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Coisotropic Submanifolds in Poisson Geometry and Branes in the Poisson Sigma Model

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Abstract. General boundary conditions ('branes') for the Poisson sigma model are studied. They turn out to be labeled by coisotropic submanifolds of the given Poisson manifold. The role played by these boundary conditions both at the classical and at the perturbative quantum level is discussed. It turns out to be related at the classical level to the category of Poisson manifolds with dual pairs as morphisms and at the perturbative quantum level to the category of associative algebras (deforming algebras of functions on Poisson manifolds) with bimodules as morphisms. Possibly singular Poisson manifolds arising from reduction enter naturally into the picture and, in particular, the construction yields (under certain assumptions) their deformation quantization.

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1. Introduction

Coisotropic submanifolds play a fundamental role in symplectic geometry as they describe systems with symmetries (Dirac's 'first-class constraints') and provide a method to generate new symplectic spaces ('symplectic reduction'). Their generalizations to Poisson manifolds also carry naturally induced foliations such that the leaf spaces ('reduced phase spaces') are again Poisson. They are the general framework to study symmetries in the Poisson world. We recall the basic facts about coisotropic submanifolds in Section 2.

The Poisson sigma model [10,18] is a topological field theory defined in terms of bundle maps from the tangent bundle of a surface to the cotangent bundle of a given Poisson manifold M. Particularly interesting is the case where the source surface is a disk, which so far has been studied only assuming particularly simple boundary conditions (viz., mapping the boundary to the zero section of T^*M); then the perturbative path integral expansion yields [4] Kontsevich's star product [14] on M, while the reduced phase space of the theory [5] is the symplectic groupoid [13,21,23] of M. A relevant problem concerns other possible boundary conditions and their role.

It turns out that coisotropic submanifolds of a Poisson manifold label the possible boundary conditions ('D-branes') of the Poisson sigma model. Something similar happens in the symplectic context where coisotropic submanifolds play an important role as D-branes for the A-model [11,17].

In Section 3 we discuss the classical Hamiltonian viewpoint. The reduced phase space of the Poisson sigma model on an interval with boundary conditions labeled by coisotropic submanifolds C_0 and C_1 is a (possibly singular) symplectic manifold endowed with a Poisson and an anti-Poisson map to the reduced phase spaces $\underline{C_0}$ and $\underline{C_1}$ of C_0 and C_1. This construction yields then a 'dual pair' which is the notion of morphism in a category, whose objects are Poisson manifolds, that seems to be natural [16] if one has quantization in mind.

Section 4, which can be read independently of Section 3, deals with the perturbative quantization of the Poisson sigma model with boundary conditions given by coisotropic submanifolds. We show that locally, under appropriate assumptions, this construction allows us (i) to deformation-quantize the (possibly singular) Poisson manifolds obtained by reduction from the given coisotropic submanifolds and (ii) to give the space of invariant functions on the intersection of two coisotropic submanifolds the structure of a bimodule for the corresponding deformed algebras. Some examples where the above procedure works are discussed in Section 5.

The construction also suggests how to modify Kontsevich's formality map from multivector fields to multidifferential operators in the presence of a given submanifold, see Section 6. This should be relevant when trying to globalize.

The nonperturbative study (probably beyond our possibilities at the moment) looks like a generalization of the Fukaya A_∞-category.

This Letter is thought of as a short overview of results that will be discussed thoroughly elsewhere [7]. To read Section 3 the reader is assumed to have had some exposure to [5], while Section 4 assumes some familiarity with [4,14]. More advanced remarks, which have no consequence for the rest of the paper, have been put in footnotes.

2. Coisotropic Submanifolds

A *Poisson manifold* (M,π) is a manifold M endowed with a bivector field π such that the bracket $\{f, g\} := \pi(\mathrm{d}f, \mathrm{d}g)$ is a Lie bracket on $C^\infty(M)$. Equivalently, the Poisson bivector field π must satisfy $[\pi, \pi]=0$ where $[\ ,\ ]$ denotes the Schouten–Nijenhuis bracket. In local coordinates, this amounts to the equations

$$\pi^{ij}\,\partial_i\pi^{kl}+\pi^{il}\,\partial_i\pi^{jk}+\pi^{ik}\,\partial_i\pi^{lj}=0. \tag{2.1}$$

The bivector field π induces a bundle map $\pi^\sharp\colon T^*M\to TM$ by

$$\langle\pi^\sharp(x)\sigma\,,\,\tau\rangle=\pi(x)(\sigma,\tau),\quad \forall x\in M,\quad \forall\sigma,\tau\in T_x^*M,$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing. Some examples of Poisson manifolds are:

Trivial case: $\pi \equiv 0$.

Symplectic case: (M, ω) is symplectic and π is the inverse of ω.

Linear case: $M = \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra, and the bracket of linear functions is defined by the Lie bracket. The resulting Poisson structure is usually called the Kostant–Kirillov Poisson structure.

In general, Poisson manifolds are foliated – by the possibly singular involutive distribution* $\pi^\sharp(T^*M)$ – and each leaf turns out to be symplectic. In the first example, each point is a symplectic leaf; in the second example, there is just one symplectic leaf, the whole manifold; in the third example symplectic leaves are the same as coadjoint orbits (and have in general varying dimensions). A submanifold C of a Poisson manifold (M, π) is said to be *coisotropic* [22] if $\pi^\sharp(N^*C) \subset TC$, where N^*C denotes the conormal bundle of C (i.e., the subbundle of $T^*_C M$ consisting of covectors that kill all vectors of TC). It follows from the Jacobi identity for π that the *characteristic distribution* $\pi^\sharp(N^*C)$ on the coisotropic submanifold C is involutive;** the corresponding foliation is called the *characteristic foliation* and we will denote by $\underline{C}$ its leaf space which we call the *reduced phase space*. Its space of 'smooth' functions may be defined also when the leaf space is not a smooth manifold by setting à la Whitney $C^\infty(\underline{C}) := C^\infty(C)^{\text{inv}}$, where the superscript denotes the invariant part (a function f on C is invariant if $X(f) = 0$ for all sections X of $\pi^\sharp(N^*C)$).

When M is symplectic, $\pi^\sharp$ yields an isomorphism between N^*C and $T^\perp C$ (the subbundle of $T_C M$ of vectors that are symplectic-orthogonal to all vectors in TC). So we recover the usual definition of coisotropic submanifolds in the symplectic case: $T^\perp C \subset TC$.

We recall a couple of examples of coisotropic submanifolds. Let M and N be Poisson manifolds and let $f\colon M \to N$ be a Poisson map (i.e., a map whose pullback is a morphism of Poisson algebras). We denote by $\bar{N}$ the Poisson manifold obtained by changing sign to the Poisson structure on N. Then

1. The graph of f is coisotropic in $M \times \bar{N}$.
2. The preimage of a symplectic leaf of N is coisotropic in M (when a submanifold).

A particular instance is when N is the dual of a Lie algebra, in which case f is an equivariant momentum map. An interesting example, to which we will return in Section 5, is the following:

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*The distribution is involutive as a consequence of the Jacobi identity (2.1). One actually has more structure; viz., T^*M is a Lie algebroid with $\pi^\sharp$ as its anchor; as for the Lie bracket on its sections, it is enough to define it on exact 1-forms for which one sets $[\mathrm{d}f, \mathrm{d}g] := \mathrm{d}\{f, g\}$. The involutive distribution $\pi^\sharp(T^*M)$ is then the canonical foliation of this Lie algebroid.

**N^*C actually turns out to be a Lie subalgebroid of T^*M with Lie algebroid structure. More precisely, conormal bundles of coisotropic submanifolds are all possible Lagrangian Lie subalgebroids of T^*M with its canonical symplectic structure. If M is integrable, coisotropic submanifolds are also in correspondence with Lagrangian subgroupoids of the symplectic groupoid of M. (see [3]).

EXAMPLE 2.1. Consider a Lie subalgebra $\mathfrak{h} \overset{\iota}{\hookrightarrow} \mathfrak{g}$, and set $M = \mathfrak{g}^*$, $N = \mathfrak{h}^*$ (with Kostant–Kirillov Poisson structure) and $f = \iota^*$. As $\{0\}$ is a symplectic leaf of $\mathfrak{h}^*$, we get the coisotropic submanifold $\mathfrak{h}^\perp := (\iota^*)^{-1}(0)$ (the annihilator of $\mathfrak{h}$) in $\mathfrak{g}^*$.

Let $\mathcal{I}$ be the ideal of functions that vanish when restricted to the submanifold C, so $C^\infty(C) = C^\infty(M)/\mathcal{I}$. Differentials of elements of $\mathcal{I}$ yield sections of N^*C. Therefore, we can also characterize coisotropic submanifolds of M as submanifolds whose vanishing ideal $\mathcal{I}$ is a Poisson subalgebra (and not just a commutative subalgebra) of $C^\infty(M)$. In Dirac's terminology, a family of functions generating $\mathcal{I}$ are called first-class constraints.

Let $N(\mathcal{I}) := \{g \in C^\infty(M) : \{g, \mathcal{I}\} \subset \mathcal{I}\}$ be the normalizer of $\mathcal{I}$. If $\mathcal{I}$ is a Poisson subalgebra, so is $N(\mathcal{I})$. Moreover, $\mathcal{I}$ is a Poisson ideal in $N(\mathcal{I})$, so $N(\mathcal{I})/\mathcal{I}$ is a new Poisson algebra. This may easily be recognized as the algebra $C^\infty(C)^{\mathrm{inv}}$ of invariant functions on C. So $\underline{C}$ is a (possibly singular) Poisson manifold.

Observe that, in the smooth case, the inclusion map $\iota\colon C \to M$ and the projection $p\colon C \to \underline{C}$ induce maps of the commutative algebras of functions that make $C^\infty(C)$ into a bimodule over $C^\infty(\underline{C})$ and $C^\infty(M)$. This clearly works also in the singular case where we have the projection $\iota^*\colon C^\infty(M) \to C^\infty(M)/\mathcal{I}$ and the inclusion $p^*\colon N(\mathcal{I})/\mathcal{I} \to C^\infty(M)/\mathcal{I}$.

We may also consider two coisotropic submanifolds C_0 and C_1. If we denote by $\underline{C_0 \cap C_1}$ the quotient of $C_0 \cap C_1$ by the intersection of the characteristic foliations, we see that $C^\infty(\underline{C_0 \cap C_1})$ is a bimodule over the commutative algebras $C^\infty(\underline{C_0})$ and $C^\infty(\underline{C_1})$. (The previous case corresponds to $C_0 = C$ and $C_1 = M$.)

The fact that these structures are compatible with the given Poisson structures gives the bimodule some extra properties that will be better understood in the following Sections.

3. Classical Hamiltonian Study of the Poisson Sigma Model

The Poisson sigma model is described at the classical Hamiltonian level by the following data: (i) a weak symplectic structure on an infinite-dimensional manifold (the 'phase space') and (ii) equations that select a coisotropic submanifold. As in every topological field theory the Hamiltonian is zero and the characteristic foliation of the coisotropic submanifold has finite codimension ('finitely many degrees of freedom').

These data depend on a given Poisson manifold as follows. Let (M, π) be a Poisson manifold. Then the phase space is the cotangent bundle T^*PM of the path space of M (open case) or the cotangent bundle T^*LM of the loop space of M (closed case) with canonical weak symplectic structure. These spaces may also be understood as the spaces of bundle maps $TI \to T^*M$ and $TS^1 \to T^*M$, respectively (where I is the interval and S^1 the circle). They may be given a Banach manifold structure by imposing certain conditions (e.g., requiring the base maps to be differentiable and the fiber maps to be continuous).

An element of these spaces is then a pair (X, ζ) where X is a (differentiable) map from I or S^1 to M and ζ is a (continuous) 1-form taking values in sections of the pulled-back bundle X^*T^*M. The coisotropic[⋆] submanifold $\mathcal{C}(M)$ is defined by the equations

$$\mathrm{d}X + \pi^\sharp(X)\,\zeta = 0. \tag{3.1}$$

The characteristic foliation is better described by choosing local coordinates $\{x^I\}_{I=1,\dots,\dim M}$, so that X and ζ are locally a set of functions X^I and of 1-forms ζ_I. Denoting by δX and $\delta\zeta$ the horizontal and vertical components of a vector field on T^*PM, an element of the characteristic distribution is given by

$$\delta X^I = \pi^{IJ}(X)\,\beta_J, \tag{3.2a}$$

$$\delta\zeta_I = -\mathrm{d}\beta_I - \partial_I\pi^{JK}\,\zeta_J\,\beta_K, \tag{3.2b}$$

where β is a (differentiable) section of X^*T^*M that, in the open case, is required to vanish on the boundary. The reduced phase space[⋆⋆] $\underline{\mathcal{C}(M)}$ has particularly interesting properties in the open case (where it is shown to be the possibly singular, source-simply-connected symplectic groupoid of M [5]).

From now we will consider only the open case and look for possible boundary conditions. Given two submanifolds C_0 and C_1 of M, we define $\mathcal{C}(M; C_0, C_1)$ to be the submanifold of $\mathcal{C}(M)$ where the base maps are paths connecting C_0 to C_1 (with this new notation we have, in particular, $\mathcal{C}(M) = \mathcal{C}(M; M, M)$). We have then [7] the following:

THEOREM 3.1. *Assume that all pairs of points of the two coisotropic submanifolds can be connected by base paths of solutions of (3.1). Then $\mathcal{C}(M; C_0, C_1)$ is coisotropic in T^*PM iff C_0 and C_1 are coisotropic in M.*

The characteristic distribution is again given by (3.2) but with the condition that β on the boundary $\partial I = \{0, 1\}$ is an element of $N^*_{X(u)}C_u$, $u = 0, 1$. (The previous case is obtained by setting $C_0 = C_1 = M$ and observing that N^*M has rank zero.) Observe that the coisotropy condition on C_t, $t = 0, 1$, ensures that $\delta X(t)$ is tangent to C_t, as required by the boundary conditions.

⋆To define $\mathcal{C}(M)$ one just needs a tensor π. One may show however ([18] for the closed and [5] for the open case) that $\mathcal{C}(M)$ is coisotropic iff π is a Poisson bivector field.

⋆⋆Using the language of Lie algebroids, one may also give the following interpretation [9, 19]: Elements of $\mathcal{C}(M)$ are precisely those bundle maps that are also morphisms of Lie algebroids, where the tangent bundles are given the canonical Lie algebroid structure and T^*M the one induced from the Poisson structure. Elements of $\underline{\mathcal{C}(M)}$ are then morphisms of Lie algebroids modulo 'homotopy.' In the closed case, we say that two morphisms $\gamma_0, \gamma_1 : TS^1 \to T^*M$ are homotopic if there exists a morphism of Lie algebroids $T(S^1 \times [0, 1]) \to T^*M$ such that its restriction to $T(S^1 \times \{u\})$ is γ_u, $u = 0, 1$. In the open case, beside the obvious replacement of S^1 by I, we put the extra condition that the restriction of the morphism to $T(\partial I \times [0, 1])$ is the zero bundle map (or, in other words, a morphism to the rank-zero Lie algebroid over M regarded as a Lie subalgebroid of T^*M.)

The characteristic foliation★ on $\underline{\mathcal{C}(M; C_0, C_1)}$ may move the endpoints of the base maps but only along the characteristic foliations of C_0 and C_1. Thus, the evaluation maps at 0 and 1 descend to the quotients and define maps $J_u\colon \underline{\mathcal{C}(M; C_0, C_1)} \to \underline{C_u}$, $u = 0, 1$. Observe that, when smooth, $\underline{\mathcal{C}(M; C_0, C_1)}$ is endowed with a symplectic structure while $\underline{C_0}$ and $\underline{C_1}$ are endowed with Poisson structures. It is then possible to prove [7] the following:

THEOREM 3.2. *J_0 and J_1 are a Poisson and an anti-Poisson map, respectively, and the J_0-fibers are symplectically orthogonal to the J_1-fibers* (*so pullbacks of functions via J_0 and J_1 Poisson commute*).

Thus, using the terminology of [12,20] (see also [2,16] and references therein), $\underline{C_0} \xleftarrow{J_0} \underline{\mathcal{C}(M; C_0, C_1)} \xrightarrow{J_1} \underline{C_1}$ is a (possibly singular) *dual pair*. Observe [15] that *dual pairs* are the morphisms of a category in which Poisson manifolds are the objects (the composition of the *dual pairs* S and S' which have the same Poisson manifold P as target and source, respectively, is obtained by symplectic reduction observing that $S \times_P S'$ is coisotropic in $S \times S'$). This structure suggests, given a Poisson manifold, to define a category★★ whose objects are the leaf spaces of its coisotropic submanifolds and whose morphisms are generated by the dual pairs obtained above.

In Section 4 we will see (cf. Theorem 4.3) that the corresponding quantum category (in the context of deformation quantization) consists of associative algebras with bimodules as morphisms.

4. Perturbative Quantization of the Poisson Sigma Model

4.1. CLASSICAL ACTION FUNCTIONAL AND SYMMETRIES

In the path integral quantization of the Poisson sigma model, one starts from a classical action functional S, a function on the space of bundle maps $T\Sigma \to T^*M$ from the tangent bundle of a surface Σ to the cotangent bundle of the Poisson manifold M. Such a bundle map $\hat{X}$ consists of a base map $X\colon \Sigma \to M$ and a linear map η for each fiber, which may be thought of as a 1-form $\eta \in \Omega^1(\Sigma, X^*T^*M)$ on Σ with values in the pull-back of the cotangent bundle. The action functional is then [10,18] $S(X, \eta) = \int_\Sigma (\langle \eta, \mathrm{d}X\rangle + \frac{1}{2}\langle \pi \circ X, \eta \wedge \eta\rangle)$. In the case of interest to us where Σ has a boundary, it is natural to consider the action functional

★The leaf space $\underline{\mathcal{C}(M; C_0, C_1)}$ may also be defined as in the second footnote on page 161, as the quotient of the space of Lie algebroid morphisms $TI \to T^*M$ by homotopies. The morphisms are however now required to have base maps connecting C_0 to C_1, and homotopies must satisfy the condition that the restriction to $T(t \times [0, 1])$, $t = 0, 1$, is a morphism of Lie algebroids with range N^*C_t.

★★There is also another category, actually a groupoid, associated to the Poisson sigma model with boundary: its objects are points in the reduced coisotropic submanifolds and the morphisms between $[x_0] \in \underline{C_0}$ and $[x_1] \in \underline{C_1}$ are the elements of $\underline{\mathcal{C}(M; C_0, C_1)}$ with $J_t([(X, \zeta)]) = [x_t]$. Composition is obtained by gluing, and the inverse by reversing I. The symplectic groupoid of M is then a subgroupoid of this.

with boundary conditions imposing that $\hat{X}$ maps the tangent bundle $T\partial\Sigma$ of the boundary to the conormal bundle N^*C of a submanifold C. With these boundary conditions, the Euler–Lagrange equations are differential equations without any boundary term, since the boundary term coming from integration by parts is

$$\int_{\partial\Sigma} \langle \eta, \delta X \rangle, \tag{4.1}$$

which vanishes for any variation δX of the base map.

If C is the whole of M, this boundary condition is the one considered in [4] and leads, in case Σ is a disk, to the construction of the Kontsevich formula for deformation quantization of M. In this case there are no conditions on the base map X and η is assumed to vanish on vectors tangent to the boundary of Σ.

If C is a coisotropic submanifold, the boundary conditions for gauge transformations of [4] can be generalized to this case. An infinitesimal gauge transformation at $\hat{X}$ is parametrized by a section $c \in \Gamma(\Sigma, X^*T^*M)$ restricting to the boundary to a section of X^*N^*C. The action functional is invariant under such a gauge transformation if C is coisotropic. Indeed, the calculation of the variation of the action of [10, 18], done for closed Σ, shows that S is invariant up to the boundary term (4.1). The infinitesimal gauge variation of the base map is $\delta X = \pi^\sharp c$, so that the boundary term vanishes if C is coisotropic.

From now on we restrict our attention to the case where Σ is a disk.

4.2. BATALIN–VILKOVISKY QUANTIZATION

The quantization of the Poisson sigma model with boundary conditions is given by path integrals $\int \exp(iS/\hbar)\mathcal{O}\mathrm{d}\hat{X}$ over the space of bundle maps $\hat{X} = (X, \eta)$ obeying the boundary conditions. The observables $\mathcal{O}$ are gauge invariant functions on this space. A class of observables of particular interest is given by evaluating functions on M at the image by X of the points of the boundary: $\mathcal{O} = \prod_{i=1}^k f_i(X(p_i))$, $p_i \in \partial\Sigma$. The condition of gauge invariance is then $\pi(\mathrm{d}f_i, c) = 0$ for $c \in N^*C$, i.e., $f_i \in N(\mathcal{I})$. Since only the value of f_i on C matters we may take $f_i \in N(\mathcal{I})/\mathcal{I} = C^\infty(\underline{C})$. Equivalently, the functions f_i are functions on C which are constant on the leaves of the foliation.

This reasoning and the results of [4], where the case $C = M$ was considered, suggest that the Batalin–Vilkovisky perturbative calculation of the path integral should yield an associative product on $C^\infty(\underline{C})$ obtained by picking three distinct points p, q, r on the boundary of the disk Σ and setting

$$(f \star g)(x) = \int_{X(r)=x} \mathrm{e}^{\frac{i}{\hbar}S(X,\eta)} f(X(p))g(X(q))\mathrm{d}X\,\mathrm{d}\eta, \tag{4.2}$$

$f, g \in C^\infty(\underline{C})$. The Batalin–Vilkovisky procedure gives a way to make sense (as a formal power series in $\hbar$) of this integral by deforming the integration domain to a Lagrangian submanifold of the odd symplectic Q-manifold of maps $\Pi T\Sigma \to$

ΠT^*M, see [4,6], giving a version of an AKSZ model [1]. This essentially amounts to replacing (X, η) by superfields $(\mathbf{X}, \boldsymbol{\eta})$, where $\mathbf{X}$ is a map $\Pi T\Sigma \to M$ to the base and $\boldsymbol{\eta}$ is a section of the pull-back $\mathbf{X}^*\Pi T^*M$. The action functional is $S = S_0 + S_\pi$, where $S_0 = \int_{\Pi T\Sigma} \langle \boldsymbol{\eta}, D\mathbf{X}\rangle \mu$ and for any multivector field α, $S_\alpha = \int_{\Pi T\Sigma} \langle \alpha \circ \mathbf{X}, \boldsymbol{\eta} \wedge \cdots \wedge \boldsymbol{\eta}\rangle \mu$. Here μ is the canonical volume form on $\Pi T\Sigma$ and D is induced by the de Rham differential on $C^\infty(\Pi T\Sigma) = \Omega^\cdot(\Sigma)$.

The boundary conditions for the case $C = M$ were discussed in [4,6]. Similar arguments apply here. The result is that the classical master equation $\{S, S\} = 0$ is obeyed if the boundary conditions are that $(\mathbf{X}, \boldsymbol{\eta})$ restricts on the boundary to a map $\Pi T\partial\Sigma \to \Pi N^*C$ for a coisotropic submanifold C. In the AKSZ formulation, the possible boundary conditions are discussed in [6]. If the source supermanifold is of the form $\Pi T\Sigma$ (Σ a manifold with boundary) and the target supermanifold Y has a QP-structure defined by an odd symplectic form $\omega = \mathrm{d}\theta$ and a solution s of the master equation, then the boundary conditions are labeled by Lagrangian submanifolds of Y where both θ and s restrict to zero (and, given such a Lagrangian submanifold L, one requires maps $\Pi T\Sigma \to Y$ to restrict on the boundary to maps $\Pi T\partial\Sigma \to L$). In the present case $Y = \Pi T^*M$, θ is the canonical 1-form $\langle p, \mathrm{d}x\rangle$ and, given a Poisson bivector field π, we set $s = \langle p, \pi^\sharp(x)p\rangle/2$ (we denote by x coordinates on M and by p coordinates on the fiber). So Lagrangian submanifolds with the above properties are the same as odd conormal bundles of coisotropic submanifolds of M.

Similar boundary conditions for the A-model are proposed in [17] where M is symplectic and N^*C is replaced by $T^\perp C$. Here the curly bracket (the BV bracket) denotes the Poisson bracket associated to the odd symplectic structure. Indeed, we have in general $\{S_\alpha, S_\beta\} = S_{[\alpha,\beta]}$, so that $\{S_\pi, S_\pi\} = 0$ for Poisson bivector fields π. The bracket with S_0 involve a boundary term from integration by parts. With our boundary conditions, $\{S_0, S_0\}$ vanishes (for any C) and $\{S_0, S_\pi\}$ vanishes for C coisotropic as the boundary term is proportional to $\int_{\partial\Sigma} \langle \pi \circ \mathbf{X}, \boldsymbol{\eta} \wedge \boldsymbol{\eta}\rangle$. The observables $\mathcal{O}$ are then cocycles for the BV differential $\{S, \ \}$.

As observed in Section 2, the conormal bundle N^*C of C is a Lagrangian Lie subalgebroid of T^*M, so ΠN^*C is a Lagrangian submanifold of ΠT^*M. One may then define more general boundary observables associated to elements of the corresponding Lie algebroid cohomology (invariant functions being the case of degree zero).

In fact, let V be a representative of a Lie algebroid cohomology class of degree k. In particular, V is a section of the kth exterior power of the normal bundle $NC = T_CM/TC$. With our choice of coordinates, we may write $V = V^{\mu_1\ldots\mu_k}\partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k}$. To it, we associate the functional

$$\mathbf{V} := V(\mathbf{X})^{\mu_1\ldots\mu_k}\, \boldsymbol{\eta}_{\mu_1} \wedge \cdots \wedge \boldsymbol{\eta}_{\mu_k}.$$

We then get observables either by evaluating $\mathbf{V}$ at a point $p \in \partial\Sigma$,

$$\mathcal{O}_V^0 := \mathbf{V}(p) = V(X(p))^{\mu_1\ldots\mu_k}\, c_{\mu_1}(p) \cdots c_{\mu_k}(p),$$

or by integrating it on the whole boundary, $\mathcal{O}^1_V := \int_{\partial\Sigma} \mathbf{V}$. It turns out that $\mathcal{O}^0_V$ and $\mathcal{O}^1_V$ are BV closed observables (of degree k and $k-1$ respectively) and that their BV cohomology classes are independent of the choices above.

More generally, one may take k coisotropic submanifolds $C_0, \ldots, C_{k-1}$ and consider Σ to be a disk whose boundary is partitioned into k intervals with the boundary condition that $\hat{X}$ maps the tangent bundle of the ith interval I_i to the conormal bundle N^*C_i. The gauge parameter c maps I_i to N^*C_i. Gauge invariant observables are obtained by evaluating functions in $C^\infty(\underline{C_i})$ at the image of points in the interior of I_i or functions in $C^\infty(\underline{C_i \cap C_{i+1}})$ evaluated at the point separating two neighboring intervals I_i and I_{i+1}, $i = 0, \ldots, k-2$. The point r separating I_{k_1} and I_0 is used to select a classical solution by the condition $X(r) = x$.

4.3. DEFORMATION OF BIMODULE STRUCTURES

In the next to simplest case $k = 2$, we then have two submanifolds C_0, C_1 and divide the boundary of the disk Σ into two intervals I_0, I_1 whose common boundary points are two points $p, q \in \partial\Sigma$. Considering path integrals with these boundary conditions and the condition that $X(q) = x \in C_0 \cap C_1$ we obtain various products between functions in $C^\infty(\underline{C_i})$ and $C^\infty(\underline{C_0 \cap C_1})$, depending on the points on $\partial\Sigma$ at which we evaluate the functions. The associativity of these products are then expected to give a deformation of the $C^\infty(\underline{C_0})$–$C^\infty(\underline{C_1})$-bimodule structure of $C^\infty(\underline{C_0 \cap C_1})$, where the deformation of the product on $C^\infty(C_i)$ is obtained from the case $k = 1$ considered above. Observe that associative algebras with bimodules as morphisms form a category which is in some sense the quantization of the category of dual pairs described at the end of Section 3.

Of course these semiclassical statements are expected to receive quantum corrections and should not be expected to hold without some additional assumptions. In fact we consider here only very simple situations in which the perturbative expansion can be computed and the statements can be checked rigorously at the level of finite-dimensional Feynman integrals.

4.4. FEYNMAN EXPANSION

We start from the case of one coisotropic submanifold and consider the case where M is an open subset of $\mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$ and the submanifold C is given by the equations

$$x^\mu = 0, \quad \mu = m+1, \ldots, n. \tag{4.3}$$

The tangent space to a point on C is then spanned by $\partial/\partial x^i$, $i = 1, \ldots, m$, and the conormal bundle by $\mathrm{d}x^\mu$, $\mu = m+1, \ldots, n$. With the convention that lower case Latin indices run over $\{1, \ldots, m\}$ and Greek indices over $\{m+1, \ldots, n\}$, the condition of coisotropy is then the condition that

$$\pi^{\mu\nu}(x^1, \ldots, x^m, 0, \ldots, 0) = 0,$$

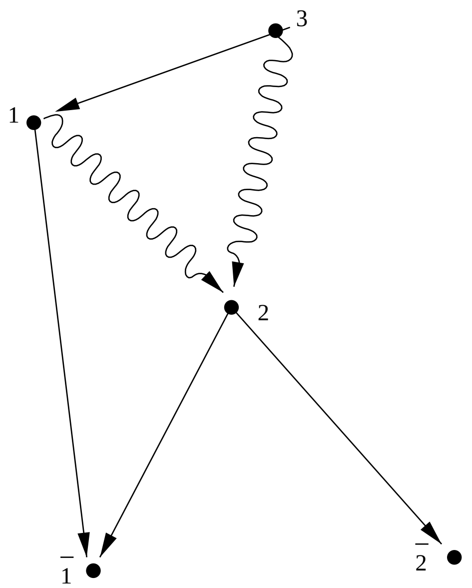

Figure 1. A simple admissible graph.

for the components of the tensor π. The coordinate functions x^μ are a system of generators for the ideal of C and the characteristic foliation is spanned by the vector fields $E^\mu = \sum_{i=1}^n \pi^{\mu i}\partial_i$ on C. The invariant functions on C are solutions of

$$E^\mu(f) = \sum_{i=1}^m \pi^{\mu i}\partial_i f = 0. \tag{4.4}$$

This condition will be modified by terms of higher order in ϵ.

The boundary conditions for the superfield are then $\mathbf{X}^\mu = 0$, $\boldsymbol{\eta}_i = 0$ on $\Pi T\Sigma$. The evaluation of the integral (4.2) in a power series in ϵ along the lines of [4] leads to a modification of the Kontsevich formulas of [14]. They can be written as follows

$$f \star g = fg + \sum_{k=1}^\infty \frac{\epsilon^k}{k!} \sum_{\Gamma\in G_{k,2}} w_\Gamma B_\Gamma(f,g), \qquad f, g \in C^\infty(C). \tag{4.5}$$

The sum is over admissible graphs Γ of order k, to which are associated a *weight* $w_\Gamma \in \mathbb{R}$ and a *bidifferential operator* B_Γ. The deformation parameter is $\epsilon = i\hbar$.

An admissible graph in $G_{k,2}$ has k vertices $1, \ldots, k$ of the first type, and 2 vertices $\bar{1}, \bar{2}$ of the second type. The edges are oriented and come in two types, say straight and wavy. There are exactly two edges emerging from each of the vertices of the first type and none from vertices of the second type. An ordering of the edges emerging from each vertex is given. Each edge may land at any vertex except at the one it emerges from. A simple example of such a graph is given in Figure 1.

The bidifferential operator associated to Γ is obtained by the following rule: to each vertex of the first type we associate a component of π and to the vertices of the second type we associate the functions f and g. To an edge from a vertex a to a vertex b we associate a partial derivative acting on the object associated to b with respect to the variable with the same index as the corresponding index of the component of π associated to a. Then we take the product and sum over Latin indices for straight lines and over Greek indices for wavy lines. Finally we evaluate the result at a point $x \in C$. For example Γ in Figure 1 gives the bidifferential operator

$$\partial_l \pi^{i\lambda} \partial_\lambda \partial_\mu \pi^{jk} \pi^{l\mu} \partial_i \partial_j f \partial_k g.$$

The sum over $\{1, \ldots, m\}$ for repeated Latin indices and over $\{m+1, \ldots, n\}$ for repeated Greek indices is understood.

The weight of Γ is

$$w_\Gamma = \frac{1}{(2\pi)^{2k}} \int_{C^+_{k,2}} \prod_{\text{edges e}} \mathrm{d}\phi_e.$$

The integral is over the configuration space $C^+_{k,2}$ of k distinct points z_i in the upper half plane and two ordered points $z_{\bar{1}} < z_{\bar{2}}$ on the real axis, modulo dilations and translation in the real direction. The differential form $\mathrm{d}\phi_e$ associated to an edge e going from a to b is $\mathrm{d}\phi(z_a, z_b)$ if the edge is straight and is $\mathrm{d}\phi(z_b, z_a)$ if it is wavy. Here $\mathrm{d}\phi(z, w)$ is the differential of the Kontsevich angle function

$$\phi(z, w) = \frac{1}{2i} \log \frac{(z-w)(z-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})} = \arg(z-w) + \arg(z-\bar{w}).$$

The ordering of factors in the product of $\mathrm{d}\phi_e$ is obtained from the ordering of vertices and the given ordering of edges emerging from each vertex.

The fact that wavy lines correspond to $\mathrm{d}\phi(z_b, z_a)$ rather than to $\mathrm{d}\phi(z_a, z_b)$ and the fact that the result of the action of the bidifferential operators are evaluated at a point $x \in C$ are the only places where the formula differs from Kontsevich's (the case $C = M$). For the readers familiar with [4] we add that the (super-)propagators $\langle \boldsymbol{\eta}_I(z) \boldsymbol{\xi}^J(w) \rangle = \delta^J_I P_I(z, w)$ in the Feynman perturbative expansion around the constant classical solution $\mathbf{X}(z) = x$, $\boldsymbol{\eta} = 0$ ($\boldsymbol{\xi}^I = \boldsymbol{X}^I - x^I$) differ for $I = \mu \in \{m+1, \ldots, n\}$ from the ones in the case $C = M$ by the boundary condition, which is that it vanishes when w rather than z is restricted to the boundary. So we have $P_i(z, w) = \mathrm{d}\phi(z, w)$ as in the case $C = M$ but $P_\mu(z, w) = \mathrm{d}\phi(w, z)$.

Note that as the differential associated to a wavy edge vanishes if it points to $\bar{1}$ or $\bar{2}$, the functions f, g are differentiated only in the tangential directions ∂_j. Therefore the bidifferential operators B_Γ are well-defined on functions on C.

4.5. STOKES' THEOREM AND ASSOCIATIVITY

As in [14], the main tool to prove properties of the product is Stokes' theorem on a compactification $\bar{C}^+_{k,m}$ of configuration spaces $C^+_{k,m}$ of k distinct points in the upper half-plane and m ordered points on the real axis modulo translations and dilations. For example, the associativity of the Kontsevich product (4.5) (the case $C = M$) is proven by evaluating the integral of the differential of a closed form (which of course vanishes) on $\bar{C}^{+}_{k,3}$ with Stokes' theorem. The sum of contributions of the faces (pieces of the boundary) yield associativity identities.

The same calculation can be applied to the case of general C of the type (4.3), but there is an important difference: the contribution from some faces (the faces with a 'bad edge') does does not vanish a priori. These are faces containing limiting configurations where a subset of the points in the upper half-plane approach a point on the real axis. These faces produce corrections to the associativity involving more general objects expressed in terms of graphs, which we proceed to describe.

For vector fields ξ, η on M introduce a differential operator $A(\xi)$ on $C^{\infty}(C)[[\epsilon]]$ by

$$A(\xi)f = \xi f + \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} \sum_{\Gamma \in G_{k+1,1}} w_\Gamma B_\Gamma(\xi) f$$

and a function

$$F(\xi, \eta) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \sum_{\Gamma \in G_{k+2,0}} w_\Gamma B_\Gamma(\xi, \eta) \in C^{\infty}(C)[[\epsilon]].$$

The definitions of w_Γ, B_Γ are the same as for $G_{k,2}$ except that graphs in $G_{k+1,1}$ have one additional vertex of the first type associated with ξ from which one line emerges and just one vertex of the second type; graphs in $G_{k+2,0}$ have two additional vertices of the first type associated to ξ and η and none of the second type. In the case $C = M$ these objects were introduced in [8] to construct global star-products on manifolds.

From now on, we make the following

ASSUMPTION 1. *$F(E^\mu, E^\nu) = 0$ for $m + 1 \leqslant \mu, \nu \leqslant n$.*

This assumption is verified in a number of examples, as we discuss below. It appears that it is possible to remove this assumption at the cost of introducing a recursive correction procedure. This will be discussed elsewhere [7].

The quantum version of the algebra of invariant functions on C is defined to be the $\mathbb{R}[[\epsilon]]$-module

$$C^{\infty}_{\epsilon}(\underline{C}) = \{ f \in C^{\infty}(C)[[\epsilon]] : A(E^\mu) f = 0 \}.$$

THEOREM 4.1. *Under Assumption 1 the product* (4.5) *restricts to an associative product on $C^{\infty}_{\epsilon}(\underline{C})$*

The proof is similar to Kontsevich's proof of associativity of his star-product and is based on Stokes' theorem. In this case new boundary components give potentially nontrivial contributions due to the fact that the 1-form associated to wavy edges does not vanish as the first argument approaches the real axis. These contributions vanish under Assumption 1 and the condition defining $C^\infty_\epsilon(\underline{C})$. Details will appear elsewhere [7].

Remark 4.2. In general, $(C^\infty_\epsilon(\underline{C}), \star)$ is not a deformation of $C^\infty(\underline{C})$. What we have is that the map

$$p\colon\ f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \mapsto f_0$$

is a ring homomorphism $(C^\infty_\epsilon(\underline{C}), \star) \to (C^\infty(\underline{C}), \cdot)$ with the property that $p((f \star g - g \star f)/\epsilon) = \{f, g\}$. It would be interesting to characterize the image of this homomorphism.

4.6. THE CASE OF TWO COISOTROPIC SUBMANIFOLDS: BIMODULES

The above calculation may be extended to the case of an arbitrary number of coisotropic submanifolds. We discuss here the simplest case of two cleanly intersecting submanifolds C_0, C_1 (one says that the intersection of C_0 and C_1 is clean if $C_0 \cap C_1$ is also a submanifold and $T(C_0 \cap C_1) = TC_0 \cap TC_1$). Again we consider path integrals of the type (4.2) and evaluate the product at a point $x \in C_0 \cap C_1$. The circle is partitioned into two parts which are sent to the two coisotropic submanifolds. It is convenient to map the disk to the first quadrant $\operatorname{Re} z \geqslant 0, \operatorname{Im} z \geqslant 0$. The parts of the boundary sent to C_0, C_1 are the positive imaginary and real axes, respectively. The point r which is sent to the point at which we evaluate the product in (4.2) is the point at infinity. We then have the option of putting the remaining points at 0 or on the real or imaginary axis, obtaining various products.

Specifically, let us consider the case where M is an open subset of $\mathbb{R}^n$ and suppose that C_q, $q = 0, 1$, is given by the equations $x^\mu = 0$, $\mu \in I_q^c$, for subsets I_0, I_1 of $\{1, \ldots, n\}$, with complements I_0^c, I_1^c. Then x^i, $i \in I_q$ form a coordinate system for C_q and the intersection $C_0 \cap C_1$ has coordinates x^i, $i \in I_0 \cap I_1$. Such a choice of coordinates is possible in the neighborhood of a point of clean intersection.

We suppose that Assumption 1 holds for both C_0 and C_1 and have therefore two algebras $C^\infty_\epsilon(\underline{C_0})$, $C^\infty_\epsilon(\underline{C_1})$. The evaluation of the path integral gives a $C^\infty_\epsilon(\underline{C_0})$–$C^\infty_\epsilon(\underline{C_1})$-bimodule $C^\infty_\epsilon(\underline{C_0 \cap C_1})$. The construction is in terms of sums over graphs and goes as follows.

The set of admissible graphs $G_{k,2}$ consists in this case of graphs with k vertices $1, \ldots, k$ of the first kind and 2 vertices $\bar{1}, \bar{2}$ of the second kind. The rules are as before except that there are four types of vertices $++$, $+-$, $-+$, $--$, rather than just two. To each such graph Γ one associates a bidifferential operator $B_\Gamma(f, g)$. The rules are the same as in the case of one submanifold, the only difference being the range of summation of the indices associated to the edges: an edge of type

$++$, $+-$, $-+$, $--$ indicates a summation over $I_0 \cap I_1$, $I_0 \cap I_1^c$, $I_0^c \cap I_1$, $I_0^c \cap I_1^c$, respectively. We also consider graphs in $G_{k+1,1}$ with one additional vertex with one outgoing edge and one vertex of the second type. They give rise to differential operators $B_\Gamma(\xi)$ depending of a vector field ξ.

The weight w_Γ of a graph Γ is obtained by integrating the product of one-forms associated to edges over configuration spaces. The one-forms $\mathrm{d}\phi_{\sigma\tau}(z,w)$, $\sigma,\tau=\pm 1$, corresponding to the different kinds of edges are obtained from the Euclidean angle function $\phi_e(z,w)=\arg(z-w)$ by reflection:

$$\phi_{\sigma\tau}(z,w)=\phi_e(z,w)+\sigma\phi_e(z,\bar{w})+\tau\phi_e(z,-\bar{w})+\sigma\tau\phi_e(z,-w).$$

If $\Gamma \in G_{k+1,1}$ the integration is over the configuration space of $k+1$ points in the first quadrant modulo dilations. The differential operator $A(\xi)=\sum_{\Gamma\in G_{k+1,1}} w_\Gamma B_\Gamma(\xi)$ is well-defined on functions on $C_0 \cap C_1$ for ξ tangent to $C_0 \cap C_1$, and we set

$$C_\epsilon^\infty(\underline{C_0 \cap C_1})=\{f \in C^\infty(C_0 \cap C_1)[[\epsilon]] : A(E^\mu)f=0, \mu \in I_0^c \cap I_1^c\}.$$

As $A(E^\mu)f=E^\mu f+O(\epsilon)$ this condition reduces modulo ϵ to the condition that $f \in C^\infty(\underline{C_0 \cap C_1})$.

If $\Gamma \in G_{k,2}$ we have two weights $w_\Gamma^0 w_\Gamma^1$ for the two module structures. The weight w_Γ^0 is obtained by integrating over the configuration space of k distinct points in the first quadrant, one point at the origin, associated to the first vertex $\bar{1}$ and one point on the positive real axis, associated to $\bar{2}$, up to dilations. The right $C_\epsilon^\infty(\underline{C_0})$-module structure is then defined by the product

$$\psi \star_0 f=\psi f+\sum_{k=1}^\infty \frac{\epsilon^k}{k!}\sum_{\Gamma\in G_{k,2}} w_\Gamma^0 B_\Gamma(\psi,f), \tag{4.6}$$

$\psi \in C_\epsilon^\infty(\underline{C_0 \cap C_1})$, $f \in C_\epsilon^\infty(\underline{C_0})$. Similarly, the weights w_Γ^1 obtained by assigning $\bar{1}$ to a point on the positive imaginary axis and $\bar{0}$ to the origin, we get the left $C_\epsilon^\infty(\underline{C_1})$-module structure

$$f \star_1 \psi=\psi f+\sum_{k=1}^\infty \frac{\epsilon^k}{k!}\sum_{\Gamma\in G_{k,2}} w_\Gamma^1 B_\Gamma(f,\psi), \tag{4.7}$$

$\psi \in C_\epsilon^\infty(\underline{C_0 \cap C_1})$, $f \in C_\epsilon^\infty(\underline{C_1})$. Applying Stokes' theorem to this situation gives our result:

THEOREM 4.3. *Let the Poisson manifold M be an open subset of $\mathbb{R}^n$ containing the origin and let C_q, $q=0,1$ be two coisotropic submanifolds given by the equation $x^\mu=0$, $\mu \in I_q^c$. Suppose that Assumption 1 holds for both C_0 and C_1. Then*

(i) *The product $\star_0$ (4.6) maps $C_\epsilon^\infty(\underline{C_0 \cap C_1}) \otimes C_\epsilon^\infty(\underline{C_0})$ to $C_\epsilon^\infty(\underline{C_0 \cap C_1})$ and is a right $C_\epsilon^\infty(\underline{C_0})$-module structure.*

(ii) *The product* $\star_1$ (4.7) *maps* $C^\infty_\epsilon(\underline{C_1}) \otimes C^\infty_\epsilon(\underline{C_0 \cap C_1})$ *to* $C^\infty_\epsilon(\underline{C_0 \cap C_1})$ *and is a left* $C^\infty_\epsilon(\underline{C_1})$*-module structure.*
(iii) *We have* $(f \star_1 \psi) \star_0 g = f \star_1 (\psi \star_0 g)$, *i.e., the two module structures combine to a bimodule structure.*
(iv) *The reduction modulo* ϵ *is a homomorphism of bimodules.*

An important special case is the case where $C_0 = M$. Then $\underline{M} = M$ and Assumption 1 is satisfied. Moreover, the algebra $C^\infty_\epsilon(M)$ is the Kontsevich deformation of the product on $C^\infty(M)$ and $C^\infty_\epsilon(\underline{C_0 \cap C_1})$ is $C^\infty(C_1)[[\epsilon]]$. In this way we get a $C^\infty_\epsilon(M)^{\mathrm{op}} \otimes C^\infty_\epsilon(\underline{C})$-module structure on $C^\infty(C)[[\epsilon]]$ for a coisotropic C obeying Assumption 1.

5. Examples

Here we discuss some cases where Assumption 1 is satisfied. In all cases we assume that M is an open subset of $\mathbb{R}^n$ and that the coisotropic submanifolds are coordinate subspaces, as in the previous section.

5.1. CODIMENSION ONE

If $C \subset M$ is any coisotropic hyperplane, Assumption 1 is satisfied since the conormal bundle is one-dimensional and F is a skew-symmetric bilinear form. So for each coisotropic hyperplane C we obtain an algebra $C^\infty_\epsilon(\underline{C})$ quantizing the algebra of invariant functions on C, a $C^\infty_\epsilon(M)^{\mathrm{op}} \otimes C^\infty_\epsilon(\underline{C})$-module $C^\infty_\epsilon(C)$ and, for each pair of coisotropic hyperplanes C_0, C_1 a bimodule $C^\infty(\underline{C_0 \cap C_1})$.

5.2. CONSTANT CASE

Let $M = \mathbb{R}^n$ with constant Poisson structure and let C be a coisotropic subspace. In this case Assumption 1 is trivially satisfied as F involves derivatives of π^{ij}. Also the condition $A(E^\mu) f = 0$ reduces to $E^\mu f = 0$ so that $C^\infty_\epsilon(\underline{C}) = C^\infty(\underline{C})[[\epsilon]]$. For example, consider the case of the standard symplectic structure on $\mathbb{R}^{2m}$. Lagrangian subspaces are coisotropic, with characteristic foliation consisting of one leaf. Thus $C^\infty_\epsilon(\underline{C})$ is the one-dimensional $\mathbb{R}[[\epsilon]]$ free module of constant functions. Taking $C_0 = M$ and $C_1 = C$ we get a trivial left module structure and $C^\infty_\epsilon(C) = C^\infty(C)[[\epsilon]]$ is a right $C^\infty_\epsilon(M)$-module. It is a formal version of the space of states in quantum mechanics.

5.3. LINEAR CASE

Let $\mathfrak{g}$ be a Lie algebra and $M = \mathfrak{g}^*$ with Kostant–Kirillov Poisson structure. The annihilator $\mathfrak{h}^\perp$ of some Lie subalgebras $\mathfrak{h}$ is then a coisotropic subspace of M (see Example 2.1 on page 160). It can be shown that Assumption 1 is satisfied in this case. As the Poisson structure is linear we may replace smooth functions by polynomial functions. Our construction gives then a quantization of $S(\mathfrak{g}/\mathfrak{h})$ as an $S(\mathfrak{g})$–$S(\mathfrak{g}/\mathfrak{h})^h$-bimodule. The quantization of $S(\mathfrak{g})$ is the Kontsevich deformation quantization $U = S_\epsilon(\mathfrak{g})$. It is isomorphic to the universal enveloping algebra of $\mathfrak{g}$ with bracket $\epsilon[\ ,\ ]$ over $\mathbb{R}[[\epsilon]]$. The quantization of $S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$ is an algebra $S_\epsilon(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$. In general $S_\epsilon(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}/\epsilon S_\epsilon(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$ is not $S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$, so we do not have a deformation quantization in general. We do, however, in the reductive case:

THEOREM 5.1. *Suppose $\mathfrak{h}$ is a Lie subalgebra of a finite dimensional Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{h}$ admits an* ad$(\mathfrak{h})$*-invariant complement. Then the algebra $S_\epsilon(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$ is isomorphic to $S(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}[[\epsilon]]$ as an $\mathbb{R}[[\epsilon]]$-module. The products $\star_0$, $\star_1$ of Subsection* 4.6 define a $U^{\mathrm{op}} \otimes S_\epsilon(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$-module structure on the space $S(\mathfrak{g}/\mathfrak{h})[[\epsilon]]$ of functions on $\mathfrak{h}^\perp$.

A very particular case where the assumptions of the Theorem are satisfied is $\mathfrak{h} = \mathfrak{g}$. In this case, $\mathfrak{h}^\perp$ is the origin of $\mathfrak{g}$, a zero of the Kostant–Kirillov Poisson structure. The construction yields then a U^{op}-module structure on $\mathbb{R}[[\epsilon]]$, that is, a character of the quantum algebra that deforms evaluation at zero.

6. Formality with Submanifolds

The integrals over configuration spaces and the (bi)differential operators considered above can be generalized to the more general setting of multivector fields and multidifferential operators. In the absence of branes, Stokes' theorem gives then identities which in [14] are formulated as the existence of an L_∞-quasi-isomorphism from the differential graded Lie algebra (DGLA) of multivector fields on $\mathbb{R}^n$ and the DGLA of multidifferential operators. This is the local part of Kontsevich's formality theorem, one of whose important applications is the globalization of the star product [8,14].

In the presence of submanifolds, one should expect a similar theorem to hold. If $C \subset M$ is a submanifold, it is natural to introduce the DGLA $\mathcal{V}(M, C) = \oplus_{j \geqslant -1} V^j(M, C)$ of *relative multivector fields*. The space $V^j(M, C)$ consists of multivector fields $\pi \in \Gamma(M, \wedge^{j+1} TM)$ whose restriction to C vanishes on $\wedge N^*C$. The Schouten–Nijenhuis bracket restricts to a bracket on relative multivector fields, which therefore form a DGLA (with trivial differential). On the other hand, let $A(M, C) = \Gamma(C, \wedge NC)$ be the graded commutative algebra of sections of the exterior algebra of the normal bundle $NC = T_C M/TC$. The Hochschild complex $C(A, A) = \oplus_j \mathrm{Hom}_{\mathbb{R}}(A^{\otimes j}, A)$ of the graded Lie algebra $A = A(M, C)$ is then a DGLA with respect to the Hochschild differential and the Gerstenhaber bracket. We then define the DGLA of *relative multidifferential operators* $\mathcal{D}(M, C)$ to be the subalgebra of $C(A, A)$ consisting of multidifferential operators.

In the case where M is an open subset of $\mathbb{R}^n$ and $C \subset M$ is given by equations $x^\mu = 0$, $\mu = m+1, \ldots n$, the Feynman rules described in Sect. 4 give rise to linear

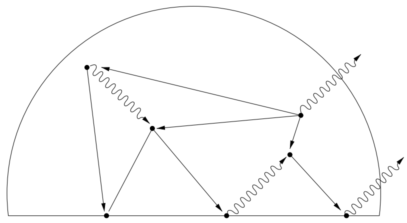

Figure 2. A graph contributing to U_4.

maps

$$U_k\colon \wedge^k \mathcal{V}(M,C) \to \mathcal{D}(M,C)[1-k], \tag{6.1}$$

defined as a sum over all admissible graphs with k vertices of the first type as in Section 4 but with arbitrary valences and number of vertices of the second type from which wavy edges are allowed to emerge.

THEOREM 6.1. *Let $C \subset M \subset \mathbb{R}^n$ be the subset of the open set M given by equations $x^\mu = 0, \mu = m+1, \ldots, n$. Then the maps U_k are the Taylor coefficients of an L_∞-morphism $U\colon \mathcal{V}(M,C) \to \mathcal{D}(M,C)$*

In the case $M = C \subset \mathbb{R}^n$, U reduces to Kontsevich's L_∞-quasi-isomorphism $\mathcal{V}(M) \to \mathcal{D}(M)$.

The DGLA $\mathcal{D}(M,C)$ is considered in disguise in [17] where it is conjectured to be formal.

If we evaluate the maps U_k on a Poisson bivector field π, with C coisotropic, we recover the objects discussed in subsection 4.5: the solution $U(\pi) = \sum \epsilon^k U_k(\pi, \ldots, \pi)/k!$ of the Maurer–Cartan equation in $D(M,C)[[\epsilon]]$ restricted to $C^\infty(C) \subset A(M,C)$ has components of degree at most 2 in NC:

$$U(\pi) = B(\pi) + \sum A(E^\mu)\partial_\mu + \frac{1}{2}\sum F(E^\mu, E^\nu)\partial_\mu \wedge \partial_\nu.$$

The star-product is $f \star g = fg + B(\pi) f \otimes g$.

An application of this generalized L_∞-morphism should be the globalization of the deformation quantization of the reduced phase space of a coisotropic submanifold. An extension of Theorem 6.1 to the case of two intersecting submanifolds should give a globalization of the bimodule structure described in Theorem 4.3 (though in general obstructions should be expected).

Let us add that we have considered here only the perturbative part of the sigma model, namely the expansion of the path integral around a trivial classical solution. The general case should lead to a generalization of the Fukaya A_∞-category (Figure 2).

Note added

The L_∞-morphism of Theorem 6.1 can actually be defined on the Lie algebra $\mathcal{V}(M)$ of all multivector fields on M, not just on $\mathcal{V}(M, C)$. It induces [7] an L_∞-quasiisomorphism $\mathcal{V}(M)/I_C \to \mathcal{D}(M, C)$ on the quotient of $\mathcal{V}(M)$ by the Lie ideal I_C consisting of multivector fields with vanishing Taylor expansion at each point of C. This quotient may be thought of as the Lie algebra of multivector fields in a formal neighborhood of C.

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