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Note

Complexity of two coloring problems in cubic planar bipartite mixed graphs

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ABSTRACT

In this note we consider two coloring problems in mixed graphs, i.e., graphs containing edges and arcs, which arise from scheduling problems where disjunctive and precedence constraints have to be taken into account. We show that they are both $\mathcal{NP}$-complete in cubic planar bipartite mixed graphs, which strengthens some results of Ries and de Werra (2008) [9].

1. Introduction

Coloring problems in undirected graphs are often used to handle scheduling problems involving incompatibility constraints. Suppose for example we are given a set of jobs with unit processing time (without preemptions) that must be processed, and for some pairs of jobs we know that they cannot be processed simultaneously; we then want to determine a schedule for these jobs respecting the incompatibility constraints. This problem can be modeled by an undirected graph: we associate a vertex with each job, and we join two vertices if the corresponding jobs cannot be processed at the same time. A vertex coloring of the resulting graph then gives a feasible schedule of the jobs.

However, in more general scheduling problems, incompatibility constraints are not the only constraints that have to be taken into account and thus the classical graph coloring model is too limited to handle this kind of problem. Consider for example a scheduling problem where in addition to the incompatibility constraints we are also given some precedence constraints, i.e., some jobs must be processed in a given partial order. In scheduling problems, precedence constraints occur frequently and have been studied in several papers (see for instance [6,7]). In order to deal with scheduling problems containing both types of constraint, we will use *mixed graphs*. Notice that if only precedence constraints occur (i.e., there are no incompatibility constraints), the problem corresponds to a vertex coloring problem in a directed graph, which can be solved in polynomial time by using longest path methods.

A mixed graph $G_M = (V, A, E)$ is a graph containing undirected edges (set E) and directed edges (set A) which we will refer to as *arcs*. Our scheduling problem which involves incompatibility and precedence constraints can be solved using the following definition of a k-coloring in such a mixed graph: a *strong mixed coloring* of a mixed graph G_M is a mapping $c: V \to C = \{0, 1, \dots\}$, such that for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$, and for each arc $(v_l, v_q) \in A$, $c(v_l) < c(v_q)$. If $|C| = k$, i.e., $C = \{0, 1, \dots, k-1\}$, we call the coloring a *strong mixed k-coloring*. Thus, by associating a vertex with each job, joining two vertices by an edge if the corresponding jobs cannot be processed simultaneously, and finally joining two

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vertices v_l, v_q by an arc (v_l, v_q) (i.e., the arc is directed from v_l to v_q) if the job corresponding to v_l must be processed before the job corresponding to v_q, we get a mixed graph. A strong mixed coloring of this graph corresponds to a schedule of the jobs with respect to both types of constraint. So mixed graphs are a helpful tool for modeling scheduling problems involving incompatibility and precedence constraints at the same time. Notice that if the mixed graph G_M contains a circuit (i.e., a closed path $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$), no strong mixed coloring exists. Thus throughout the paper, when working on strong mixed colorings in mixed graphs G_M, we assume that G_M contains no circuit.

In this paper we consider, besides the strong mixed coloring, another type of coloring in mixed graphs. We define a *weak mixed coloring* of a mixed graph G_M as a mapping $c: V \to C = \{0, 1, \ldots\}$, such that for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$ and for each arc $(v_l, v_q) \in A$, $c(v_l) \leq c(v_q)$. As before, if $|C| = k$, i.e., $C = \{0, 1, \ldots, k-1\}$, we call the coloring a *weak mixed k-coloring*. This type of coloring may be used if we want to model precedence constraints which impose that some jobs must not be processed after some other jobs (but possibly at the same time). Notice that a strong mixed k-coloring is a special case of a weak mixed k-coloring. In fact a strong mixed k-coloring in G_M corresponds to a weak mixed k-coloring in G'_M, where G'_M is obtained from G_M by adding an edge $[v_i, v_j]$ for each arc (v_i, v_j) (and thus obtaining multiple edges/arcs). In this paper we will only consider simple mixed graphs.

For both types of coloring we can state the following decision problems.

Strong mixed graph coloring problem $S(G_M, k)$:

Instance: A mixed graph $G_M = (V, A, E)$ and a positive integer k.
Question: Can the vertices of G_M be colored using at most k colors so that, for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$, and for each arc $(v_l, v_q) \in A$, $c(v_l) < c(v_q)$?

Weak mixed graph coloring problem $W(G_M, k)$:

Instance: A mixed graph $G_M = (V, A, E)$ and a positive integer k.
Question: Can the vertices of G_M be colored using at most k colors so that, for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$, and for each arc $(v_l, v_q) \in A$, $c(v_l) \leq c(v_q)$?

Both problems have been studied by several authors (see for instance [3–5,8,9]). In [3,4], $S(G_M, k)$ is considered in mixed trees as well as in mixed series–parallel graphs, and the authors provide polynomial-time algorithms. In [5], lower and upper bounds are given on the minimum number of colors necessary for a general mixed graph G_M to admit a strong mixed coloring. In [8] and [9], $S(G_M, k)$ and $W(G_M, k)$ are considered in special classes of mixed graphs, and their complexity status is determined. More precisely, in [9], it is shown that both $S(G_M, k = 3)$ and $W(G_M, k = 3)$ are $\mathcal{NP}$-complete if G_M is a bipartite mixed graph of maximum degree 3 or if G_M is a planar bipartite mixed graph with maximum degree 4. In this paper, we strengthen these results by showing that $S(G_M, k = 3)$ and $W(G_M, k = 3)$ are $\mathcal{NP}$-complete in cubic planar bipartite mixed graphs.

For all graph theoretical terms not defined here, the reader is referred to [1].

2. Complexity results

2.1. Strong mixed graph coloring problem

In order to show the $\mathcal{NP}$-completeness of $S(G_M, 3)$, we will use a transformation from the List Coloring Problem (*LiCol*) which is defined as follows.

LiCol(G)

Instance: An undirected graph $G = (V, E)$ together with sets of feasible colors $L(v)$ for all vertices $v \in V$.
Question: Does there exist a proper vertex coloring of G (i.e., adjacent vertices get different colors) such that every vertex is colored with a feasible color from $L(v)$?

In [2], this problem has been shown to be $\mathcal{NP}$-complete if the total number of available colors is three (i.e., $|L| = 3$) and if $G \in \mathcal{G}$, where $\mathcal{G}$ is a special class of cubic planar bipartite graphs which we will describe hereafter. In order to show the $\mathcal{NP}$-completeness, the authors use a transformation from the $\mathcal{NP}$-complete problem Cubic Planar Monotone 1-in-3SAT (*CPM*1*in*3*SAT*). The transformation is the following. They associate with each vertex v of the cubic planar bipartite graph $G' = (V', E')$ (built on an instance of *CPM*1*in*3*SAT*), which represents a variable, a list $L(v) = \{0, 1\}$. Next, they replace each vertex C in G' representing a clause $C = u \vee v \vee w$ as well as its incident edges $[u, C]$, $[v, C]$, $[w, C]$ by the gadget shown in Fig. 1 and associate with each vertex in the gadget a list of feasible colors. Thus they obtain another cubic planar bipartite graph $G = (V, E)$. The set $\mathcal{G}$ is exactly the family of these graphs G obtained by applying the transformation mentioned above.

Theorem 1. *$S(G_M, 3)$ is $\mathcal{NP}$-complete if G_M is a mixed cubic planar bipartite graph.*

Proof. Consider an instance of *LiCol*(G) with $G = (V, E) \in \mathcal{G}$ and with the lists of feasible colors as shown in Fig. 1. First notice that we can replace the parallel edges by the gadgets shown in Fig. 2 (all vertices for which no list is indicated in the gadgets will get the list $\{0, 1, 2\}$). Indeed, in any list-coloring l of the new graph $G' = (V', E')$ (i.e., the graph obtained from G by replacing the parallel edges by the gadgets), we must have $l(x') = 1$ and $l(y') = 0$, thus $l(x) \in \{0, 2\}$ and $l(y) \in \{1, 2\}$ in a gadget G_1, and $l(x') = 2$ and $l(y') = 0$, thus $l(x) \in \{0, 1\}$ and $l(y) \in \{1, 2\}$ in a gadget G_2. So clearly G is list-colorable if and only if G' is list-colorable, since one can color properly the vertices in the gadgets G_i, $i = 1, 2$, which are different from

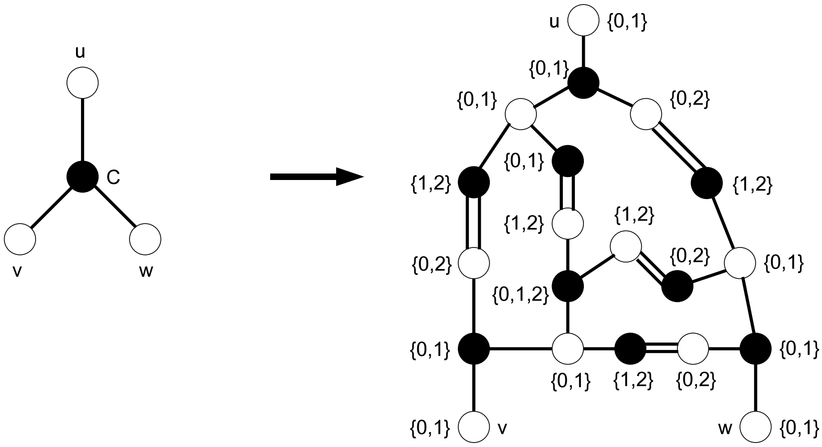
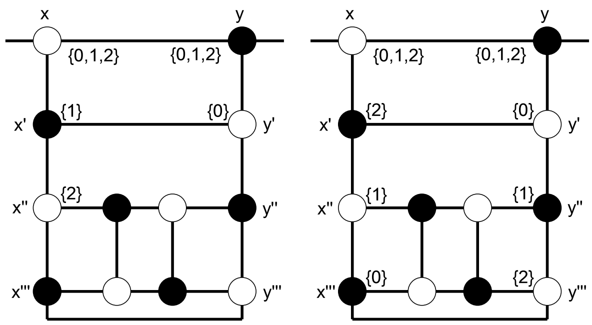


Fig. 1. The gadget replacing a vertex C corresponding to a clause $C = u \vee v \vee w$ (see [2]).



(a) Gadget G_1. (b) Gadget G_2.

Fig. 2. The gadget replacing parallel edges between x and y (a) if $L(x) = \{0, 2\}$ and $L(y) = \{1, 2\}$; (b) if $L(x) = \{0, 1\}$ and $L(y) = \{1, 2\}$.

x, y, x' and y', using at most colors 0, 1 and 2. Furthermore, notice that G' is still cubic planar bipartite, and let us denote by $\mathcal{G}'$ the class of graphs obtained from a graph $G \in \mathcal{G}$ by replacing parallel edges by the gadgets shown in Fig. 2. Thus we conclude that the *LiCol* problem remains $\mathcal{NP}$-complete if the considered graph G' is in $\mathcal{G}'$ and the total number of available colors is three.

We will explain now how to replace each vertex in G' in order to get a cubic planar bipartite mixed graph $G_M = (V, A, E)$ with the property that G_M admits a strong mixed 3-coloring if and only if G' is list-colorable:

(i) Each vertex x with list $|L(x)| = 3$ remains unchanged.
(ii) Each vertex x with list $L(x) = \{0, 1\}$ is replaced by the gadget shown in Fig. 3, which we shall call the $\{0, 1\}$-*gadget*.
(iii) Each vertex x with list $L(x) = \{1, 2\}$ is replaced by the gadget obtained from the $\{0, 1\}$-gadget by inverting the arcs; we shall call it the $\{1, 2\}$-*gadget*.
(iv) In each gadget G_1, we transform the edges $[x', y']$ and $[x', x'']$ into arcs (y', x') and (x', x''). Furthermore, in each gadget G_2, we transform the edges $[x', x'']$, $[x'', x''']$, $[y', y'']$ and $[y'', y''']$ into arcs (x'', x'), (x''', x''), (y', y'') and (y'', y''').

We denote by $G_M = (V, A, E)$ the resulting mixed graph. Notice that G_M is cubic planar bipartite (the bipartition is represented by the black and white vertices in the figures). Now let us fix $k = 3$; thus we get an instance of $S(G_M, 3)$. We will show now that G_M admits a strong mixed 3-coloring if and only if G' is list-colorable.

Suppose that $S(G_M, 3)$ has a positive answer and denote by c the corresponding strong mixed 3-coloring using colors 0, 1 and 2. Consider any $\{0, 1\}$-gadget in G_M corresponding to a vertex x in G'. First notice that $c(x_1), c(x_2), c(x_3) \in \{0, 1\}$ since all three vertices are incident to an outgoing arc ((x_1, a_1) for x_1, (x_2, a_2) for x_2 and (x_3, a_2) for x_3). Thus if $c(x_1) = 2$ (resp. $c(x_2) = 2$ or $c(x_3) = 2$), there would be no feasible color for a_1 (resp. a_2) since we must have $c(x_1) < c(a_1)$ (resp. $c(x_2) < c(a_2)$ and $c(x_3) < c(a_2)$). Next we will show that $c(x_1) = c(x_2) = c(x_3)$. Suppose that $c(x_1) = 0$. Then $c(d) = 1$ (d cannot get color 2 otherwise there is no feasible color for b) and thus $c(x_2) = 0$. Furthermore $c(b) = 2$ (since $c(d) = 1$), which implies that $c(a_1) = c(a_2) = 1$ and hence $c(x_3) = 0$. Similarly, if $c(x_1) = 1$, then $c(d) = 0$ and thus $c(x_2) = 1$. We have $c(a_1) = 2$, implying $c(g) = 1$ and $c(f) = 0$, which forces $c(x_3) = 1$. We conclude that, in any strong mixed 3-coloring of G_M, the vertices x_1, x_2, x_3 of a $\{0, 1\}$-gadget get the same color $c^* \in \{0, 1\}$. Notice that necessarily we must have $c(y), c(z), c(t) \neq c^*$.

Now, using similar arguments, one can show that, in any strong mixed 3-coloring of G_M, the vertices x_1, x_2, x_3 of a $\{1, 2\}$-gadget get the same color $c^{**} \in \{1, 2\}$. Again we necessarily have $c(y), c(z), c(t) \neq c^{**}$.

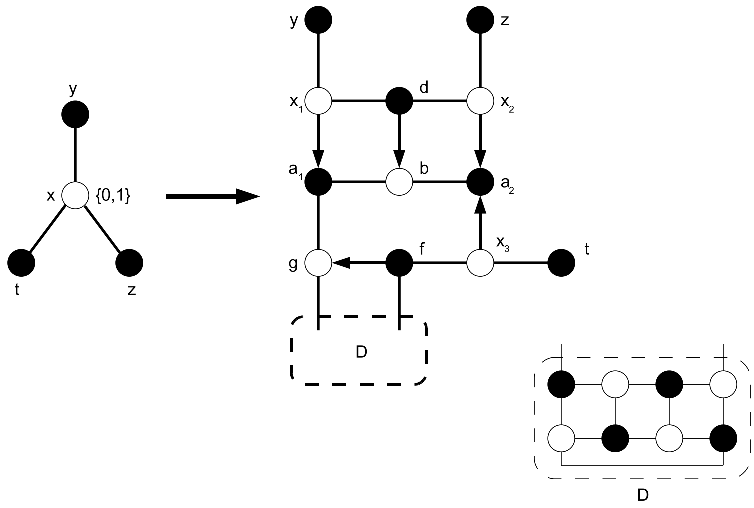


Fig. 3. The $\{0, 1\}$-gadget replacing a vertex x with $L(x) = \{0, 1\}$.

In any gadget G_1, we necessarily have $c(x'') = 2, c(x') = 1, c(y') = 0$ since there is a path (y', x', x''). Furthermore, in any gadget G_2, $c(y') = 0, c(y'') = 1$ and $c(y''') = 2$ since there is a path (y', y'', y'''), and $c(x') = 2, c(x'') = 1$ and $c(x''') = 2$ since there is a path (x''', x'', x').

We are now able to provide a positive answer for the *LiCol* problem in G'. In fact, in the graph G', we color:

(i) each vertex x with list $L(x) = \{0, 1\}$ with the color c^* of the vertices x_1, x_2, x_3 in the $\{0, 1\}$-gadget replacing x in G_M;
(ii) each vertex x with list $L(x) = \{1, 2\}$ with the color c^{**} of the vertices x_1, x_2, x_3 in the $\{1, 2\}$-gadget replacing x in G_M;
(iii) each vertex x with $|L(x)| = 1$ or $|L(x)| = 3$ with the color $c(x)$ of x in G_M.

By the discussion above we conclude that this will give a feasible list-coloring of G'.

Conversely, let us suppose now that the *LiCol* problem in G' has a positive answer and let us denote by l_c the corresponding list-coloring. We will get a positive answer for $S(G_M, 3)$ by doing the following operations.

(i) For each vertex x in G' with $L(x) = \{0, 1\}$, we color the vertices x_1, x_2, x_3 in the corresponding $\{0, 1\}$-gadget in G_M with $l_c(x)$. If $l_c(x) = 0$, then vertices a_1, a_2, d and f will get color 1, and vertices b and g will get color 2. The remaining yet uncolored vertices of the gadget can be colored properly by giving color 2 to the white vertices and color 1 to the black vertices. Similarly, if $l_c(x) = 1$, then vertices a_1 and a_2 will get color 2, vertices d and f will get color 0, and vertices b and g will get color 1. Again the remaining yet uncolored vertices of the gadget can be colored properly by giving color 1 to the white vertices and color 2 to the black vertices.
(ii) For each vertex x in G' with $L(x) = \{1, 2\}$, we color the vertices x_1, x_2, x_3 in the corresponding $\{1, 2\}$-gadget in G_M with $l_c(x)$. Using similar arguments as in the previous case, we can show that all vertices of the $\{1, 2\}$-gadget can be properly colored using colors 0, 1 and 2.
(iii) Each vertex x with $|L(x)| = 1$ or $|L(x)| = 3$ will keep its color in G_M.

Clearly, these operations give us a feasible strong mixed 3-coloring of G_M, and thus we have a positive answer for $S(G_M, 3)$. Since G_M can be obtained from G' in polynomial time, we conclude that $S(G_M, 3)$ is $\mathcal{NP}$-complete. □

Notice that this $\mathcal{NP}$-completeness result is best possible in the sense that, if we consider $S(G_M, 3)$ with G_M having maximum degree 2 or $S(G_M, 2)$, the problem becomes polynomially solvable. Indeed it has been shown in [9] that $S(G_M, 2)$ can be solved in polynomial time. Furthermore, the problem can be solved in polynomial time for graphs of maximum degree 2 (and in fact also more generally for all series–parallel graphs (see [4])).

2.2. Weak mixed graph coloring problem

Now let us consider $W(G_M, 3)$. We have the following result.

Theorem 2. *$W(G_M, 3)$ is $\mathcal{NP}$-complete if G_M is a cubic planar bipartite mixed graph.*

Proof. We use a reduction from $S(G'_M, 3)$, which we just showed to be $\mathcal{NP}$-complete if G'_M is a cubic planar bipartite mixed graph (see Theorem 1).

Consider a cubic planar bipartite mixed graph $G'_M = (V', A', E')$. We replace each arc $(x, y) \in A'$ by the gadget shown in Fig. 4 (the graph D is the same as in Fig. 3). The resulting mixed graph G_M is clearly cubic planar bipartite. We will now show that G_M admits a weak mixed 3-coloring if and only if G'_M admits a strong mixed 3-coloring.

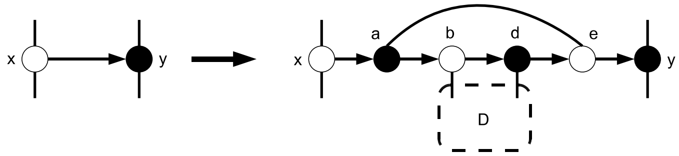


Fig. 4. The gadget replacing an arc (x, y). The graph D is shown in Fig. 3.

Suppose that $S(G'_M, 3)$ has a positive answer and denote by c' the corresponding strong mixed 3-coloring. We get a feasible weak mixed 3-coloring c of G_M by proceeding as follows. In G_M we color each vertex x which does also belong to G'_M with color $c(x) = c'(x)$. In each gadget (which replaces an arc of G'_M) we color the vertices a, b, d and e as follows: $c(a) = c(b) = c(x)$ and $c(d) = c(e) = c(y)$. So we will have $c(u) \leq c(v)$ for each arc (u, v) in G_M. Notice that the remaining yet uncolored vertices in the gadget can be properly colored using colors 0, 1 and 2 by giving color $c(b)$ to all white vertices and color $c(d)$ to all black vertices. Thus we obtain a feasible weak mixed graph 3-coloring of G_M.

Conversely, suppose that $W(G_M, 3)$ has a positive answer and denote by c the corresponding weak mixed 3-coloring. Then we get a feasible strong mixed 3-coloring c' of G'_M by proceeding as follows: in G'_M we color each vertex x with color $c'(x) = c(x)$. This necessarily gives us a feasible strong mixed 3-coloring of G'_M. In fact in each gadget of G_M which replaces an arc (x, y) of G'_M, vertices x and y are colored such that $c'(x) < c'(y)$. Indeed vertices a and e have different colors since they are linked by an edge. Furthermore, since there is a path (a, b), (b, d), (d, e) from a to e, we must have $c(a) \leq c(e)$. We conclude that $c(a) < c(e)$ and so we necessarily have $c(x) < c(y)$. Thus we get a feasible strong mixed 3-coloring of G'_M. □

Again we can claim that this $\mathcal{NP}$-completeness result is best possible. Indeed, in [9] it has been shown that $W(G_M, 2)$ is polynomially solvable. Furthermore, a mixed graph having maximum degree 2 consists of a family of disjoint mixed chains and mixed cycles. Mixed cycles can be colored optimally (see [9]), and the case of mixed chains is trivial.

3. Conclusion

In this note, we have shown that both the strong mixed graph coloring problem and the weak mixed graph coloring problem are $\mathcal{NP}$-complete in planar cubic bipartite mixed graphs, which strengthens some results of [9]. In order to detect some more polynomially solvable cases for the strong mixed graph coloring problem, it would be interesting to analyze for instance mixed bipartite graphs containing directed subgraphs with a special structure. For the weak mixed graph coloring problem, it would be interesting to analyze for instance mixed bipartite graphs containing undirected subgraphs with a special structure.

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