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Multivariate extremes and the aggregation of dependent risks: examples and counter-examples

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Abstract Properties of risk measures for extreme risks have become an important topic of research. In the present paper we discuss sub- and superadditivity of quantile based risk measures and show how multivariate extreme value theory yields the ideal modeling environment. Numerous examples and counter-examples highlight the applicability of the main results obtained.

Keywords Multivariate extreme value theory · Multivariate regular variation · Risk aggregation · Spectral measure · Subadditivity · Tail dependence · Value-at-Risk

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1 Introduction

In Embrechts et al. (2002), the following example was worked out. Suppose X_1, X_2 are independent random variables (rvs) each with a Pareto distribution function (df)

$$P(X_i > x) = x^{-1/2}, \quad x \geq 1,$$

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i.e., the X_i's have infinite mean. Consider $X = X_1 + X_2$ and let $\alpha \in (0, 1)$, then

$$F^{-1}_{X_1+X_2}(\alpha) > F^{-1}_{X_1}(\alpha) + F^{-1}_{X_2}(\alpha), \tag{1.1}$$

i.e., quantiles act superadditively. This rather trivial example has far-reaching consequences in finance, where the X_i's correspond to profits or losses, over a given fixed (holding) period, in particular markets/instruments. A quantile of such a fixed period position is referred to as Value-at-Risk (VaR); see also Definition 2.1 below. Hence Eq. 1.1 can be rewritten as

$$\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2). \tag{1.2}$$

The above example (Eq. 1.1) and its numerous generalizations form an important topic of research in Quantitative Risk Management (QRM) as for instance discussed in McNeil et al. (2005), Chapter 6. It also has important consequences within (re)insurance when modeling catastrophic risks; see Ibragimov et al. (2008).

Understanding the practical relevance of situations where Eq. 1.2 holds, or indeed where subadditivity ("$\leq$" in Eq. 1.2) holds are crucial within the regulatory framework (so-called Basel I and II) of financial institutions; see Chapter 1 in McNeil et al. (2005) and the references therein. Indeed, under the Basel II framework, the quantile risk measure $\mathrm{VaR}_\alpha(X)$ corresponds to regulatory (risk) capital that a financial institution has to hold in order to be able to carry the risky position X on its books. Furthermore, the quantity

$$D_\alpha(X_1, X_2) = \mathrm{VaR}_\alpha(X_1 + X_2) - \mathrm{VaR}_\alpha(X_1) - \mathrm{VaR}_\alpha(X_2)$$

can be seen as a measure of *diversification*. Alternatively, the quantity

$$C_\alpha(X_1, X_2) = \frac{\mathrm{VaR}_\alpha(X_1 + X_2)}{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)}$$

is referred to as a measure of *concentration* within the Basel II framework. Consequently, a deeper understanding concerning the possible values of either $D_\alpha(X_1, X_2)$ and $C_\alpha(X_1, X_2)$ across a wide family of dfs relevant for QRM practice is important.

This paper presents several results on this topic for arbitrary dimensions $n \geq 2$ and dependence structures, and this within the unifying framework of multivariate extreme value theory (MEVT). The MEVT approach to the above problems is by no means new. We found however that a summary of these results keeping financial applications in mind would be highly useful. Through many concrete examples and counter-examples we show that care has to be taken concerning possible constraints/properties of the dfs of the underlying risk factors. In a wider context of QRM, these same techniques are becoming increasingly important in the analysis of high risk scenarios, see for instance Balkema and Embrechts (2007), and therefore will become part of the standard toolkit of QRM.

The paper is organized as follows. Section 2 recalls the basic notion of multivariate regular variation and its link to questions like Eq. 1.1. In Section 3 we discuss three examples where Eq. 1.1 may or may not hold, stressing in particular the important difference between one-sided and two-sided risk dfs. For positive rvs, Section 4 uses the notion of spectral measure to derive additivity-type results under general portfolio assumptions. Sections 5 and 6 study the link with tail dependence concepts, whereas Section 7 concludes.

2 Value-at-Risk and multivariate regular variation

In this section we introduce multivariate regular variation, which provides a natural framework to discuss diversification of a portfolio under the risk measure VaR. Throughout the paper, we use the language of MEVT. For the latter, several approaches exist, like the more geometric one as presented in Balkema and Embrechts (2007) or Barbe (2003). In these contributions, the geometry of the level sets of the underlying multivariate densities plays a crucial role. We choose to base our discussion on the notion of spectral measure and MEVT results in this context. Part IV in Balkema and Embrechts (2007) compares some of the different approaches and highlights these in the context of high-threshold exceedances, which is akin to VaR estimation.

Definition 2.1 (Value-at-Risk) Let X be a rv with df F. The *Value-at-Risk* with respect to the level $\alpha \in (0,1)$ is defined as the generalized inverse of F, $\mathrm{VaR}_\alpha(X) = F^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}$.

In all relevant situations, α is typically close to 1. We say that VaR is *asymptotically subadditive* for $X_1, \ldots, X_n$, if

$$\lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} \leq 1, \tag{2.1}$$

provided the limit exists. VaR is called *asymptotically superadditive* for $X_1, \ldots, X_n$ if "$\geq$" in Eq. 2.1 holds. We assume the reader to be familiar with univariate EVT and in particular univariate regular variation; see for instance Embrechts et al. (1997) for an introduction. The following definition introduces multivariate regular variation and also the *limiting constant* q, which is of main interest in this paper; standard textbooks on multivariate EVT are for instance Resnick (1987, 2007), Beirlant et al. (2004), de Haan and Ferreira (2006) and Balkema and Embrechts (2007). A brief and very readable introduction to the field is found in Mikosch (2004).

Definition 2.2 (Multivariate regular variation) A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)'$ is *multivariate regularly varying* with index $-\beta < 0$, if there exists a probability measure μ, a measurable function b: $(0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} b(t) = \infty$ and a scalar $q = q(b) > 0$ such that for all $r > 0$,

$$\lim_{t\to\infty} \; t\,P\left(\|\boldsymbol{X}\| > rb(t),\; \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \in G \right) = qr^{-\beta}\mu(G),$$

for any Borel set $G \subset \aleph^{n-1}_{\|\cdot\|} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1 \right\}$. We write $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$.

The definition of multivariate regular variation is independent of the explicit choice of the norm $\|\cdot\|$ on $\mathbb{R}^n$. This comes from the fact that all norms on $\mathbb{R}^n$ are equivalent; see Lemma 2.1 in Hult and Lindskog (2002) for details. Note that the limiting constant q depends on the index $-\beta < 0$ and on the norm $\|\cdot\|$ chosen.

The goal of this paper is to analyze the properties of the limiting constant q for random vectors $\boldsymbol{X}$ with identically distributed marginals (this assumption can be relaxed using change of norms techniques; see Section 4) and with a dependence structure within the framework of multivariate regular variation. It follows from Definition 2.2 that for $\boldsymbol{X} = (X_1, \ldots, X_n)' \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$,

$$q(\beta, \|\cdot\|) = \lim_{x\to\infty} \frac{P(\|\boldsymbol{X}\| > x)}{P(X_1 > x)} > 0;$$

see Barbe et al. (2006), formula (9) and Remark 1 in Resnick (2004). An interesting choice of norm is the l_1-norm $\|\cdot\|_1$ on $\mathbb{R}^n_+$, to study the sum $X_1 + \ldots + X_n$ of n risky positions. However, also more general loss functions, say Ψ, are considered in practice.

Lemma 2.3 *Let* $\boldsymbol{X} = (X_1, \ldots, X_n)' \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, *with identically distributed marginals. If for a measurable function* $\Psi : \mathbb{R}^n \to \mathbb{R}$,

$$\lim_{x\to\infty} \frac{P(\Psi(\boldsymbol{X}) > x)}{P(X_1 > x)} = q_\Psi \in (0, \infty), \tag{2.2}$$

then

$$\lim_{\alpha\nearrow 1} \frac{\mathrm{VaR}_\alpha(\Psi(\boldsymbol{X}))}{\mathrm{VaR}_\alpha(X_1)} = q_\Psi^{1/\beta}.$$

Proof Consider $F_\Psi(x) = P(\Psi(\boldsymbol{X}) \le x)$ and $F(x) = P(X_1 \le x)$. Using Eq. 2.2 and the regular variation properties of X_1, one shows that

$$\lim_{\alpha\nearrow 1} \frac{F_\Psi^{\leftarrow}(\alpha)}{F^{\leftarrow}(\alpha)} = \lim_{x\to\infty} \frac{x}{F^{\leftarrow}(F_\Psi(x))} = q_\Psi^{1/\beta}.$$

The details are straightforward and therefore omitted. □

Remark 2.4 Equation 2.2 holds for example if $\Psi(\boldsymbol{X}) = \|\boldsymbol{X}\|$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ or if $\Psi(\boldsymbol{X}) = \sum_{i=1}^n X_i$ for $X_1, \ldots, X_n$ i.i.d.; see Barbe et al. (2006), formula (9) and Embrechts et al. (1997), Corollary 1.3.2, respectively.

3 Three examples

Many examples show that VaR properties for rvs with doubly infinite support are not easy to handle, particularly in the case of infinite mean models; see for instance Nešlehová et al. (2006), Chavez-Demoulin et al. (2006), Ibragimov and Walden (2007). To illustrate this, we give three basic examples:

Example 3.1 For $n \geq 2$, let $X_1, \ldots, X_n$ be i.i.d. rvs, regularly varying with index $-\beta < 0$. In this case, it is well-known that asymptotic subadditivity holds if and only if $\beta \geq 1$. This follows from Lemma 2.3, yielding

$$\lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} = n^{1/\beta - 1} > 1, \quad \text{for } \beta < 1,$$

because the limiting constant q_Ψ in Eq. 2.2 is equal to n for $\Psi(\boldsymbol{X}) = \sum_{i=1}^n X_i$; see Corollary 1.3.2 in Embrechts et al. (1997).

When allowing for dependence, one has to be more careful when analyzing additivity properties of VaR; see for instance Example 6.4, Fig. 4 below. As shown in Balkema and Embrechts (2007) and Barbe (2003), for distributions with a density, these questions are closely linked with the properties of the level sets. In the next example, we consider elliptically distributed random vectors.

Definition 3.2 (Elliptical distribution) A random vector $\mathbf{X}$ has an *elliptical distribution* with mean $\boldsymbol{\mu} \in \mathbb{R}^n$ and dispersion matrix Σ, if there exist R, A and $\mathbf{U}$ satisfying $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + RA\mathbf{U}$, with

a) $R \geq 0$, a non-negative rv;
b) $\mathbf{U}$ uniformly distributed on the unit sphere $\aleph_{\|\cdot\|_2}^{n-1} = \{\boldsymbol{z} \in \mathbb{R}^n, \|\boldsymbol{z}\|_2 = 1\}$, independent of R, and
c) $A \in \mathbb{R}^{n \times n}$ with $AA' = \Sigma$.

Example 3.3 Theorem 6.8 in McNeil et al. (2005) states that for $\boldsymbol{X} = (X_1, \ldots, X_n)'$ elliptically distributed, we have for all $\alpha \in [\frac{1}{2}, 1)$,

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i).$$

That is, in the elliptical world, subadditivity of VaR holds true for finite *and* infinite mean models.

What is the reason for this discrepancy between Example 3.1 and Example 3.3 ? For $\beta > 1$ (finite mean case) the asymptotic VaR is subadditive in both models. However, for $\beta < 1$, we are in the infinite mean regime and the asymptotic VaR behaves very differently in the models analyzed. The reason for this difference is connected with the behavior of the joint df (or more precisely, the *spectral measure*; see Section 4) and can not be explained by the marginal dfs alone. We will discuss risk aggregation in the light of dependence structures describing interdependencies in the *joint tail(s)* of the distribution.

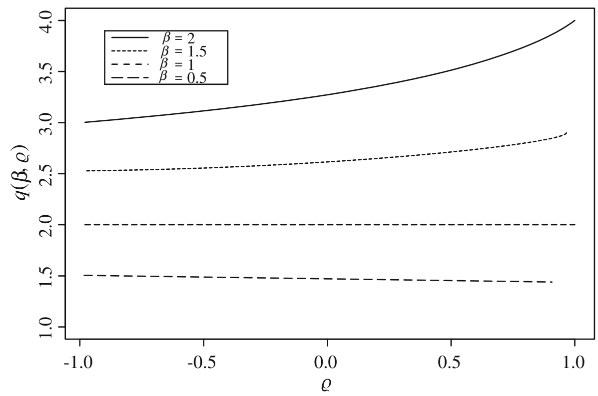
In Example 3.3 we learned that subadditivity of VaR holds for every elliptical distribution. However, asymptotic subadditivity of VaR fails for infinite mean models as soon as we weaken the influence of the negative tails by restricting for example to the positive quadrant of the elliptical distribution (Fig. 1).

Example 3.4 Let $\mathbf{X} = RA\mathbf{U}$ be a bivariate elliptical random vector with $R \in \mathrm{RV}_{-\beta}$, $\beta > 0$,

$$A = \begin{pmatrix} 1 & 0 \\ \varrho & \sqrt{1-\varrho^2} \end{pmatrix},$$

and $\mathbf{U}$ uniformly distributed on the unit sphere $\aleph^1_{\|\cdot\|_2}$, i.e., $\mathbf{U} = (\cos W, \sin W)'$, with $W \sim \mathrm{Unif}(-\pi, \pi)$. We are interested in the behavior of $\mathbf{X} = (X_1, X_2)'$, restricted to the positive quadrant. We thus consider $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \widetilde{X}_2)' = \mathbf{X} | \{\mathbf{X} \geq \mathbf{0}\}$, where the inequality has to be interpreted componentwise. We consider

Fig. 1 The limiting constant q in Eq. 3.1 as a function of ϱ for different values of β.



$q(\beta, \|\cdot\|_1)$ as a function of β and ϱ. Using the Dominated Convergence Theorem in the last step below, we get

$$\begin{aligned}
q(\beta,\varrho) &= \lim_{x\to\infty} \frac{P(\widetilde{X}_1+\widetilde{X}_2>x)}{P(\widetilde{X}_1>x)} \\
&= \lim_{x\to\infty} \frac{P\left(R\left((1+\varrho)\cos W+\sqrt{1-\varrho^2}\sin W\right)>x\middle|W\in[-\arcsin\varrho,\pi/2]\right)}{P\left(R\cos W>x|W\in[-\arcsin\varrho,\pi/2]\right)} \\
&= \lim_{x\to\infty} \frac{\int_{-\arcsin\varrho}^{\pi/2} P\left(R>x/\left((1+\varrho)\cos w+\sqrt{1-\varrho^2}\sin w\right)\right)\,dw}{\int_{-\arcsin\varrho}^{\pi/2} P\left(R>x/\cos w\right)\,dw} \\
&= \frac{\int_{-\arcsin\varrho}^{\pi/2}\left((1+\varrho)\cos w+\sqrt{1-\varrho^2}\sin w\right)^{\beta}\,dw}{\int_{-\arcsin\varrho}^{\pi/2}\cos^{\beta} w\,dw}. \qquad (3.1)
\end{aligned}$$

Proposition 3.5 *Let $q(\beta,\varrho)$ be defined as in Example 3.4, then*

a) *for all $\varrho\in[-1,1]$, $q(\beta,\varrho)\le 2^{\beta}$ if $\beta\ge 1$ and $q(\beta,\varrho)\ge 2^{\beta}$ if $\beta\le 1$;*
b) $\lim_{\varrho\to-1} q(\beta,\varrho)=1+\beta$, *and*
c) $\lim_{\varrho\to 1} q(\beta,\varrho)=2^{\beta}$.

Proof Define for $f\in L^{\beta}([-\pi/2,\pi/2])$,

$$\zeta_\beta(f)=\left(\int_{-\alpha}^{\pi/2} f^{\beta}(w)\,dw\right)^{1/\beta},$$

with a fixed $\alpha=\arcsin\varrho\in[-\pi/2,\pi/2]$ and $0<\beta<\infty$. From Eq. 3.1 and some standard trigonometric transformations we get

$$q(\beta,\sin\alpha)^{1/\beta}=\frac{\zeta_\beta\left(\cos(\cdot)+\sin(\alpha+\cdot)\right)}{\zeta_\beta\left(\cos(\cdot)\right)}.$$

Applying Minkowski's inequality for $\beta\ge 1$, we have

$$q(\beta,\sin\alpha)^{1/\beta}\le 1+\frac{\zeta_\beta\left(\sin(\alpha+\cdot)\right)}{\zeta_\beta\left(\cos(\cdot)\right)}=2\quad\text{for}\quad\beta\ge 1.$$

For $\beta\le 1$, the "$\le$" turns into a "$\ge$" by Theorem 198 in Hardy et al. (1934). This proves part a). Part b) follows from Eq. 3.1 and part c) is a consequence of the comonotonicity of X_1 and X_2 or can be calculated explicitly using Eq. 3.1. □

By part a) of Proposition 3.5 the following corollary follows from Lemma 2.3:

Corollary 3.6 *Let $\mathbf{X} \in \mathrm{MRV}_2(-\beta)$, $\beta > 0$, be an elliptical random vector as in Example 3.4, and $\widetilde{\mathbf{X}}$ the random vector $\mathbf{X}$ restricted to the positive quadrant, then VaR is asymptotically subadditive for $\widetilde{\mathbf{X}}$ if $\beta \geq 1$ and asymptotically superadditive if $\beta \leq 1$.*

The three examples elaborated in this section show that, besides the dependence structure and the tail behavior of the marginal dfs, it is also important to differentiate between rvs with one-sided and two-sided support.

For the infinite mean multivariate *t*-distribution, subadditivity of VaR holds due to the dependence properties in the upper left and lower right corner. High values of one risk are compensated by low values of the other risk, turning VaR into a coherent risk measure for such infinite mean models. Of course this has important consequences in risk management. Risk managers should be aware of this property for elliptical distributions, particularly when the compensation of high losses by high gains turns out to be an inappropriate characteristic of the considered risk class.

4 Spectral measures for positive rvs

In the following we consider multivariate regularly varying $\mathbb{R}^n_+$-valued random vectors. Operations between vectors should be interpreted componentwise. Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ be an arbitrary norm. Denote the positive part of the unit sphere with respect to the norm $\|\cdot\|$ by $\aleph^{n-1}_{+,\|\cdot\|} = \{z \in \mathbb{R}^n_+ \mid \|z\| = 1\}$. Note that we write $\aleph^{n-1}_{+,\|\cdot\|}$ for the positive part of $\aleph^{n-1}_{\|\cdot\|}$. For $\mathbb{R}^n_+$-valued random vectors $\boldsymbol{X}$, Theorem 1 in Resnick (2004) or Theorem 6.1 in Resnick (2007) states that multivariate regular variation of $\boldsymbol{X}$ in the sense of Definition 2.2 is equivalent to the existence of a *Radon measure* ν_β such that

$$\lim_{t\to\infty} t\, P(\boldsymbol{X}/b(t) \in B) = \nu_\beta(B),$$

for all $B \subset [0,\infty]^n \setminus \{\mathbf{0}\}$ relatively compact with $\nu_\beta(\partial B) = 0$. The term *Radon* means that ν_β is finite for all compact subsets of $[0,\infty]^n \setminus \{\mathbf{0}\}$. Resnick (2007) calls ν_β the *limit measure* and, after normalization of the marginal dfs, ν_1 is referred to as the *exponent measure* in de Haan and Ferreira (2006). Choosing

$$B = \left\{z \in [0,\infty]^n \mid \|z\| > r,\, z/\|z\| \in G\right\},$$

for $r > 0$ and a Borel set $G \in \aleph^{n-1}_{+,\|\cdot\|}$, we get from Definition 2.2,

$$q(\beta, \|\cdot\|) r^{-\beta} \mu(G) = \nu_\beta \left\{z \in [0,\infty]^n \mid \|z\| > r,\, z/\|z\| \in G\right\}.$$

For $\beta = 1$ and $r = 1$, this defines the *spectral measure* $S_{\|\cdot\|}$ by

$$S_{\|\cdot\|}(G) = \nu_1 \left\{z \in [0,\infty]^n \mid \|z\| > 1,\, z/\|z\| \in G\right\}.$$

Following Barbe et al. (2006), we have $q(\beta, \|\cdot\|) = \nu_1 \left\{z \in [0,\infty]^n \mid \|z^{1/\beta}\| > 1\right\}$, and therefore the following theorem.

Theorem 4.1 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be a $\mathbb{R}^n_+$-valued random vector with identically distributed marginals, then*

$$q(\beta, \|\cdot\|) = \lim_{x\to\infty} \frac{P(\|\boldsymbol{X}\| > x)}{P(X_1 > x)} = \int_{\aleph^{n-1}_{+,\|\cdot\|}} \|\boldsymbol{z}^{1/\beta}\|^{\beta} \; S_{\|\cdot\|}(d\boldsymbol{z}).$$

Proof Barbe et al. (2006) give an explicit proof when $\|\cdot\|$ is the l_1-norm in $\mathbb{R}^n$. The same proof holds true for general norms in $\mathbb{R}^n$, as was certainly noticed by these authors. We therefore refrain from giving the details. □

Theorem 4.1 shows that $\beta = 1$ plays an important role in this context. Regardless of our choice of the norm, we have $q(\beta = 1, \|\cdot\|) = S_{\|\cdot\|}(\aleph^{n-1}_{+,\|\cdot\|})$. If we consider the l_1-norm, we can give the following result.

Corollary 4.2 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be a $\mathbb{R}^n_+$-valued random vector with identically distributed marginals. Let $\|\cdot\|_1$ be the l_1-norm in $\mathbb{R}^n$, then*

$$n \le q(\beta, \|\cdot\|_1) \le n^{\beta}, \quad \text{for } \beta \ge 1,$$
$$n \ge q(\beta, \|\cdot\|_1) \ge n^{\beta}, \quad \text{for } \beta \le 1.$$

Proof Proposition 2.2 in Barbe et al. (2006) states that $q(\beta, \|\cdot\|_1)$ is increasing in β. Further, $q(1, \|\cdot\|_1) = S_{\|\cdot\|_1}\left(\aleph^{n-1}_{+,\|\cdot\|_1}\right) = n$, because $S_{\|\cdot\|_1}/n$ is a probability measure. This proves the LHS of the statements. For the RHS, consider the functional

$$\widetilde{\zeta}_{\beta}(f) = \left(\int_{\aleph^{n-1}_{+,\|\cdot\|_1}} f^{\beta}(\boldsymbol{z}) \; S_{\|\cdot\|_1}(d\boldsymbol{z})\right)^{1/\beta},$$

for non-negative functions $f \in L^{\beta}\left(\aleph^{n-1}_{+,\|\cdot\|_1}, S_{\|\cdot\|_1}\right)$. Note that for $\beta \ge 1$, by Minkowski's inequality (note the slight abuse of notation),

$$(q(\beta, \|\cdot\|_1))^{1/\beta} = \widetilde{\zeta}_{\beta}\left(\sum_{i=1}^{n} z_i^{1/\beta}\right) \le \sum_{i=1}^{n} \widetilde{\zeta}_{\beta}\left(z_i^{1/\beta}\right) = \sum_{i=1}^{n}\left(\int_{\aleph^{n-1}_{+,\|\cdot\|_1}} z_i \; S_{\|\cdot\|_1}(d\boldsymbol{z})\right)^{1/\beta} = n.$$

For $\beta \le 1$ the "$\le$" turns into a "$\ge$" by Theorem 198 in Hardy et al. (1934). □

Theorem 4.3 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be a $\mathbb{R}^n_+$-valued random vector with identically distributed marginals, then VaR_{α} is asymptotically subadditive for $\boldsymbol{X}$ if $\beta \ge 1$ and asymptotically superadditive if $\beta \le 1$.*

Proof Lemma 2.3 and Corollary 4.2 yield the result. □

Asymptotic subadditivity for bivariate regularly varying random vectors with $\beta \ge 1$ has already been proven in Daníelsson et al. (2005), Proposition 1.

Remark 4.4 Note that all components of $\boldsymbol{X}$ in Theorem 4.3 need to be positive. If this assumption is not fulfilled, subadditivity also for infinite mean models may occur, for example for elliptical distributed random vectors; see Example 3.3.

The norm $\|\boldsymbol{z}\|_1 = |z_1| + \cdots + |z_n|$ is a natural choice, because it allows for the study of sums of dependent, positive risks and in particular for an analysis of the additivity properties of VaR; see Theorem 4.3. Sometimes however, spectral measures with respect to other norms are chosen; for instance in Stărică (1999) and Hult and Lindskog (2002), the spectral measure with respect to $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively, is more convenient in their context.

Also when one deals with elliptical types of distributions, where (after a linear transformation of $\boldsymbol{X}$) the spectral measure with respect to the Euclidean norm $\|\cdot\|_2$ is uniformly distributed on $\aleph^{n-1}_{\|\cdot\|_2}$, a change of measure could be appropriate. It is thus important to express the spectral measure with respect to one norm in terms of the spectral measure with respect to another norm. This can always be done; see for instance formula (8.38) in Beirlant et al. (2004), which we formulate in the following lemma.

Lemma 4.5 *Let $S_{\|\cdot\|}$ and $S_{\|\cdot\|'}$ be the spectral measure with respect to the norms $\|\cdot\|$ and $\|\cdot\|'$, respectively, then*

$$S_{\|\cdot\|}(G) = \int_{\aleph^{n-1}_{+,\|\cdot\|'}} \mathbb{1}_{\{z/\|z\| \in G\}} \|z\| \; S_{\|\cdot\|'}(dz),$$

for any Borel set $G \subset \aleph^{n-1}_{+,\|\cdot\|}$.

We call a $\mathbb{R}^n_+$-valued multivariate regularly varying random vector asymptotically independent, if the spectral measure $S_{\|\cdot\|}$ is concentrated on the points $\boldsymbol{e}_i/\|\boldsymbol{e}_i\|$, $i = 1, \ldots, n$, with $\boldsymbol{e}_i$ the ith basis vector of the canonical basis in $\mathbb{R}^n$; it is called asymptotically fully dependent, if the spectral measure $S_{\|\cdot\|}$ is concentrated on $\mathbb{1}/\|\mathbb{1}\|$, with $\mathbb{1} = (1, \ldots, 1)'$; see Resnick (2004). Note that by Lemma 4.5 asymptotic independence as well as asymptotic full dependence is well-defined.

Proposition 4.6 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be an asymptotically independent $\mathbb{R}^n_+$-valued random vector with identically distributed marginals, then $q(\beta, \|\cdot\|) = \sum_{i=1}^n \|\boldsymbol{e}_i\|^\beta$ and in particular, if $\|\boldsymbol{e}_i\| = 1$ for all $i = 1, \ldots, n$, $q(\beta, \|\cdot\|) = n$.*

Proof Theorem 4.1 yields

$$q(\beta, \|\cdot\|) = \sum_{i=1}^n \| (\boldsymbol{e}_i/\|\boldsymbol{e}_i\|)^{1/\beta} \|^\beta S_{\|\cdot\|}(\boldsymbol{e}_i/\|\boldsymbol{e}_i\|) = \sum_{i=1}^n \|\boldsymbol{e}_i\|^{\beta-1} S_{\|\cdot\|}(\boldsymbol{e}_i/\|\boldsymbol{e}_i\|),$$

with $S_{\|\cdot\|}(\boldsymbol{e}_i/\|\boldsymbol{e}_i\|) = \|\boldsymbol{e}_i\| S_{\|\cdot\|_1}(\boldsymbol{e}_i) = \|\boldsymbol{e}_i\|$, by Lemma 4.5 and because $S_{\|\cdot\|_1}/n$ is a probability measure. □

This proposition generalizes Lemma 2.1 in Davis and Resnick (1996) to arbitrary norms; see also Lemma 3.1 in Jessen and Mikosch (2006), where the result from Davis and Resnick (1996) is generalized to rvs with doubly infinite support.

Proposition 4.7 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be an asymptotically fully dependent $\mathbb{R}^n_+$-valued random vector with identically distributed marginals, then $q(\beta, \|\cdot\|) = \|\mathbb{1}\|^\beta$.*

Proof Theorem 4.1 yields

$$q(\beta, \|\cdot\|) = \|\,(\mathbb{1}/\|\mathbb{1}\|)^{1/\beta}\,\|^\beta S_{\|\cdot\|}(\mathbb{1}/\|\mathbb{1}\|) = \|\mathbb{1}\|^{\beta-1} S_{\|\cdot\|}(\mathbb{1}/\|\mathbb{1}\|),$$

with $S_{\|\cdot\|}(\mathbb{1}/\|\mathbb{1}\|) = \|\mathbb{1}/n\| S_{\|\cdot\|_1}(\mathbb{1}/n) = \|\mathbb{1}\|$, by Lemma 4.5 and because $S_{\|\cdot\|_1}/n$ is a probability measure. □

Proposition 4.8 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be a $\mathbb{R}^n_+$-valued random vector with identically distributed marginals. Let $S_{\|\cdot\|_\infty}$ be the spectral measure with respect to $\|\cdot\|_\infty$, the maximum-norm in $\mathbb{R}^n$, then*

$$q(\beta, \|\cdot\|_\infty) = S_{\|\cdot\|_\infty}\left(\aleph^{n-1}_{+,\|\cdot\|_\infty}\right) = \int_{\aleph^{n-1}_{+,\|\cdot\|}} \bigvee_{i=1}^n z_i \; S_{\|\cdot\|}(dz).$$

Proof Note that $\|z^{1/\beta}\|^\beta_\infty = 1$ on $\aleph^{n-1}_{+,\|\cdot\|_\infty}$, for all $\beta > 0$. Hence, the first equality follows from Theorem 4.1. Using Lemma 4.5 the second equality follows. □

The following well-known result characterizes asymptotic independence and full dependence.

Corollary 4.9 *Let $\boldsymbol{X} \in \mathrm{MRV}_n(-\beta)$, $\beta > 0$, be a $\mathbb{R}^n_+$-valued random vector with identically distributed marginals, then*

i) *$\boldsymbol{X}$ is asymptotically independent if and only if*

$$\int_{\aleph^{n-1}_{+,\|\cdot\|}} \bigvee_{i=1}^n z_i \; S_{\|\cdot\|}(dz) = n.$$

ii) *$\boldsymbol{X}$ is asymptotically fully dependent if and only if*

$$\int_{\aleph^{n-1}_{+,\|\cdot\|}} \bigvee_{i=1}^n z_i \; S_{\|\cdot\|}(dz) = 1.$$

Proof The "⇒"-part is straightforward from the definition of asymptotic independence and full dependence, but also a consequence of Propositions 4.6, 4.7 and 4.8. For the converse, see Beirlant et al. (2004), Section 8.2.7. □

By Proposition 4.8 and Corollary 4.9, it suffices to evaluate $q(\beta, \|\cdot\|_\infty)$ in order to test for asymptotic independence and full dependence, respectively.

5 Tail dependence and asymptotic independence

In Sections 5 and 6, we will discuss further examples and counter-examples for subadditivity of VaR. We restrict ourselves to the bivariate case and only sums of rvs are considered. Since the marginal dfs have equal asymptotic behavior in the different infinite mean models in Examples 3.1 and 3.3, the asymptotic VaR behavior for the sum of the risks must follow from the different dependence structures (copulas, spectral measures). We exemplify this issue through the notions of asymptotic dependence coefficients in the (four) tails of the underlying bivariate distribution.

Definition 5.1 Let $(X_1, X_2)'$ be a bivariate random vector, with marginal dfs F_{X_1} and F_{X_2}. The *positive upper* (λ_u^+)*, positive lower* (λ_l^+)*, negative upper* (λ_u^-) and *negative lower* (λ_l^-) *tail dependence coefficients* are defined as

$$\lambda_u^+ = \lim_{\alpha \nearrow 1} P\left(X_2 > F_{X_2}^{\leftarrow}(\alpha) | X_1 > F_{X_1}^{\leftarrow}(\alpha)\right),$$

$$\lambda_l^+ = \lim_{\alpha \searrow 0} P\left(X_2 \le F_{X_2}^{\leftarrow}(\alpha) | X_1 \le F_{X_1}^{\leftarrow}(\alpha)\right),$$

$$\lambda_u^- = \lim_{\alpha \nearrow 1} P\left(X_2 > F_{X_2}^{\leftarrow}(\alpha) | X_1 \le F_{X_1}^{\leftarrow}(1-\alpha)\right),$$

$$\lambda_l^- = \lim_{\alpha \searrow 0} P\left(X_2 \le F_{X_2}^{\leftarrow}(\alpha) | X_1 > F_{X_1}^{\leftarrow}(1-\alpha)\right),$$

provided the limits exist in [0, 1].

A sufficient condition for the existence of the tail dependence coefficient is bivariate regular variation of $(f(X_1), f(X_2))'$ for some strictly increasing transformation f; see Mikosch (2006) and references therein for weaker conditions on $(X_1, X_2)'$.

Note that for $(X_1, X_2)' \in \mathrm{MRV}_2(-\beta)$, $\beta > 0$, a $\mathbb{R}_+^2$-valued random vector with identically distributed marginals, we have

$$\lambda_u^+ = \lim_{x\to\infty} P(X_2 > x | X_1 > x) = 2 - \lim_{x\to\infty} \frac{P(\max(X_1, X_2) > x)}{P(X_1 > x)} = 2 - q(\beta, \|\cdot\|_\infty),$$

and hence by Proposition 4.8, $\lambda_u^+ = 2 - S_{\|\cdot\|_\infty}(\aleph^1_{+,\|\cdot\|_\infty})$.

Proposition 5.2 *Let* $(X_1, X_2)' \in \mathrm{MRV}_2(-\beta)$*,* $\beta > 0$*, be a* $\mathbb{R}_+^2$*-valued random vector with identically distributed marginals, then*

$$\lambda_u^+ = 0 \quad \Longleftrightarrow \quad (X_1, X_2)' \text{ asymptotically independent.}$$

Proof By Proposition 4.8 and Corollary 4.9, asymptotic independence of the random vector $(X_1, X_2)'$ is equivalent to $2 = q(\beta, \|\cdot\|_\infty)$. This is equivalent to $\lambda_u^+ = 0$. □

Remark 5.3 The concept of positive tail dependence is well-known and often used in risk management, in particular for describing so-called spillover events;

see for instance Straetmans (1998). However, negative tail dependence λ_u^-, λ_l^-, i.e., the probability that high values of X_1 are compensated by low values of X_2 and vice versa, did not draw risk managers' attention so far. We will see its importance in the sequel. High negative tail dependence might not always be reasonable in reality. If it is not appropriate that high losses are compensated by high gains with positive probability, then more conservative models should be considered. Recently Zhang (2008) introduced negative tail dependence in order to define a novel dependence measure called *total tail dependence*, which is a (2×2)–matrix with components $\lambda_u^+, \lambda_l^+, \lambda_u^-, \lambda_l^-$.

The tail dependence coefficients do not depend on the marginal dfs and thus can be written in terms of the corresponding copulas.

Proposition 5.4 *Let F_{X_1} and F_{X_2} from Definition* 5.1 *be continuous dfs and C the corresponding copula, then*

$$\lambda_u^+ = \lim_{\alpha \nearrow 1} \frac{1 - 2\alpha + C(\alpha, \alpha)}{1 - \alpha}, \tag{5.1}$$

$$\lambda_l^+ = \lim_{\alpha \searrow 0} \frac{C(\alpha, \alpha)}{\alpha}, \tag{5.2}$$

$$\lambda_u^- = 1 - \lim_{\alpha \nearrow 1} \frac{C(1 - \alpha, \alpha)}{1 - \alpha}, \tag{5.3}$$

$$\lambda_l^- = 1 - \lim_{\alpha \searrow 0} \frac{C(1 - \alpha, \alpha)}{\alpha}. \tag{5.4}$$

Proof See for instance Joe (1997), Section 2.1.10 or McNeil et al. (2005), Section 5.2.3 for the proof of Eqs. 5.1 and 5.2. The proof of Eqs. 5.3 and 5.4 is completely analogous. □

In the case of regularly varying elliptical distributions, the four tail dependence coefficients can be calculated explicitly.

Proposition 5.5 *Let $\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + RA\mathbf{U}$ be a bivariate elliptically distributed regularly varying random vector with index $-\beta < 0$, as defined in Definition* 3.2, *then*

$$\lambda_u^+ = \lambda_l^+ = \frac{\int_{a_+}^{\pi/2} \cos^\beta t \, dt}{\int_0^{\pi/2} \cos^\beta t \, dt}, \tag{5.5}$$

$$\lambda_u^- = \lambda_l^- = \frac{\int_{a_-}^{\pi/2} \cos^\beta t \, dt}{\int_0^{\pi/2} \cos^\beta t \, dt}, \tag{5.6}$$

with $a_+ = (\pi/2 - \arcsin \varrho)/2$, $a_- = (\pi/2 + \arcsin \varrho)/2$, *and where* $\varrho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ *with* $(\sigma_{ij})_{1 \le i, j \le 2} = \Sigma = AA'$.

Proof Equation 5.5 has been proven in Hult and Lindskog (2002). By considering the map $\mathbf{X} \mapsto D\mathbf{X}$, with

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Eq. 5.6 follows from Eq. 5.5. □

Even for rvs with a positive linear correlation coefficient, λ_u^- can be significantly larger than zero. Another important consequence of the previous proposition is that there exists *no* elliptical distribution without negative tail dependence and with heavy (i.e., regularly varying) tails, provided $\varrho < 1$.

Remark 5.6 The class of *Archimedean* copulas contains several dependence structures important for practical purposes; see Nelsen (2007) Section 4.1, for a definition of an Archimedean copula and further results. An interesting connection between Archimedean copulas and so-called l_1-norm symmetric distributions is established by Nešlehová and McNeil (2008). One can show that bivariate dfs with continuous marginals and with certain Archimedean copulas (e.g., with strict generator; see Nelsen 2007) have no negative tail dependence, that is $\lambda_u^- = \lambda_l^- = 0$. Therefore, they stand in violent contrast to elliptical distributions, where (unless in the comonotonic case) λ_u^- and λ_l^- are always positive.

For every elliptical copula one can always find a (strict) Archimedean copula with the same positive upper tail dependence coefficient λ_u^+. However, the asymptotic VaRs behave very differently; see also Embrechts et al. (2008). We hence conclude that the positive upper tail dependence coefficient in an infinite mean model is not able to explain the sub-/superadditive behavior of VaR.

In the next section we show that a crucial role is indeed played by λ_u^- and λ_l^- whenever X_1 and X_2 have doubly infinite support.

6 Tail dependence and subadditivity

The simplest model incorporating independence as well as co- and countermonotonicity is the *Fréchet family*. Therefore, we combine the independent copula $C_{0,0}(u, v) = uv$, the comonotonic copula $C_{1,0}(u, v) = u \wedge v = \min(u, v)$ and the countermonotonic copula $C_{0,1}(u, v) = (u + v - 1)^+$; see Nelsen (2007), Exercise 2.4. Let C_{p_1,p_2} be a convex combination of these copulas,

$$C_{p_1,p_2}(u, v) = p_1(u \wedge v) + p_2(u + v - 1)^+ + (1 - p_1 - p_2)uv, \tag{6.1}$$

for $p_1 \in [0, 1]$ and $p_2 \in [0, 1 - p_1]$. The copula family C_{p_1,p_2} is referred to as the *Fréchet family*. Let X_1, X_2 be two identically distributed regularly varying rvs with index $-\beta < 0$ and with marginal df F_β (i.e., $\bar{F}_\beta(x) = x^{-\beta}L(x)$, with

L slowly varying) and copula C_{p_1,p_2}. The bivariate df of $(X_1, X_2)'$ is then given by

$$F_{X_1,X_2}(x_1, x_2) = C_{p_1,p_2}(F_\beta(x_1), F_\beta(x_2)). \tag{6.2}$$

In Fig. 2 we give a typical random sample from the copula C_{p_1,p_2} in Eq. 6.1 and the bivariate df in Eq. 6.2.

In the following, we only look at symmetric marginals, i.e., where $X \stackrel{d}{=} -X$. In order to investigate subadditivity properties of VaR for the df Eq. 6.2, we consider q_Ψ as a function of p_1, p_2 and with $\Psi(\boldsymbol{X}) = X_1 + X_2$,

$$q_\Psi(\beta, (p_1, p_2)) = \lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)}.$$

In the symmetric case, q_Ψ can be calculated explicitly.

Proposition 6.1 *Let $(X_1, X_2)'$ be a bivariate random vector defined by Eq. 6.2 with identically distributed, symmetric, regularly varying marginals with index $-\beta < 0$, then*

$$q_\Psi(\beta, (p_1, p_2)) = 2^\beta p_1 + 2(1 - p_1 - p_2).$$

Proof Due to the linearity of Eq. 6.1 it is sufficient to check the independent, the comonotonic and the countermonotic case separately. For X_1, X_2 independent, we have $q_\Psi(\beta, (0, 0)) = 2$, which does not depend on β. For X_1, X_2 comonotonic and $X_1 \stackrel{d}{=} X_2$, we have $X_1 = X_2$ P-a.s. and hence $q_\Psi(\beta, (1, 0)) = \lim_{x\to\infty}(x/2)^{-\beta}/x^{-\beta} = 2^\beta$. For X_1, X_2 countermonotonic and $X_1 \stackrel{d}{=} X_2$, we have $X_1 = -X_2$ P-a.s. and hence $q_\Psi(\beta, (0, 1)) = 0$. □

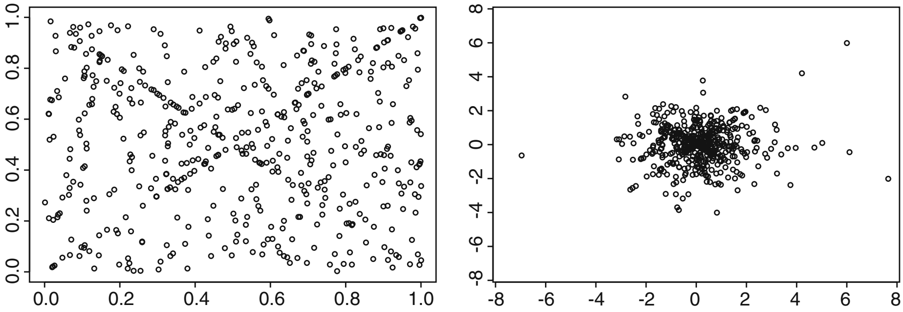


Fig. 2 *Left panel*: 500 realizations of the copula C_{p_1,p_2} in Eq. 6.1 with parameters $p_1 = p_2 = 0.1$. *Right panel*: the realizations are transformed according to Eq. 6.2 where F_β is a *t* distribution with 6 degrees of freedom.

Note that the positive upper tail dependence coefficient of the model Eq. 6.2 is given by

$$\lambda_u^+ = \lim_{u \nearrow 1} \frac{1 - 2u + C(u, u)}{1 - u} = p_1.$$

Equivalently, we have $\lambda_l^+ = p_1$ and $\lambda_u^- = \lambda_l^- = p_2$. Thus, we have the following corollary.

Corollary 6.2 *Let $(X_1, X_2)'$ be a bivariate random vector with a copula from the Fréchet family defined by Eq. 6.1 and identically distributed, symmetric, regularly varying marginals with index $-\beta < 0$, then $q_\Psi(\beta, (\lambda_u^+, \lambda_u^-)) = 2^\beta \lambda_u^+ + 2(1 - \lambda_u^+ - \lambda_u^-)$, with $\lambda_u^+, \lambda_u^- \geq 0$ and $\lambda_u^+ + \lambda_u^- \leq 1$.*

If $\beta \geq 1$, then of course, asymptotic subadditivity always holds. This follows from the fact that $q_\Psi(\beta, (\lambda_u^+, \lambda_u^-)) = 2^\beta \lambda_u^+ + 2(1 - \lambda_u^+ - \lambda_u^-) \leq 2^\beta (\lambda_u^+ + (1 - \lambda_u^+ - \lambda_u^-)) \leq 2^\beta$, together with Lemma 2.3, yielding that

$$\lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha(X_1 + X_2)}{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)} \leq 1.$$

In the case $\beta < 1$, $q_\Psi(\beta, (\lambda_u^+, \lambda_u^-))$ can be smaller or larger than 2^β, thus depending on the values λ_u^+, λ_u^-, subadditivity may hold or fail. To analyze this infinite mean model in more detail, we exclude the trivial case $\lambda_u^+ = 1$ and introduce the *relative negative tail dependence coefficient*

$$\gamma = \frac{\lambda_u^-}{1 - \lambda_u^+}.$$

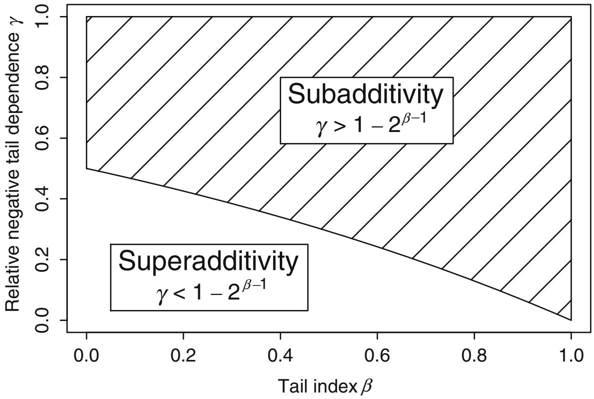
Because $\lambda_u^+ + \lambda_u^-$ is always smaller than 1, γ takes values only in $[0, 1]$ and is interpreted as the amount of negative tail dependence, relative to the possible maximal negative tail dependence coefficient $1 - \lambda_u^+$. We then have the following theorem.

Theorem 6.3 *Let $(X_1, X_2)'$ be a bivariate random vector with a copula from the Fréchet family defined by Eq. 6.1 with $p_1 < 1$ and identically distributed, symmetric, regularly varying marginals with index $-\beta < 0$. Then asymptotic subadditivity of VaR holds if and only if $\gamma \geq 1 - 2^{\beta - 1}$.*

Proof This follows immediately from Corollary 6.2 and Lemma 2.3. □

Theorem 6.3 provides a simple criterion for asymptotic subadditivity in the case of the Fréchet family model Eq. 6.2. Only for sufficiently small values of γ superadditivity occurs. For large values of γ subadditivity always holds. The interpretation of this behavior is that if the negative tail dependence coefficient is sufficiently large, then positive extreme values in one coordinate are compensated by negative extreme values in the other coordinate. This effect can be so strong that we obtain asymptotic subadditivity. In Fig. 3, we

Fig. 3 This figure shows the impact of negative tail dependence on subadditivity properties of VaR for infinite mean models. Inside of the *hatched area* subadditivity holds, whereas outside superadditivity holds.



plot the range, where subadditivity occurs as a function of $\beta \in [0, 1]$ and the relative negative tail dependence coefficient γ.

Several authors mention the substantial influence of the transition from a finite to an infinite mean model on the additivity properties of VaR; see for instance Nešlehová et al. (2006), Embrechts et al. (2008), Ibragimov and Walden (2007) and Jang and Jho (2007). An early statement in the finance literature that *diversification does not in general reduce its effects on the dispersion of the portfolio return* is found already in 1972 in Fama and Miller (1972); these authors based their conclusion on properties of Lévy-stable dfs.

Theorem 6.3 also shows that $\beta = 1$ plays a fundamental role for independent rvs and more generally if (in the Fréchet model) the relative negative tail dependence coefficient γ is 0. However, as soon as $\gamma > 0$, it is likely that high losses are compensated by high gains and therefore the transition from sub- to superadditivity will be located at β strictly smaller than 1; see again Fig. 3.

Theorem 6.3 of course does not hold in general (outside the Fréchet family model). However, it gives first insight in the still open problem of the characterization of asymptotic sub- and superadditivity in an infinite mean model. A more in depth analysis would presumably need to be based on properties of the level sets of the densities. Consider as an example a bivariate distribution with identical t_ν marginals and a t_4-copula with dependence parameter $\varrho = 0$. This is an example from the so-called class of meta-t_4-distributions. McNeil et al. (2005), Paragraph 5.1.3 contains a general definition of meta-distributions, as well as a discussion of applications to risk management. A simulation with 10^5 realizations then shows that the transition from sub- to superadditivity is located at a value ν_0 in the interval $(0.8, 0.9)$, significantly below 1. If Theorem 6.3 would hold in general, it would indicate a (theoretical) transition located at $\nu_0 = \log_2(1 - \gamma) + 1 \approx 0.877 \in (0.8, 0.9)$, in agreement with our empirical result. This simulation-based statement we have included for illustrative purposes, because to the best of our knowledge one does not have a satisfactory explanation for this anomaly at the moment.

In Section 3, we mentioned that in an infinite mean model sub- as well as superadditivity of VaR may occur in a completely arbitrary (or somewhat chaotic) way. In the following example we construct such an (artificial) model.

Example 6.4 Corollary 6.2 shows that there exist bivariate rvs where the positive upper tail dependence coefficient is not the main driver leading to sub- or superadditivity of VaR.

Indeed, by choosing $\beta = 1/2$ (infinite mean model) and for instance

$$p_1 = \lambda_u^+, \tag{6.3}$$

$$p_2 = \left(1 - \lambda_u^+\right) \sin^2\left(4\pi \frac{\lambda_u^+}{1 - \lambda_u^+}\right), \tag{6.4}$$

the plot of $q_\Psi(\beta, \lambda_u^+)$ in Fig. 4 clearly shows that there is no obvious connection between the positive upper tail dependence coefficient and subadditivity of VaR. Asymptotic subadditivity occurs if and only if $q_\Psi(\beta = 1/2, \lambda_u^+) \leq \sqrt{2}$ (horizontal line). One can always choose $p_2 = \lambda_u^-$ such that for an arbitrary value of $\lambda_u^+ < 1$, $q_\Psi(\beta, \lambda_u^+)$ is greater *or* smaller than $\sqrt{2}$. Sub- as well as superadditivity occurs in a completely arbitrary way.

Note that in Theorem 6.3 we assume the rvs to have doubly infinite support. We give an explicit counter-example, when this assumption is not fulfilled.

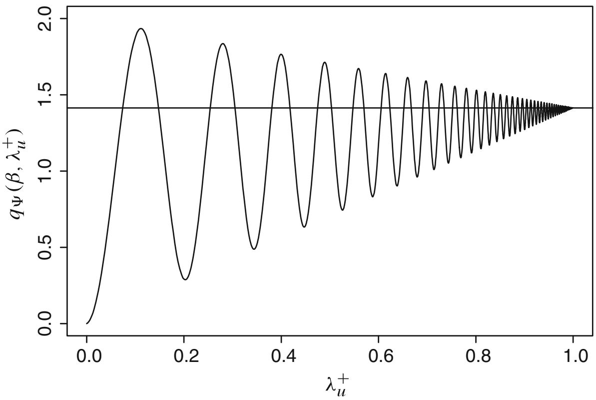
Example 6.5 Let X_1, X_2 be two countermonotonic (i.e., $\gamma = 1$) Pareto distributed rvs with marginal dfs

$$F_{X_1}(x) = F_{X_2}(x) = 1 - x^{-1/2}, \quad x \geq 1.$$

Using countermonotonicity of X_1 and X_2, we deduce that the df of $X_1 + X_2$ for all $x \geq 8$ is

$$F_{X_1+X_2}(x) = \sqrt{1 + 4(1 - \sqrt{1+x})/x}\,;$$

Fig. 4 $q_\Psi(\beta, \lambda_u^+)$ as a function of λ_u^+ for the Fréchet family copula Eq. 6.1 with parameters p_1 and p_2 from Eqs. 6.3 and 6.4, respectively. Subadditivity occurs below the *horizontal line*, superadditivity above.



see for instance Strassburger and Pfeifer (2005), Lemma 5.1. Hence,

$$F_{X_1+X_2}(x) = 1 - 2x^{-1/2} + O\left(x^{-3/2}\right), \quad x \to \infty,$$

and therefore

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)} = 2.$$

Thus, by Lemma 2.3, asymptotic subadditivity does not hold.

For $X_1^{'}, X_2^{'} \overset{i.i.d.}{\sim}$ Pareto(1/2), for $x \geq 2$,

$$F_{X_1^{'}+X_2^{'}}(x) = 1 - 2\frac{\sqrt{x-1}}{x}.$$

Hence,

$$F_{X_1^{'}+X_2^{'}}(x) = 1 - 2x^{-1/2} + O\left(x^{-3/2}\right), \quad x \to \infty,$$

and therefore $X_1 + X_2$ in Example 6.5 and $X_1^{'} + X_2^{'}$ are tail-equivalent. Hence, full diversification in the sense of countermonotonicity is as bad as independence (and therefore worse than no diversification in the sense of comonotonicity). For infinite mean models, diversification clearly goes the wrong way. More generally, we have the following proposition; see for instance Davis and Resnick (1996), Lemma 2.1 or Albrecher et al. (2006), Corollary 3.2.

Proposition 6.6 *Let $X_1, X_2 > 0$ be two identically distributed regularly varying rvs with index $-\beta < 0$ and with $\lambda_u^+ = 0$, i.e., such that X_1, X_2 are tail-independent in the positive upper tail, then*

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{P(X_1 > x)} = 2.$$

This proposition is indeed a special case of Proposition 4.6. It has the nice interpretation that the sum of tail-independent (in the positive upper tail) rvs behaves asymptotically as if the summands were independent. According to the above remarks, Proposition 6.6 yields the following result. Let $X_1, X_2, X_1^{'}, X_2^{'} > 0$ be identically distributed regularly varying rvs with X_1, X_2 countermonotonic and $X_1^{'}, X_2^{'}$ independent, then $\mathrm{VaR}_\alpha(X_1 + X_2) \sim \mathrm{VaR}_\alpha(X_1^{'} + X_2^{'})$, for $\alpha \to 1$. That is, the VaR of the sum of highly "diversified" positive rvs is asymptotically equal to the VaR of the sum of independent rvs for α large.

7 Conclusion

Under the current regulatory guidelines for banking and insurance, risk diversification, concentration and aggregation play a prominent role. Within and between "nice" risk categories, with elliptical dfs, say, these concepts are easily modeled and particular solutions can be readily worked out. However, for skew, heavy-tailed risk rvs, diversification and aggregation have to be handled with care.

In this paper we show that the interplay between existence, non-existence of a finite moment, one- or two-sidedness and symmetry versus asymmetry of the underlying risk dfs have to be carefully balanced in order to be able to conclude sub- or superadditivity of quantile based risk measures like Value-at-Risk.

We have highlighted in the paper that MEVT offers the canonical language for analyzing from an asymptotic point of view questions of the above type for heavy-tailed dfs. That answers to these questions are relevant for practice can for instance be seen in applied publications like Moscadelli (2004) and Aas et al. (2007). Though we obtained a better understanding of the diversification-concentration-aggregation issue for VaR, many questions still remain unsolved and further research is no doubt needed, especially for two-sided skew rvs. For two-sided rvs the sum operator is not a norm but only a so-called gauge function and this makes their analysis much more delicate; see for instance Balkema and Embrechts (2007) for details on MEVT in this context.

The following recent preprint came to our attention during the refereeing process, Mainik and Rüschendorf (2008), in which statistical estimation procedures for problems presented in Section 4 are discussed.

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