Discretization of Compact Riemannian Manifolds Applied to the Spectrum of Laplacian

TATIANA MANTUANO⋆
*Université de Neuchâtel, Institut de Mathématiques, rue Emile-Argand 13, 2007 Neuchâtel,
Switzerland. e-mail: tatiana.mantuano@unine.ch*

Abstract. For $\kappa \geqslant 0$ and $r_0 > 0$ let $\mathcal{M}(n, \kappa, r_0)$ be the set of all connected, compact n-dimensional Riemannian manifolds (M^n, g) with Ricci $(M, g) \geqslant -(n-1)\kappa g$ and Inj $(M) \geqslant r_0$. We study the relation between the kth eigenvalue $\lambda_k(M)$ of the Laplacian associated to (M^n, g), $\Delta = -\text{div}(\text{grad})$, and the kth eigenvalue $\lambda_k(X)$ of a combinatorial Laplacian associated to a discretization X of M. We show that there exist constants $c, C > 0$ (depending only on n, κ and r_0) such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and X a discretization of M, $c \leqslant \frac{\lambda_k(M)}{\lambda_k(X)} \leqslant C$ for all $k < |X|$. Then, we obtain the same kind of result for two compact manifolds M and $N \in \mathcal{M}(n, \kappa, r_0)$ such that the Gromov–Hausdorff distance between M and N is smaller than some $\eta > 0$. We show that there exist constants $c, C > 0$ depending on η, n, κ and r_0 such that $c \leqslant \frac{\lambda_k(M)}{\lambda_k(N)} \leqslant C$ for all $k \in \mathbb{N}$.

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1. Introduction

Since the work of Buser in [6], it is known that in order to understand the spectrum of the Laplacian associated to a compact Riemannian manifold, $\Delta = -\text{div}(\text{grad})$, it may be very powerful to discretize the manifold. Using this technique, Buser considered manifolds with Ricci curvature and injectivity radius bounded below and gave an uniform estimate of the spectrum, depending only on these bounds (see Theorem 6.2 in [6]). The estimate of the kth eigenvalue turned out to be very precise for large k's (i.e. for k larger than a constant proportional to the volume of the manifold). However, for the beginning of the spectrum, the result is not strong enough to decide whether the eigenvalues may be close to zero or not.

Since, this question has been investigated by Brooks [2, 3], Burger [4], and Buser himself in [7]. In these papers, the manifolds were especially, closely related to Cayley graphs of groups or to Schreier graphs associated to a family of covering spaces (in another context see also the work of Kanai [13, 14]).

The point of view that will interest us here is the one taken up by Chavel in his book [8], where the question of discretization is very well explained and where he studies in particular the case of isoperimetric inequalities ([8], Chapter V) and Sobolev inequalities ([8], Chapter VI). This book will be the main reference for this paper.

The purpose of this note is to compare the spectrum of a compact Riemannian manifold (M^n, g) to the spectrum of the combinatorial Laplacian of an associated discretization X (defined as in [8]). More precisely, if $\mathcal{M}(n, \kappa, r_0)$ denotes the set of all compact n-dimensional Riemannian manifolds with Ricci curvature and injectivity radius uniformly bounded below (i.e. with Ricci $(M, g) \geqslant -(n-1)\kappa g$, $\kappa \geqslant 0$ and $\text{Inj}(M) \geqslant r_0 > 0$, see Definition 3.6), we will show in Theorem 3.7 that there exist positive constants c, C (depending only on n, κ, r_0) such that for all manifolds M in $\mathcal{M}(n, \kappa, r_0)$ and X a discretization associated to M, we have

$$c \leqslant \frac{\lambda_k(M)}{\lambda_k(X)} \leqslant C$$

for all $k < |X|$. Remark that $|X|$ behaves as the volume of M.

After defining precisely in Section 2 the notion of discretization and the Laplacian related to it, Section 3 will be concerned with the proof of this result.

In Section 4, as a corollary of our result, we present a very simple proof of Theorem 1 of Brooks in [2], which says that the first nonzero eigenvalue of a tower of covering spaces of a compact manifold goes to zero if and only if the Cheeger constant of the discretizations associated to the covering spaces does. In fact, we prove that the kth eigenvalue of a tower of covering spaces of a compact manifold goes to zero if and only if the kth eigenvalue of the discretizations associated to the covering spaces does, which implies obviously the result of Brooks (see Theorem 4.1). Note that our proof avoids integral geometry and some not obvious considerations on the boundary of Dirichlet's fundamental domains.

In Section 5, we compare the spectrum of two compact Riemannian manifolds $M \in \mathcal{M}(m, \kappa, r_0)$ and $N \in \mathcal{M}(n, \kappa, r_0)$ which are close with respect to the Gromov–Hausdorff distance (see Theorem 5.1). In particular, as $m = n$ we show in Corollary 5.2 that we have an uniform control $c \leqslant \frac{\lambda_k(M)}{\lambda_k(N)} \leqslant C$ for all k and where c, $C > 0$ depend on n, κ, r_0 and on the Gromov–Hausdorff distance between M and N (for the behaviour of the spectrum under convergence of manifolds with respect to the Gromov–Hausdorff distance see Section 7 of [10]).

We conclude this note with Section 6, where we give an example to show that the assumption on the injectivity radius is essential in Theorem 5.1; the spectra of two manifolds with Ricci curvature bounded below and arbitrarily Gromov–Hausdorff close may strongly differ.

2. Spectrum of Roughly Isometric Graphs

Let $X = (V, E)$ be a finite graph, where V denotes the set of vertices and E the set of edges, and consider the path metric on this graph so that it becomes a metric

space (see [8], p. 140). Denote by $N(v)$ the set of neighbours of $v \in V$, that is to say the set of vertices at distance 1 from v for the path metric. We will refer to $m(v)$ as the number of neighbours of $v \in V$ and to ν_X as an upper bound for $m(v)$ (i.e. ν_X is such that for all $v \in V$, $m(v) \leqslant \nu_X$).

For such a finite graph, we can define a combinatorial Laplacian as in [15] (Section 4.2) and the spectrum of this Laplacian will be denoted by

$$\mathrm{Spec}(X) = \{0 = \lambda_0(X) \leqslant \lambda_1(X) \leqslant \cdots \leqslant \lambda_l(X)\}$$

where $l = |X| - 1$ and $|X|$ denotes the number of vertices of the graph. We have the following variational characterization of $\mathrm{Spec}(X)$ (see [1], p. 268). For a function $f : V \to \mathbb{R}$, consider the Rayleigh quotient of f,

$$R(f) = \frac{\|\mathrm{d}f\|^2}{\|f\|^2}$$

where $\|\mathrm{d}f\|^2 = \sum_{v \in V} |\mathrm{d}f|^2(v) = \sum_{v \in V} \sum_{w \in N(v)} |f(w) - f(v)|^2$ and $\|f\|^2 = \sum_{v \in V} f^2(v)$. Then, for any $(k+1)$-dimensional vector subspace $W^{(k+1)}$ of the vector space $\mathcal{F}(V) = \{f : V \to \mathbb{R}\}$, the kth eigenvalue of X satisfies

$$\lambda_k(X) \leqslant \sup\{R(f) : f \in W^{(k+1)} , \ f \neq 0\}$$

So, if $\Lambda(W)$ denotes the supremum of Rayleigh quotients of nonzero functions in W (i.e. $\Lambda(W) = \sup\{R(f) : f \in W , \ f \neq 0\}$) and if E_{k+1} denotes the set of all $(k+1)$-dimensional vector subspaces of $\mathcal{F}(V)$, then we have

$$\lambda_k(X) = \inf\{\Lambda(W) : W \in E_{k+1}\}$$

Moreover, let us recall that a rough isometry is an application between metric spaces $\Phi : (X_1, d_1) \to (X_2, d_2)$ such that there exist some constants $a \geqslant 1, b \geqslant 0$, $\tau \geqslant 0$ satisfying

$$\forall x_1, y_1 \in X_1 , \quad a^{-1} d_1(x_1, y_1) - b \leqslant d_2(\Phi(x_1), \Phi(y_1)) \leqslant a d_1(x_1, y_1) + b$$

and $\cup_{x \in X_1} B(\Phi(x), \tau) = X_2$ (see [8], p. 142). The constants a, b and τ will be referred as the constants of rough isometry.

THEOREM 2.1. *Let X and Y be finite, connected graphs. Then, for each rough isometry between X and Y, there exist positive constants c and C depending only on ν_X, ν_Y and on the constants of rough isometry such that*

$$c \leqslant \frac{\lambda_k(X)}{\lambda_k(Y)} \leqslant C$$

for all $k < \min\{|X|, |Y|\}$.

Note that the constants c and C are independent of k, $|X|$ and $|Y|$.

Proof. It suffices to prove that it exists $C > 0$ such that

$$\lambda_k(X) \leqslant C \lambda_k(Y). \tag{1}$$

We proceed in two steps.

First, we show that it exists a constant $c' > 0$ independent of k, such that if $\lambda_k(Y) \leqslant c'$ then (1) is true for some $C > 0$. Let X and Y denote also the set of vertices of the respective graphs. Let $\Phi \colon X \to Y$ be a rough isometry. To each $f \colon Y \to \mathbb{R}$, we associate

$$\Phi^* f = f \circ \Phi \colon X \to \mathbb{R}$$

It can be shown that, there are positive constants c_1, c_2, c_3, c_4 depending only on ν_X, ν_Y and on the constants of rough isometry such that

$$\|\Phi^* f\|^2 \leqslant c_1 \|f\|^2, \tag{2}$$

$$\|d(\Phi^* f)\|^2 \leqslant c_2 \|df\|^2, \tag{3}$$

$$\|f\|^2 \leqslant c_3 \|df\|^2 + c_4 \|\Phi^* f\|^2 \tag{4}$$

(see [8], lemma VI.5.2 and VI.5.4).

Then, consider $f_0, \ldots, f_k \colon Y \to \mathbb{R}$ eigenfunctions associated to $\lambda_0(Y), \ldots,$ $\lambda_k(Y)$ and the corresponding functions on X, $\Phi^* f_0, \ldots, \Phi^* f_k \colon X \to \mathbb{R}$. Denote by V the subspace spanned by the f_i's, $V = \langle f_0, \ldots, f_k \rangle$ and by W the corresponding space, $W = \langle \Phi^* f_0, \ldots, \Phi^* f_k \rangle$.

If $\lambda_k(Y) \leqslant c' = (2c_3)^{-1}$ then W is of dimension $k + 1$: indeed, let $g = \sum_{i=0}^{k} a_i \Phi^* f_i$ with at least one nonzero a_i. In fact, g can be rewritten as $g = \Phi^* f$ with $f = \sum_{i=0}^{k} a_i f_i$. So, f is a nonzero function of V and satisfies $\|df\|^2 \leqslant \lambda_k(Y) \|f\|^2$. Then by (4)

$$\|g\|^2 = \|\Phi^* f\|^2 \geqslant c_4^{-1}(\|f\|^2 - c_3 \|df\|^2) \geqslant c_4^{-1} \|f\|^2 (1 - c_3 \lambda_k(Y)) \tag{5}$$

In particular, as $\lambda_k(Y) \leqslant c' = (2c_3)^{-1}$, the function g is not zero and this implies that W is $(k+1)$-dimensional. Moreover, under the same assumption on $\lambda_k(Y)$ and using (3) and (5), we have $R(\Phi^* f) \leqslant 2c_2 c_4 R(f)$. Finally, we get

$$\lambda_k(X) \leqslant \sup\{R(g) \mid g \in W - \{0\}\}$$
$$= \sup\{R(\Phi^* f) \mid f \in V - \{0\}\}$$
$$\leqslant 2c_2 c_4 \lambda_k(Y)$$

It remains to show that (1) is still true if $\lambda_k(Y) > c' = (2c_3)^{-1}$. But by this assumption, we have $\lambda_k(X) \leqslant \lambda_k(X) \lambda_k(Y) c'^{-1}$. So, in order to conclude, we need an upper bound on $\lambda_k(X)$. For each $f \colon X \to \mathbb{R}$, we have

$$\|df\|^2 = \sum_{x \in X} \sum_{y \in N(x)} |f(x) - f(y)|^2$$
$$\leqslant 2 \sum_{x \in X} \sum_{y \in N(x)} \left(|f(x)|^2 + |f(y)|^2 \right)$$
$$\leqslant 4 \nu_X \|f\|^2$$

This implies that $R(f) \leqslant 4\nu_X$ for all $f \neq 0$ and $\lambda_k(X) \leqslant 4\nu_X$. Finally, we get $\lambda_k(X) \leqslant 4\nu_X c'^{-1} \lambda_k(Y)$. □

3. Comparison of Spectra Between a Manifold and Its Discretization

Let (M^n, g) be a connected, compact Riemannian manifold. Consider the Laplacian associated to M

$$\Delta f = -\text{div}(\text{grad} f)$$

and denote its spectrum by

$$\text{Spec}(M) = \{0 = \lambda_0(M) < \lambda_1(M) \leqslant \ldots\}$$

The characterization of the eigenvalues we will use subsequently is given by Rayleigh quotients and Min-Max Theorem (see [16], p. 269, Min-Max Theorem). So for $F : M \to \mathbb{R}$ define the Rayleigh quotient of F to be

$$R(F) = \frac{\|\text{d}F\|^2}{\|F\|^2} = \frac{\int_M \|\text{d}F(x)\|^2 \text{d}V(x)}{\int_M F(x)^2 \text{d}V(x)}$$

where dV denotes the volume form on (M, g). Then, for any $(k + 1)$-dimensional vector subspace $W^{(k+1)}$ of the vector space $\mathcal{C}^\infty(M)$ of smooth functions on M, the kth eigenvalue of M satisfies

$$\lambda_k(M) \leqslant \sup\{R(F) : F \in W^{(k+1)}, F \neq 0\}$$

So, if $\Lambda(W)$ denotes the supremum of Rayleigh quotients of nonzero functions in W (i.e. $\Lambda(W) = \sup\{R(F) : F \in W, F \neq 0\}$) and if E_{k+1} denotes the set of all $(k + 1)$-dimensional vector subspaces of $\mathcal{C}^\infty(M)$, then we have

$$\lambda_k(M) = \inf\{\Lambda(W) : W \in E_{k+1}\}$$

Now, we associate a graph to a Riemannian manifold following [8] (Section V.3.2). Let (M^n, g) be a connected compact n-dimensional Riemannian manifold. A discretization of M, of mesh $\varepsilon > 0$, is a graph $X = (V, E)$ such that the set V of vertices is a maximal ε-separated subset of M (so it verifies that for any v, $w \in V$, $v \neq w$, we have $d(v, w) \geqslant \varepsilon$ and $\cup_{v \in V} B(v, \varepsilon) = M$). Moreover, the graph structure of X is entirely determined by the collection of neighbours that we define as follows. For each $v \in V$, $w \in V$ is a neighbour of v, if $0 < d(v, w) < 3\varepsilon$ (see [8], p. 140). We denote by $N(v)$ the set of neighbours of v.

Furthermore, X is roughly isometric to M (see [8], p. 147). So we will use on X the metric induced by M rather than the path metric.

Fix once and for all ε smaller than $\frac{1}{2}\text{Inj}(M)$. Denote by $\kappa \geqslant 0$ a constant such that $\text{Ricci}(M, g) \geqslant -(n - 1)\kappa g$. Then, by the Bishop–Gromov comparison theorem, v_X is bounded above by a constant depending only on n, κ and ε. So we can assume $v_X = v(n, \kappa, \varepsilon)$. Furthermore, using Croke's Inequality (cf. [8], p. 136) and Bishop's comparison theorem, we have

$$\frac{1}{V_{-\kappa}(\varepsilon)}\text{Vol}(M) \leqslant |V| \leqslant \frac{2^n}{\varepsilon^n c(n)}\text{Vol}(M)$$

where $V_{-\kappa}(\varepsilon)$ denotes the volume of the ball of radius ε in the simply connected space of constant sectional curvature $-\kappa$ and of dimension n.

The goal of this section is to compare $\mathrm{Spec}(X)$ and $\mathrm{Spec}(M)$ using the same idea as in the case of roughly isometric graphs. So we have to associate functions on M to functions on V and vice versa. This leads us to use the smoothing and the discretization applications considered in Chavel's book. Let us recall the main definitions and results from [8] (Section VI.5) we will need.

First, to go from the discretization $X = (V, E)$ to the manifold M, we need to smooth discrete functions.

DEFINITION 3.1.   Let $\{\phi_v\}_{v \in V}$ be a partition of unity subordinate to the cover $\{B(v, 2\varepsilon)\}_{v \in V}$ of M. Then for each $f : V \to \mathbb{R}$, the smoothing of f is defined by $F = \mathcal{S}f : M \to \mathbb{R}$

$$(\mathcal{S}f)(x) = \sum_{v \in V} \phi_v(x) f(v)$$

LEMMA 3.2.   *There exist positive constants c_1 and c_2 depending only on n, κ and on the mesh of the discretization such that*

$$\|\mathcal{S}f\|^2 \leqslant c_1 \|f\|^2, \tag{6}$$
$$\|\mathrm{d}(\mathcal{S}f)\|^2 \leqslant c_2 \|\mathrm{d}f\|^2. \tag{7}$$

Proof.   See [8], Section VI.5.2.   □

Then, to go in the other direction, we want to discretize smooth functions.

DEFINITION 3.3.   For each $F : M \to \mathbb{R}$ the discretization of F is defined by $f = \mathcal{D}F : V \to \mathbb{R}$

$$\mathcal{D}F(v) = \frac{1}{\mathrm{Vol}(B(v, 3\varepsilon))} \int_{B(v, 3\varepsilon)} F(x) \mathrm{d}V(x)$$

where $\mathrm{d}V$ denotes the volume form on (M, g).

LEMMA 3.4.   *There exist positive constants C_1 and C_2 depending only on n, κ and on the mesh of the discretization such that*

$$\|\mathcal{D}F\|^2 \leqslant C_1 \|F\|^2 \tag{8}$$
$$\|\mathrm{d}(\mathcal{D}F)\|^2 \leqslant C_2 \|\mathrm{d}F\|^2 \tag{9}$$

Proof.   See [8], Section VI.5.1.   □

Finally, compose $\mathcal{S}$ and $\mathcal{D}$ and look at how it differs from the identity.

LEMMA 3.5. *There exist positive constants c_3 and C_3 depending only on n, κ and on the mesh of the discretization such that*

$$\|f - \mathcal{DS}f\|^2 \leqslant c_3\|\mathrm{d}f\|^2 \tag{10}$$

$$\|F - \mathcal{SDF}\|^2 \leqslant C_3\|\mathrm{d}F\|^2 \tag{11}$$

Proof. see [8], Sections VI.5.4 and VI.5.5. □

We now state and prove the main technical theorem of this note.

DEFINITION 3.6. For $\kappa \geqslant 0$ and $r_0 > 0$ define $\mathcal{M}(n, \kappa, r_0)$ as the set of all connected, compact n-dimensional Riemannian manifolds (M^n, g) with Ricci curvature and injectivity radius uniformly bounded below, i.e. with $\mathrm{Ricci}(M, g) \geqslant -(n-1)\kappa g$ and $\mathrm{Inj}(M) \geqslant r_0$.

THEOREM 3.7. *There exist positive constants c, C such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2}r_0$), we have*

$$c \leqslant \frac{\lambda_k(M)}{\lambda_k(X)} \leqslant C$$

for all $k < |X|$.
The constants c and C depend only on n, κ and on the mesh of X.

In particular, if we fix the mesh equal to $\frac{1}{4}r_0$, then these constants depend only on the local geometry of M, i.e. on n, κ and r_0. Note moreover that all constants are independent of k.

Theorem 3.7 is a direct consequence of Theorem 3.8 and Theorem 3.9. In fact, the proof of Theorem 3.7 goes in two steps as for the discrete case. The first step deals in some sense with small eigenvalues; we show that Theorem 3.7 is true for eigenvalues smaller than some constant, using Chavel's results (see Theorem 3.8). The second step of the proof of Theorem 3.7 consists in showing that Theorem 3.7 is true even if the eigenvalues are 'big', that is to say bigger than the constants appearing in Theorem 3.8. The proof here is really different from the previous one and uses basic facts on eigenvalues of Laplacian (see Theorem 3.9).

THEOREM 3.8. *There exist positive constants a, A, c' and C' such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2}r_0$), we have*

(i) *If $\lambda_k(X) \leqslant a$, then $\dfrac{\lambda_k(M)}{\lambda_k(X)} \leqslant c'$,*

(ii) *If $\lambda_k(M) \leqslant A$, then $\dfrac{\lambda_k(X)}{\lambda_k(M)} \leqslant C'$*

for all $k < |X|$.
The constants a, A, c' and C' depend only on n, κ and on the mesh of X.

Proof. (i) The idea of the proof is exactly the same as in Theorem 2.1, using the above lemma in order to bound Rayleigh quotients. Let $M \in \mathcal{M}(n, \kappa, r_0)$ and $X = (V, E)$ a discretization with mesh $< \frac{1}{2}r_0$. Then, consider $f_0, \ldots, f_k : V \to \mathbb{R}$ eigenfunctions associated to the first $k + 1$ eigenvalues of X. Denote by W the subspace spanned by these eigenfunctions. Smooth each f_i to obtain $F_0 = \mathcal{S}f_0$, $\ldots, F_k = \mathcal{S}f_k : M \to \mathbb{R}$ and $\mathcal{S}W$ the subspace spanned by the F_i's.

Then, if $\lambda_k(X) < a = \frac{1}{4}c_3^{-1}$, $\mathcal{S}W$ is $(k + 1)$-dimensional. In order to prove this fact, let F be an element of $\mathcal{S}W$, $F = \sum_{i=0}^{k} a_i F_i$ with at least one non-zero coefficient. In fact, F is the smoothing of a discrete $f \in W$ such that $f = \sum_{i=0}^{k} a_i f_i$ and by (10), we have $\|f - \mathcal{DS}f\|^2 \leqslant c_3 \|df\|^2$. So the norm of F satisfies (by (8))

$$\|F\| \geqslant C_1^{-\frac{1}{2}} \|\mathcal{D}F\| \geqslant C_1^{-\frac{1}{2}}(\|f\| - \|f - \mathcal{DS}f\|) \geqslant C_1^{-\frac{1}{2}}\left(\|f\| - c_3^{\frac{1}{2}}\|df\|\right) \quad (12)$$

But f is a nonzero function belonging to the subspace W, so it satisfies $\|df\| \leqslant \sqrt{\lambda_k(X)}\|f\|$ and by assumption on $\lambda_k(X)$ and by (12), we get

$$\|F\| \geqslant \frac{1}{2}C_1^{-\frac{1}{2}}\|f\| \tag{13}$$

As the eigenfunctions of X are linearly independent, this shows that the dimension of $\mathcal{S}W$ is the same as the dimension of W, that is to say $k + 1$. Moreover, under the same assumption on $\lambda_k(X)$ and using (13), we obtain

$$R(\mathcal{S}f) \leqslant c'R(f)$$

for all $f \in W - \{0\}$. This leads now to the conclusion. Using the Min-Max Theorem, we have

$$\begin{aligned}
\lambda_k(M) &\leqslant \sup\{R(F)|F \in \mathcal{S}W - \{0\}\} \\
&\leqslant c' \sup\{R(f)|f \in W - \{0\}\} \\
&= c'\lambda_k(X)
\end{aligned}$$

and this ends the proof of the theorem. The second part of the proof can be carried out exactly in the same way, because of the symmetry of the results concerning the smoothing and the discretization of functions. So it would not bring more informations to do it here. $\square$

THEOREM 3.9. *Let a and A as in Theorem 3.8. Then, there exist positive constants c'' and C'' such that for all $M \in \mathcal{M}(n, \kappa, r_0)$ and for any discretization X of M (with mesh $< \frac{1}{2}r_0$), we have*

(i) *If $\lambda_k(X) > a$, then $\dfrac{\lambda_k(M)}{\lambda_k(X)} \leqslant c''$.*

(ii) *If $\lambda_k(M) > A$, then $\dfrac{\lambda_k(X)}{\lambda_k(M)} \leqslant C''$,*

for all $k < |X|$.

The constants c'' and C'' depend only on n, κ and on the mesh of X.

Proof. (i)   As $\lambda_k(X) > a$, then $\lambda_k(M) < a^{-1}\lambda_k(X)\lambda_k(M)$. So, it suffices to show that $\lambda_k(M) \leqslant \lambda_1^\kappa(\frac{\varepsilon}{2})$, where $\lambda_1^\kappa(\frac{\varepsilon}{2})$ denotes the first nonzero eigenvalue of the Dirichlet problem on the ball of radius $\frac{\varepsilon}{2}$ in the simply connected space of constant sectional curvature $-\kappa$ and of same dimension as M. We prove this result as follows. For each vertex v_i of X, $i \geqslant 1$, we can consider $f_i : M \to \mathbb{R}$ the first eigenfunction of the Dirichlet problem for the ball $\overline{B}(v_i, \frac{\varepsilon}{2})$ extended by 0 outside the ball. By Cheng's comparison Theorem $R(f_i) \leqslant \lambda_1^\kappa\left(\frac{\varepsilon}{2}\right)$ (see [9], p. 74).

Now, consider V_i the subspace spanned by $f_1, \ldots, f_i$. As the balls of radius $\varepsilon/2$ are disjoint, the f_i's are orthogonal and so V_i is of dimension i. We can apply Min-Max Theorem and get for all $k < |X|$

$$
\begin{aligned}
\lambda_k(M) &\leqslant \sup\{R(f) : f \in V_{k+1}, f \neq 0\} \\
&= \sup\left\{ \frac{\sum a_i^2 \|df_i\|^2}{\sum a_i^2 \|f_i\|^2} : f = \sum a_i f_i \in V_{k+1}, f \neq 0 \right\} \\
&\leqslant \sup\left\{ \frac{\sum a_i^2 \lambda_1^\kappa\left(\frac{\varepsilon}{2}\right)\|f_i\|^2}{\sum a_i^2 \|f_i\|^2} : f = \sum a_i f_i \in V_{k+1}, f \neq 0 \right\} \\
&\leqslant \lambda_1^\kappa\left(\frac{\varepsilon}{2}\right)
\end{aligned}
$$

Finally, we get $\lambda_k(M) < a^{-1}\lambda_1^\kappa(\frac{\varepsilon}{2})\lambda_k(X)$.

(ii) As $\lambda_k(M) > A$, then $\lambda_k(X) \leqslant A^{-1}\lambda_k(M)\lambda_k(X)$. We have seen in Theorem 2. that $\lambda_k(X) \leqslant 4\nu_X$. So we get $\lambda_k(X) \leqslant 4A^{-1}\nu_X\lambda_k(M)$. □

4. Application to the Spectrum of a Tower of Coverings

As a first application, we will discuss the following theorem.

THEOREM 4.1   *Let (M^n, g) be a compact n-dimensional Riemannian manifold. Let $\{M_i\}_{i \geqslant 1}$ be a family of finite-sheeted covering spaces of M with induced Riemannian metric. Let Γ_i be the Schreier graph of the subgroup $\pi_1(M_i)$ of $\pi_1(M)$. Then, there exist constants $c, C > 0$ such that for all $k < |\Gamma_i|$*

$$
c \leqslant \frac{\lambda_k(M_i)}{\lambda_k(\Gamma_i)} \leqslant C.
$$

In particular, for all k

$$
\lambda_k(M_i) \to 0 \quad when \quad i \to \infty \iff \lambda_k(\Gamma_i) \to 0 \quad when \quad i \to \infty.
$$

For $k = 1$, it is exactly the result of Brooks (see [2], Theorem 1).

We prove the result in two steps. First, we associate to M_i a discretization X_i of sufficiently small mesh, in order to compare $\text{Spec}(M_i)$ to $\text{Spec}(X_i)$ (as in Section 3). Secondly, we show that X_i and Γ_i are roughly isometric, which allows us to apply

Theorem 2.1 to $\mathrm{Spec}(\Gamma_i)$ and $\mathrm{Spec}(X_i)$, so that we obtain the desired result between $\mathrm{Spec}(M_i)$ and $\mathrm{Spec}(\Gamma_i)$.

The first step is really a direct application of Theorem 3.7 and is stated in Theorem 4.2.

THEOREM 4.2 *Let M and $\{M_i\}_{i \geqslant 1}$ be as in Theorem 4.1. Let X be a discretization of M (with mesh $< \frac{1}{2}\mathrm{Inj}(M)$) and lift it to M_i to obtain a discretization X_i of M_i. Then, there exist positive constants c and C independent of i such that for all $k < |X_i|$.*

$$c \leqslant \frac{\lambda_k(M_i)}{\lambda_k(X_i)} \leqslant C$$

In particular, for all k

$$\lambda_k(M_i) \to 0 \quad when \quad i \to \infty \iff \lambda_k(X_i) \to 0 \quad when \quad i \to \infty$$

Proof. If $(M^n, g) \in \mathcal{M}(n, \kappa, r_0)$, then $M_i \in \mathcal{M}(n, \kappa, r_0)$, as M_i is provided with the induced Riemannian metric. Moreover, X_i is a discretization of M_i with same mesh of X smaller than $\frac{1}{2}r_0$. So we can apply Theorem 3.7 to each pair (M_i, X_i) and get constants independent of i. $\qquad\square$

In the second step, we have to compare the Schreier graph Γ_i appearing in Theorem 4.1 to the discretization X_i of M_i appearing in Theorem 4.2. This is the next result.

THEOREM 4.3 *Let $\{(X_i, \Gamma_i)\}_{i \geqslant 1}$ be as in Theorems 4.1 and 4.2. Then, there exist positive constants c and C independent of i such that for all $k < |\Gamma_i|$*

$$c \leqslant \frac{\lambda_k(X_i)}{\lambda_k(\Gamma_i)} \leqslant C$$

In particular, for all k

$$\lambda_k(X_i) \to 0 \quad when \quad i \to \infty \iff \lambda_k(\Gamma_i) \to 0 \quad when \quad i \to \infty$$

Proof. Geometrically, the graph Γ_i corresponds to the lift of a graph Γ in M, where Γ consists of a unique point (see [1], p. 254, for a definition of Schreier graphs). As M is compact, Γ and X are roughly isometric (see [8], p. 147). Let us call Γ, Γ_i and X, X_i the set of vertices of the respective graphs too and $\Phi : \Gamma \to X$ a rough isometry such that $d(g, \Phi(g)) < \varepsilon$. We can lift Φ and get $\Phi_i : \Gamma_i \to X_i$ in the following way. If $g \in \Gamma_i$, then by construction of X_i, there exists $\Phi_i(g) = x$ with $\pi_i(x) = \Phi(\pi_i(g))$ and $d(x, g) < \varepsilon$ (where π_i denotes the canonical projection of M_i onto M). Clearly, Φ_i is a rough isometry with same constants of rough isometry as Φ. So we can apply Theorem 2.1 to each pair (X_i, Γ_i) and get constants independent of i. $\qquad\square$

Note that, as any two discretizations of a compact manifold are roughly isometric, then we can replace Γ_i by the lift of any discretization of M and Theorem 4.1 is still true.

5. Gromov–Hausdorff Close Manifolds have Comparable Spectra

Another application of Theorem 3.7 is the following result.

THEOREM 5.1. *Let $(M^m, g_M) \in \mathcal{M}(m, \kappa, r_0)$ and $(N^n, g_N) \in \mathcal{M}(n, \kappa, r_0)$. Suppose that the Gromov–Hausdorff distance between M and N is smaller than $\eta > 0$. Then, there exist positive constants c and C (depending only on η, κ, r_0 and on the dimensions) and $K > 0$ (proportional to the volume of M and N) such that for all $k < K$*

$$c \leqslant \frac{\lambda_k(M)}{\lambda_k(N)} \leqslant C$$

and there exist c' and C' (depending on the dimensions, η, κ and r_0) such that for all $k \geqslant K$

$$c' \leqslant \frac{\lambda_k(M)^m}{\lambda_k(N)^n} \leqslant C'$$

COROLLARY 5.2. *Let M and N be two same dimensional compact Riemannian manifolds, i.e. $M, N \in \mathcal{M}(n, \kappa, r_0)$. Suppose that the Gromov–Hausdorff distance between M and N is smaller than $\eta > 0$. Then, there exist positive constants c and C (depending only on η, n, κ and r_0) such that for all k*

$$c \leqslant \frac{\lambda_k(M)}{\lambda_k(N)} \leqslant C$$

Proof of Theorem 5.1. Recall (from Section 3) that if X_M is a discretization of $M \in \mathcal{M}(m, \kappa, r_0)$ of mesh $\varepsilon < \frac{1}{2}r_0$, then

$$\frac{1}{V_{-\kappa}(\varepsilon)} \mathrm{Vol}(M) \leqslant |X_M| \leqslant \frac{2^m}{\varepsilon^m c(m)} \mathrm{Vol}(M)$$

Let X_M be a discretization of $M \in \mathcal{M}(m, \kappa, r_0)$ and X_N a discretization of $N \in \mathcal{M}(n, \kappa, r_0)$ with same mesh $\varepsilon < \frac{1}{2}r_0$. Moreover, choose $\varepsilon < \frac{1}{2}r_0$ such that

$$\min\{|X_M|, |X_N|\} \geqslant \max\left\{ \mathrm{Vol}(M)\frac{2^{m+2}}{c(m)r_0^m}, \mathrm{Vol}(N)\frac{2^{n+2}}{c(n)r_0^n} \right\}$$

(ε depend only on m, n, κ and r_0).

Then, the proof is done in two steps. First, if $k < \min\{|X_M|, |X_N|\}$, we will apply Theorem 2.1 and Theorem 3.7 (as will follow) and second, for $k \geqslant \min\{|X_M|, |X_N|\}$, we will use a result of Buser (Theorem 6.2 in [6]).

Consider the case $k < \min\{|X_M|, |X_N|\}$. By definition of the Gromov–Hausdorff distance (see [12]) there exist Z a Riemannian manifold and two isometric embeddings $f: M \to Z$, $g: N \to Z$ such that $\cup_{x \in M} B(f(x), \eta) \supset g(N)$ and $\cup_{y \in N} B(g(y), \eta) \supset f(M)$. Then, X_M and X_N are roughly isometric via $\Phi: X_M \to X_N$ defined as follows. For each $x \in X_M$ there exist $x' \in N$ and $x'' \in X_N$ such that $d(f(x), g(x')) < \eta$ and $d(x', x'') < \varepsilon$. Then define $\Phi(x) = x''$ so that $d(f(x), g(\phi(x))) < \eta + \varepsilon$. Then, we clearly have that

$$d(x, y) - 2(\eta + \varepsilon) < d(\Phi(x), \Phi(y)) < d(x, y) + 2(\eta + \varepsilon)$$

Moreover, if $z \in X_N$ then there exist $y \in M$ and $x \in X_M$ such that $d(g(z), f(y)) < \eta$ and $d(x, y) < \varepsilon$. This implies that

$$d(\Phi(x), z) \leqslant d(g(\Phi(x)), f(x)) + d(f(x), f(y)) + d(f(y), g(z)) < 2(\eta + \varepsilon)$$

and so $\cup_{x \in X_M} B(\Phi(x), 2(\eta + \varepsilon)) \supset X_N$. Note that the constants of rough isometry depend only on ε and η. To conclude this first part, it suffices to apply Theorem 3.7 to (M, X_M) and (N, X_N) and Theorem 2.1 to (X_M, X_N).

Now, consider the case $k \geqslant \min\{|X_M|, |X_N|\}$. In this case and by assumption on ε, we can apply Theorem 6.2 of [6] to M and N which leads to the result

$$\left(\frac{k}{\mathrm{Vol}(M)}\right)^{\frac{2}{m}} c_1 \leqslant \lambda_k(M) \leqslant \left(\frac{k}{\mathrm{Vol}(M)}\right)^{\frac{2}{m}} c_2,$$

where c_1 and c_2 are constants depending on m, κ and r_0. Similarly,

$$\left(\frac{k}{\mathrm{Vol}(N)}\right)^{\frac{2}{n}} c_3 \leqslant \lambda_k(N) \leqslant \left(\frac{k}{\mathrm{Vol}(N)}\right)^{\frac{2}{n}} c_4,$$

where c_3 and c_4 are constants depending on n, κ and r_0. Putting both inequalities together, we get that there exist constants c_5 and c_6 depending only on the dimensions, κ and r_0 such that

$$c_5 \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(N)}\right)^2 \leqslant \frac{\lambda_k(M)^m}{\lambda_k(N)^n} \leqslant c_6 \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(N)}\right)^2$$

Furthermore, we have seen in Section 3 that there exist constants c_7, c_8, c_9 and c_{10} depending only on the dimensions, κ and r_0 such that

$$c_7 |X_M| \leqslant \mathrm{Vol}(M) \leqslant c_8 |X_M|,$$
$$c_9 |X_N| \leqslant \mathrm{Vol}(N) \leqslant c_{10} |X_N|.$$

As we have shown in the first part of the proof the discretizations are roughly isometric (the constants of rough isometry depend only on ε and η). This implies that there exist constants c_{11} and c_{12} depending on ε and η such that

$$c_{11} \leqslant \frac{|X_M|}{|X_N|} \leqslant c_{12}.$$

These last inequalities give us upper and lower bounds (depending only on η, m, n, κ and r_0) for $\mathrm{Vol}(M)/\mathrm{Vol}(N)$.

Finally, we get that there exist constants $c, C > 0$ depending only on η, m, n, κ and r_0 such that for all k

$$c \leqslant \frac{\lambda_k(M)^m}{\lambda_k(N)^n} \leqslant C.$$

In particular, the constants do not depend on k. $\qquad\square$

6. An Example

Let us now discuss an example in order to show that the assumption on the injectivity radius in Theorem 5.1 is necessary. Consider M_ε the manifold obtained by taking two hyperbolic cylinders, gluing them together at both ends and 'smoothing the angles'.

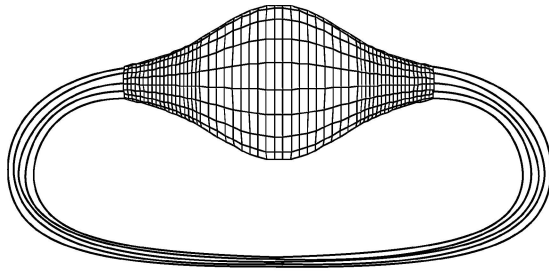More precisely, for $0 < \varepsilon < 1$ take

$$M_\varepsilon = [-1 - \rho_\varepsilon; 1 + \rho_\varepsilon] \times \mathbb{S}^1/_\sim$$

where $\sim$ identifies the ends (i.e. $(\rho_1, t_1) \sim (\rho_2, t_2)$ if and only if $\rho_1 = \rho_2 = \pm(1+\rho_\varepsilon)$ and $t_1 = t_2$) and $\rho_\varepsilon = \mathrm{arccosh}(\varepsilon^{-\frac{1}{2}})$. Provide M_ε with the Riemannian metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + f(\rho)^2 \mathrm{d}t^2$$

where

$$f(\rho) = \begin{cases} \varepsilon \cosh(\rho + \rho_\varepsilon + 1) & \text{if } \rho \in [-1 - \rho_\varepsilon; -1], \\ \frac{\sqrt{\varepsilon}}{8}\left((1+\sqrt{1-\varepsilon})\rho^4 - 2(1+3\sqrt{1-\varepsilon})\rho^2 + 9 + 5\sqrt{1-\varepsilon}\right) & \text{if } \rho \in [-1; 1], \\ \varepsilon \cosh(-\rho + \rho_\varepsilon + 1) & \text{if } \rho \in [1; 1 + \rho_\varepsilon]. \end{cases}$$

The manifold M_ε

Then, M_ε has sectional curvature uniformly bounded (i.e. independently of ε) and injectivity radius comparable to ε. Moreover, as M_ε admits an involution and if $D_\varepsilon = \{(\rho, t) : \rho \in [-1 - \rho_\varepsilon; 0], t \in \mathbb{R}/\mathbb{Z}\}$, then $\lambda_1(M_\varepsilon)$ is either the first nonzero eigenvalue of D_ε for the Neumann problem $\lambda_1^N(D_\varepsilon)$ or the first eigenvalue of D_ε for the Dirichlet problem $\lambda_1^D(D_\varepsilon)$, where D_ε is provided with the same

Riemannian metric as M_ε. But D_ε can be provided with an hyperbolic metric $ds'^2 = d\rho^2 + \varepsilon^2 \cosh^2(\rho + \rho_\varepsilon + 1)dt^2$ and we can easily show that there are positive constants c_1 and c_2 independent of ε such that

$$c_1 ds^2 \leqslant ds'^2 \leqslant c_2 ds^2.$$

Then, applying a result of Dodziuk (see [11], Proposition 3.3), we get a constant c_3 independent of ε such that

$$\lambda_1^{N,D}(D_\varepsilon, ds^2) \geqslant c_3 \lambda_1^{N,D}(D_\varepsilon, ds'^2)$$

But, we know that for thin hyperbolic cylinders $\lambda_1^{N,D}(D_\varepsilon, ds'^2) \geqslant \frac{1}{4}$ (see [5], p. 35). Finally, we have shown that $\lambda_1(M_\varepsilon) \geqslant c > 0$ where c is independent on ε.

Moreover, M_ε is Hausdorff–Gromov close to the circle S_ε of length $2\rho_\varepsilon + 2$ and $\lambda_1(S_\varepsilon)$ goes to zero when ε goes to zero too. So

$$\frac{\lambda_1(M_\varepsilon)}{\lambda_1(S_\varepsilon)} \to \infty \quad \text{when } \varepsilon \to 0$$

Then, this quotient cannot be bounded and the theorem is not true for the family $(M_\varepsilon, S_\varepsilon)$, because the injectivity radius of M_ε is not uniformly bounded below.

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