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MINIMAL SYSTEMS OF ARBITRARY LARGE MEAN TOPOLOGICAL DIMENSION

BY

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ABSTRACT

Let G be a countable amenable group and P a polyhedron. The mean topological dimension $\mathrm{mdim}(X,G)$ of a subshift $X \subset P^G$ is a real number satisfying $0 \leq \mathrm{mdim}(X,G) \leq \dim(P)$, where $\dim(P)$ denotes the usual topological dimension of P. We give a construction of minimal subshifts $X \subset P^G$ with mean topological dimension arbitrarily close to $\dim(P)$.

1. Introduction

Mean topological dimension is a numerical topological invariant for actions of amenable groups introduced by M. Gromov. It is a useful tool for studying spaces with infinite topological dimension or entropy; see [Gro], [LiW], [Li].

Let G be a countable group. A G**-space** (or G**-system**) is a couple (X,G), where X is a compact metrisable space endowed with a continuous G-action. One says that the G-space (X,G) **is embeddable** (or **embeds**) in the G-space (Y,G), if there exists a G-equivariant topological embedding from X to Y.

Let K be a compact metrisable space. The **full** G**-shift** with **space of symbols** K is the G-space given by the left action of G on the product space $K^G = \{(x_g) \colon x_g \in K\}$ defined by

$$g'(x_g)_{g\in G} = (x_{gg'})_{g\in G}$$

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for all $g' \in G$ and $(x_g)_{g\in G} \in K^G$. Note that K^G is compact and metrisable since it is a countable product of compact metrisable spaces. A closed G-invariant subset of K^G is called a **subshift**.

It can be shown (see Theorem 2.7) that if G is amenable, then the mean topological dimension $\mathrm{mdim}(X, G)$ of a subshift $X \subset K^G$ is at most $\dim(K)$, where $\dim(K)$ denotes the usual topological dimension of K. If P is a **polyhedron**, i.e. a topological space homeomorphic to a finite simplicial complex, then one has $\mathrm{mdim}(P^G, G) = \dim(P)$. Hence, for all subshifts $X \subset P^G$, we have $0 \le \mathrm{mdim}(X, G) \le \dim(P)$.

A G-space (X, G) is said to be **minimal** if all its orbits are dense in X. A. Jaworski proved in [Jaw, Corollary IV.2.1] that if G is an abelian group, then every minimal G-space of finite topological dimension embeds into the G-shift on $[0, 1]^G$. In relation to this result, J. Auslander asked in [Aus] whether every minimal $\mathbf{Z}$-space embeds in the $\mathbf{Z}$-shift on $[0, 1]^{\mathbf{Z}}$. This question had remained open for many years. In [LiW], E. Lindenstrauss and B. Weiss answered this question negatively by constructing a minimal $\mathbf{Z}$-space X with mean dimension $\mathrm{mdim}(X, \mathbf{Z}) > \mathrm{mdim}([0, 1]^{\mathbf{Z}}, \mathbf{Z}) = 1$. The authors of [LiW] assert also (but give no detailed proof) that for every $0 \le t \le \infty$, there exists a minimal $\mathbf{Z}$-space with mean topological dimension t.

The main result of this paper is the construction of minimal G-spaces with arbitrary large mean topological dimension in the case when G is an infinite countable amenable group. More precisely, we prove the following:

Theorem 1.1: *Let G be an infinite countable amenable group and P a polyhedron. Let $\rho \in [0, 1[$. Then there exists a minimal subshift $X \subset P^G$ with mean topological dimension $\mathrm{mdim}(X, G) \ge \rho \dim(P)$, where $\dim(P)$ denotes the topological dimension of P.*

As the construction of the minimal system of Theorem 1.1 can also be done by taking the Hilbert cube $[0, 1]^{\mathbf{N}}$ as space of symbols, we obtain a minimal system with infinite mean topological dimension. We deduce:

Theorem 1.2: *Let G be an infinite countable amenable group. Then there exists a minimal action of G on a compact metrisable space X, such that (X, G) is not embeddable in K^G, whenever K is a compact metrisable space of finite topological dimension.*

The referee has mentioned that a similar construction as the one given in the proof of Theorem 1.1, but using Ornstein-Weiss quasi-tiling techniques ([OrW, Section I.2]), should give systems (resp., minimal systems) with arbitrary mean dimension.

In the case when G is a residually finite countable amenable group, constructions of systems (not minimal, in general) with arbitrary mean dimension are given in [CoK]. See also [Kri] for constructions of minimal systems with large mean dimension in this case.

The paper is organized as follows. In Section 2, we recall the definition and some results of mean topological dimension for actions of amenable groups. The main result in this section is Proposition 2.8 which gives a lower bound of the mean dimension for a certain type of subshifts. The notion of syndetic subset of a group is defined in Section 3. It plays a fundamental role in the construction of minimal G-spaces (see Lemme 3.1). Some auxiliary lemmas used in the proof of Theorem 1.1 are also given. In Section 4, we prove Theorems 1.1 and 1.2.

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2. Amenable groups and mean topological dimension

2.1. Amenability. There are several equivalent definitions of amenable groups in the literature. The one we give here is a characterization due to Følner [Føl]. For a more complete description of this class of groups see, for example, [Gre] or [Pat].

Given subsets A, L of a group G, we define the **outer envelope** $A^{+L} \subset G$ of A by

$$A^{+L} = \bigcup_{a \in A} La = LA.$$

A group G is said to be **amenable** if for all $\epsilon > 0$ and all finite subsets $L \subset G$, there exists a finite subset $F \subset G$ such that $|F^{+L}| \leq (1+\epsilon)|F|$.

Let us denote by $\mathcal{F}(G)$ the set of all nonempty finite subsets of G. Observe that a countable group is amenable if and only if there is a sequence (F_n) in $\mathcal{F}(G)$ such that:

$$\lim_{n \to \infty} \frac{|{F_n}^{+L}|}{|F_n|} = 1$$

for all finite subsets $L \subset G$. Such a sequence is called a **Følner sequence** of G.

Observe that if (F_n) is Følner so is $(F'_n g_n)$ for every subsequence (F'_n) of (F_n) and for all sequences (g_n) of elements in G.

Recall that the class of amenable groups contains all finite groups, abelian groups, it is closed by taking subgroups, quotients, extensions and inductive limits. All finitely generated groups of subexponential growth are amenable. A basic example of a nonamenable group is the free group of rank 2.

The next lemma says that in a countable amenable group G, there always exists an increasing Følner sequence with limit G:

LEMMA 2.1: *Let (F_n) be a Følner sequence of a countable group G. Then there exist a sequence (g_n) of elements in G and a subsequence $(F_{\varphi(n)})$ such that (F'_n) defined by $F'_n = F_{\varphi(n)} g_n$, $n \in \mathbf{N}$, satisfies the following conditions:*

(i) *(F'_n) is a Følner sequence of G;*
(ii) *$F'_n \subset F'_{n+1}$ for all $n \in \mathbf{N}$;*
(iii) *$\bigcup_{n\in\mathbf{N}} F'_n = G$.*

Proof. This result follows from the general fact: if (F_n) is a Følner sequence of G and L a finite subset of G then, if n is large enough, there exists $g \in G$ such that $L \subset F_n g$. It can be proved as follows. Let $n \in \mathbf{N}$. Remark that we have

$$\{g \in F_n \colon Lg \cap (G \setminus F_n) \neq \varnothing\} \subset \bigcup_{l \in L} F_n \setminus (l^{-1} F_n).$$

Now, using the fact that

$$|F_n \setminus (l^{-1} F_n)| = |l F_n \setminus F_n| \leq |L F_n \setminus F_n|$$

for all $l \in L$, we deduce

$$|\{g \in F_n \colon Lg \cap (G \setminus F_n) \neq \varnothing\}| \leq |L||L F_n \setminus F_n|.$$

Since (F_n) is a Følner sequence, we deduce that the right member of the latter inequality is smaller than $|F_n|$ if n is large enough. Hence, the set $\{g \in F_n \colon Lg \cap (G \setminus F_n) = \varnothing\}$ is nonempty for n large enough.

Now, it is easy to construct by induction a sequence (F'_n) satisfying (i), (ii), (iii). As G is countable, one can write $G = \bigcup_n E_n$ where $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of finite subsets of G. Let us suppose that we have already built $F'_n = F_{\varphi(n)} g_n$. By the above observation, there exist an integer $\varphi(n+1) > \varphi(n)$ and $g_{n+1} \in G$ such that $E_{n+1} \cup F'_n \subset F_{\varphi(n+1)} g_{n+1}$. Define $F'_{n+1} = F_{\varphi(n+1)} g_{n+1}$. This finishes the construction since $(F_{\varphi(n)} g_n)$ is a Følner sequence. ∎

Lemma 2.2: *Let (F_n) be a Følner sequence of G, $A \in \mathcal{F}(G)$ and $A_n \subset F_n^{+A}$. Then $(F_n \cup A_n)$ is a Følner sequence.*

Proof. Let $L \in \mathcal{F}(G)$. For all $n \in \mathbf{N}$, one has

$$\frac{|L(F_n \cup A_n)|}{|F_n \cup A_n|} \leq \frac{|LF_n|}{|F_n|} + \frac{|L(A_n \setminus F_n)|}{|F_n|} \leq \frac{|LF_n|}{|F_n|} + |L|\frac{|AF_n \setminus F_n|}{|F_n|}.$$

As (F_n) is a Følner sequence, the right term of the above inequalities tends to 1 as n tends to infinity. ∎

2.2. Mean topological dimension. Let (X, G) be a G-space and suppose G is amenable. We recall the definition of the mean topological dimension of (X, G) together with some results used in the proof of Theorem 1.1. For more details see [Gro], [LiW], [Coo], [CoK].

Let $\alpha = (U_i)_{i \in I}$ and $\beta = (V_j)_{j \in J}$ be finite open covers of X. The **join** of α and β is the finite open cover $\alpha \vee \beta$ of X defined by

$$\alpha \vee \beta = (U_i \cap V_j)_{(i,j) \in I \times J}.$$

One says that β is **finer** than α and one writes $\beta \succ \alpha$ if for all $j \in J$, there exists $i \in I$ such that $V_j \subset U_i$.

The **order** of α is defined by

$$\operatorname{ord}(\alpha) = -1 + \max_{x \in X} \big(|\{i \in I \colon x \in U_i\}| \big).$$

Definition of $\mathbf{D}(\boldsymbol{\alpha})$ ([HuW]): Let α be a finite open cover of X. One defines the integer $D(\alpha)$ by

$$\mathrm{D}(\alpha) = \min_{\beta} \operatorname{ord}(\beta),$$

where β runs over all finite open covers of X such that $\beta \succ \alpha$.

Recall that the **topological dimension** of X is defined by

$$\dim(X) = \sup_{\alpha} \mathrm{D}(\alpha),$$

where α runs over all finite open covers of X.

Definition of $\mathbf{dim}_{\boldsymbol{\epsilon}}\boldsymbol{(X, d)}$*:* Let d be a compatible metric on X and $\epsilon > 0$. A map f from X in a set E is said to be **ϵ-injective** if one has $d(x_1, x_2) < \epsilon$ for all $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$.

One defines $\dim_\epsilon(X, d)$ by

$$\dim_\epsilon(X, d) = \min_{K} \dim(K),$$

where the minimum is taken over all compact metrisable spaces K such that there exists a continuous ϵ-injective map $f\colon X \to K$.

LEMMA 2.3: *Let (X', d') be a compact metric space such that there exists a continuous map $\varphi\colon X \to X'$ which satisfies*

$$d(x_1, x_2) \le d'(\varphi(x_1), \varphi(x_2)) \quad \textit{for all } x_1, x_2 \in X.$$

Then one has $\dim_\epsilon(X, d) \le \dim_\epsilon(X', d')$.

Proof. If $f\colon X' \to K$ is ϵ-injective, so is $f \circ \varphi\colon X \to K$. ∎

PROPOSITION 2.4: *Let P be a polyhedron and let ρ be a metric on P which is compatible with the topology. For each $n \in \mathbf{N}$, let ρ_n denote the metric on P^n defined by*

$$\rho_n(x, y) = \max_{1 \le i \le n} \rho(x_i, y_i) \quad \textit{for all } x = (x_i), y = (y_i) \in P^n.$$

Then there is a constant $\epsilon_0 = \epsilon_0(P, \rho)$ which does not depend on n such that

$$\dim_\epsilon(P^n, \rho_n) = n \dim(P)$$

for all $\epsilon \le \epsilon_0$.

Proof. See [CoK, Coro. 2.8]. ∎

Definition of $\mathbf{mdim(}\boldsymbol{X, G}\mathbf{)}$*:* Let $\alpha = (U_i)_{i \in I}$ be a finite open cover of X and $F \in \mathcal{F}(G)$. Denote by α_F the finite open cover of X defined by

$$\alpha_F = \bigvee_{g \in F} g^{-1}\alpha,$$

where $g^{-1}\alpha = (g^{-1}U_i)_{i \in I}$. One defines the real $D(\alpha, G)$ by

$$\mathrm{D}(\alpha, G) = \lim_{n \to \infty} \mathrm{D}(\alpha_{F_n})/|F_n|,$$

where (F_n) is a Følner sequence of G. It can be proved that this limit exists, is finite and doesn't depend on the choice of the Følner sequence. The **mean topological dimension** $\mathrm{mdim}(X, G)$ of the G-space (X, G) is defined by

$$\mathrm{mdim}(X, G) = \sup_\alpha \mathrm{D}(\alpha, G),$$

where α runs over all finite open covers of X.

Metric approach of $\mathbf{mdim(X, G)}$. For $F \in \mathcal{F}(G)$, define the metric d_F on X by

$$d_F(x,y) = \max_{g\in F} d(gx, gy) \quad \text{ for all } x, y \in X.$$

It is clear that the metric d_F is compatible with the topology of X. We define the real $\mathrm{mdim}_\epsilon(X, d, G)$ by

$$\mathrm{mdim}_\epsilon(X,d,G) = \lim_{n\to\infty} \frac{\dim_\epsilon(X, d_{F_n})}{|F_n|}, \quad \text{for all } \epsilon > 0,$$

where (F_n) is a Følner sequence in G. Again, it can be shown that this limit exists, is finite and does not depend on the choice of the Følner sequence.

Theorem 2.5: *We have* $\mathrm{mdim}(X, G) = \lim_{\epsilon\to 0} \mathrm{mdim}_\epsilon(X, d, G)$.

Proof. See [CoK, Th. 3.3]. ∎

A subset $Y \subset X$ is said to be G**-invariant** if $gY \subset Y$ for all $g \in G$. In this case, G acts continuously of Y by restriction.

Proposition 2.6: *Let Y be a closed G-invariant subset of X. Then one has* $\mathrm{mdim}(Y, G) \leq \mathrm{mdim}(X, G)$.

Proof. ([CoK, Proposition 3.4]) Let $A \in \mathcal{F}(G)$ and $\epsilon > 0$. If $f \colon X \to K$ is ϵ-injective with respect to the metric d_A, then so is the restriction of f to Y. Therefore, we have $\dim_\epsilon(Y, d_A) \leq \dim_\epsilon(X, d_A)$. Thus, if (F_n) is a Følner sequence, we get

$$\begin{aligned}\mathrm{mdim}_\epsilon(Y,d,G) &= \lim_{n\to\infty} \frac{\dim_\epsilon(Y, d_{F_n})}{|F_n|} \leq \lim_{n\to\infty} \frac{\dim_\epsilon(X, d_{F_n})}{|F_n|} \\ &= \mathrm{mdim}_\epsilon(X,d,G).\end{aligned}$$

Letting ϵ tend to 0, we obtain $\mathrm{mdim}(Y, G) \leq \mathrm{mdim}(X, G)$ by using Theorem 2.5. ∎

Theorem 2.7: *Let K be a compact metrisable space and (K^G, G) the full G-shift. Then one has*

$$\mathrm{mdim}(K^G, G) \leq \dim(K).$$

If K is a polyhedron, then $\mathrm{mdim}(K^G, G) = \dim(K)$.

Proof. See [CoK, Cor. 4.2 and Cor. 5.5]. ∎

For $J \subset G$, we define the **density** $\delta(J) \in [0,1]$ of J in G by

$$\delta(J) = \sup_{(F_n)} \limsup_{n\to\infty} \frac{|J \cap F_n|}{|F_n|},$$

where (F_n) runs over all Følner sequences of G.

The next proposition is used in the proof of the main theorem for estimating the mean topological dimension. This result is originally due to Lindenstraus and Weiss for **Z**-actions [LiW, Prop. 3.3].

Proposition 2.8: *Let G be a countable amenable group. Let $X \subset P^G$ be a closed subshift, where P is a polyhedron or the Hilbert cube $[0,1]^{\mathbf{N}}$. Suppose that there exist $\bar{x} = (\bar{x}_g)_{g\in G} \in X$ and a subset $J \subset G$ satisfying the following condition*

$$\pi_{G\setminus J}(x) = \pi_{G\setminus J}(\bar{x}) \Rightarrow x \in X,$$

for all $x \in P^G$.

Then, in the case when P is a polyhedron, we have

$$\mathrm{mdim}(X,G) \geq \delta(J)\dim(P).$$

If P is the Hilbert cube and $\delta(J) > 0$, we have $\mathrm{mdim}(X,G) = \infty$.

Proof. Consider a compatible metric ρ on P. As G is countable, there is a family $(\alpha_g)_{g\in G}$ of positive reals such that $\alpha_{1_G} = 1$ and $\sum_{g\in G} \alpha_g < \infty$. Consider the metric d on P^G defined by

$$d(x,y) = \sum_{g\in G} \alpha_g \rho(x_g, y_g) \quad \text{for all } x = (x_g)_{g\in G}, y = (y_g)_{g\in G} \in P^G.$$

Observe that d is compatible with respect to the product topology on P^G and that one has

$$\rho(x_{1_G}, y_{1_G}) \leq d(x,y), \tag{2.1}$$

for all $x, y \in P^G$. Let $F \in \mathcal{F}(G)$. Recall that the metric d_F is defined by

$$d_F(x,y) = \max_{g\in F} d(gx, gy) \quad \text{for all } x, y \in P^G.$$

Let ρ_F be the metric on P^F defined by

$$\rho_F(u,v) = \max_{g\in F} \rho(u_g, v_g), \tag{2.2}$$

for all $u = (u_g)_{g \in F},\ v = (v_g)_{g \in F} \in P^F$. Consider the embedding $\psi_F \colon P^F \to P^G$ which associates to each $u \in P^F$ the element $x \in P^G$ defined by

$$x_g = \begin{cases} u_g & \text{if } g \in F \\ \bar{x}_g & \text{if } g \in G \setminus F. \end{cases}$$

Inequality (2.1) and equality (2.2) imply

$$\rho_F(u, v) \le d_F(\psi_F(u), \psi_F(v)) \quad \text{for all } u, v \in P^F. \tag{2.3}$$

Let (F_n) be a Følner sequence of G and fix $n \in \mathbf{N}$. Define $J_n = J \cap F_n$. The properties satisfied by $\bar{x}$ and J imply $\psi_{J_n}(u) \in X$ for all $u \in P^{J_n}$. Using inequality (2.3) and the inclusion $J_n \subset F_n$, we deduce

$$\rho_{J_n}(u, v) \le d_{J_n}(\psi_{J_n}(u), \psi_{J_n}(v)) \le d_{F_n}(\psi_{J_n}(u), \psi_{J_n}(v))$$

for all $u, v \in P^{J_n}$. By Lemma 2.3, we obtain

$$\dim_\epsilon(P^{J_n}, \rho_{J_n}) \le \dim_\epsilon(X, d_{F_n}), \tag{2.4}$$

for all $\epsilon > 0$.

Case 1: P is a polyhedron. It follows from Proposition 2.4 that there exists $\epsilon_0 > 0$ such that:

$$\dim_\epsilon(P^{J_n}, \rho_{J_n}) = |J_n| \dim(P)$$

for $\epsilon \le \epsilon_0$. Together with inequality (2.4) we obtain

$$|J_n| \dim(P) \le \dim_\epsilon(X, d_{F_n}).$$

By definition of mdim_ϵ we deduce

$$\mathrm{mdim}_\epsilon(X, d, G) = \lim_{n \to \infty} \frac{\dim_\epsilon(X, d_{F_n})}{|F_n|} \ge \limsup_{n \to \infty} \frac{|J \cap F_n|}{|F_n|} \dim(P).$$

Using Theorem 2.5 and the fact that (F_n) were an arbitrary Følner sequence, we deduce

$$\mathrm{mdim}(X, G) = \lim_{\epsilon \to 0} \mathrm{mdim}_\epsilon(X, d, G) \ge \delta(J) \dim(P).$$

Case 2: $P = [0, 1]^{\mathbf{N}}$. We will prove that for all $k \in \mathbf{N}$, there is $\epsilon_k > 0$ such that

$$\delta(J)k \le \mathrm{mdim}_{\epsilon_k}(X, d, G). \tag{2.5}$$

This will show $\mathrm{mdim}(X,G)=\infty$ since $\mathrm{mdim}_{\epsilon_k}(X,d,G)\leq \mathrm{mdim}(X,G)$ (see Theorem 2.5). Let $k\in\mathbf{N}$ and $P_k=[0,1]^{\{0,1,\ldots,k-1\}}\subset[0,1]^{\mathbf{N}}$ endowed with the metric $\rho_k=\rho_{|P_k}$. Let $\epsilon>0$. Using Lemma 2.3 and inequality (2.4), we obtain

$$\dim_\epsilon((P_k)^{J_n},(\rho_k)_{J_n})\leq \dim_\epsilon(P^{J_n},\rho_{J_n})\leq \dim_\epsilon(X,d_{F_n}).$$

Proposition 2.4 says that there is $\epsilon_k>0$ (which does not depend on n) such that $\dim_{\epsilon_k}((P_k)^{J_n},(\rho_k)_{J_n})=k|J_n|$. We deduce

$$k|J_n|\leq \dim_{\epsilon_k}(X,d_{F_n}).$$

Inequality (2.5) follows. ∎

3. Syndetic subsets of a group

Let G be a group and recall that $\mathcal{F}(G)$ denotes the set of all nonempty finite subsets of G.

A subset $S\subset G$ is called **syndetic** if there exists a finite subset $F\subset G$ such that $G=FS$, where $FS=\{fs\colon f\in F,\ s\in S\}$. Finite indexed subgroups of G are examples of syndetic subsets of G. Other examples are given in Lemma 3.3.

Recall that an element x in a G-space X is said to be **almost periodic** if for any neighborhood V of x, there exists a syndetic set $S\subset G$ such that $Sx\subset V$. There is a well–known characterization of minimality for G-spaces.

Lemma 3.1: *A G-space (X,G) is minimal if and only if X is the orbit closure of an almost periodic point.*

Proof. See [Aus, Th. 7, p. 11]. ∎

Let $F\in\mathcal{F}(G)$ and $\Omega\subset G$. A family of translates $(Ft)_{t\in T}$, where $\varnothing\neq T\subset G$, is called a *maximal disjoint family* in Ω if

(i) $Ft\subset\Omega$ for all $t\in T$;
(ii) $Ft\cap Ft'=\varnothing$ for all $t\neq t'$ in T;
(iii) $Fg\subset\Omega\Rightarrow\exists t\in T$ such that $Fg\cap Ft\neq\varnothing$.

Lemma 3.2: *Let $F\in\mathcal{F}(G)$, $\Omega\subset G$ and suppose $\{g\in G\colon Fg\subset\Omega\}\neq\varnothing$. Then there exists $S\subset G$ such that $(Fs)_{s\in S}$ is a maximal disjoint family in Ω.*

Proof. We suppose $1_G\in F$. Now, if $(Ft)_{t\in T}$ is a maximal disjoint family in Ω, then $T\subset\Omega$. If Ω is finite or countable, it is easy to construct such a family.

In general, we use Zorn's Lemma. Let $\mathcal{M}$ be the set of all nonempty subsets $T \subset \Omega$ such that $(Ft)_{t\in T}$ is a disjoint family in Ω. Remark that $\mathcal{M} \neq \varnothing$ by hypothesis. The set $\mathcal{M}$ is partially ordered by inclusion and it is clear that $\mathcal{M}$ is an inductive set. By Zorn's Lemma, there is a maximal element S in $\mathcal{M}$. Hence $(Fs)_{s\in S}$ is a maximal disjoint family. ∎

LEMMA 3.3: *Let $(Fs)_{s\in S}$ be a maximal disjoint family in G. Then S is syndetic.*

Proof. Let $g \in G$. As $(Fs)_{s\in S}$ is a maximal disjoint family in G, there is $s \in S$ such that $Fg \cap Fs \neq \varnothing$. Hence $G = (F^{-1}F)S$. ∎

LEMMA 3.4: *Let $(C_n)_{n\in\mathbf{N}}$ be an increasing sequence of subsets of G such that $\bigcup_{n\in\mathbf{N}} C_n = G$. Let $F \in \mathcal{F}(G)$ and $S \subset G$. Suppose that $(Fs)_{s\in S_n}$, where $S_n = \{s \in S\colon Fs \subset C_n\}$, is a maximal disjoint family in C_n, for all $n \in \mathbf{N}$. Then $(Fs)_{s\in S}$ is maximal disjoint in G.*

Proof. Let s, s' be distinct points in S. As (C_n) is increasing and $\bigcup_n C_n = G$, there is an integer n such that s, s' are contained in S_n. It follows that $Fs \cap Fs' = \varnothing$ since $(Fs)_{s\in S_n}$ is a disjoint family. Thus $(Fs)_{s\in S}$ is a disjoint family. It is also a maximal disjoint family since for each $g \in G$, the set Fg is contained in some C_n and hence $Fg \cap Fs \neq \varnothing$ for an element $s \in S_n$ by maximality of $(Fs)_{s\in S_n}$. ∎

For $A \subset G$ define $\widetilde{A} = AA^{-1} = \{gh^{-1}\colon g, h \in A\}$.

LEMMA 3.5: *Let $(Ft)_{t\in T}$ be a maximal disjoint family in $\Omega^{+\widetilde{F}}$, where $\Omega \subset G$. Let g be an element in G such that $Fg \cap Ft = \varnothing$ for all $t \in T$. Then we have $Fg \cap \Omega = \varnothing$.*

Proof. If $Fg \cap \Omega \neq \varnothing$, then $Fg \subset FF^{-1}\Omega = \Omega^{+\widetilde{F}}$ which contradicts the fact that $(Ft)_{t\in T}$ is a maximal disjoint family in $\Omega^{+\widetilde{F}}$. ∎

Given a set A and a subset $R \subset G$, we denote the set of all functions from R to A by

$$A^R = \{x = (x_g)_{g\in R}\colon x_g \in A\}.$$

For $S \subset R$ and $x \in A^R$, let $\pi_S\colon A^R = A^S \times A^{R\setminus S} \to A^S$ be the canonical projection.

Suppose that A contains an element $*$ called **star**. For $x \in A^R$, we define $S(x) \subset R$ by

$$S(x) = \{g \in R \colon x_g = *\},$$

which represents the set of the positions of the stars contained in (the image of) x. If R is a nonempty finite subset of G, one defines the rational $s(x)$ by

$$s(x) = |S(x)|/|R| \tag{3.1}$$

which represents the frequency of stars contained in x.

For disjoint sets A an B, one writes $A \sqcup B$ their union.

Lemma 3.6: *Let $C_1 \subset C_2 \cdots \subset C_n = C$, $T_1, T_2, \dots, T_{n-1}$ and B be finite subsets of G such that the sets $C_i t_i$, $i \in I = \{1, \dots, n-1\}$, $t_i \in T_i$, are disjoint and contained in $C \setminus B$. Let x be an element of A^C such that $x_g = *$ for all $g \in C \setminus (B \cup \bigsqcup_{i \in \{1,\dots,n-1\}, t_i \in T_i} C_i t_i)$. Then one has*

$$s(x) \geq \frac{|C \setminus B|}{|C|} m,$$

where $m = \min_{i \in I, t_i \in T_i} s(\pi_{C_i t_i}(x))$.

Let $\epsilon > 0$. A subset Y in a metric space (X, d) is said to be **ϵ-dense** if for all $x \in X$, there is $y \in Y$ such that $d(x, y) \leq \epsilon$. Recall that any compact metrisable space has a finite ϵ-dense subset.

Lemma 3.7: *Let (K, d_K) be a compact metrisable space and G be a countable group. Let $\epsilon > 0$ and d be a metric on K^G compatible with the product topology. Then there exist a finite set $F \subset G$ and $\delta > 0$ such that for all x in K^G and for any δ-dense subset $A_\delta \subset K$, there exists $a \in {A_\delta}^F$ such that*

$$\pi_F(y) = a \Rightarrow d(x, y) \leq \epsilon,$$

for all $y \in K^G$.

Proof. Let $\epsilon > 0$. By the definition of the product topology on K^G, there is a $\delta > 0$ and a finite set $F \subset G$ such that

$$\sup_{g \in F} d_K(x_g, y_g) \leq \delta \Rightarrow d(x, y) \leq \epsilon$$

for all $x, y \in K^G$. The result follows. ∎

4. Proof of Theorems 1.1 and 1.2

This section is devoted to the construction of a G-space X satisfying the conclusion of Theorem 1.1. First we give the construction of this space, then we calculate a lower bound of its mean dimension and finally we prove minimality of the action.

For $C \subset G$ and $g \in G$, one denotes by $\Phi_{C,g}\colon A^C \to A^{Cg}$ the natural bijection defined by

$$\Phi_{C,g}(x)_{cg} = x_c$$

for all $x \in A^C$ and $c \in C$.

4.1. Construction of X. Let d_K be a compatible metric on K (where K is a polyhedron or the Hilbert cube $[0,1]^{\mathbf{N}}$) and d be a metric on K^G compatible with the product topology. Fix a decreasing sequence (ϵ_n) of positive reals which tends to 0 and a Følner sequence $F'_1 \subset F'_2 \subset \cdots$ satisfying $\bigcup_n F'_n = G$ (see Lemma 2.1). For each $n \in \mathbf{N}$ let $p_n \in \mathbf{N}$ and $\delta_n > 0$ satisfying the conclusion of Lemma 3.7 with $\epsilon = \epsilon_n$, $\delta = \delta_n$ and $F = F'_{p_n}$. Up to reindex the sequence (F'_n), we suppose that $p_n = n$ for all $n \in \mathbf{N}$.

Let $\rho \in [0,1[$. We will construct recursively a decreasing sequence $X_n \subset K^G$ in a way such that the G-invariant closed set $X = \overline{\bigcup_{g\in G} g(\bigcap_n X_n)} \subset K^G$ will be a minimal subshift with mean topological dimension $\mathrm{mdim}(X,G) \geq \rho \dim(K)$ if K is a polyhedron and $\mathrm{mdim}(X,G) = \infty$ if $K = [0,1]^{\mathbf{N}}$.

For convenience denote by $\widehat{K} = K \cup \{*\}$ the set obtained by adjoining to K an element $* \notin K$.

Step 1: Take $C_1 = F'_1$ and let $x_1 \in \widehat{K}^{C_1}$ with $s(x_1) > \rho$ (see (3.1) for the definition of s). Define $X_1 = \{x \in K^G \colon x_g = (x_1)_g \text{ for all } g \in C_1 \setminus S(x_1)\}$.

Step 2: Let K_{δ_1} be a δ_1-dense finite subset of K. As G is infinite, there is $R_1 \subset G$, $1_G \in R_1$, with cardinality $|R_1| = 1 + |K_{\delta_1}|^{|S(x_1)|}$ and such that $C_1 r$, $r \in R_1$, are disjoint. Observe that $|K_{\delta_1}|^{|S(x_1)|}$ represents the cardinality of all possibilities there exist for replacing all stars in the image of x_1 with elements of K_{δ_1}; denote by $V_1 \subset K^{C_1}$ the set of all these possibilities. Let $k_2 > 1$ be an integer large enough (it will be fixed later) so that $B_1 = \bigsqcup_{r\in R_1} C_1 r \subset F'_{k_2}$ and set $F_2 = F'_{k_2}$. Define $A_2^{(0)}, A_2^{(1)} \subset G$ by

$$A_2^{(0)} = F_2, \quad A_2^{(1)} = (F_2)^{+\widetilde{C_1}}.$$

Let $(C_1t)_{t\in T'^{(1)}_2}$ be a maximal disjoint family in $A^{(1)}_2 \setminus B_1$ and observe that $(C_1t)_{t\in T^{(1)}_2}$, where $T^{(1)}_2 = T'^{(1)}_2 \cup R_1$, is a maximal disjoint family in $A^{(1)}_2$. Define

$$C_2 = F_2 \cup \bigsqcup_{t\in T^{(1)}_2} C_1t.$$

Let x_2 the element in $\widehat{K}^{C_2}$ defined by the following conditions:

- the restriction of x_2 to C_1g is equal to x_1 for all $g \in T'^{(1)}_2 \cup \{1_G\}$, i.e.
$$\pi_{C_1g}(x_2) = \Phi_{C_1,g}(x_1) \quad \text{for all} \;\; g \in T'^{(1)}_2 \cup \{1_G\};$$
- for each $r \in R_1 \setminus \{1_G\}$ choose an element $v \in V_1$ in a one to one correspondence and define the restriction of x_2 to C_1r by
$$\pi_{C_1r}(x_2) = \Phi_{C_1,r}(v);$$
- otherwise define $(x_2)_g = *$.

By Lemma 3.6 applied to $T_1 = T'^{(1)}_2$ and $B = B_1$, we have

$$s(x_2) \geq \frac{|C_2 \setminus B_1|}{|C_2|} m,$$

where $m = s(x_1) > \rho$. Now, choose and fix k_2 large enough (recall $F_2 = F'_{k_2}$ and $\lim_{n\to\infty} |F'_n| = \infty$) so that we have

$$s(x_2) > \rho$$

and

$$|C_2^{+F_1}| \leq (1 + 1/2)|C_2|$$

(the latter condition is possible for k_2 large enough by applying Lemma 2.2 to the Følner sequence (F'_n) since $C_2 \setminus F'_{k_2} \subset \bigsqcup_{t\in T^{(1)}_2} C_1t \subset (F'_{k_2})^{+\widetilde{C_1}}$). Define

$$X_2 = \{x \in K^G \colon x_g = (x_2)_g \;\; \text{for all} \;\; g \in C_2 \setminus S(x_2)\}.$$

The induction step goes as follows. Let us suppose that step n was done, where $n \geq 2$. We will obtain x_{n+1} from x_n in a similar way as we obtained x_2 from x_1.

Step $n+1$: Let K_{δ_n} be a δ_n-dense finite subset of K. As G is infinite, there is $R_n \subset G$, $1_G \in R_n$, with cardinality $|R_n| = 1 + |K_{\delta_n}|^{|S(x_n)|}$ such that C_nr, $r \in R_n$, are disjoint. Observe that $|K_{\delta_n}|^{|S(x_n)|}$ represents the cardinality of all possibilities there exist for replacing all stars in the image of x_n with elements of K_{δ_n}; denote by $V_n \subset K^{C_n}$ the set of all these possibilities. Let $k_{n+1} > k_n$ be an integer large enough (it will be fixed later) so that $B_n = \bigsqcup_{r\in R_n} C_n r \subset F'_{k_{n+1}}$ and set $F_{n+1} = F'_{k_{n+1}}$. Define $A^{(0)}_{n+1}, A^{(1)}_{n+1}, \dots, A^{(n)}_{n+1} \subset G$ recursively by

$$A^{(0)}_{n+1} = F_{n+1}, \quad \text{and} \quad A^{(i)}_{n+1} = \widetilde{C}_i \widetilde{C}_{i-1} \dots \widetilde{C}_1 F_{n+1} \quad (= (A^{(i-1)}_{n+1})^{+\widetilde{C}_i})$$

for $i = 1, \dots, n$.

Let $(C_n t)_{t\in T'^{(n)}_{n+1}}$ be a maximal disjoint family in $A^{(n)}_{n+1} \setminus B_n$. Observe that $(C_n t)_{t\in T^{(n)}_{n+1}}$, where $T^{(n)}_{n+1} = T'^{(n)}_{n+1} \cup R_n$, is a maximal disjoint family in $A^{(n)}_{n+1}$. Then, recursively, for $i = 1, \dots, n-1$, choose a maximal disjoint family $(C_{n-i} t)_{t\in T^{(n-i)}_{n+1}}$ in

$$A^{(n-i)}_{n+1} \setminus \Bigg(\bigcup_{t\in T^{(n)}_{n+1}} C_n t \cup \bigcup_{t\in T^{(n-1)}_{n+1}} C_{n-1} t \cup \dots \cup \bigcup_{t\in T^{(n-i+1)}_{n+1}} C_{n-i+1} t \Bigg).$$

Define

$$C_{n+1} = F_{n+1} \cup \Bigg(\bigcup_{t\in T^{(n)}_{n+1}} C_n t \cup \bigcup_{t\in T^{(n-1)}_{n+1}} C_{n-1} t \cup \dots \cup \bigcup_{t\in T^{(1)}_{n+1}} C_1 t \Bigg).$$

Let x_{n+1} the element in $\widehat{K}^{C_{n+1}}$ defined by the following conditions:

- the restriction of x_{n+1} to $C_n g$ is equal to x_n for all $g \in T'^{(n)}_{n+1} \cup \{1_G\}$, i.e.
$$\pi_{C_n g}(x_{n+1}) = \Phi_{C_n,g}(x_n) \quad \text{for all } g \in T'^{(n)}_{n+1} \cup \{1_G\};$$
- for each $r \in R_n \setminus \{1_G\}$ choose an element $v \in V_n$ in a one to one correspondence and define the restriction of x_{n+1} to $C_n r$ by
$$\pi_{C_n r}(x_{n+1}) = \Phi_{C_n,r}(v);$$
- the restriction of x_{n+1} to $C_i g$ is equal to x_i for all $g \in T^{(i)}_{n+1}$, i.e.
$$\pi_{C_i g}(x_{n+1}) = \Phi_{C_i,g}(x_i) \quad \text{for all } g \in T^{(i)}_{n+1}$$
for all $i = 1, \dots, n-1$;
- otherwise define $(x_{n+1})_g = *$.

By Lemma 3.6 applied to $T_n = T'^{(n)}_{n+1}$, $T_i = T^{(i)}_{n+1}$ for $i \in \{1, \dots, n-1\}$ and $B = B_n$, we obtain

$$s(x_{n+1}) \geq \frac{|C_{n+1} \setminus B_n|}{|C_{n+1}|} m,$$

where $m = \min_{i=1,\dots,n} s(x_i) > \rho$ (since by induction hypothesis, we have $s(x_i) > \rho$ for $i = 1, \dots, n$). It follows that if we choose k_{n+1} large enough, the cardinality of C_{n+1} ($\supset F'_{k_{n+1}}$) will also be large enough so that

$$s(x_{n+1}) > \rho \tag{4.1}$$

and

$$|C_{n+1}^{+F_n}| \leq \left(1 + \frac{1}{n+1}\right) C_{n+1}. \tag{4.2}$$

The last condition is possible if we take k_{n+1} large enough by Lemma 2.2 since $C_{n+1} \setminus F'_{k_{n+1}} \subset (F'_{k_{n+1}})^{+\widetilde{C_n}}$.

Define

$$X_{n+1} = \{x \in K^G \colon x_g = (x_{n+1})_g \quad \text{for all} \quad g \in C_{n+1} \setminus S(x_{n+1})\}.$$

This finishes the construction of the sequence (X_n). Observe that (X_n) is a decreasing sequence of no empty subsets of K^G. Set $Y = \bigcap_n X_n$ and define $X \subset K^G$ by

$$X = \overline{\bigcup_{g \in G} gY}.$$

It is clear that X is a G-invariant compact metrisable space. It remains to show that $\operatorname{mdim}(X, G) \geq \rho \dim(K)$ and that the induced G-shift action on X is minimal.

This is the aim of the next two paragraphs.

4.2. Lower bound for the mean topological dimension of (X, G). Set $S_n = S(x_n)$ for all $n \in \mathbf{N}^*$. By definition of (X_n), the elements of $Y = \bigcap_n X_n$ are defined by

$$y \in Y \Leftrightarrow \pi_{C_n \setminus S_n}(y) = \pi_{C_n \setminus S_n}(x_n) \quad \text{for all} \quad n \geq 1. \tag{4.3}$$

Define $J = \bigcap_n (S_n \cup (G \setminus C_n))$. Observe that $J \cap C_n = S_n$ for all $n \geq 1$ and that (C_n) is an increasing Følner sequence for G (see inequality (4.2)) and $G = \bigcup_n C_n$. Choose an arbitrary element $y_0 \in Y$. Then, equation (4.3) says:

$$y \in Y \Leftrightarrow \pi_{G \setminus J}(y) = \pi_{G \setminus J}(y_0) \tag{4.4}$$

since the subsets $C_n \setminus S_n$ increase to $G \setminus J$. Using Proposition 2.8 and inequality (4.1), one obtains

$$\operatorname{mdim}(X, G) \geq \limsup_n \frac{|J \cap C_n|}{|C_n|} \dim(K) = \limsup_n \frac{|S_n|}{|C_n|} \dim(K) \geq \rho \dim(K)$$

if K is a polyhedron, and $\mathrm{mdim}(X,G)=\infty$ if $K=[0,1]^{\mathbf{N}}$ and $\rho>0$.

4.3. Proof of the minimality of X. The property satisfied by Y which implies minimality of $X=\overline{\bigcup_{g\in G} gY}$ is the following. Let $\epsilon>0$. Then, for all $y\in Y$ and for all finite subsets F of G, there exists a syndetic set $S\subset G$ such that $\pi_F(sy)$, $s\in S$, are ϵ-closed functions in K^F.

We begin by proving that the centers of the constructed translates of C_n are syndetic in G. Recall that each C_n, $n\geq 2$, is the union of F_n and of the union of the sets of the families $(C_1t)_{t\in T_n^{(1)}},\dots,(C_{n-1}t)_{t\in T_n^{(n-1)}}$ (disjoint families and the most part of these translates are contained in F_n).

Set $L_n^{(n-1)}=T_n^{(n-1)}$ and define for $i=1,\dots,n-2$

$$L_n^{(i)}=T_n^{(i)}\cup\bigcup_{j=i+1}^{n-1} L_j^{(i)}T_n^{(j)}.$$

It is convenient to think of the set $T_n^{(i)}$ as the set of centers of the (disjoint) translates C_it, $t\in T_n^{(i)}$, in C_n and of $L_j^{(i)}T_n^{(j)}$ as the set of the centers of the (disjoint) translates of C_i in $C_jt\subset C_n$ for $t\in T_n^{(j)}$, $i<j\leq n-1$. Observe that for a fixed i, the sets of the family $(C_it)_{t\in L_n^{(i)}}$ are disjoint in C_n by construction.

Define for all $k\in\mathbf{N}^*$

$$L^{(k)}=\bigcup_{n\geq k+1} L_n^{(k)}$$

(the set of all centers of the constructed translates of C_k in G). Note that it is an increasing union. The next lemma, together with Lemma 3.3, shows that $L^{(k)}$ is syndetic in G for all $k\geq 1$.

Lemma 4.1: *The family $(C_kt)_{t\in L^{(k)}}$ is a maximal disjoint family in G for all $k\in\mathbf{N}^*$. More precisely, the family $(C_kt)_{t\in L_n^{(k)}}$ is a maximal disjoint family in C_n for all $1\leq k\leq n-1$ and $n\geq 2$.*

Proof. We prove by induction on $n\geq 2$ that one has the implication

$$(H_n)\qquad (C_kg\cap C_kt=\varnothing\quad\forall t\in L_n^{(k)})\ \Rightarrow\ C_kg\cap C_n=\varnothing$$

for all $g\in G$ and $k\in\{1,\dots,n-1\}$. The result will then follows, according to Lemma 3.4. Property (H_2) comes from Lemma 3.5 since the family $(C_1t)_{t\in L_2^{(1)}}$ (here $L_2^{(1)}=T_2^{(1)}$) is a maximal disjoint family in $F_2^{+\widetilde{C}_1}$ and $C_2=F_2\cup\bigcup_{t\in T_2^{(1)}} C_1t$, by construction. Now suppose that (H_l) is true for

$l = 2, \ldots, n$ and let us prove (H_{n+1}). Let $g \in G$, $k \in \{1, \ldots, n\}$ and suppose

$$(4.5) \qquad C_k g \cap C_k t = \varnothing \quad \text{for all } t \in L_{n+1}^{(k)} = T_{n+1}^{(k)} \cup \bigcup_{j=k+1}^{n} L_j^{(k)} T_{n+1}^{(j)}.$$

Observe that the recurrence hypothesis (H_l) (for $l = 2, \ldots, n$) and assertion (4.5) imply in particular

$$(4.6) \qquad C_k g \cap C_j t = \varnothing \quad \text{for all } t \in T_{n+1}^{(j)} \text{ and } k \le j \le n.$$

Since one has

$$C_{n+1} = F_{n+1} \cup \left(\bigcup_{t \in T_{n+1}^{(n)}} C_n t \cup \bigcup_{t \in T_{n+1}^{(n-1)}} C_{n-1} t \cup \cdots \cup \bigcup_{t \in T_{n+1}^{(1)}} C_1 t \right),$$

it remains to show that $C_k g$ do not intersect F_{n+1} and do not intersect the sets $C_i t$, $t \in T_{n+1}^{(i)}$, for $i = 1, \ldots, k-1$. By definition, the family $(C_k t)_{t \in T_{n+1}^{(k)}}$ is a maximal disjoint family in

$$A_{n+1}^{(k)} \setminus \left(\bigcup_{t \in T_{n+1}^{(n)}} C_n t \cup \bigcup_{t \in T_{n+1}^{(n-1)}} C_{n-1} t \cup \cdots \cup \bigcup_{t \in T_{n+1}^{(k+1)}} C_{k+1} t \right)$$

where

$$A_{n+1}^{(k)} = (A_{n+1}^{(k-1)})^{+\widetilde{C}_k}.$$

It follows that $C_k g \cap A_{n+1}^{(k-1)} = \varnothing$. In fact, suppose that $C_k g \cap A_{n+1}^{(k-1)} \neq \varnothing$. Then $C_k g \subset C_k C_k^{-1} A_{n+1}^{(k-1)} = A_{n+1}^{(k)}$. Using (4.6) we obtain

$$C_k g \subset A_{n+1}^{(k)} \setminus \left(\bigcup_{t \in T_{n+1}^{(n)}} C_n t \cup \bigcup_{t \in T_{n+1}^{(n-1)}} C_{n-1} t \cup \cdots \cup \bigcup_{t \in T_{n+1}^{(k+1)}} C_{k+1} t \right).$$

This is in contradiction with the fact that $(C_k t)_{t \in T_{n+1}^{(k)}}$ is a maximal disjoint family in the right set of the above inclusion together with (4.5). In particular, $C_k g$ intersects neither F_{n+1}, nor $C_1 t$ (for $t \in T_{n+1}^{(1)}$), $\ldots$, $C_{k-1} t$ (for $t \in T_{n+1}^{(k-1)}$). One deduces $C_k g \cap C_{n+1} = \varnothing$. Hence (H_{n+1}). ∎

Now we are ready to prove the minimality of X. Observe first that by the definition of (X_n), all elements of Y coincide on $G \setminus J$ (see (4.4)). Let $y, y' \in Y$ and $\epsilon > 0$. We want to show that there is a syndetic set $S \subset G$ such that

$$(4.7) \qquad d(y, sy') < \epsilon \quad \text{for all} \quad s \in S.$$

In fact, choose an integer n such that $\epsilon_n < \epsilon$. By construction of x_{n+1} (see step $n+1$ of the construction), and since $C_n \supset F_n \supset F_n'$, it follows that there is an $r \in R_n$ such that

$$d(y, ry') \leq \epsilon_n$$

(since $\pi_{C_n}(y) = \pi_{C_n}(ry')$). Now, observe that $\pi_{C_n}(rty') = \pi_{C_n}(ry')$ for all $t \in L^{(n+1)}$ by construction (in fact all elements in Y coincide on $C_n rt \subset G \setminus J$, for all $r \in R_n$ and $t \in L^{(n+1)}$). We deduce inequality (4.7) with $S = rL^{(n+1)}$. But $L^{(n+1)}$ is syndetic in G by Lemma 4.1. Thus S is syndetic.

Fix $y_0 \in Y$. Inequality (4.7) proves, in particular, that all elements in Y are almost periodic points (take $y = y'$) but also that Y is contained in $\overline{Gy_0}$. Hence $X = \overline{\bigcup_{g \in G} gY} = \overline{Gy_0}$ which proves minimality of X by Lemma 3.1. This finishes the proof of Theorem 1.1.

To obtain Theorem 1.2, take $K = [0,1]^{\mathbf{N}}$ in the preceding construction to obtain a minimal G-space with infinite mean dimension and conclude by using Propositions 2.6 and 2.8.

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