A Note on Moments of Dividends

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Abstract We reconsider a formula for arbitrary moments of expected discounted dividend payments in a spectrally negative Lévy risk model that was obtained in Renaud and Zhou (2007, [4]) and in Kyprianou and Palmowski (2007, [3]) and extend the result to stationary Markov processes that are skip-free upwards.

Keywords dividends, barrier strategies, stationary Markov process, scale function

2000 MR Subject Classification 91B30; 60G51; 62P05

In two recent papers, Renaud and Zhou[4] and Kyprianou and Palmowski[3] independently showed that the kth moment of the expected discounted dividend payments in a spectrally negative Lévy risk model with initial surplus u and horizontal dividend barrier b ≥ u is given by

\[ V_k(u; b) = k! \frac{W_{k\delta}(u)}{W_{k\delta}(b)} \prod_{i=1}^{k} \frac{W_{i\delta}(b)}{W'_{i\delta}(b)}. \]  

(1)

Here \( \delta \geq 0 \) is a constant discount rate and \( W_q(x) \) is the scale function of the underlying Lévy process (with Laplace exponent \( \psi \)) defined through its Laplace transform

\[ \int_0^\infty e^{-\lambda x} W_q(x) \, dx = 1/(\psi(\lambda) - q). \]

In this short note we show that formula (1) can be established in a more general framework by direct probabilistic reasoning. Concretely, assume that the surplus process \( U(t) \) is a stationary Markov process that has no jumps upwards and has the strong Markov property. Let \( T_{(0,a)} = \inf\{t \geq 0 \mid U(t) \notin (0,a)\} \) and define for \( 0 \leq u_1 \leq u_2 \) the function

\[ C_\delta(u_1, u_2) = \mathbb{E}_{u_1}[e^{-\delta T_{(0,u_2)}}; U(T_{(0,u_2)}) = u_2], \]

which is the Laplace transform of the upper exit time out of the interval \((0, u_2)\) when starting in \( u_1 \), i.e. \( U(0) = u_1 \). As discussed in [1], one immediately deduces from the absence of upward jumps and the strong Markov property that

\[ C_\delta(u_1, u_3) = C_\delta(u_1, u_2) C_\delta(u_2, u_3), \quad 0 \leq u_1 \leq u_2 \leq u_3. \]

Thus, there exists a positive increasing function \( h_\delta(x) \) such that

\[ C_\delta(u_1, u_2) = \frac{h_\delta(u_1)}{h_\delta(u_2)} \]

for \( 0 \leq u_1 \leq u_2 \).

(Note that in the particular situation where \( U(t) \) is a spectrally negative Lévy process, \( h_\delta(x) \) can be identified with the scale function \( W_\delta(x) \)). Since the function \( h_\delta(x) \) is unique only up to

Manuscript received February 2, 2011.

∗Supported by the Swiss National Science Foundation Project (No. 200021-124635/1).
a constant factor, we can choose $u_0$ and set $h_\delta(u_0) = 1$, giving
\[
h_\delta(u) = \begin{cases} C_\delta(u, u_0), & u < u_0, \\ 1/C_\delta(u_0, u), & u > u_0. \end{cases}
\]
Dividends are paid according to the barrier strategy with horizontal barrier $b$, that is, any potential excess of the surplus beyond $b$ is paid as dividends. Let $D_u(t)$ denote the aggregate dividends paid up to time $t$, and let $\tau$ be the time of ruin. Then the present value of all dividends up to ruin is
\[
D_u = \int_0^\tau e^{-st} dD_u(t).
\]
The $k$th moment of $D_u$ is denoted by $V_k(u; b) = \mathbb{E}_u(D^k_u)$.

Analogously to Proposition 2 of Renaud and Zhou\cite{4}, it immediately follows from $(e^{-st}D_u)^k = e^{-k\delta t}D^k_u$ and the strong Markov property of $U(t)$ applied at the upper exit time of the interval $(0, b)$ that
\[
V_k(u; b) = C_{k\delta}(u; b)V_k(b; b) = \frac{h_{k\delta}(u)}{h_{k\delta}(b)}V_k(b; b), \quad 0 \leq u \leq b. \tag{2}
\]

Related to an idea of Gerber and Shiu\cite{2}, consider next the difference between the total discounted dividends when starting in $U(0) = b$ and $U(0) = b - \epsilon$, respectively, for a sufficiently small $\epsilon > 0$. If $U(0) = b - \epsilon$, then the dividend barrier will be reached “shortly”. At that time, the process that starts at $b$ has led to a total dividend of $\epsilon$, and after this time the trajectories of the two processes are identical. Hence we have the approximate relationship $D_b - D_{b-\epsilon} \approx \epsilon$ and subsequently $D^k_b - D^k_{b-\epsilon} \approx D^k_b - (D_b - \epsilon)^k = \epsilon kD^{k-1}_b + o(\epsilon)$. Taking expectations and the limit $\epsilon \to 0$, we arrive at
\[
\frac{dV_k(u; b)}{du} \bigg|_{u=b-} = kV_{k-1}(b; b). \tag{3}
\]
From (2) and (3) we obtain the recursive formula
\[
V_k(b; b) = k \frac{h_{k\delta}(b)}{h_{k\delta}(b)} V_{k-1}(b; b).
\]
From this and $V_0(u; b) = 1$ we obtain
\[
V_k(b; b) = k! \prod_{i=1}^k \frac{h_{i\delta}(b)}{h_{i\delta}(b)}.
\]
Substitution in (2) yields
\[
V_k(u; b) = k! \frac{h_{k\delta}(u)}{h_{k\delta}(b)} \prod_{i=1}^k \frac{h_{i\delta}(b)}{h_{i\delta}(b)}, \quad 0 \leq u \leq b,
\]
which extends (1) to stationary Markov processes that are skip-free upwards.

References
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