [F_2 : N_i] + 1 = 2^i + 1$.

So we can conclude that g_{j-1} is not in $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$ and therefore there exists a generator h of N_i that is in $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$, but not in $\langle g_i, \dots, g_{j-2}\rangle\Gamma(N_i)$. As there exists such a generator for all $2^i + 1$ values of j and N_i only has $2^i + 1$ generators we can conclude that h is unique. Therefore $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$ contains $j - i$ generators of N_i. □

Proof Proposition 6.2.7. In order to classify the generators of $\Gamma(N_i)$ we classify the generators of N_j with $i < j \leq i + 2^i + 1$. We classify the generators of N_j as follows:

1. For every generator h of N_i that is not in $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$ there are two sets of 2^{j-i-1} generators of N_j with the following properties:
   - generators from the same set are equal modulo $\Gamma(N_{i+1})$,
   - generators from any of the two sets are equal to h modulo $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$, and
   - generators from the two different sets differ by $h_1'^{-1}h_2'$ modulo $\Gamma(N_{i+1})$, where h_1' and h_2' are the two generators of N_{i+1} lift-corresponding to h.
2. There are 2^{j-i-1} generators of N_j which are equal to g_i^2 modulo $\Gamma(N_{i+1})$.
3. For every generator $h \neq g_i$ of N_i in $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$ there are 2^{j-i-1} generators of N_j which are either equal to $[g_i, h]$, if $h \in N_{i+1}$, or equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$, if $h \notin N_{i+1}$.
4. Every remaining generator of N_j lies in $\Gamma(N_{i+1})$.

We will finish the proof by showing that case 1 is empty for $j = i + 2^i + 1$, which implies the proposition.

First we suppose that $j = i + 1$. Note $g_i \in \langle g_i\rangle\Gamma(N_i)$ and due to Lemma 6.2.9 this is the only generator of N_i in $\langle g_i\rangle\Gamma(N_i)$. So it suffices to show that every generator of N_{i+1} that is not g_i^2 is in case 1. Due to Remark 6.2.6 every other generator h' of N_{i+1} lift-corresponds to a generator $h \in N_i$. If $h \in N_{i+1}$, then we have two set $\{h\}$ and $\{g_ihg_i^{-1}\}$ of 2^{j-i-1} generators. If $h \in N_i \setminus N_{i+1}$, then we have two set $\{g_ih\}$ and $\{g_ih^{-1}\}$ of 2^{j-i-1} generators. Note that in both cases one of these sets is $\{h'\}$, so h' is in case 1, because of Remark 6.2.6.

By induction we may assume to have this classification for the generators of N_{j-1}. We will show that the generators of N_j are in the same case as the lift-corresponding generator in N_{j-1}, except when that lift-corresponding generator is in case 1, then that generator could also be in cases 3 or 4.

Case 1 is shown by Lemma 6.2.8.

Next we show case 2. To show that there are also 2^{j-i-1} generators of N_j which are equal to g_i^2 modulo

$\Gamma(N_{i+1})$, we may assume by induction that there are 2^{j-i-2} generators of N_{j-1} which are equal to g_i^2 modulo $\Gamma(N_{i+1})$. As $g_i^2 \in \Gamma(N_i) < N_j$ we have that for each such generator k of N_{j-1} there are two generators of N_j that lift-correspond to k, namely k and $g_{j-1}kg_{j-1}^{-1}$. Therefore there are 2^{j-i-1} generators of N_j which are equal to g_i^2 modulo $\Gamma(N_{i+1})$.

Now we show case 3. Let h be a generator of N_i that is not g_i, but is in $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$. We need to show that there are 2^{j-i-1} generators of N_j which are either equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$, if $h \in N_{i+1}$, or equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$, if $h \notin N_{i+1}$.

Here we have two cases to consider: either h lies in $\langle g_i, g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_i)$ or it does not. If h is in $\langle g_i, g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_i)$, then by induction there are 2^{j-i-2} generators of N_{j-1} which are either all equal to $[g_i, h]$ or all equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$, depending on h. As all these generators lie in $\Gamma(N_i)$ we have that each one lift-corresponds to two generators of N_j, the initial generator of N_{j-1} and its conjugate by g_{j-1}. This gives 2^{j-i-1} generators of N_j which are either equal to $[g_i, h]$, if $h \in N_{i+1}$, or equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$, if $h \notin N_{i+1}$.

Now we consider the other case. Due to Lemma 6.2.9 there is only on generator of N_i in $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$ that is not in $\langle g_i, g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_i)$, let h be that generator. Let h'_1 and h'_2 be the generators of N_{i+1} lift-corresponding to h such that g_{j-1} is equal to h'_1 modulo $\langle g_i, g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_i)$.

By induction we know there are 2^{j-i-2} generators of N_{j-1} which are equal to g_{j-1} modulo $\Gamma(N_{i+1})$. Let k be such a generator. If $k \neq g_{j-1}$, then $g_{j-1}k$ and $g_{j-1}k^{-1}$ are generators of N_j which lie in $\Gamma(N_{i+1})$. We also know that there are 2^{j-i-2} generators of N_{j-1} which are equal to $g_{j-1}h_1'^{-1}h'_2$ modulo $\Gamma(N_{i+1})$. Let k be such a generator, then both $g_{j-1}k$ and $g_{j-1}k^{-1}$ are equal to $h_1'^{-1}h'_2$ modulo $\Gamma(N_{i+1})$.
So it suffices to show that $h_1'^{-1}h'_2$ is equal to $[g_i, h]$ if $h \in N_{i+1}$ and equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$ if $h \notin N_{i+1}$. Note that h'_1 and h'_2 lie in N_{i+1}, so $h_1'^{-1}h'_2 \equiv h'_1h'_2 \equiv h'_1h_2'^{-1} \equiv h'_2h_1'^{-1}$ modulo $\Gamma(N_{i+1})$. If $h \in N_{i+1}$, then without loss of generality we have that $h'_1 = g_ihg_i^{-1}$ and $h'_2 = h$, so $h'_1h_2'^{-1} = [g_i, h]$. If $h \in N_i \setminus N_{i+1}$, then without loss of generality we have that $h'_1 = g_ih$ and $h'_2 = g_ih^{-1}$, so $h'_1h'_2 = g_ihg_i^2g_i^{-1}h^{-1} \equiv g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$.

Finally we show case 4, which says that the remaining generators of N_j lie in $\Gamma(N_{i+1})$. Let k' be a remaining generator of N_j and let k be its lift-corresponding generator of N_{j-1}.
By induction we may assume that k satisfies one of the four cases. In cases 2 and 3 we have that $k \in \Gamma(N_i)$, so then k' is also in either case 2 or 3 respectively. If k satisfies case 4, then $k \in \Gamma(N_{i+1})$, so k' is either k or its conjugate by g_{j-1}, both of which lie in $\Gamma(N_{i+1})$.

So suppose that k is in case 1, which means k is equal to h modulo $\langle g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_{i+1})$, where h is a generator of N_i not in $\langle g_i, g_{i+1}, \dots, g_{j-2}\rangle\Gamma(N_{i+1})$. As k' is a remaining generator we have that h is not in $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_{i+1})$. In fact k must be equal to g_{j-1} modulo $\Gamma(N_{i+1})$. Therefore k' is either $g_{j-1}k$ or $g_{j-1}k^{-1}$ and both lie in $\Gamma(N_{i+1})$.

Due to Lemma 6.2.8 we know that $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i)$ contains exactly $j-i$ generators of N_i. So for $j = i + 2^i + 1$ we have that $\langle g_i, g_{i+1}, \dots, g_{j-1}\rangle\Gamma(N_i) = N_i$. Therefore there are no generators of N_{i+2^i+1} in case 1.

In order to conclude the prove consider the proven classification for $j = i + 2^i + 1$:
Note that Corollary 6.2.5 we have that $N_{i+2^i+1} = \Gamma(N_i)$.
Due to Lemma 6.2.9 we know that every generator of N_i is in $\langle g_i, \dots, g_{j-1}\rangle\Gamma(N_i)$. So there are no generators in case 1. For every generator $h \neq g_i$ of N_i which is not in N_{i+1}, there are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to $g_i^2[g_i, h]$ modulo $\Gamma(N_{i+1})$ and for every generator h of N_i which is in N_{i+1}, there are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$, because of case 3.
There are 2^{2^i} generators of $\Gamma(N_i)$ which are equal to g_i^2 modulo $\Gamma(N_{i+1})$, because of case 2.
Every other generator of $\Gamma(N_i)$ must be in case 4, so they lie in $\Gamma(N_{i+1})$. □

Proposition 6.2.10. *Let i and j be such that $i < j \leq i + 2^i + 1$ and let g_{j+2^i} be equivalent to $[g_i, g_{j-1}]$ modulo $\Gamma(N_{i+1})$ for every n with $i + 2^i + 1 < n < j$. For every generator h of N_j such that $h \notin \Gamma(N_i)$, the group N_{j+2^i+1} has 2^{2^j} generators which are all equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$ and all other generators of N_{j+2^i+1} lie in $\Gamma(N_{i+1})$.*

Proof. First we take $j = i+1$, let h be a generator of N_{i+1} which is not in $\Gamma(N_i)$ and due to Remark 6.2.6 we can take h' to be the lift-corresponding generator of N_i, so h is equal to h', $g_ih'g_i^{-1}$, g_ih' or $g_ih'^{-1}$. Due to Proposition 6.2.7 there are 2^{2^i} generators of N_{j+2^i} which are either all equal to $[g_i, h']$ or all equal to $g_i^2[g_i, h']$ modulo $\Gamma(N_{i+1})$. Since $g_{i+2^i+1} = g_i^2$ we get 2^{2^j} generators of N_{j+2^i+1} which are equal to $[g_i, h']$ modulo $\Gamma(N_{i+1})$. So it suffices to show that $[g_i, h'] \equiv [g_i, h]$ modulo $\Gamma(N_{i+1})$.

Due to Proposition 6.2.4 we can apply Proposition 6.2.3 to N_i and N_{i+1}. As either $h \in h'\Gamma(N_i)$ or $h \in h'g_i^{-1}\Gamma(N_i)$ we conclude that $[g_i, h]$ is either equal to $[g_i, h']$ or $[g_i, h'g_i^{-1}] = g_i h' g_i^{-1} g_i^{-1} g_i h'^{-1} = [g_i, h']$ modulo $\Gamma(N_{i+1})$ and therefore there are $2^{2^{i+1}}$ generators which are all equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$.

Now every other generator h of N_{i+2^i+2} is either $g_{i+2^i+1}^2 \in \Gamma(N_{i+1})$ or lift-corresponds to a generator of N_{i+2^i+1} that is equal to 1 or g_i^2 modulo $\Gamma(N_{i+1})$. As $g_{i+2^i+1} = g_i^2$ we have that $h \in \Gamma(N_{i+1})$. This proves the proposition for $j = i + 1$.

By induction we may assume that the proposition is true for $j - 1$. Let h be a generator of N_j which is not in $\Gamma(N_i)$. Note that h is not $g_{j+2^i}^2 \in \Gamma(N_{i+1})$. So due to Remark 6.2.6 there exists a lift-corresponding generator h' of N_{j-1} such that h is equal to h', $g_{j-1}h'g_{j-1}^{-1}$, $g_{j-1}h'$ or $g_{j-1}h'^{-1}$. By induction there are $2^{2^{j-1}}$ generators of N_{j+2^i} which are all equal to $[g_i, h']$ modulo $\Gamma(N_{i+1})$.
Note that due to Proposition 6.2.3 and Proposition 6.2.4 the map $\varphi\colon \frac{M}{\Gamma(N)} \to \frac{[N,M]\Gamma(M)}{\Gamma(M)} : x \mapsto [g, x]$ is an isomorphism.
If $h' \in N_j$, then h is equal to h' modulo $\Gamma(N_i)$ and $[g_i, h]$ is equal to $[g_i, h']$ modulo $\Gamma(N_{i+1})$, because φ is an isomorphism. As $[g_i, h'] \in N_{j+2^i+1}$ these $2^{2^{j-1}}$ generators of N_{j+2^i} lift-correspond to 2^{2^j} generators of N_{j+2^i+1} which are all equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$.
If $h' \notin N_j$, then h is equal to $g_{j-1}h'$ modulo $\Gamma(N_i)$. As φ is an isomorphism we have that $[g_i, h] \equiv [g_i, g_{j-1}][g_i, h'] \equiv g_{j+2^i}[g_i, h']$ modulo $\Gamma(N_{i+1})$. As $[g_i, h'] \notin N_{j+2^i+1}$ these $2^{2^{j-1}}$ generators of N_{j+2^i} lift-correspond to 2^{2^j} generators of N_{j+2^i+1} which are all equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$.

Now it suffices to show that every other generator of N_{j+2^i+1} lies in $\Gamma(N_{i+1})$. Let k be such a generator. If $k = g_{j+2^i}^2$, then $k \in \Gamma(N_{i+1})$. Otherwise there exist generator k' of N_{j-1} that lift-corresponds to k, as noted in Remark 6.2.6.
If $k' \in \Gamma(N_{i+1})$, then k is either k' or $g_{j-1}k'g_{j-1}^{-1}$ and therefore lies in $\Gamma(N_{i+1})$. If $k' \notin \Gamma(N_{i+1})$, then by induction there exists a generator h of N_{j-1} such that $h \notin \Gamma(N_i)$ and k' is equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$.

As $k' \notin \Gamma(N_{i+1})$, but $k \in \Gamma(N_{i+1})$, we have that $k = g_{j+2^i}k'$ or $k = g_{j+2^i}k'^{-1}$. As $k = g_{j+2^i}k'^{\pm 1}$ we have that $k \equiv [g_i, g_{j-1}][g_i, h'^{\pm 1}]$. As φ is an isomorphism we have that $k \equiv [g_i, g_{j-1}h'^{\pm 1}]$. Note that $g_{j-1}h'^{\pm 1}$ is a generator of N_j, because h' is a generator of N_{j-1} that is not in N_j.
We assumed that k is not one of the 2^{2^j} generator of N_j that is equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$ with h a generator of N_j that is not in $\Gamma(N_i)$. Therefore we have that $h \in \Gamma(N_i)$. Now due to Corollary 6.2.5 we have that $k \in [N_i, \Gamma(N_i)]\Gamma(N_{i+1}) = N_{i+2^i+1} = \Gamma(N_{i+1})$.
So all other generators of N_{j+2^i+1} lie in $\Gamma(N_{i+1})$, which concludes this corollary. $\square$

In Proposition 6.2.10 had the condition that g_{j+2^i} is equivalent to $[g_i, g_{j-1}]$ modulo $\Gamma(N_{i+1})$. Now we will show that g_n can be taken such that this condition is always satisfied.

Corollary 6.2.11. *There exist a choice of g_n such that $g_n \equiv [g_i, g_{n-2^i-1}]$ modulo $\Gamma(N_{i+1})$ for any i and n such that $i + 2^i + 1 < n < i + 1 + 2^{i+1} + 1$.*

Proof. We fixed $g_0 = a$ and $g_1 = ab$. For $n \geq 2$ we take i to be the biggest integer such that $i + 2^i + 1 \leq n$. If $n = i + 2^i + 1$, then $g_n = g_i^2$. If $n > i + 2^i + 1$, then $i < n - 2^i - 1 \leq i + 2^i + 1$. By induction we can assume that g_m is equivalent to $[g_i, g_{m-2^i-1}]$ modulo $\Gamma(N_{i+1})$ for every $i + 2^i + 1 < m < n$. Then due to Proposition 6.2.10 there exist $2^{2^{n-2^i-1}}$ generators that are equal to $[g_i, g_{n-2^i-1}]$ modulo $\Gamma(N_{i+1})$. So we can choose g_n to be one of these generators. $\square$

Now we can rephrase Proposition 6.2.10.

Corollary 6.2.12. *For every i and j such that $i < j \leq i + 2^i + 1$ and every generator h of N_j such that $h \notin \Gamma(N_i)$, the group N_{j+2^i+1} has 2^{2^j} generators which are all equal to $[g_i, h]$ modulo $\Gamma(N_{i+1})$ and all other generators of N_{j+2^i+1} lie in $\Gamma(N_{i+1})$.*

6.3 Metric properties of the sequence

In this section we will define a new metric on $\Box_{(N_n)}F_2$, using the generating sets of the subgroups N_n defined in the previous section. Then we will use this metric to show that $\Box_{(N_n)}F_2$ embeds coarsely into a Hilbert space.
First we define a pseudo-metric d_T^m on N_n/N_m relative to a maximal spanning tree T in F_2/N_n and $n < m \leq n + 2^n + 1$. Consider the elements of N_n corresponding to a loop in F_2/N_n that only crosses one edge that is not in T. This set provides a generating set of N_n. Now d_T^m is the word metric on N_n/N_m according to this generating set. For any other element $x \in F_2/N_m$ there exists a unique lift of T containing x. The neutral

element $e \in F_2/N_n$ gets lifted to an element $y \in N_n/N_m$. Now we can identify x with y, i.e. $d_T^m(x, y) = 0$. This defines a pseudo-metric on F_2/N_m.

In order to control these pseudo-metrics, we will show that the isomorphism given in Proposition 6.2.3 is isometric for these pseudo-metrics. Note that λ is the left multiplication action on any group, i.e. $\lambda_x(y) = xy$.

Proposition 6.3.1. *Let α be the automorphism of F_2 such that $\alpha(a) = b$ and $\alpha(b) = a$. For every i and j with $i < j < m \leq i + 2^i + 1$ and every $x \in F_2$ we have that the following two maps are isometric isomorphisms:*

$$\varphi_x\colon \left(\frac{N_j}{N_m}, d^m_{\lambda_x(T_j)}\right) \to \left(\frac{[N_i,N_j]\Gamma(N_{i+1})}{[N_i,N_m]\Gamma(N_{i+1})}, d^{m+2^i+1}_{\lambda_x(T_{j+2^i+1})}\right)\colon y \mapsto [\lambda_x(g_i), y] \quad \textit{and}$$
$$\varphi'_x\colon \left(\frac{N_j}{N_m}, d^m_{\lambda_x\circ\alpha(T_j)}\right) \to \left(\frac{[N_i,N_j]\Gamma(N_{i+1})}{[N_i,N_m]\Gamma(N_{i+1})}, d^{m+2^i+1}_{\alpha\circ\lambda_x(T_{j+2^i+1})}\right)\colon y \mapsto [\alpha\circ\lambda_x(g_i), y].$$

Proof. Let $x \in F_2$, let φ be either φ_x or φ'_x and σ be equal to either λ_x or $\lambda_x \circ \alpha$ respectively. Then $\varphi\colon \left(\frac{N_j}{N_m}, d^m_{\sigma(T_j)}\right) \to \left(\frac{[N_i,N_j]\Gamma(N_{i+1})}{[N_i,N_m]\Gamma(N_{i+1})}, d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}\right)\colon y \mapsto [\sigma(g_i), y]$. Due to Proposition 6.2.3 and Proposition 6.2.4 we know that $\frac{N_{i+1}}{\Gamma(N_i)} \to \frac{[N_i,N_{i+1}]\Gamma(N_{i+1})}{\Gamma(N_{i+1})}\colon x \mapsto [\sigma(g_i), x]$ is an isomorphism. In that map N_j and N_m get mapped to $[N_i,N_j]\Gamma(N_{i+1})$ and $[N_i,N_m]\Gamma(N_{i+1})$ respectively, therefore φ is also an isomorphism. To show that it is isometric it suffices to check that for x and y in $\frac{N_j}{N_m}$ we have that $d^m_{\sigma(T_j)}(x, y) = 1$ if and only if $d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}(\varphi(x), \varphi(y)) = 1$, because both metrics are word metrics.

Suppose that $d^m_{\sigma(T_j)}(x, y) = 1$, then without loss of generality $\sigma^{-1}(x^{-1}y)$ is a generator of N_j/N_m which is non-trivial and therefore the corresponding generator of N_j does not lie in $\Gamma(N_i)$. Due to Corollary 6.2.12 we know that there exists a generator of $[N_i,N_j]\Gamma(N_{i+1}) = N_{j+2^i+1}$ which is equal to $[g_i, \sigma^{-1}(x^{-1}y)] = \sigma^{-1}([\sigma(g_i), x^{-1}y])$ modulo $\Gamma(N_{i+1})$. As φ is an isomorphism, we have that $\varphi(x) \neq \varphi(y)$ and $[\sigma(g_i), x^{-1}y] = [\sigma(g_i), x]^{-1}[\sigma(g_i), y]$, so we can conclude that $1 \leq d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}(\varphi(x), \varphi(y)) = d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}([\sigma(g_i), x], [\sigma(g_i), y]) \leq 1$, so $d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}(\varphi(x), \varphi(y)) = 1$.
Suppose that $d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}(\varphi(x), \varphi(y)) = d^{m+2^i+1}_{\sigma(T_{j+2^i+1})}([\sigma(g_i), x], [\sigma(g_i), y]) = 1$. Then without loss of generality we know that $\sigma^{-1}([\sigma(g_i), x])^{-1}\sigma^{-1}([\sigma(g_i), y]) = [g_i, \sigma^{-1}(x^{-1}y)]$ is a non-trivial generator of the group $[N_i,N_j]\Gamma(N_{i+1})/[N_i,N_m]\Gamma(N_{i+1})$. Due to Corollary 6.2.12 we know that there exists a generator h of N_j with $h \notin \Gamma(N_i)$ and $\varphi(hN_m) = [g_i, \sigma^{-1}(x^{-1}y)]$. So $[g_i, h^{-1}\sigma^{-1}(x^{-1}y)] = 1$ in $[N_i,N_j]\Gamma(N_{i+1})/[N_i,N_m]\Gamma(N_{i+1})$ and as φ is an isomorphism we have that $h^{-1}\sigma^{-1}(x^{-1}y) = 1$ in N_j/N_m. Therefore $d^m_{\sigma(T_j)}(x, y) = 1$, because h is a generator of N_j.
So φ is an isometric isomorphism. □

Now we want to define a new distance on $\square_{(N_i)}F_2$ which is coarsely equivalent to the usual distance and isometrically embeds into ℓ^1. This new distance is a linear combination of the pseudo-metrics d_T^m. First we add these pseudo-metrics such that the result is invariant for the left multiplication action.

Let α be the automorphism on F_2 that switches its generators a and b and let $A_{n,m}$ be equal to $\{g : g \text{ generator of } N_n, g \notin N_m\}$. For every n and m with $n < m \leq n + 2^n + 1$ we take d_n^m such that for every x and y in F_2/N_m we have that

$$d_n^m(x, y) = \frac{2^{n-m}}{|A_{n,m}|}\sum_{z\in F_2/N_m}\left(d^m_{\lambda_z(T_n)}\left(C_x^{\lambda_z(T_n)}, C_y^{\lambda_z(T_n)}\right) + d^m_{\lambda_z(\alpha(T_n))}\left(C_x^{\lambda_z(\alpha(T_n))}, C_y^{\lambda_z(\alpha(T_n))}\right)\right). \tag{6.1}$$

Here C_x^T is the element of N_n/N_m that is contained in the same lift of T as x.
Now we will prove some properties of these pseudo-metrics. First we show that it is invariant for the left multiplication action.

Proposition 6.3.2. *For every n and m with $n \leq m \leq n + 2^n + 1$, every x in F_2/N_m and every y in N_n/N_m we have that $d_n^m(x, xy) = d_n^m(e, y)$.*

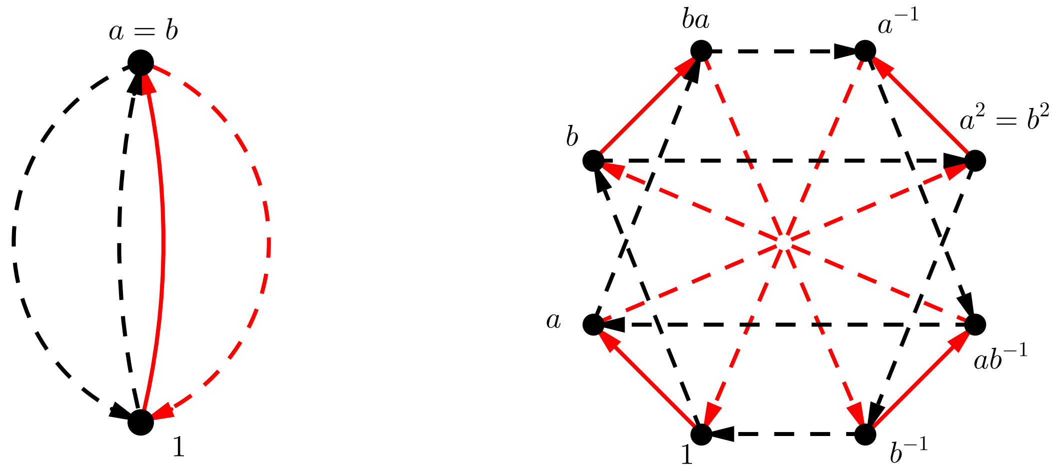


Figure 6.4: On the left we have the graphs of F_2/N_1 with the maximal spanning tree T_1 in full lines for $g_0 = a$. On the right we have the graph of F_2/N_3 with the lifts of T_1. These lifts of T_1 are called slices. When we contract these slices to the unique vertex that corresponds to an element of N_1/N_3 we find the Cayley graph of N_1/N_3 with the generating set corresponding to T_1. In Eq. (6.1) we have C_x^T, which is the element of N_1/N_3 to which the slice of x gets contracted.
Note: you could also say that C_x^T is the slice itself and the distance is induced by the contractions.

Proof. This proposition is shown by the following computation:

$$\begin{aligned}
d_n^m(x,xy) &= \frac{2^{n-m}}{|A_{n,m}|}\sum_{z\in F_2/N_m}\left(d^m_{\lambda_z(T_n)}\left(C_x^{\lambda_z(T_n)}, C_{xy}^{\lambda_z(T_n)}\right) + d^m_{\lambda_z(\alpha(T_n))}\left(C_x^{\lambda_z(\alpha(T_n))}, C_{xy}^{\lambda_z(\alpha(T_n))}\right)\right) \\
&= \frac{2^{n-m}}{|A_{n,m}|}\sum_{z\in F_2/N_m}\left(d^m_{\lambda_{x^{-1}z}(T_n)}\left(C_e^{\lambda_{x^{-1}z}(T_n)}, C_y^{\lambda_{x^{-1}z}(T_n)}\right)\right. \\
&\quad \left. + \, d^m_{\lambda_{x^{-1}z}(\alpha(T_n))}\left(C_e^{\lambda_{x^{-1}z}(\alpha(T_n))}, C_y^{\lambda_{x^{-1}z}(\alpha(T_n))}\right)\right) \\
&= \frac{2^{n-m}}{|A_{n,m}|}\sum_{z\in F_2/N_m}\left(d^m_{\lambda_z(T_n)}\left(C_e^{\lambda_z(T_n)}, C_y^{\lambda_z(T_n)}\right) + d^m_{\lambda_z(\alpha(T_n))}\left(C_e^{\lambda_z(\alpha(T_n))}, C_y^{\lambda_z(\alpha(T_n))}\right)\right) \\
&= d_n^m(e,y).
\end{aligned}$$

□

Next we show that these pseudo-metrics are 1-Lipschitz compared to the graph metric.

Proposition 6.3.3. *For every n and m with $n \leq m \leq n + 2^n + 1$ and every x and y in F_2/N_m we have that $d_n^m(x,y) \leq d(x,y)$, where d is the graph distance on F_2/N_m.*

Proof. As d is a graph distance, it suffices to show the proposition for $d(x,y) = 1$. Let x and y be such elements with $d(x,y) = 1$, let e be the projection of the edge between x and y on F_2/N_n and let T be any maximal spanning tree of F_2/N_n. Then we have that $d_T^m(x,y) = 1$ if and only if e is not and edge of T and the generator of N_n that corresponds to $e \notin T$ is not in N_m. If we shift T_n and $\alpha(T_n)$ by every element of F_2/N_n, then the edge e gets mapped to every edge of F_2/N_n. So $d_n^m(x,y) = \frac{2^{n-m}}{|A_{n,m}|}|A_{n,m}|[N_n, N_m] = 1$ □

Proposition 6.3.4. *For every i, m and k with $i < m < k \leq i + 2^i + 1$ and every x in F_2/N_k and every $y \neq e$ in N_m/N_k we have that $d^{k+2^i+1}_{m+2^i+1}(x, x[g_i, y]) = 2d_m^k(x, xy)$.*

Proof. Due to Proposition 6.2.4 we have that $[N_m : N_{m+2^i+i}] = 2^{2^i+1}$ and due to Corollary 6.2.12 we know that $|A_{m+2^i+1,k+2^i+1}| = 2^{2^i}|A_{m,k}|$. Combining these two results with Proposition 6.3.1 and Proposition 6.3.2

gives the following result:

$$\begin{aligned}
d^{k+2^i+1}_{m+2^i+1}(x,x[g_i,y]) &= d^{k+2^i+1}_{m+2^i+1}(e,[g_i,y]) \\
&= \frac{2^{m+2^i+1-k-2^i-1}}{|A_{m+2^i+1,k+2^i+1}|} \sum_{z\in F_2/N_{k+2^i+1}} \left(d^{k+2^i+1}_{\lambda_z(T_{m+2^i+1})} \left(C_e^{\lambda_z(T_{m+2^i+1})}, C_{[g_i,y]}^{\lambda_z(T_{m+2^i+1})}\right)\right. \\
&+ \left. d^{k+2^i+1}_{\lambda_z(\alpha(T_{m+2^i+1}))} \left(C_e^{\lambda_z(\alpha(T_{m+2^i+1}))}, C_{[g_i,y]}^{\lambda_z(\alpha(T_{m+2^i+1}))}\right)\right) \\
&= \frac{2^{m-k}2^{-2^i}}{|A_{m,k}|} \sum_{z\in F_2/N_{k+2^i+1}} \left(d^{k+2^i+1}_{\lambda_z(T_{m+2^i+1})}(e,[g_i,y]) + d^{k+2^i+1}_{\lambda_z(\alpha(T_{m+2^i+1}))}(e,[g_i,y])\right) \\
&= \frac{2^{m-k}}{|A_{m,k}|} 2^{-2^i} \sum_{z\in F_2/N_{k+2^i+1}} \left(d^k_{\lambda_z(T_m)}(e,y) + d^k_{\lambda_z(\alpha(T_m))}(e,y)\right) \\
&= \frac{2^{m-k}}{|A_{m,k}|} 2 \sum_{z\in F_2/N_k} \left(d^k_{\lambda_z(T_m)} \left(C_e^{\lambda_z(T_m)}, C_y^{\lambda_z(T_m)}\right)\right. \\
&+ \left. d^k_{\lambda_z(\alpha(T_m))} \left(C_e^{\lambda_z(\alpha(T_m))}, C_y^{\lambda_z(\alpha(T_m))}\right)\right) \\
&= 2d^k_m(e,y) \\
&= 2d^k_m(x,xy),
\end{aligned}$$

which proves the proposition. □

Proposition 6.3.5. *For every n and for every x in F_2/N_{n+1} we have that $d_n^{n+1}(x,xg_n) \geq \operatorname{girth}(F_2/N_n)$.*

Proof. For $n=0$ we have $d_0^1(x,xg_0) = 1 = \operatorname{girth}(F_2/N_0)$ and for $n=1$ we have $d_1^2(x,xg_1) = 2 = \operatorname{girth}(F_2/N_1)$. So let $n \geq 2$ and take i the biggest integer such that $n \geq i+2^i+1$.
First we look at the cases where $n > i+2^i+1$. Due to Corollary 6.2.11 we were able to take $g_n = [g_i, g_{n-2^i-1}]$, so due to Proposition 6.3.4 we have $d_n^{n+1}(x,xg_n) = d_n^{n+1}(x,x[g_i,g_{n-2^i-1}]) = 2d^{n-2^i}_{n-2^i-1}(x,xg_n)$. By induction we can conclude that $d_n^{n+1}(x,xg_n) \geq 2\operatorname{girth}(F_2/N_{n-2^i-1})$. So it suffices to show that $2\operatorname{girth}(F_2/N_{n-2^i-1}) \geq \operatorname{girth}(F_2/N_n)$. Note that $2\operatorname{girth}(F_2/N_i) = \operatorname{girth}(F_2/\Gamma(N_i))$, so due to Corollary 6.2.5 we can conclude that $2\operatorname{girth}(F_2/N_{n-2^i-1}) \geq 2\operatorname{girth}(F_2/N_{i+1}) \geq \operatorname{girth}(F_2/N_{i+2^{i+1}+2}) \geq \operatorname{girth}(F_2/N_n)$.
So now we only need to consider the case where $n = i+2^i+1$. Due to Proposition 6.3.2 we have that $d_n^{n+1}(x,xg_n) = d_n^{n+1}(e,g_n) = \frac{[F_2:N_{n+1}]}{|A_{n,n+1}|}$, due to Proposition 6.2.7 we have that $|A_{n,n+1}| = 2^{2^i}|A_{i,i+1}|$ and due to Proposition 6.2.4 we have that $[F_2:N_{n+1}] = 2^{2^i+1}[F_2:N_{i+1}]$. So by induction we can conclude that

$$d_n^{n+1}(x,xg_n) = \frac{[F_2:N_{n+1}]}{|A_{n,n+1}|} = 2\frac{[F_2:N_{i+1}]}{|A_{i,i+1}|} = 2d_i^{i+1}(e,g_i) \geq 2\operatorname{girth}(F_2/N_i) = \operatorname{girth}(F_2/N_n).$$

□

Proposition 6.3.6. *For every i and j such that $i < j < i+2^i+1$ and every x and y in F_2/N_{i+2^i+1} with $e \neq x^{-1}y \in N_j/N_{i+2^i+1}$ we have the following:*

- *We have that $d(x,y) \leq \operatorname{girth}(F_2/N_i)$ if and only if $d_j^{i+2^i+1}(x,y) \leq \operatorname{girth}(F_2/N_i)$.*
- *If $d(x,y) \leq \operatorname{girth}(F_2/N_i)$, then $d_j^{i+2^i+1}(x,y) = d(x,y)$.*

This proof is similar to the proof of Proposition 1 of [Khu14].

Proof. Note that $N_{i+2^i+1} = \Gamma(N_i)$, which is shown in Corollary 6.2.5.
Let x and y be in $F_2/\Gamma(N_i)$ and let γ be a geodesic from x to y. Consider $\bar{\gamma}$ the projection of γ onto F_2/N_i. If $\bar{\gamma}$ does not cross the same edge twice, then the projection of γ onto F_2/N_j does not cross the same edge twice, nor does it cross two edges that project to the same edge in F_2/N_i. So for every $z \in F_2/\Gamma(N_i)$ we have that

$$d^{i+2^i+1}_{\lambda_z(T_j)}\left(C_x^{\lambda_z(T_j)}, C_y^{\lambda_z(T_j)}\right) = \sum_{(x_1,x_2)\text{ edge in }\gamma} d^{i+2^i+1}_{\lambda_z(T_j)}\left(C_{x_1}^{\lambda_z(T_j)}, C_{x_2}^{\lambda_z(T_j)}\right)$$

and we have that

$$d^m_{\lambda_z(\alpha(T_n))}\left(C_x^{\lambda_z(\alpha(T_n))}, C_y^{\lambda_z(\alpha(T_n))}\right) = \sum_{(x_1,x_2)\text{ edge in }\gamma} d^m_{\lambda_z(\alpha(T_n))}\left(C_{x_1}^{\lambda_z(\alpha(T_n))}, C_{x_2}^{\lambda_z(\alpha(T_n))}\right).$$

Therefore we can conclude that $d_j^{i+2^i+1}(x,y) = \sum_{(x_1,x_2) \text{ edge in } \gamma} d_j^{i+2^i+1}(x_1,x_2) = (\text{length of } \gamma) = d(x,y)$.

If $d(x,y) \leq \operatorname{girth}(F_2/N_i)$, then $\bar{\gamma}$ does not cross the same edge twice. So it suffices to show that $d_j^{i+2^i+1}(x,y) \leq \operatorname{girth}(F_2/N_i)$ implies $d(x,y) \leq \operatorname{girth}(F_2/N_i)$ when $\bar{\gamma}$ does cross the same edge twice.

Let $\bar{x}$ and $\bar{y}$ be the projections of x and y respectively onto F_2/N_i. Consider $\bar{\gamma}$ without double edges, this is a path between $\bar{x}$ and $\bar{y}$ plus same additional disconnected loops. If there are no additional loops, then the path between $\bar{x}$ and $\bar{y}$ can be lifted to a path from x to y that is shorter than γ, because the projection of γ had double edges.
So $\bar{\gamma}$ contains a cycle γ' of single edges. Now the contribution of edges on γ' to the value $d_j^{i+2^i+1}(x,y)$ does not get undone, so $d_j^{i+2^i+1}(x,y) \geq (\text{length of } \gamma') \geq \operatorname{girth}(F_2/N_i)$. □

Corollary 6.3.7. *For every i and j such that $i < j < i + 2^i + 1$ and every x and y in $F_2/\Gamma(N_i)$ with $e \neq x^{-1}y \in N_j/\Gamma(N_i)$ we have $d_j^{i+2^i+1}(x,y) \geq \operatorname{girth}(F_2/N_i)$.*

Now we are almost ready to define the alternative metric on $\square_{(N_n)}F_2$. For any n and m with $n < m \leq n + 2^n + 1$ we define another pseudo-metric $\tilde{d}_n^m$ on F_2/N_m. For $n = 0$ we take $\tilde{d}_0^m = d_0^m$. For $n = 1$ and $m = 2$ we take $\tilde{d}_1^2 = d_1^2$. For $n = 1$ and $m = 3$ we take $\tilde{d}_1^3 = \frac{1}{2}d_1^2 + \frac{1}{2}d_2^3$. For $n = 1$ and $3 < m \leq 5$ we take $\tilde{d}_1^3 = \frac{1}{8}d_1^2 + \frac{1}{8}d_2^3 + \frac{3}{4}d_3^m$.
For $n \geq 2$ take i the biggest integer such that $i+2^i+1 \leq m$. If $m = i+2^i+1$, then $\tilde{d}_n^m = d_n^m$. If $m = i+2^i+2$ and $n = m-1$, then $\tilde{d}_n^m = d_n^m$. If $m = i+2^i+2$ and $n < m-1$, then $\tilde{d}_n^m = \frac{1}{2}d_n^{m-1} + \frac{1}{2}d_{m-1}^m$. Now let $m > i+2^i+2$. If $n > i + 2^i + 1$ then due to Proposition 6.3.4 we can lift the metric $\tilde{d}_{n-2^i-1}^{m-2^i-1}$ to a metric $\tilde{d}_n^m$ on F_2/N_m. If $n = i+2^i+1$, then we can take $\tilde{d}_n^m = \frac{1}{4}d_n^{n+1} + \frac{3}{4}\tilde{d}_{n+1}^m$. If $n < i+2^i+1$, then $\tilde{d}_n^m = \frac{1}{8}d_n^{i+2^i+1} + \frac{1}{8}d_{i+2^i+1}^{i+2^i+2} + \frac{3}{4}\tilde{d}_{i+2^i+2}^m$.

Proposition 6.3.8. *For every n and m such that $n < m \leq n + 2^n + 1$ and for every x and y in F_2/N_m with $x^{-1}y \in N_n/N_m$ and $x \neq y$ we have $\tilde{d}_n^m(x,y) \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}$.*

Proof. If $m = 1$ and therefore $n = 0$, then $d_n^m(x,y) = d_0^1(x,y) = 1 \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_1)}$ for $x \neq y$. For every other value of m we can take i to be the biggest integer such that $i + 2^i + 1 \leq m$. Let x and y in F_2/N_m with $x^{-1}y$ in N_n/N_m.

Note that for any $k \in \mathbb{N}$ we have that $2\operatorname{girth}(F_2/N_k) = \operatorname{girth}(F_2/\Gamma(N_k))$, $\operatorname{girth}(F_2/N_{k+1}) \geq \operatorname{girth}(F_2/N_k)$ and due to Corollary 6.2.5 we know that $N_{k+2^k+1} = \Gamma(N_k)$. So for any k and ℓ with $k + 2^k + 1 < \ell \leq k + 2^{k+1} + 2$ we have

$$2\operatorname{girth}(F_2/N_k) \geq \operatorname{girth}(F_2/N_{\ell-2^k-1}) \geq \frac{1}{2}\operatorname{girth}(F_2/N_\ell),$$

because $2\operatorname{girth}(F_2/N_{\ell-2^k-1}) \geq 2\operatorname{girth}(F_2/N_{k+1}) = \operatorname{girth}(F_2/N_{k+2^{k+1}+2}) \geq \operatorname{girth}(F_2/N_\ell)$.

If $m = i + 2^i + 1$, then $\tilde{d}_n^m = d_n^m$. Due to Corollary 6.3.7 we have that

$$\tilde{d}_n^m(x,y) \geq \operatorname{girth}(F_2/N_i) = \operatorname{girth}(F_2/N_m)/2 \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

If $m = i + 2^i + 2$, then we have two different cases $n = m - 1$ and $n < m - 1$. The first case is shown in Proposition 6.3.5 as $\tilde{d}_n^m = d_n^m$. In the second case we have $\tilde{d}_n^m = \frac{1}{2}d_n^{m-1} + \frac{1}{2}d_{m-1}^m$. So for $x^{-1}y$ not in N_{m-1}/N_m we have that $\tilde{d}_n^m(x,y) \geq \frac{1}{2}\operatorname{girth}(F_2/N_i) \geq \frac{1}{8}\operatorname{girth}(F_2/N_m) \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}$, due to Corollary 6.3.7. For $x^{-1}y$ in N_{m-1}/N_m on the other hand we have that $\tilde{d}_n^m(x,y) \geq \frac{1}{2}\operatorname{girth}(F_2/N_n) \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}$ due to Proposition 6.3.5.
Now we may suppose that $m > i + 2^i + 2$. If $n > i + 2^i + 1$, then due to Proposition 6.3.1 we can write $x^{-1}y$ as $[g_i, z]$ with $z \in N_{n-2^i-1}/N_{m-2^i-1}$. Since $\tilde{d}_n^m$ is the lift of $\tilde{d}_{n-2^i-1}^{m-2^i-1}$ as in Proposition 6.3.4, we have that $\tilde{d}_n^m(x,y) = \tilde{d}_n^m(x, x[g_i,z]) = 2\tilde{d}_{n-2^i-1}^{m-2^i-1}(x,xz)$. Therefore

$$\tilde{d}_n^m(x,y) \geq \frac{1}{8}\sqrt{\operatorname{girth}(F_2/N_{m-2^i-1})} \geq \frac{1}{8}\sqrt{\operatorname{girth}(F_2/N_m)/2} \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

If $n = i + 2^i + 1$, then $\tilde{d}_n^m = \frac{1}{4}d_n^{n+1} + \frac{3}{4}\tilde{d}_{n+1}^m$. If $x^{-1}y \notin N_{n+1}/N_m$, then due to Proposition 6.3.5 we have

$$\tilde{d}_n^m(x,y) \geq \frac{1}{4}\operatorname{girth}(F_2/N_n) \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

If on the other hand $x^{-1}y \in N_{n+1}/N_m$, then we have that

$$\tilde{d}_n^m(x,y) \geq \frac{3}{4}\tilde{d}_{n+1}^m(x,y) \geq \frac{1}{8}\sqrt{\frac{9}{16}\operatorname{girth}(F_2/N_m)/2} \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)},$$

because $\tilde{d}_{n+1}^m(x,y) \geq \frac{1}{8}\sqrt{\operatorname{girth}(F_2/N_m)/2}$.
At last if $n < i + 2^i + 1$, then $\tilde{d}_n^m = \frac{1}{8}d_n^{i+2^i+1} + \frac{1}{8}d_{i+2^i+1}^{i+2^i+2} + \frac{3}{4}\tilde{d}_{i+2^i+2}^m$. If $x^{-1}y \notin N_{i+2^i+1}/N_m$, then due to Corollary 6.3.7 we have

$$\tilde{d}_n^m(x,y) \geq \frac{1}{8}\operatorname{girth}(F_2/N_i) \geq \frac{1}{8}\sqrt{\operatorname{girth}(F_2/N_m)/4} \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

If $x^{-1}y \in N_{i+2^i+1}/N_m$ and $x^{-1}y \notin N_{i+2^i+2}/N_m$, then due to Proposition 6.3.5 we

$$\tilde{d}_{i+2^i+2}^m(x,y) \geq \frac{1}{8}\operatorname{girth}(F_2/N_{i+2^i+2}) \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

Finally if $x^{-1}y \in N_{i+2^i+2}/N_m$, then as before

$$\tilde{d}_n^m(x,y) \geq \frac{3}{4}\tilde{d}_{i+2^i+1}^m(x,y) \geq \frac{1}{8}\sqrt{\frac{9}{16}\operatorname{girth}(F_2/N_m)/2} \geq \frac{1}{16}\sqrt{\operatorname{girth}(F_2/N_m)}.$$

This proves the proposition. □

Now we are ready to define this new metric on $\Box_{(N_n)}F_2$. However we will immediately use this metric to prove that this box space embeds into a Hilbert space.

Theorem 6.3.9. *The box space $\Box_{N_n}F_2$ embeds into a Hilbert space.*

Proof. First we define an equivalent metric $(d_n)_n$ on $\Box_{N_n}F_2$. For $n \geq 2$ we can take the biggest i such that $i + 2^i + 1 \leq n$. For x and y in F_2/N_n take $d_n(x,y) = \frac{1}{2}d_i^{i+2^i+1}(\bar{x},\bar{y}) + \frac{1}{2}\tilde{d}_{i+2^i+1}^n(x,y)$ where $\bar{x}$ and $\bar{y}$ the projections of x and y respectively onto F_2/N_{i+2^i+1}.
Note that d_n is a linear combination of pseudo-metrics d_T^m on F_2/N_m with $m \in \mathbb{N}$ and T a maximal spanning tree on some quotient F_2/N_k with $k < m \leq k + 2^k + 1$. These pseudo-metrics are isometrically equivalent to the Cayley graph of $\mathbb{Z}_2^d$ with the standard generating set for some d. Therefore it embeds into ℓ^1 and its linear combinations embeds into ℓ^1 as well. So we have that $(\Box_{N_n}F_2, (d_n)_n)$ embeds isometrically into ℓ^1.
As ℓ^1 embeds into a Hilbert space, it suffices to show that $(d_n)_n$ is coarsely equivalent to the graph metric. Indeed, then $\Box_{(N_n)}F_2$ is coarsely equivalent to $(\Box_{N_n}F_2, (d_n)_n)$, which coarsely embeds into ℓ^1 and therefore coarsely embeds into a Hilbert space. Which would prove the theorem.

So we only need to show that $(d_n)_n$ is indeed coarsely equivalent to the usual graph metric. Due to Proposition 6.3.3 we have by induction that $d_n \leq d$, so as $\operatorname{girth}(F_2/N_n) \to \infty$ as $n \to \infty$, it suffices to prove that if $d(x,y) \leq \frac{1}{2}\operatorname{girth}(F_2/N_i)$, then $d_n(x,y) \geq \frac{1}{2}d(x,y)$ and if $d(x,y) > \frac{1}{2}\operatorname{girth}(F_2/N_i)$, then either $d_n(x,y) > \frac{1}{4}\operatorname{girth}(F_2/N_i)$ or $d_n(x,y) \geq \frac{1}{32}\sqrt{\operatorname{girth}(F_2/N_n)}$.
If $d(x,y) \leq \frac{1}{2}\operatorname{girth}(F_2/N_i)$, then $d(x,y) = d(\bar{x},\bar{y}) \leq \frac{1}{2}\operatorname{girth}(F_2/N_i)$ where $\bar{x}$ and $\bar{y}$ are the projections of x and y onto F_2/N_{i+2^i+1}, so due to Proposition 6.3.6 we have that $d_n(x,y) \geq \frac{1}{2}d_i^{i+2^i+1}(\bar{x},\bar{y}) = \frac{1}{2}d(\bar{x},\bar{y}) = \frac{1}{2}d(x,y)$.
Now suppose that $d(x,y) > \frac{1}{2}\operatorname{girth}(F_2/N_i)$ and $d_n(x,y) \leq \frac{1}{4}\operatorname{girth}(F_2/N_i)$. Then it suffices to show that $d_n(x,y) \geq \frac{1}{32}\sqrt{\operatorname{girth}(F_2/N_n)}$. As $d_n(x,y) \leq \frac{1}{4}\operatorname{girth}(F_2/N_i)$ we have that $d_i^{i+2^i+1}(\bar{x},\bar{y}) \leq \frac{1}{2}\operatorname{girth}(F_2/N_i)$, so $d(\bar{x},\bar{y}) = d_i^{i+2^i+1}(\bar{x},\bar{y}) \leq \operatorname{girth}(F_2/N_i)$ due to Proposition 6.3.6. Take z such that $d(x,zy) = d(\bar{x},\bar{y})$. Then we can make the following computations:

$$\begin{aligned}
d_n(x,y) &= \frac{1}{2}d_i^{i+2^i+1}(\bar{x},\bar{y}) + \frac{1}{2}\tilde{d}_{i+2^i+1}^n(x,y) \\
&\geq \frac{1}{2}d(\bar{x},\bar{y}) - \frac{1}{2}\tilde{d}_{i+2^i+1}^n(x,zy) + \frac{1}{2}\tilde{d}_{i+2^i+1}^n(zy,y) \\
&\geq \frac{1}{2}\tilde{d}_{i+2^i+1}^n(zy,y).
\end{aligned}$$

Due to Proposition 6.3.8 we can conclude that $d_n(x,y) \geq \frac{1}{32}\sqrt{\operatorname{girth}(F_2/N_n)}$.
Therefore $(d_n)_n$ is coarsely equivalent to the usual graph metric and so $\Box_{(N_i)}F_2$ embeds coarsely into a Hilbert space. □

Chapter 7

The asymptotic dimension of box spaces of virtually nilpotent groups

This chapter is based on [DT18], joint work with Tointon. Here we prove that the box spaces of virtually nilpotent groups have finite asymptotic dimension.

The chapter is organised as follows. In section 7.1 we present some basic facts about box spaces and about asymptotic dimension; in section 7.2 we compute the asymptotic dimension of certain box spaces in terms of the asymptotic dimensions of the groups they are constructed from; and then finally, in section 7.3, we bound the asymptotic dimension of box spaces of groups of polynomial growth in terms of the growth rate and deduce Theorem 2.2.5.

7.1 Background

In this section we collect together various results about asymptotic dimension.

Recall that asymptotic dimension has several equivalent alternative definitions.

Proposition 7.1.1 ([BD08, Theorem 19]). *Let X be a metric space. Then the following are equivalent:*

- $\operatorname{asdim}(X) \leq n$*,*
- *for every $R > 0$ there exists $S > 0$ and a covering $\mathcal{U}$ of X such that $\mathcal{U}$ has R-multiplicity at most $n+1$ and* $\operatorname{diam}(U) \leq S$ *for every $U \in \mathcal{U}$,*
- *for every $R > 0$ there exist families $\mathcal{U}_0, \dots, \mathcal{U}_n$ such that the union of these families is a uniformly bounded covering of X and every U and V in the same family are R-disjoint, for every $u \in U$ and $v \in V$ we have that $d(u, v) > R$.*

Lemma 7.1.2 ([BD01, Finite Union Theorem]). *Let X be a metric space and let $X_1, \dots, X_n$ be a finite partition of X. Then* $\operatorname{asdim} X = \max(\operatorname{asdim} X_i)$*.*

Let $\mathcal{U}$ be a family of metric spaces. We say that $\mathcal{U}$ has asymptotic dimension at most n *uniformly*, and write $\operatorname{asdim}\mathcal{U} \leq_{\text{unif}} n$, if for every $R > 0$ there exists $S > 0$ such that $(R, S)\text{-}\dim X \leq n$ for every $X \in \mathcal{U}$. This definition is particularly useful to us in light of the following result.

Lemma 7.1.3 ([BD01, Theorem 1]). *Let X be a metric space, and let $\mathcal{U}$ be a family of subspaces that covers X. Suppose that* $\operatorname{asdim}\mathcal{U} \leq_{\text{unif}} n$*, and that for every $k \in \mathbb{N}$ there exists $F_k \subset X$ with* $\operatorname{asdim} F_k \leq n$ *such that the family $\{Y \backslash F_k : Y \in \mathcal{U}\}$ is k-disjoint. Then* $\operatorname{asdim} X \leq n$*.*

Corollary 7.1.4. *If X is a coarse disjoint union of metric spaces $(X_n)_{n=1}^{\infty}$ then* $\operatorname{asdim}(X_n) \leq_{\text{unif}} m$ *if and only if* $\operatorname{asdim} X \leq m$*.*

We also record the following trivial fact as a lemma for ease of later reference.

Lemma 7.1.5. *Let X be a metric space, and let $\mathcal{U}$ be a family of metric spaces each of which is isometric to a subspace of X. Then* $\operatorname{asdim}\mathcal{U} \leq_{\text{unif}} \operatorname{asdim} X$*.*

The following result is presumably well known, although we could not find a reference.

Lemma 7.1.6. *Let G be a finitely generated infinite group, and let B be a coarse disjoint union of the balls $B_G(e, r)$ as r ranges over the natural numbers. Then* $\operatorname{asdim} B = \operatorname{asdim} G$*.*

Proof. The fact that $\operatorname{asdim} B \leq \operatorname{asdim} G$ follows from Corollary 7.1.4 and Lemma 7.1.5.

To prove that $\operatorname{asdim} G \leq \operatorname{asdim} B$ it suffices to show that $(R,S)-\dim G \leq (R,S)-\dim B$ for every $R, S > 0$. To that end, fix $R, S > 0$ and suppose that $(R,S) - \dim B = n \in \mathbb{Z}$, so that there exist R-disjoint families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of B that cover B such that $\operatorname{diam}(U) \leq S$ for every $U \in \mathcal{U}_j$ and $j \in \{0, 1, ..., n\}$.

We partition G into sets $U_0, \ldots, U_n$ as follows. First, enumerate the elements of G as $x_1, x_2, x_3, \ldots$ in such a way that $|x_m|$ is non-decreasing. We will specify for the x_m in turn which set U_j will contain x_m. Note that for each m and each $r \geq |x_m|$ there is a copy of x_m lying in the component $B_G(e, r)$ of B.

There exists i_1 and an infinite sequence $r_{1,1} < r_{1,2} < \ldots$ such that for each $r_{1,j}$ the copy of x_1 in the component $B_G(e, r_{1,j})$ lies in a set belonging to $\mathcal{U}_{i_1}$. We declare that $x_1 \in U_{i_1}$. Similarly, there exists i_2 and an infinite subsequence $r_{2,1} < r_{2,2} < \ldots$ of $r_{1,1} < r_{1,2} < \ldots$ such that for each $r_{2,j}$ the copy of x_2 in the component $B_G(e, r_{2,j})$ lies in a set belonging to $\mathcal{U}_{i_2}$. We declare that $x_2 \in U_{i_2}$. Continuing in this way, for each m in turn there exists i_m and an infinite subsequence $r_{m,1} < r_{m,2} < \ldots$ of $r_{m-1,1} < r_{m-1,2} < \ldots$ such that for each $r_{m,j}$ the copy of x_m in the component $B_G(e, r_{m,j})$ lies in a set belonging to $\mathcal{U}_{i_m}$. We declare that $x_m \in U_{i_m}$.

It follows from the definition of the $\mathcal{U}_i$ that each U_i can be partitioned into subsets of diameter at most S that are R-disjoint, and this completes the proof. □

7.2 The asymptotic dimension of arbitrary box spaces

Yamauchi [Yam17, Theorem 1.3] shows that a coarse disjoint union of a sequence of graphs with girth tending to infinity has asymptotic dimension either infinite or at most 1. The following is an adaptation of his argument. We are grateful to Rufus Willett (private communication) for pointing it out to us.

Proposition 7.2.1. *Let G be an infinite, residually finite, finitely generated group and let N_n be a filtration. Then* $\operatorname{asdim}\left(\square_{(N_n)} G\right)$ *is either infinite or equal to* $\operatorname{asdim} G$.

Proof. We may assume that $\operatorname{asdim}\left(\square_{(N_n)} G\right) < \infty$, and so $\operatorname{asdim}\left(\square_{(N_n)} G\right) = m$ for some $m \in \mathbb{N}$. We first prove that $\operatorname{asdim}\left(\square_{(N_n)} G\right) \leq \operatorname{asdim} G$.

By definition, for every $k \in \mathbb{N}$ there exists $S_k \in \mathbb{N}$ such that (k, S_k)-$\dim\left(\square_{(N_n)} G\right) \leq m$. For every such k there therefore exist k-disjoint families $\mathcal{U}_0^k, \mathcal{U}_1^k, \ldots, \mathcal{U}_m^k$ of subsets of $\square_{(N_n)} G$, such that $\operatorname{diam}(U) \leq S_k$ for every $U \in \mathcal{U}_j^k$ and such that $\bigcup_{j=1}^m \mathcal{U}_j^k$ covers $\square_{(N_n)} G$. By Proposition 1.4.9 we can take a sequence $i_k \to \infty$ such that for every $i \geq i_k$ the balls of radius $\max\{k, S_k\}$ in G/N_i are isometric to balls of radius $\max\{k, S_k\}$ in G. Without loss of generality we may assume that $(i_k)_k$ is a non-decreasing sequence.

If $U \in \mathcal{U}_j^k$ satisfies $U \subset G/N_i$ for some i then U is contained in a ball of radius at most S_k inside G/N_i, and if this i is at least i_k then U is isometric to a subspace of G. Lemma 7.1.5 therefore implies that

$$\operatorname{asdim}\left(\bigcup_{j,k} \{U \in \mathcal{U}_j^k : U \subset G/N_i \text{ for some } i \geq i_k\}\right) \leq_{\text{unif}} \operatorname{asdim} G. \tag{7.1}$$

Now if $i \geq i_k$ then G/N_i is at a distance greater than S_k from its complement in $\square_{(N_n)} G$, and so if $U \in \mathcal{U}_j^k$ intersects G/N_i non-trivially then in fact $U \subset G/N_i$. This implies that for every $i \geq i_k$ the set G/N_i is covered by the sets $U \in \mathcal{U}_j^k$ with $U \subset G/N_i$ and $j \in \{0, \ldots, m\}$. This means that, defining families $\mathcal{Y}_j^k$ of subsets of $\square_{(N_n)} G$ for $j \in \{0, \ldots, m\}$ and $k \in \mathbb{N}$ via

$$\mathcal{Y}_j^k = \{U \in \mathcal{U}_j^k : U \subset G/N_i \text{ for some } i \in [i_k, i_{k+1})\},$$

and then defining families $\mathcal{Y}_j^{\geq k}$ via

$$\mathcal{Y}_j^{\geq k} = \bigcup_{k' \geq k} \mathcal{Y}_j^{k'},$$

for each k the set $\bigcup_{i=i_k}^\infty G/N_i$ is covered by the families $\mathcal{Y}_j^{\geq k}$ with $j \in \{0, \ldots, m\}$. In particular, setting $F = \bigcup_{i=1}^{i_1 - 1} G/N_i$ and $X_j = \bigcup_{Y \in \mathcal{Y}_j^{\geq 1}} Y$ we have

$$\square_{(N_n)} G = F \cup \bigcup_{j=0}^m X_j. \tag{7.2}$$

Note that for each j we have $\operatorname{asdim} \mathcal{Y}_j^{\geq 1} \leq_{\text{unif}} \operatorname{asdim} G$ by (7.1). Note also that each family $\mathcal{Y}_j^{\geq k}$ is k-disjoint by the definitions of $\mathcal{U}_j^k$ and i_k. Finally, note that if we write $F_k = \bigcup_{i=1}^{i_k - 1} G/N_i$ then we have $\mathcal{Y}_j^{\geq k} = \{Y \backslash F_k : Y \in \mathcal{Y}_j^{\geq 1}\}$. Since F_k is finite, and hence of asymptotic dimension 0, it therefore follows from Lemma 7.1.3 that $\operatorname{asdim} X_j \leq \operatorname{asdim} G$, and then from (7.2) and Lemma 7.1.2 that

$$\operatorname{asdim}\left(\square_{(N_n)} G\right) \leq \operatorname{asdim} G,$$

as desired.

Conversely, since N_n is a filtration there is a subspace B of $\Box_{(N_n)}G$ that is isometric to a coarse disjoint union of the balls $B_G(e,R)$ as R ranges over the natural numbers. It follows from Lemma 7.1.6 that $\operatorname{asdim} B = \operatorname{asdim} G$, and since B is a subspace of $\Box_{(N_n)}G$ this implies that

$$\operatorname{asdim}\left(\Box_{(N_n)}G\right) \geq \operatorname{asdim} G,$$

which completes the proof. □

7.3 Coarse disjoint unions of groups of polynomial growth

Given a group G with a fixed finite generating set S we write $B_G(x,R)$ for the ball of radius R about the element $x \in G$ in the Cayley graph $\operatorname{Cay}(G,S)$. The group G is said to have *polynomial growth of degree* d if there exists $C > 0$ such that $|B_G(e,r)| \leq Cr^d$ for every $r \geq 1$. It is well known and easy to check that this notion does not depend on the choice of finite generating set.

We say that a family $(G_\alpha)_{\alpha\in A}$ of groups with fixed generating sets S_α has *uniform polynomial growth of degree at most* d if there exists $C > 0$ such that $|B_{G_\alpha}(e,r)| \leq Cr^d$ for every $r \geq 1$ for every $\alpha \in A$.

Proposition 7.3.1. *Let G_n be a sequence of finite groups with generating sets S_n, and suppose that the sequence $(\operatorname{Cay}(G_n,S_n))_{n=1}^\infty$ has uniform polynomial growth of degree at most d. Let X be a coarse disjoint union of the $\operatorname{Cay}(G_n,S_n)$. Then $\operatorname{asdim} X \leq 4^d$.*

Proof. We start with the standard observation that a polynomial growth bound implies a so-called *doubling* condition, as used by Gromov [Gro81] in his proof of his polynomial-growth theorem, for example. Specifically, let $K = 4^d + 1$, let $R > 0$, and take $S_0 = 4^{m+1}R$ with m such that $(K/4^d)^m \geq CR^d$. Then we claim that for every n there exists an R_n such that $R \leq R_n \leq \frac{S_0}{4}$ and $|B_{G_n}(4R_n)| \leq K|B_{G_n}(R_n)|$. Indeed, if this were not the case then $K^i|B_{G_n}(R)| < |B_{G_n}(4^iR)| \leq C4^{id}R^d$ for every i with $4^iR \leq S_0$, and so setting $i = m$ would imply that $C4^{md}R^d > K^m|B_{G_n}(R)| \geq C4^{md}R^d|B_{G_n}(R)|$, contradicting the growth assumption.

Following Ruzsa [Ruz99], for every $n \in \mathbb{N}$ with $\operatorname{diam}(G_n) > R$ we take X_n maximal in G_n such that $B_{G_n}(x,R_n)$ and $B_{G_n}(y,R_n)$ are disjoint for every x and y in X_n. We then define $F_R = \bigcup_{n\,:\,\operatorname{diam}(G_n)\leq R} G_n$, take

$$\mathcal{U} = \{F_R\} \cup \bigcup_{n\,:\,\operatorname{diam}(G_n)>R} \{B_{G_n}(x,2R_n) : x \in X_n\},$$

and set $S = \max\{S_0, \operatorname{diam} F_R\}$.

First we show that $\mathcal{U}$ is a covering of $\bigsqcup_n G_n$. Let $z \in G_m$ for some m. If $\operatorname{diam}(G_m) \leq R$, then $z \in F_R$. If $\operatorname{diam}(G_m) > R$, then as X_m is maximal, there exists an $x \in X_m$ such that $B_{G_m}(z,R_m) \cap B_{G_m}(x,R_m)$ is non-empty, so $z \in B_{G_m}(x,2R_m)$.

Next we note that $\operatorname{diam}(U) \leq S$ for every $U \in \mathcal{U}$. For $U = F_R$ this is true by definition of S. On the other hand, for $U \in \mathcal{U}$ with $U \neq F_R$ then $U \subset G_m$ for some m and $\operatorname{diam}(U) = 4R_m \leq S_0 \leq S$.

Finally, we show that $\mathcal{U}$ has R-multiplicity at most K. The R-multiplicity of $\mathcal{U}$ in G_n with $\operatorname{diam}(G_n) \leq R$ is $1 \leq K$, so take $z \in G/N_m$ with m such that $\operatorname{diam}(G_m) \geq R$. Now for every $B_{G_m}(x,2R_m) \in \mathcal{U}$ which has an element at a distance at most R to z, we have that $x \in B_{G_m}(z,2R_m+R) \subset B_{G_m}(z,3R_m)$. Now consider $B_{G_m}(z,3R_m) \cap X_m$. As $B_{G/N_m}(x,R_m)$ and $B_{G/N_m}(y,R_m)$ are disjoint for any x and y in X_m, we have that $|B_{G_m}(z,3R_m) \cap X_m| \leq \frac{|B_{G_m}(z,4R_m)|}{|B_{G_m}(z,R_m)|} \leq K$. Therefore the R-multiplicity of $\mathcal{U}$ is at most $K = 4^d+1$, and so $\operatorname{asdim} \bigsqcup_n G_n \leq 4^d$ by Proposition 7.1.1. □

Corollary 7.3.2. *Let G be a finitely generated virtually nilpotent group. Then $\Box_f G$ has finite asymptotic dimension.*

Proof. As G is virtually nilpotent there exist constants k and C such that $|B_G(e,r)| \leq Cr^k$ for every $r \geq 1$. This also means that $|B_{G/N}(e,r)| \leq Cr^k$ for every $r \geq 1$ for every $N \lhd G$, and so the result follows from Proposition 7.3.1. □

Proof of Theorem 2.2.5. Since $\Box_{(N_n)}G$ is a subspace of $\Box_f G$ we have $\operatorname{asdim}\Box_{(N_n)}G \leq \operatorname{asdim}\Box_f G < \infty$ by Corollary 7.3.2. Proposition 7.2.1 therefore implies that $\operatorname{asdim}\Box_{(N_n)}G = \operatorname{asdim}(G)$. Since G is virtually polycyclic, the result therefore follows from (2.1). □

Appendix A

Large girth and asymptotic dimension

This appendix is based on some unpublished work. Here we show that sequences of large girth do not have asymptotic dimension 2. This work was never made public because of the publication of [Yam17].

A.1 Preliminaries

In this appendix we restrict to a specific kind of metrization of the disjoint union. We define the metric relative to a sequence ℓ_n in $\mathbb{R}^+$.

Definition A.1.1. *Let (X_n, d_n) be a sequence of finite metric spaces and let ℓ_n be a sequence in $\mathbb{R}^+$ such that $\ell_n \geq \operatorname{diam}(X_n)$ and $\ell_n \to \infty$ as $n \to \infty$. Then the metrized disjoint union $X = \coprod_n X_n$ corresponding to the sequence ℓ_n is the disjoint union with the metric d defined such that $d(x,y) = d_n(x,y)$ if $x, y \in X_n$ and $d(x,y) = \ell_n + \ell_m$ if $x \in X_n$, $y \in X_m$ and $n \neq m$.*

In this appendix we also use an alternative definition of asyptotic dimension.

Definition A.1.2. *Let X be a metric space and let $n \in \mathbb{N}$, then $\operatorname{asdim} X \leq n$ if and only if for every ε there is a uniformly cobounded, ε-Lipschitz map $\varphi\colon X \to K$ where K is a simplicial complex of dimension n.*

This definition of asymptotic dimension is equivalent to usual definitions. This is shown in Theorem 19 of [BD08]. This property will not depend on the sequence ℓ_n we used to define the metriced disjoint union.

Proposition A.1.3. *Let $(X_n)_n$ be a sequence of finite metric spaces and let $X = \coprod_n X_n$. Then for any $m \in \mathbb{N}$ we have that $\operatorname{asdim}(X) \leq m$ if and only if for every $\varepsilon > 0$ there exists an increasing map $\rho\colon \mathbb{N} \to \mathbb{R}$ such that $\rho(n) \to +\infty$ as $n \to \infty$, there exist simplicial complexes K_n of dimension m and there exist maps $\varphi_n\colon X_n \to K_n$ such that for every n and every $x, y \in X_n$ we have that $\rho(d(x,y))) \leq d(\varphi_n(x), \varphi_n(y)) \leq \varepsilon d(x,y)$.*

Proof. If $\operatorname{asdim}(X) \leq m$, then by definition for every ε there exists a uniformly cobounded, ε-Lipschitz map $\varphi\colon X \to K$ where K is a simplicial complex of dimension m. For any n let φ_n be the restriction of φ to X_n. These maps are ε-Lipschitz and they are uniformly cobounded in a uniform way. This proves one implication of the proposition.
For the other implication let $\varepsilon > 0$. Then there exists an increasing map $\rho\colon \mathbb{N} \to \mathbb{R}$ such that $\rho(k) \to +\infty$ as $k \to \infty$, there exist simplicial complexes K_n of dimension m and there exist maps $\varphi_n\colon X_n \to K_n$ such that for every n and every $x, y \in X_n$ we have that $\rho(d(x,y))) \leq d(\varphi_n(x), \varphi_n(y)) \leq \varepsilon d(x,y)$.
To construct K take a point k. For every n consider $\lfloor \varepsilon \ell_n \rfloor - 1$. For n such that $\lfloor \varepsilon \ell_n \rfloor - 1$ is larger than zero connect each vertex of K_n to k via a path of length $\lfloor \varepsilon \ell_n \rfloor - 1$, as shown in Fig. A.1. Now define φ for every $x \in X_n$, if $\lfloor \varepsilon \ell_n \rfloor - 1$ is equal to 0 or -1, then $\varphi(x) = k$, otherwise take $\varphi(x) = \varphi_n(x)$ as K_n is embedded in K.
Now we need to show that φ is ε-Lipschitz and uniformly cobounded. To show that is ε-Lipschitz, take $x \in X_i$ and $y \in X_j$. If $i = j$, then $d(\varphi(x), \varphi(y)) \leq \varepsilon d(x,y)$, either because $d(\varphi(x), \varphi(y)) = d(\varphi_i(x), \varphi_i(y))$ or because $d(\varphi(x), \varphi(y)) = 0$. If $i \neq j$, then $d(\varphi(x), \varphi(y)) \leq 1 + (\lfloor \varepsilon \ell_i \rfloor - 1) + (\lfloor \varepsilon \ell_j \rfloor - 1) + 1 = \lfloor \varepsilon \ell_i \rfloor + \lfloor \varepsilon \ell_j \rfloor \leq \varepsilon(\ell_i + \ell_j)$.
Now it suffices to show that φ is uniformly cobounded. To do so define $\rho'(x) = \rho(x) - 2$ and let $x \in X_i$ and $y \in X_j$. If $i = j$, then as ρ is an increasing map we have that $\rho'(d(x,y)) \leq \rho(d(x,y))$ and therefore $\rho'(d(x,y)) \leq d(\varphi_i(x), \varphi_j(y)) = d(\varphi(x), \varphi(y))$. Now suppose that $i \neq j$. Note that $\rho(r) \leq \varepsilon r$ for every $r \in \mathbb{N}$ and there exist vertices (0-dimensional simplices) $v_x \in K_i$ and $v_y \in K_j$ such that the geodesic between $\varphi(x)$ and

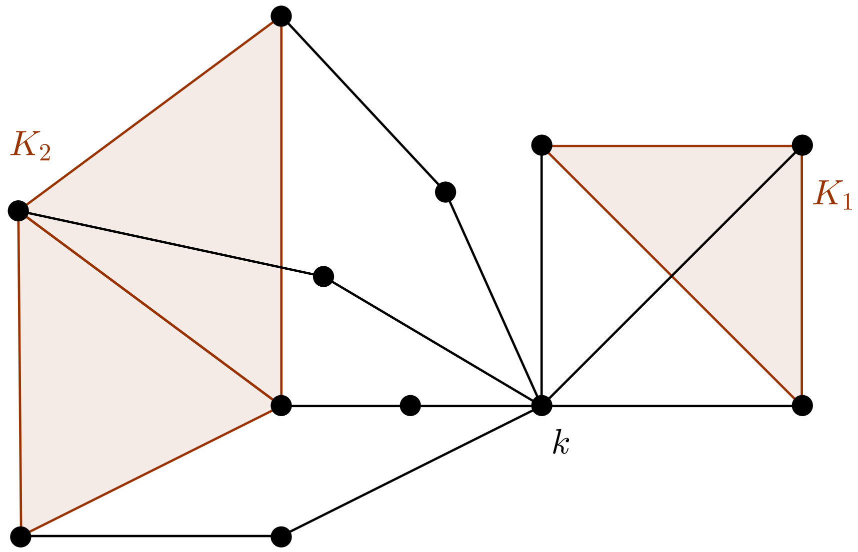


Figure A.1: Contruction of the simplicial complex K in the proof of Proposition A.1.3 with $\lfloor \varepsilon\ell_1 \rfloor = 2$ and $\lfloor \varepsilon\ell_2 \rfloor = 3$.

$\varphi(y)$ passes through v_x and v_y. Then we can make the following computation:

$$\begin{aligned}
\rho'(d(x,y)) &= \rho(\ell_i + \ell_j) - 2 && d(x,y) = \ell_i + \ell_j \\
&\leq \varepsilon(\ell_i + \ell_j) - 2 && \forall r \in \mathbb{N} \colon \rho(r) \leq \varepsilon r \\
&\leq \lfloor \varepsilon\ell_i \rfloor + \lfloor \varepsilon\ell_j \rfloor && \forall r \in \mathbb{R} \colon r - 1 \leq \lfloor r \rfloor \\
&= d(v_x, k) + d(k, v_y) && v_x \text{ (res. } v_y\text{) is vertex in } K_i \text{ (res. } K_j\text{)} \\
&\leq d(\varphi(x), \varphi(y)) && v_x,\ v_y \text{ and } k \text{ lie on the geodesic between } \varphi(x) \text{ and } \varphi(y).
\end{aligned}$$

So φ is uniformly cobounded, which proves the theorem. □

A.2 Sequences of graphs with large girth

In this section we will prove Theorem 2.2.6. We will do this using Proposition A.1.3. We will transform maps on a simplicial complex of dimension 2 such that the image lies in its 1-skeleton. First we show that you can deform the ε-Lipschitz map such that if x gets mapped in a simplex of dimension 2 and $x \sim y$ in X, then y gets mapped into the closure of that simplex. This will be used to transform the map one simplex at a time.

Lemma A.2.1. *Let X be a finite connected graph, let K be a simplicial complex of dimension 2, let $0 < \varepsilon < \frac{1}{4}$, let $\rho \colon \mathbb{N} \to \mathbb{R}$ and let $\varphi \colon X \to K$ be such that $\rho(d(x,y))) \leq d(\varphi_n(x), \varphi_n(y)) \leq \varepsilon d(x,y)$. Then there exists a map $\varphi' \colon X \to K$ with the following properties for every $x, y \in X$:*

- *$\rho(d(x,y))) - 4\varepsilon \leq d(\varphi'_n(x), \varphi'_n(y)) \leq 5\varepsilon d(x,y)$,*
- *if $\varphi'(x)$ lies in a 2 dimensional simplex $\sigma_2 \subset K$ and $d_X(x,y) = 1$, then $\varphi'(y)$ lies in the closure of σ_2.*

Proof. Let $x \in X$. If there exists a v that lies in a simplex in K of dimension 0 such that $d(\varphi(x), v) \leq 2\varepsilon$, then take $\varphi'(x) = v$. If instead there exists a v that lies in a simplex of dimension 1 such that $d(\varphi(x), v) \leq \varepsilon$, then $\varphi'(x) = v'$ where v' is such a v that minimizes $d(\varphi(x), v)$. If none of the above cases can be applied, then $\varphi'(x) = \varphi(x)$.
Now we will show that φ' is well-defined. For the first case suppose there exist two points v_1 and v_2 in K such that $d(\varphi(x), v_i) \leq 2\varepsilon < \frac{1}{2}$ for $i \in \{1, 2\}$. Then $d(v_1, v_2) < 1$, so if both lie in a simplex of dimension 0, then $v_1 = v_2$.
In the second case we know that for every simplex σ_1 of dimension 1 there exist a unique point in σ_1 that minimize its distance towards $\varphi(x)$. So suppose there exist $v_1 \neq v_2$ such that they both minimize the distance towards $\varphi(x)$. Then $\varphi(x)$ lies in a simplex of dimensional 2, which is an equilateral triangle. So using that $d(\varphi(x), v_i) \leq \varepsilon$ for $i \in \{1, 2\}$, we find a corner v of the triangle such that $d(\varphi(x), v) \leq 2\varepsilon$.
Note that $d(\varphi(x), \varphi'(x)) \leq 2\varepsilon$ for every x. So for every $x, y \in X$ with $x \neq y$ we have that $\rho(d(x,y))) - 4\varepsilon \leq d(\varphi_n(x), \varphi_n(y)) - 4\varepsilon \leq d(\varphi'_n(x), \varphi'_n(y))$ and as $d(x,y) \geq 1$ we have that $d(\varphi'_n(x), \varphi'_n(y)) \leq d(\varphi_n(x), \varphi_n(y)) + 4\varepsilon \leq$

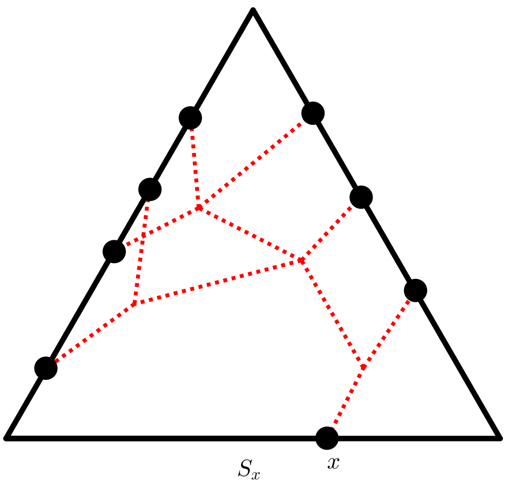
$5\varepsilon d(x,y)$. If instead $x = y$, then we already have $\rho(d(x,y))) - 4\varepsilon \leq \rho(d(x,y))) \leq 0 = d(\varphi_n'(x), \varphi_n'(y)) \leq \varepsilon d(x,y) \leq 5\varepsilon d(x,y)$.
Let $x \in X$ be such that $\varphi'(x)$ lies in a 2 dimensional simplex $\sigma_2 \subset K$ and let y be adjacent to x in X. Then it suffices to show that $\varphi'(y)$ lies in the closure of σ_2. As $\varphi'(x)$ lies in a 2 dimensional simplex, we know that $\varphi'(x) = \varphi(x)$ and $d(\varphi(x), v) > \varepsilon$ for every v in a simplex of dimension less than 2, so $\varphi(y) \in \sigma_2$, because $d(\varphi(x), \varphi(y)) \leq \varepsilon$. So $\varphi'(y)$ lies in the closure of σ_2.
This shows that φ' satisfies the necessary conditions. □

Due to Lemma A.2.1 we can try to deform the map φ by keeping the images that already map onto the 1-skeleton of K and then solving the rest one simplex at a time. We will however not entirely succeed, we will have to deform φ when it maps to the 1-skeleton of K, but we will solve this problem after solving it for a single simplex of dimension 2.

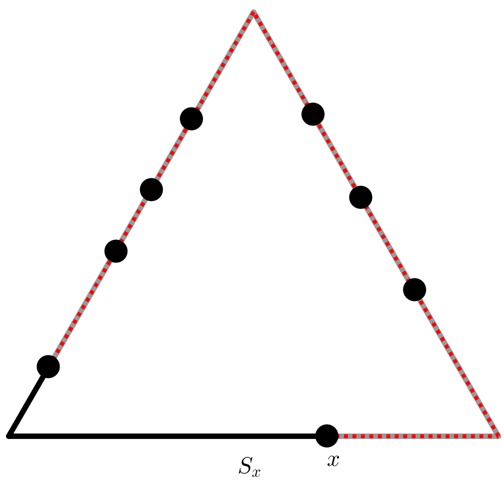
Lemma A.2.2. *Let $\varepsilon > 0$, let T be a tree, let σ_2 be a simplicial simplex of dimension 2, let $\varphi\colon T \to \overline{\sigma_2}$ be a ε-Lipschitz map, let $x \in T$ such that $\varphi(x) \in \partial\sigma_2$ and let $M = \overline{\partial\sigma_2} \setminus S_x$, where S_x is a simplex of dimension 1 such that $\varphi(x) \in \overline{S_x}$. Then there exists a map $\varphi'\colon T \to \overline{\partial\sigma_2}$ with the following properties:*

- *It is 10ε-Lipschitz,*
- *$\varphi'(x) = \varphi(x)$,*
- *for every v in M and every $y \in T$ we have that $d(\varphi'(y), v) \leq 10\, d(\varphi(y), v)$,*
- *for every $y \in T$ if $\varphi'(y) \in S_x$, then $d(\varphi'(y), \varphi(x_1)) = 10\varepsilon d(y, x_1)$ and $\varphi'(y)$ lies on the segment between $\varphi(x_1)$ and the corner of the triangle that is the closest to $\varphi(x_1)$.*

An graphical representation of Lemma A.2.2 can be found in Fig. A.2



(a) Representation of φ



(b) Representation of φ'

Figure A.2: The graphical representation of the maps φ and φ' coming from Lemma A.2.2.

Proof of Lemma A.2.2. Take v_1, v_2 and v_3 to be the corners of the triangle $\overline{\sigma_2}$ with $\varphi(x)$ on the line segment between v_1 and v_2 and closer to v_2. (see Fig. A.2) Let $\alpha\colon [0,3] \to \overline{\partial\sigma_2}$ be such that on $[0,1]$ it is the standard path from v_1 to v_2, on $[1,2]$ it is the standard path from v_2 to v_3 and on $[2,3]$ it is the standard path from v_3 to v_1.
Let $\Psi\colon T \to [0,3]\colon y \mapsto \min\left(3, \alpha^{-1}(\varphi(x)) + 10\varepsilon d(y,x), \inf_{v \in M} \alpha^{-1}(v) + 10\, d(\varphi(y), v)\right)$ and let $\varphi'\colon T \to \overline{\partial\sigma_2}\colon y \mapsto \alpha(\Psi(y))$.

First we show that φ' is 10ε-Lipschitz. To do so it suffices to show that $|\Psi(y_1) - \Psi(y_2)| \leq 10\varepsilon d(y_1, y_2)$ for every $y_1, y_2 \in T$. Without loss of generality we can assume that $\Psi(y_2) \leq \Psi(y_1)$. Now we can make the following

computation:

$$\begin{aligned}
\Psi(y_1) &= \min\Big(3, \alpha^{-1}(\varphi(x)) + 10\varepsilon d(y_1, x), \inf_{v\in M} \alpha^{-1}(v) + 10\, d(\varphi(y_1), v)\Big) \\
&\leq \min\Big(3, \alpha^{-1}(\varphi(x)) + 10\varepsilon d(y_2, x) + 10\varepsilon d(y_2, y_1), \\
&\qquad \inf_{v\in M} \alpha^{-1}(v) + 10\, d(\varphi(y_2), v) + 10\, d(\varphi(y_2), \varphi(y_1))\Big) \\
&\leq \min\Big(3, \alpha^{-1}(\varphi(x)) + 10\varepsilon d(y_2, x), \inf_{v\in M} \alpha^{-1}(v) + 10\, d(\varphi(y_2), v)\Big) + 10\varepsilon d(y_2, y_1) \\
&= \Psi(y_2) + 10\varepsilon d(y_2, y_1).
\end{aligned}$$

So we can conclude that $|\Psi(y_1) - \Psi(y_2)| \leq 10\varepsilon d(y_1, y_2)$.
Secondly we show that $\varphi'(x) = \varphi(x)$. It suffices to show that $\Psi(x) = \alpha^{-1}(\varphi(x))$. As for every $v \in M$ we have $\alpha^{-1}(v) \geq \alpha^{-1}(\varphi(x))$, we can conclude that $\Psi(x) = \alpha^{-1}(\varphi(x)) + 10\varepsilon d(x, x) = \alpha^{-1}(\varphi(x))$.
Next we show that for every v in M and every $y \in T$ we have that $d(\varphi'(y), v) \leq 10\, d(\varphi(y), v)$. So let $v \in M$ and $y \in T$. It suffices to show that $|\Psi(y) - \alpha^{-1}(v)| \leq 10\, d(\varphi(y), v)$. We have $\Psi(y) \leq \inf_{v'\in M} \alpha^{-1}(v') + 10\, d(\varphi(y), v') \leq \alpha^{-1}(v) + 10\, d(\varphi(y), v)$. So now we only need to show that $\Psi(y) \geq \alpha^{-1}(v) - 10\, d(\varphi(y), v)$. Clearly $3 \geq \alpha^{-1}(v) \geq \alpha^{-1}(v) - 10\, d(\varphi(y), v)$. As $\varphi(x)$ is closer to v_2 than v_1 we have that $\alpha^{-1}(\varphi(x)) + 10d(v, \varphi(x)) \geq \alpha^{-1}(v)$, so we can make the following computation:

$$\begin{aligned}
\alpha^{-1}(\varphi(x)) + 10\varepsilon d(y, x) &\geq \alpha^{-1}(\varphi(x)) + 10\, d(\varphi(y), \varphi(x)) \\
&\geq \alpha^{-1}(\varphi(x)) + 10\, d(v, \varphi(x)) - 10\, d(\varphi(y), v) \\
&\geq \alpha^{-1}(v) - 10\, d(\varphi(y), v).
\end{aligned}$$

Similarly for every $v' \in M$ we have that $\alpha^{-1}(v') + 10d(v, v') \geq \alpha^{-1}(v)$, so we can make the following computation:

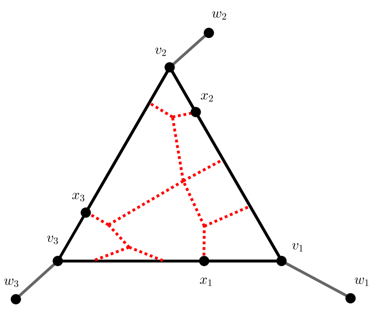
$$\begin{aligned}
\alpha^{-1}(v') + 10\, d(\varphi(y), v') &\geq \alpha^{-1}(v') + 10\, d(v, v') - 10\, d(\varphi(y), v) \\
&\geq \alpha^{-1}(v) - 10\, d(\varphi(y), v).
\end{aligned}$$

Combining these results we find that $\Psi(y) \geq \alpha^{-1}(v) - 10\, d(\varphi(y), v)$.
Finally we need to show that for every $y \in T$ if $\varphi'(y) \in S_x$, then $d(\varphi'(y), \varphi(x_1)) = 10\varepsilon d(y, x_1)$ and $\varphi'(y)$ lies on the segment between $\varphi(x_1)$ and the corner of the triangle that is the closest to $\varphi(x_1)$. So take $y \in T$ such that $\varphi'(y) \in S_x$. As $\alpha^{-1}(v) \geq 1$ for every $v \in M$, we know that $\Psi(y) = \alpha^{-1}(\varphi(x)) + 10\varepsilon d(y, x) \leq 1$. Therefore $d(\varphi'(y), \varphi(x_1)) = 10\varepsilon d(y, x_1)$.
This proves the lemma. □

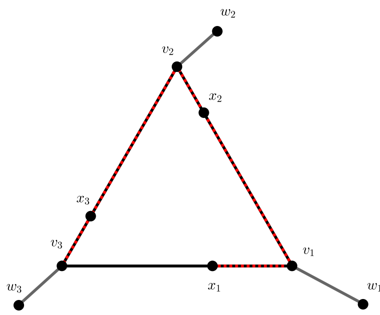
We will not be able transform the map one triangle at a time, the vertex x in Lemma A.2.2 might get mapped somewhere else when seen as an element of another triangle. So we will transform the map such that we can still choose where to map x.

Lemma A.2.3. *Let $\varepsilon > 0$, let T be a tree, let σ_2 be a simplicial simplex of dimension 2, let $\varphi\colon T \to \overline{\sigma_2}$ be a ε-Lipschitz map, let A be a non-empty subset of $\{1, 2, 3\}$, let S_i be the edges of the triangle $\overline{\sigma_2}$ where $i \in \{1, 2, 3\}$ and let x_i be vertices in T such that $\varphi(x_i) \in S_i$ for every $i \in A$.*
Let $v_i \in \overline{\sigma_2}$ be such that $d(\varphi(x_i), v_i) \leq \frac{1}{2}$ and v_i is a corner of $\overline{\sigma_2}$. Let K be $\partial\sigma_2$, where for every i you attach a line segment of length $d(\varphi(x_i), v_i)$ at the point v_i with one end and let w_i be the other end. Then there exists a map $\varphi'\colon T \to K$ with the following properties:

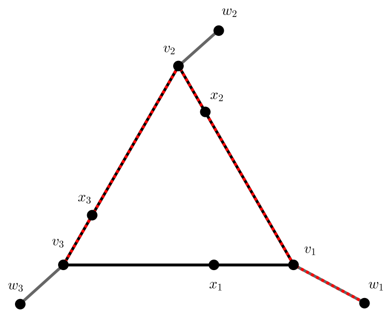
- *It is $10^3\varepsilon$-Lipschitz,*
- *for every i we have $\varphi'(x_i) = w_i$,*
- *for every i and every $y \in T$ with $\varphi(y) \in S_i$ we have that either $\varphi'(y)$ lies on $\overline{\partial\sigma_2}$ or it lies on the segment between v_i and w_i,*
- *for every i and every $y \in T$ if $\varphi'(y)$ lies on the segment between v_i and w_i, then $d(v_i, \varphi(y)) \leq \frac{3}{5}$ and $d(v_i, \varphi'(y)) \leq d(v_i, \varphi(y))$,*
- *for every corner v of the triangle $\overline{\sigma_2}$ and every $y \in T$ we have $d(\varphi'(y), v) \leq 10^4 d(\varphi(y), v)$.*



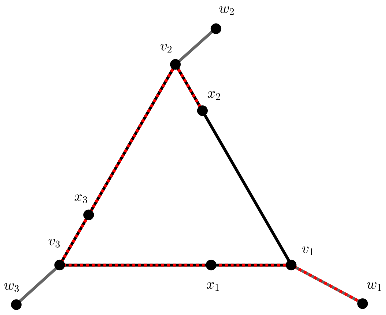
(a) Representation of φ



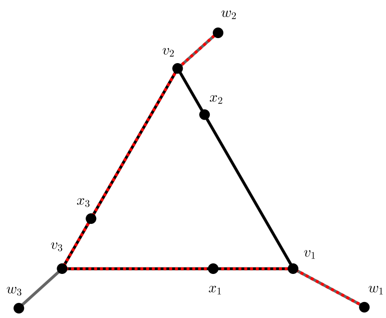
(b) Representation of φ_1'



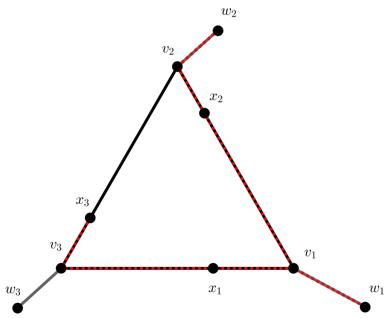
(c) Representation of φ_1



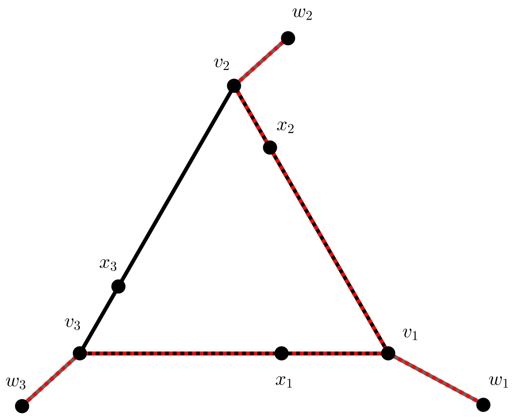
(d) Representation of φ_2'



(e) Representation of φ_2



(f) Representation of φ_3'



(g) Representation of $\varphi_3 = \varphi'$

Figure A.3: The graphical representation of the maps φ, φ_i', φ_i and φ' coming from Lemma A.2.3.

Proof. If $1 \in A$, then we can use Lemma A.2.2 to transform φ into φ_1' (the φ' in Lemma A.2.2), now we will define φ_1 very similar: Let $x \in T$ if $\varphi_1'(x) \notin S_1$, then $\varphi_1(x) = \varphi_1'(x)$. If instead $\varphi_1'(x) \in S_1$, then as $\varphi(x)$ lies in the line segment between v_1 and $\varphi(x_1)$ and $d(\varphi(x_1), v_1) = d(v_1, w_1)$, there exists a unique point v' on the line segment between v_1 and w_1 such that $d(v', v_1) = d(\varphi_1'(x), v)$. Then we take $\varphi_1(x) = v'$.
If $1 \neq A$, then we take $\varphi_1 = \varphi$. To continue we can do the same thing to create φ_2 and φ_3. A graphical representation can be found in Fig. A.3. Finally we can take $\varphi_3 = \varphi'$.

First we show that φ' is $10^3\varepsilon$-Lipschitz. Due to Lemma A.2.3 we know that φ_1 is 10ε-Lipschitz, that φ_2 is $10^2\varepsilon$-Lipschitz and that $\varphi_3 = \varphi'$ is $10^3\varepsilon$-Lipschitz.
Secondly we show that for every $i \in A$ we have $\varphi'(x_i) = w_i$. If $i = 1$, then due to Lemma A.2.2 $\varphi_1'(x_1) = \varphi(x_1)$ and therefore $\varphi_1(x_1) = w_1$. As $\varphi_1(x_1) \notin \overline{\sigma_2}$ we have that $\varphi_1(x_1) = \varphi_3(x_1) = \varphi'(x_1)$. If $i = 2$, then due to Lemma A.2.2 we have that $\varphi(x_2) = \varphi_1'(x_2) = \varphi_1(x_2)$. As before $w_2 = \varphi_2(x_2)$ and therefore $w_2 = \varphi'(x_2)$. If $i = 3$, then as before we have that $\varphi(x_3) = \varphi_2'(x_3) = \varphi_2(x_3)$ due to Lemma A.2.2 and $w_3 = \varphi_3(x_3) = \varphi'(x_3)$.
Next we show that for every i and every $y \in T$ with $\varphi(y) \in S_i$ we have that either $\varphi(y)$ lies on $\overline{\partial\sigma_2}$ or it lies on the segment between v_i and w_i. Suppose that for some i and some $y \in T$ we have that $\varphi(y) \in S_i$, but $\varphi'(y)$ lies on the segment between v_j and w_j where $i \neq j$. Due to Lemma A.2.2 we have that $d(\varphi'(y), w_j) \geq 10\varepsilon d(y, x_j) \geq 10 d(\varphi(y), \varphi(x_j))$. As $\varphi(y)$ lies on another edge of the triangle $\overline{\sigma_2}$, we have that $d(\varphi(y), \varphi(x_j)) \geq \frac{1}{2} d(v_j, \varphi(x_j))$. So $d(\varphi'(y), w_j) \geq d(v_j, \varphi(x_j))$, therefore $\varphi'(y)$ does not lie on the segment between v_j and w_j, which contradict our assumption.
Next we show that if $\varphi'(y)$ lies on the segment between v_i and w_i, then $d(v_i, \varphi(y)) \leq \frac{3}{5}$ and $d(v_i, \varphi'(y)) \leq d(v_i, \varphi(y))$. Due to Lemma A.2.2 we know that $d(\varphi'(y), w_i) \geq 10\varepsilon d(y, x_i)$ and $d(w_i, v_i) = d(\varphi(x_i), v_i) \leq \frac{1}{2}$. So $d(v_i, \varphi(y)) \leq d(v_i, \varphi(x_i)) + d(\varphi(x_i), \varphi(y)) \leq \frac{1}{2} + \varepsilon d(y, x_i) \leq \frac{1}{2} + \frac{1}{10} d(\varphi'(y), w_i) \leq \frac{1}{2} + \frac{1}{10} d(w_i, v_i) \leq \frac{3}{5}$ and $d(v_i, \varphi'(y)) = d(v_i, w_i) - d(w_i, \varphi'(y)) \leq d(v_i, \varphi(x_i)) - 10\varepsilon d(y, x_i) \leq d(v_i, \varphi(x_i)) - 10 d(\varphi(y), \varphi(x_i)) \leq d(v_i, \varphi(y))$.
Finally let v be a corner of the triangle $\overline{\sigma_2}$ and let $y \in T$, we need to show that $d(\varphi'(y), v) \leq 10^4 d(\varphi(y), v)$. Note that v always lies in the set M in Lemma A.2.2 and transforming φ_i' to φ_i only changes the distance towards v if $\varphi'(x)$ lies on the line segment between v_i and w_i, which can happen for at most one value of i and if it happens it only changes the distance by at most a factor of 3. Therefore $d(\varphi'(x), v) \leq 3 \cdot 10^3 d(\varphi(x), v) \leq 10^4 d(\varphi(x), v)$. □

Theorem A.2.4. *Let X be a finite connected graph, let K be a simplicial complex of dimension 2, let K' be the 1-skeleton of K, let $0 < \varepsilon < \frac{1}{2}$ and let $\varphi\colon X \to K$ be a ε-Lipschitz map such that*
if $d(x, y) \geq \operatorname{girth}(X)/2$, then $d(\varphi(x), \varphi(y)) \geq 2$
and if $x \in X$ such that $\varphi(x)$ lies in a simplex σ_2 in K of dimension 2 and $y \in X$ such that $d(x, y) = 1$, then $\varphi(y)$ lies in the closure of σ_2.
Then there exists a $10^7\varepsilon$-Lipschitz map $\varphi'\colon X \to K'$ such that $d(\varphi'(x), \varphi(x)) \leq 2$.

Proof. First we want to isolate the element of X that map into a simplex of dimension 2. Let X_2 be the set of elements $x \in X$ such that $\varphi(x)$ lies in a simplex of dimension 2. Note that for every $x, y \in X_2$ such that $d(x, y) = 1$ we have that $\varphi(x)$ and $\varphi(y)$ lie in the same simplex. So when we restrict φ to a component of X_2 the image is contained in a simple simplex. Let τ be the set of components of X_2. To determine the points x_i in Lemma A.2.3 we work edge by edge. We define ν to be the set of non-empty $\varphi^{-1}(\overline{\sigma_1})$ where σ_1 is a simplex in K of dimension 1. As X is finite we know that ν is finite as well, so we can write $\nu = \{\nu_1, \nu_2, \ldots, \nu_m\}$. Note τ is finite as well.

Now we want to have a notion of crossing an edge. We say $T \in \tau$ crosses ν_i to $T' \in \tau$, if there exists a component C_ν of ν_i such that we can take $v \in T$, $v' \in T'$ and $u, u' \in C_\nu$ such that v is adjacent to u and v' is adjacent to u'.
First we show that for every i and every $\tau' \subset \tau$ either there is no element of τ' crossing ν_i to an other element of τ' or there exists an element of τ' that crosses ν_i to only one other element of τ'.
To show this suppose that there exists such a crossing form T_1 to T_2, but for every element of τ' there exists more than one such a crossing. Then T_2 also crosses to an other component T_3 and T_3 crosses to some T_4. As there are only finitely many elements of τ there must exist an n_1 and n_2 such that $T_{n_1} = T_{n_2}$. Note that this provides a cycle in X without backtracking such that the diameter of its image is smaller than 2, as all T_n are connected components of X_2 and they are connected with each other via components of ν_i. If this were to exist, then we can take a sub path of this cycle of length $\operatorname{girth}(X)/2$, as the length of this cycle must be bigger than $\operatorname{girth}(X)$. Let x and y be the end points of this path. Then $d(\varphi(x), \varphi(y)) < 2$, so $d(x, y) < \operatorname{girth}(X)/2$, but this means we can add a path from y to x that is shorter than $\operatorname{girth}(X)/2$, which provides a cycle that is shorter than the girth.

Next we can put the elements of τ into a sequence. For every i we can take $T_{i,1} \in \tau$ such that it crosses ν_i only once to an element of τ. Then we can take $T_{i,2} \in \tau \setminus \{T_{i,1}\}$ such that it crosses ν_i only once to an element of $\tau \setminus \{T_{i,1}\}$. We can continue until it is impossible to take such an element of τ. We will get to such a point as we already established that there exist only finitely many elements of τ that have an element connecting to an element of ν_i. So we find a sequence $T_{i,1}, \ldots, T_{i,m_i}$. Then we can add the other elements of τ

that contain a vertex that connects to a vertex in ν_i. So we find a sequence $T_{i,1},\dots,T_{i,m_i},T_{i,m_i+1},\dots,T_{i,m_i'}$. Putting these sequences together we find a sequence $T_{1,1},\dots,T_{1,m_1'},T_{2,1},\dots,T_{2,m_2'},\dots,T_{m,m_m}$. For each $T_{i,j}$ with $j \le m_i$ we can uniquely define $x_{i,j}$ as follows: We know that $T_{i,j}$ is chosen to be such that it crosses ν_i only once to an other element of τ that is not $T_{i,j'}$ with $j' < j$, then we take $x_{i,j}$ to be the vertex on ν_i such that it is adjacent to a vertex in $T_{i,j}$ and there exists a vertex in the component of ν_i containing $x_{i,j}$ such that it is adjacent to an element of $T_{i,j'}$ with $j' > j$.
For every T denote the unique simplex of K that contains $\varphi(T)$ by σ_T. Note that for every $T \in \tau$ there exists at least one combination (i,j) such that $T = T_{i,j}$, because X is connected. Also note that there can be at most three such combination (i,j), one for every edge of σ_T. Finally note that for these combinations correspond to vertices $x_{i,j}$ we have that $\varphi(x_{i,j})$ lies on $\overline{\partial\sigma_T}$.
For a fixed $T \in \tau$ consider the union of the following sets: T, $\{x_{i,j} \mid T = T_{i,j}\}$ and for every i the components of ν_i that contain a vertex adjacent to a vertex in T, but do not contain $x_{i,j}$ with $T = T_{i,j}$. Denote this set by U_T and define the map $\varphi_T \colon U_T \to \overline{\sigma_T} \colon x \mapsto \varphi(x)$.
If for $T \in \tau$ there exists an i and a $j \le m_i$ such that $T = T_{i,j}$, then due to Lemma A.2.3 we can take a map $\varphi_T' \colon U_T \to \sigma^+$ where σ^+ is $\partial\sigma_T$ with some line segments attached to it such that φ_T' is $10^3\varepsilon$-Lipschitz, $x_{i,j}$ are mapped to the ends of the line segments if $T = T_{i,j}$, the distance toward corners of the triangle $\overline{\sigma_T}$ gets at most 10^4 times bigger and if $\varphi_T(x)$ lies on the edge containing $x_{i,j}$ then either $\varphi_T'(x)$ lies in $\overline{\partial\sigma_T}$ or it lies on the line segment containing $\varphi_T'(x_{i,j})$.
If for $T \in \tau$ there does not exist an i and a $j \le m_i$ such that $T = T_{i,j}$. Then there still exists an $x \in \overline{\partial\sigma_T}$, so due to Lemma A.2.2 we can take a $10^3\varepsilon$-Lipschitz map $\varphi_T' \colon U_T \to \overline{\partial\sigma_T}$ such that $\varphi_T'(x) = \varphi_T(x)$ and the distance toward corners of the triangle $\overline{\sigma_T}$ gets at most 10 times bigger, so less than 10^4 times.
Now we will define φ' for every $x \in X$. If $x \in U_T$ for any $T \in \tau$, then we take $\varphi'(x) = \varphi_T'(x)$. Now we can work backwards from $i = m$ and $j = m_m'$ to $i = j = 1$ and define $\varphi'(x)$ if $x \in U_{T_{i,j}}$ and $\varphi_{T_{i,j}}'(x)$ lies on the line segment between $\varphi_T'(x_{i,j})$ and corner $v_{i,j}$ of $\overline{\sigma_{T_{i,j}}}$. We claim that there exists a $j' > j$ such that $x_{i,j} \in U_{T_{i,j'}}$, so $\varphi'(x_{i,j})$ is already defined, therefore we can pick a geodesic between $v_{i,j}$ and $\varphi'(x_{i,j})$ and choose $\varphi'(x)$ on this geodesic respectively to the position of $\varphi_T'(x)$ on the line segment between $\varphi_T'(x_{i,j})$ and $v_{i,j}$. Now if $\varphi'(x)$ is not jet defined, then we take $\varphi'(x) = \varphi(x)$.

At last it suffices to show that φ' is well defined, $10^7\varepsilon$-Lipschitz and $d(\varphi'(x),\varphi(x)) \le 2$ for every $x \in X$.
First we show that for every i and j there exists a j' such that $x_{i,j} \in U_{T_{i,j'}}$. Consider the component of ν_i that contains $x_{i,j}$ and take j' the biggest such that $T_{i,j'}$ contains a vertex adjacent to a vertex of this component. Then by definition this component is a subset of $U_{T_{i,j'}}$.
Secondly we show that φ' is uniquely defined for every $x \in X$. If $x \in X_2$, then there is only one $T \in \tau$ such that $x \in U_T$, so $\varphi'(x)$ is uniquely defined. If $\varphi(x)$ lies in a simplex of dimension 0, then for every i and j with $x \in U_{T_{i,j}}$ we have $d(\varphi_T'(x),\varphi(x)) \le 10^4 d(\varphi(x),\varphi(x)) = 0$. So $\varphi'(x) = \varphi(x)$.
If $\varphi(x)$ lies in a simplex of dimension 1, then there exists an i such that $x \in \nu_i$. Take C_x to be the component of ν_i containing x and take $j_1 > j_2 > \dots$ such that T_{i,j_k} contains an element that is adjacent to an element of C_x for every k. Note that for $k > 1$ we have T_{i,j_k} crosses ν_i to T_{i,j_1}, so either $x = x_{i,j_k}$ or $x \notin U_{T_{i,j_k}}$. Either way $\varphi'(x)$ is defined using $\varphi_{T_{i,j_1}}'$.
Before we show that φ' is $10^7\varepsilon$-Lipschitz and $d(\varphi'(x),\varphi(x)) \le 2$ for every $x \in X$, we need to show that for every i and j and every v in a simplex of K of dimension 1 we have $d(\varphi'(x_{i,j}),v) \le 10^4 d(\varphi(x_{i,j}),v)$ and there exists a simplex σ_2 of dimension 2 such that $\varphi'(x_{i,j})$ and $\varphi(x_{i,j})$ lie in $\overline{\sigma_2}$.
By induction we may assume that for all bigger j this is true. For this j we know there exists a $j' > j$ such that $x_{i,j} \in U_{T_{i,j'}}$. Then $\varphi_{T_{i,j'}}'(x_{i,j})$ either lies in $\overline{\partial\sigma_{T_{i,j'}}}$ or on the line segment containing $\varphi_{T_{i,j'}}'(x_{i,j'})$. In the first case we know that $\varphi'(x_{i,j}) = \varphi_{T_{i,j'}}'(x_{i,j})$ and $d(\varphi_{T_{i,j'}}'(x_{i,j}),v) \le 10^4 d(\varphi(x_{i,j}),v)$. So $d(\varphi'(x_{i,j}),v) \le 10^4 d(\varphi(x_{i,j}),v)$ and $\varphi_{T_{i,j'}}'(x_{i,j})$ lies in $\overline{\partial\sigma_{T_{i,j'}}}$. In the second case, if $\varphi_{T_{i,j'}}'(x_{i,j})$ lies on the line segment containing $\varphi_{T_{i,j'}}'(x_{i,j'})$, then by induction we know that $d(\varphi'(x_{i,j'}),v) \le 10^4 d(\varphi(x_{i,j'}),v)$ and there exists a simplex σ_2 of dimension 2 such that $\varphi'(x_{i,j'})$ and $\varphi(x_{i,j'})$ lie in $\overline{\sigma_2}$. So both $\varphi'(x_{i,j})$ and $\varphi'(x_{i,j})$ also lie in $\overline{\sigma_2}$. Now $d(\varphi'(x_{i,j}),v) \le 2 + d(\varphi(x_{i,j}),v)$. So we only need to show that $d(\varphi_{T_{i,j'}}'(x_{i,j}),v) \le 10^4 d(\varphi(x_{i,j}),v)$ for $d(\varphi(x_{i,j}),v) \le \frac{2}{9999} < \frac{2}{5}$. Due to Lemma A.2.3 we know that v is contained in the line segment containing $\varphi_{T_{i,j'}}'(x_{i,j'})$ and therefore $d(\varphi_{T_{i,j'}}'(x_{i,j}),v) \le d(\varphi(x_{i,j}),v)$. As by induction $d(\varphi'(x_{i,j'}),v) \le 10^4 d(\varphi(x_{i,j'}),v)$ we find that $d(\varphi'(x_{i,j}),v) \le 10^4 d(\varphi_{T_{i,j'}}'(x_{i,j}),v) \le 10^4 d(\varphi(x_{i,j}),v)$.
Next we show that φ' is $10^7\varepsilon$-Lipschitz. As X is a graph, it suffices to show that for every x and y with $d(x,y) = 1$ we have that $d(\varphi'(x),\varphi'(y)) \le 10^7\varepsilon$.
If $\varphi(x)$ and $\varphi(y)$ lie in different simplexes of dimension 0 or 1, then there exist a vertex v such that $\{v\}$ is a simplex in K and $d(v,\varphi(x)), d(v,\varphi(y)) \le 2\varepsilon$. Due to Lemma A.2.3 we know for every $T \in \tau$ that if $x \in U_T$, then $d(v,\varphi_T'(x)) \le d(v,\varphi(x))$ or $\varphi_T'(x) \in K'$ either way $d(v,\varphi_T'(x)) \le 10^4 d(v,\varphi(x))$. The same of true for $y \in U_T$. So $d(\varphi'(y),\varphi'(x)) \le d(v,\varphi'(x)) + d(v,\varphi'(y)) \le 10^4 d(v,\varphi(x)) + 10^4 d(v,\varphi(y)) \le 2 \cdot 10^4\varepsilon + 2 \cdot 10^4\varepsilon \le 10^7\varepsilon$.
If $\varphi(x)$ or $\varphi(y)$ lies in simplex of dimension 2, then there exists a $T \in \tau$ that contains x or y, in fact both x and

y lie in U_T. Then $d(\varphi'_T(x), \varphi'_T(y)) \leq 10^3\varepsilon$, since φ'_T is $10^3\varepsilon$-Lipschitz. As we know that for every corner v and every i and j we have $d(\varphi'(x_{i,j}), v) \leq 10^4 d(\varphi(x_{i,j}), v)$ we can conclude that $d(\varphi'(x), \varphi'(y)) \leq 10^7\varepsilon$.
If $\varphi(x)$ and $\varphi(y)$ lie in the same simplexes of dimension 0 or 1, then there exists an i such that x and y lie in ν_i. As x and y are adjacent, they lie in the same component of ν_i. If that component does not have any element adjacent to an element in X_2, then $d(\varphi'(x), \varphi'(y)) = d(\varphi(x), \varphi(y)) \leq \varepsilon < 10^7\varepsilon$. If that component has an element adjacent to an element in X_2, then we can take j the biggest such that $T_{i,j}$ has an element adjacent to an element of this component. Then $x_{i,j}$ does not lie in this component, so $x, y \in U_{T_{i,j}}$. As before $d(\varphi'(x), \varphi'(y)) \leq 10^7\varepsilon$. So φ' is $10^7\varepsilon$-Lipschitz.
Finally if suffices to show that for every $x \in X$ we have that $d(\varphi'(x), \varphi(x)) \leq 2$. If there exists no $T \in \tau$ such that $x \in U_T$, then $\varphi'(x) = \varphi(x)$. If however we can take $T \in \tau$ such that $x \in U_T$, then either $\varphi'(x) \in \overline{\partial\sigma_T}$ or there exist i and j such that $\varphi'_T(x)$ lies on the line segment containing $\varphi'_T(x_{i,j})$. In the first case $d(\varphi(x), \varphi'(x)) \leq 1 \leq 2$. In the second case $d(\varphi(x), \varphi(x_{i,j})) \leq 1$ and there exist a simplex σ_2 in K of dimension 2 such that $\varphi(x_{i,j}))$ and $\varphi'(x_{i,j}))$ lie in $\overline{\partial\sigma_2}$. As $\varphi'_T(x)$ lies on the line segment containing $\varphi'_T(x_{i,j})$, we have that $\varphi'(x)$ also lies in $\overline{\partial\sigma_2}$. So $d(\varphi(x), \varphi'(x)) \leq d(\varphi(x), \varphi(x_{i,j})) + d(\varphi(x_{i,j}), \varphi'(x)) \leq 1 + 1 = 2$. □

Proof of Theorem 2.2.6. Suppose that $\operatorname{asdim} X \leq 2$, then it suffices to show that $\operatorname{asdim} X \leq 1$.
Let $\varepsilon > 0$ with $\varepsilon \leq 1$. Then due to Proposition A.1.3 there exist simplicial complexes K_n of dimension 2, a map $\rho\colon \mathbb{N} \to \mathbb{R}$ and maps $\varphi_n\colon X_n \to K_n$ such that $\rho(n) \to +\infty$ as $n \to \infty$ and $\rho(d(x,y))) \leq d(\varphi_n(x), \varphi_n(y)) \leq 2 \cdot 10^{-8}\varepsilon d(x,y)$ for every n and every $x, y \in X_n$.
Due to Lemma A.2.1 we know that for every n there exists a map $\tilde{\varphi}_n\colon X_n \to K_n$ such that for every $x, y \in X_n$ we have $\rho(d(x,y))) - 1 \leq d(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) \leq 10^{-7}\varepsilon d(x,y)$ and if $d(x,y) = 1$ and $\tilde{\varphi}_n(x)$ lies in a simplex of dimension 2, then $\tilde{\varphi}_n(y)$ lies in the closure of that simplex.
Let K'_n be the 1-skeleton of K_n. Then due to Theorem A.2.4 we can take $\varphi'_n\colon X_n \to K'_n$ to be ε-Lipschitz with $d(\tilde{\varphi}_n(x), \varphi'_n(x))$ for every $x \in X_n$, if $\rho(\operatorname{girth}(X_n)/2) \geq 2$. As the sequence X_n has large girth, there exists an N such that for $n \geq N$ we have that $\rho(\operatorname{girth}(X_n)/2) \geq 2$. For $n < N$ we can take $\varphi'_n\colon X_n \to K'_n\colon x \mapsto v$ where v is an arbitrary element of K'_n.
Set $C = \max\limits_{n<N}(\operatorname{diam}(X_n))$. Also take $\rho'\colon \mathbb{N} \to \mathbb{R}$ such that $\rho'(k) = \rho(k) - \max(\rho(C), 5)$.
For $n < N$ and for every $x, y \in X_n$ we have $\rho'(d(x,y)) \leq \rho'(C) \leq 0 = d(\varphi'_n(x), \varphi'_n(y)) \leq \varepsilon d(x,y)$.
For $n \geq N$ we know there exists an ε-Lipschitz map $\varphi'_n\colon X_n \to K'_n$ such that $d_{K_n}(\tilde{\varphi}_n(x), \varphi'_n(x)) \leq 2$. Then for every $x, y \in X_n$ we can conclude with the following computation:

$$
\begin{aligned}
\rho'(d(x,y)) &\leq \rho(d(x,y)) - 5 \\
&\leq d(\tilde{\varphi}_n(x), \tilde{\varphi}_n(y)) - 4 \\
&\leq d(\tilde{\varphi}_n(x), \varphi'_n(x)) + d(\varphi'_n(x), \varphi'_n(y)) + d(\varphi'_n(y), \tilde{\varphi}(y)) - 4 \\
&\leq d(\varphi'_n(x), \varphi'_n(y))
\end{aligned}
$$

As we already know that φ'_n is ε-Lipschitz, so due to Proposition A.1.3 we can conclude the theorem. □

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