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On SLE Martingales in Boundary WZW Models

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Abstract. Following Bettelheim et al. (Phys Rev Lett 95:251601, 2005), we consider the boundary WZW model on a half-plane with a cut growing according to the Schramm–Loewner stochastic evolution and the boundary fields inserted at the tip of the cut and at infinity. We study necessary and sufficient conditions for boundary correlation functions to be SLE martingales. Necessary conditions come from the requirement for the boundary field at the tip of the cut to have a depth two null vector. Sufficient conditions are established using Knizhnik–Zamolodchikov equations for boundary correlators. Combining these two approaches, we show that in the case of $G=SU(2)$ the boundary correlator is an SLE martingale if and only if the boundary field carries spin 1/2. In the case of $G=SU(n)$ and the level $k=1$, there are several situations when boundary one-point correlators are SLE$_\kappa$-martingales. If the boundary field is labelled by the defining n-dimensional representation of $SU(n)$, we obtain $\varkappa=2$. For n even, by choosing the boundary field labelled by the (unique) self-adjoint fundamental representation, we get $\varkappa=8/(n+2)$. We also study the situation when the distance between the two boundary fields is finite, and we show that in this case the SLE$_\varkappa$ evolution is replaced by SLE$_{\varkappa,\rho}$ with $\rho=\varkappa-6$.

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0. Introduction

Random conformally invariant curves often appear in the scaling limit of interfaces in 2D statistical models at critical points, (see [2,4,8,16] for reviews) and such curves, if they have a Markov property, are described by the Schramm–Loewner evolution (SLE). Specifically, let a random conformally invariant Markov curve γ_t start at the origin of the upper half plane $\mathbb{H}$. The parameter $t \geq 0$ can be regarded as the time of evolution. The seminal result of Schramm [14] states that the dynamics of the tip z_t of the curve is given by the law $z_t = g_t^{-1}(\sqrt{\varkappa}\xi_t)$, where $g_t(z)$ is the uniformizing conformal map which maps the slit domain $\mathbb{H}/\gamma_t$ back to $\mathbb{H}$ and which satisfies the following stochastic differential equation:

$$\mathrm{d}g_t(z) = \frac{2\mathrm{d}t}{g_t(z) - \sqrt{\varkappa}\xi_t}, \qquad g_0(z) = z. \tag{1}$$

Here ξ_t is the normalized Brownian process on $\mathbb{R}$, starting at the origin, i.e., $\xi_0 = 0$, and $\mathbb{E}[d\xi_t \mathrm{d}\xi_t] = \mathrm{d}t$. The parameter $\varkappa > 0$ is the diffusion coefficient of the Brownian motion, and thus it is also an important parameter of the SLE trace.

The interplay between SLE and boundary conformal field theory has been studied in detail in the case of minimal models [1] (see also [2] for a review). Consider the boundary minimal model $\mathcal{M}(p, p')$ (p and p' are co-prime integers such that $p' > p \geq 2$) on the slit domain $\mathbb{H}/\gamma_t$, where γ_t is an SLE trace. Insert the boundary changing operators, ϕ and $\phi^\dagger$, at the tip z_t of γ_t and at $z = \infty$, respectively. This insertion introduces two different boundary conditions, one on the semi-axis from $-\infty$ to z_t, and the other one on the semi-axis from z_t to $+\infty$. Let $\mathcal{O}$ stand for a set of primary operators at fixed points in the bulk. It was observed in [1] that the normalized boundary correlation function

$$\mathcal{M}_t = \frac{\langle \phi(z_t) \mathcal{O} \phi^\dagger(\infty) \rangle}{\langle \phi(z_t) \phi^\dagger(\infty) \rangle} \tag{2}$$

is an SLE martingale. That is, it is conserved in mean under SLE$_\varkappa$, $\mathbb{E}[\frac{d}{dt}\mathcal{M}_t] = 0$, provided that $\varkappa = 4p'/p$ or $\varkappa = 4p/p'$ and ϕ is the primary operator $\phi_{1,2}^{p,p'}$ or $\phi_{2,1}^{p,p'}$, respectively.

Because analytic properties of CFT correlation functions are well understood (see, e.g. [7]), existence of martingales of type (2) can be exploited in computation of various SLE related probabilities, see e.g. [2]. This is a motivation to search for new martingales in non-minimal boundary CFTs. For instance, the case of models with parafermionic symmetry was considered in [13]. For the $SU(2)$ WZW model, some results in this direction were obtained in [3,11]. The aim of this paper is to better understand and extend the results of [3].

The paper is organized as follows. In Section 1, we show that a boundary correlation function of the WZW model with a boundary field ϕ_Λ inserted at the tip of an SLE trace is an SLE$_\varkappa$ martingale if a certain descendant of ϕ_Λ is a level two null vector with respect to the Kac-Moody algebra $\hat{\mathfrak{g}}_k$. In comparison to the minimal models, one has to assume in addition that the evolution of the SLE trace is accompanied with a random gauge transformation of the bulk fields [3]. The randomness of the gauge transformation is described by a Brownian motion on the group with a coupling constant τ. This is an additional parameter which must be adjusted to the value of $\varkappa$.

In Section 2, we analyse necessary conditions for the null vector ensuring the martingale property of the correlation function. We show that, for a given Lie algebra $\mathfrak{g}$, these conditions are satisfied for more than two different values of k (and thus there can be more than two different values of $\varkappa$) only if $\dim \mathfrak{g} = 3$.

Furthermore, for $\mathfrak{g}=\mathfrak{su}(2)$, we show that Λ must be the fundamental representation (i.e. corresponding to spin 1/2), and $\varkappa=\frac{4(k+2)}{k+3}$ unless $k=1$ (if $k=1$, $\varkappa$ is not fixed). This confirms the conclusions of [3]. For $\mathfrak{g}=\mathfrak{su}(n)$ with $n>2$, we show that when Λ is the fundamental representation, the necessary conditions imply $k=1$ and $\varkappa=2$. For non-fundamental representations Λ and $k=1$, the necessary conditions imply $\varkappa=\frac{8}{n+2}$ provided that the Casimir operator C_Λ acquires a certain value. We show that this condition holds for all even n for a self-conjugate Λ of a specific form.

In Section 3, we use the Knizhnik–Zamolodchikov equations to derive a sufficient condition ensuring the martingale property. More precisely, we show that the correlation function is a martingale if it is contained in the kernel of a certain matrix. For $\mathfrak{g}=\mathfrak{su}(2)$ and $k>1$, we observe that under the necessary conditions of Section 2 the matrix in question vanishes, and the necessary conditions turn out to be sufficient.

In Section 4, we consider explicit expressions for boundary correlation functions with one bulk field. We study the situation when the $\mathfrak{g}$-invariant submodule is two-dimensional, but the corresponding space of conformal blocks is one-dimensional due to the fusion rules at the level $k=1$. We show that, for the weights Λ allowed by the necessary conditions and the corresponding values of $\varkappa$ found in Section 2, the one-point boundary correlators are indeed SLE_κ martingales.

In Section 5, we consider the case when the second boundary operator is inserted at a finite distance from the origin. We show that the corresponding boundary correlator is an $\mathrm{SLE}_{\varkappa,\rho}$ martingale if $\rho=\varkappa-6$ and the null vector condition of Section 2 holds. We use the KZ equation to derive a sufficient condition similar to that found in Section 3.

1. SLE Martingales in WZW

Let $\mathfrak{g}$ be a simple Lie algebra. We study the boundary $\hat{\mathfrak{g}}_k$ WZW model on the slit domain $\mathbb{H}/\gamma_t$, where γ_t is an $\mathrm{SLE}_\varkappa$ trace. Consider a boundary correlation function with N primary fields in the bulk, where the field $\phi_{\lambda_i}(z_i)$ $(i=1,\ldots,N,$ $\Im(z_i)>0)$ has a conformal weight h_i and carries an irreducible $\mathfrak{g}$ representation of a highest weight λ_i. The boundary condition changing operators, ϕ_Λ and ϕ_{Λ^*}, are inserted at the tip z_t of γ_t and at $z=\infty$. The boundary correlation function [5] for this set of fields is a certain chiral conformal block (the choice of a particular conformal block depends on the boundary conditions) for the theory on the complex plane $\mathbb{C}$ with additional primary fields corresponding to conjugate representations λ_i^* placed at the mirror image points $\bar{z}_i$,

$$\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}\equiv\frac{\langle\phi_\Lambda(z_t)\phi_{\lambda_1}(z_1)\ldots\phi_{\lambda_N}(z_N)\,\phi_{\lambda_1^*}(\bar{z}_1)\ldots\phi_{\lambda_N^*}(\bar{z}_N)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}{\langle\phi_\Lambda(z_t)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}. \tag{3}$$

Here, the numerator takes values in the $\mathfrak{g}$-invariant subspace of the tensor product $V_\Lambda\otimes V_{\lambda_1}\otimes\ldots\otimes V_{\Lambda^*}$. The denominator takes values in the $\mathfrak{g}$-invariant subspace of

$V_\Lambda \otimes V_{\Lambda^*}$, which by the Schur's lemma is one-dimensional, and so the denominator is a scalar. The $\mathfrak{g}$-invariance of the correlation function is expressed by the equation,

$$\left(\sum_{i=0}^{2N+1} t_i^a\right)\langle\phi_{\{\lambda\}}\{z_i\}\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}=0, \tag{4}$$

where t^a form an orthonormal basis of $\mathfrak{g}$, and t_i^a is a matrix representing t^a in the ith tensor factor in the representation of a highest weight λ_i, t_0^a acts on $\phi_\Lambda(z_t)$ and t_{2N+1}^a acts on $\phi_{\Lambda^*}(\infty)$.

It is convenient to introduce the conformal map $w_t(z)=g_t(z)-\sqrt{\varkappa}\xi_t$. The dynamics of the tip of the SLE trace is then given by $z_t=w_t^{-1}(0)$. The map $w_t(z)$ satisfies the stochastic differential equation:

$$\mathrm{d}w_t(z)=\frac{2\mathrm{d}t}{w_t(z)}-\sqrt{\varkappa}\mathrm{d}\xi_t \tag{5}$$

with initial condition $w_0(z)=z$. The map $w_t(z)$ maps the initial configuration of fields on the slit domain $\mathbb{H}/\gamma_t$ into a configuration on the upper half plane $\mathbb{H}$. The boundary condition changing operators ϕ_Λ and ϕ_{Λ^*} are now inserted at $w=0$ and $w=\infty$, and the bulk primary fields ϕ_{λ_i} are positioned at the points $w_i\equiv w_t(z_i)$ which are moving as t increases. For the theory on $\mathbb{H}$, it is well known [5] that the mirror images of bulk fields are located at the complex conjugate points, that is $w_{i+N}=\overline{w_i}, i=1,\ldots,N$. Note that solutions of Equation (5) satisfy the reflection property, $\overline{w_t(z)}=w_t(\bar{z})$. Therefore, in (3), we also have pairs of conjugate points $z_i, \bar{z}_i$.

Transforming a primary field ϕ of conformal weight h by conformal map $w_t(z)$ yields a factor $\left(\frac{\partial w}{\partial z}\right)^h$, hence we can rewrite the correlation function (3) in the new coordinates:

$$\langle\phi_{\{\lambda\}}\{z_i\}\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}=\left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\langle\phi_{\{\lambda\}}\{w_i\}\rangle_{\Lambda,0}^{\Lambda^*,\infty}, \tag{6}$$

where $w_{i+N}=\bar{w}_i, z_{i+N}=\bar{z}_i$.

Let us determine the increment of (6) when t is increased by $\mathrm{d}t$. Equation (5) implies that the prefactor changes as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial w_i}{\partial z_i}\right)^h=h\left(\frac{\partial w_i}{\partial z_i}\right)^{h-1}\partial_t\frac{\partial w_i}{\partial z_i}=h\left(\frac{\partial w_i}{\partial z_i}\right)^{h-1}\partial_{z_i}\left(\frac{2}{w_i}\right)=-\frac{2h}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^h. \tag{7}$$

If we were considering a minimal model, the increment of a bulk field would have been given by

$$\mathrm{d}\phi_{\lambda_i}(w_i)=\mathcal{G}_i\phi_{\lambda_i}(w_i), \tag{8}$$

where, by (5), we would have had $\mathcal{G}_i=\mathrm{d}w_i\partial_{w_i}=(\frac{2\mathrm{d}t}{w_i}-\sqrt{\varkappa}\mathrm{d}\xi_t)\partial_{w_i}$. In the case of the WZW model, the fields are Lie group valued, and one can introduce an additional random motion in the target space. The following modification was proposed in [3]:

$$\mathcal{G}_i = \left(\frac{2\mathrm{d}t}{w_i} - \sqrt{\varkappa}\mathrm{d}\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\sum_{a=1}^{\dim\mathfrak{g}}\left(\mathrm{d}\theta^a\, t_i^a\right), \tag{9}$$

where $\mathrm{d}\theta^a$ are normalized generators of a $\mathbb{R}^{\dim\mathfrak{g}}$-valued Brownian motion, i.e.,

$$\mathbb{E}(\mathrm{d}\theta^a\mathrm{d}\theta^b) = \delta_{ab}\,\mathrm{d}t. \tag{10}$$

Note that a Brownian motion on a Lie group G is defined by the following stochastic differential equation:

$$\mathrm{d}g = \left(\sqrt{\tau}\sum_a \mathrm{d}\theta^a t^a + \frac{\tau}{2}\sum_a t^a t^a\,\mathrm{d}t\right)g. \tag{11}$$

Here, the second term on the right hand side is taking care of the exponential map between the Lie algebra and the Lie group. For instance, in the case of $G = U(1)$, we have $g = \exp(i\sqrt{\tau}\xi_t)$, where ξ_t is the one-dimensional Brownian motion. Then, using the standard Ito calculus, we obtain

$$\mathrm{d}g = \left(i\sqrt{\tau}\mathrm{d}\xi_t - \frac{\tau}{2}\,\mathrm{d}t\right)g.$$

Equation (11) suggests the following alternative writing of Equation (9):

$$\mathcal{G}_i = \mathrm{d}t\left(\frac{2}{w_i}\partial_{w_i} - \frac{\tau C_i}{2w_i^2}\right) - \sqrt{\varkappa}\mathrm{d}\xi_t\partial_{w_i} + \left(\frac{\sqrt{\tau}}{w_i}\sum_a \mathrm{d}\theta^a\, t_i^a + \frac{\tau}{2w_i^2}\sum_a t_i^a t_i^a\,\mathrm{d}t\right), \tag{12}$$

where C_i is the value of the quadratic Casimir operator $\sum_a t_i^a t_i^a$ is the representation with highest weight λ_i. In operator (12), the first two terms correspond to the SLE developing on the upper half-plane, and the third term describes the Brownian motion on the group.

Returning to the analysis of the boundary correlation function, let us introduce the following operator:

$$\Theta = \sum_{i=1}^{2N}\left(\frac{2}{w_i}\partial_{w_i} - \frac{2h_i}{w_i^2}\right) + \frac{\varkappa}{2}\sum_{i,j=1}^{2N}\partial_{w_i}\partial_{w_j} + \frac{\tau}{2}\sum_{i,j=1}^{2N}\frac{T_{ij}}{w_i w_j}, \tag{13}$$

where $T_{ij} = T_{ji} \equiv \sum_a t_i^a\, t_j^a$. Let us show that the correlator $\langle\phi_{\{\lambda\}}\{z_i\}\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}$ is an SLE$_\varkappa$ martingale if and only if its w-image is annihilated by Θ,

$$\Theta\,\langle\phi_{\{\lambda\}}\{w_i\}\rangle_{\Lambda,0}^{\Lambda^*,\infty} = 0. \tag{14}$$

Indeed, substituting (9) in (8), using the Ito formula, and taking into consideration (6) and (7), we find

$$\begin{aligned}
&\left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{-h_i}\right)\mathbb{E}\left[\mathrm{d}\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}\right]\\
&=-\sum_{i=1}^{2N}\frac{2h_i\mathrm{d}t}{w_i^2}\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}+\mathbb{E}\left[\mathrm{d}\left\langle\phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}\right]\\
&=\left(-\sum_{i=1}^{2N}\frac{2h_i\mathrm{d}t}{w_i^2}+\mathbb{E}\left[\sum_{i=1}^{2N}\mathcal{G}_i+\frac{1}{2}\sum_{i,j=1}^{2N}\mathcal{G}_i\mathcal{G}_j\right]\right)\left\langle\phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}\\
&=\mathrm{d}t\,\Theta\left\langle\phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}.
\end{aligned}\tag{15}$$

Recall (see, e.g., [7]) that, if $X=\prod_i\phi_i(w_i)$, then $\langle(L_{-n}\phi)(z)X\rangle=\mathcal{L}_{-n}\langle\phi(z)X\rangle$ for $n\geq 1$ and $\langle(J_{-n}^a\phi)(z)X\rangle=\mathcal{J}_{-n}^a\langle\phi(z)X\rangle$ for $n\geq 0$, where

$$\mathcal{L}_{-n}=\sum_i\left(\frac{(n-1)h_i}{(w_i-z)^n}-\frac{1}{(w_i-z)^{n-1}}\partial_{w_i}\right),\qquad \mathcal{J}_{-n}^a=-\sum_i\frac{t_i^a}{(w_i-z)^n}.\tag{16}$$

Therefore, the martingale condition (14) can be rewritten as follows:

$$\begin{aligned}
0&=\left(-2\mathcal{L}_{-2}+\frac{1}{2}\varkappa\mathcal{L}_{-1}^2+\frac{1}{2}\tau\sum_a\mathcal{J}_{-1}^a\mathcal{J}_{-1}^a\right)\left\langle\phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}\\
&=\frac{\langle\psi(0)\phi_{\lambda_1}(w_1)\ldots\phi_{\lambda_N^*}(w_{2N})\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}{\langle\phi_\Lambda(0)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}},
\end{aligned}\tag{17}$$

where

$$\psi=\left(-2L_{-2}+\frac{1}{2}\varkappa L_{-1}^2+\frac{1}{2}\tau\sum_{a=1}^{\dim\mathfrak{g}}J_{-1}^aJ_{-1}^a\right)\phi_\Lambda.\tag{18}$$

Thus, a sufficient condition for the correlation function in question to be a covariant $\mathrm{SLE}_\varkappa$ martingale is the requirement that ψ be a level two null vector.

2. Null Vectors and Necessary Conditions

In this section, we analyse in detail the null vector property of ψ defined by (18). It is equivalent to two equations, $J_1^a\psi=0$ and $J_2^a\psi=0$. Recall that the Kac-Moody and Virasoro generators satisfy the following commutation relations:

$$[L_m,L_{m'}]=(m-m')L_{m+m'}+\frac{c}{12}m(m^2-1)\delta_{m+m',0},\tag{19}$$

$$[L_m,J_{m'}^a]=-m'J_{m+m'}^a,\tag{20}$$

$$[J_m^a,J_{m'}^b]=\sum_c if_{abc}J_{m+m'}^c+km\delta_{ab}\delta_{m+m',0},\tag{21}$$

where k is the level and c is the central charge given by

$$c = \frac{k \dim \mathfrak{g}}{k + h^V}. \tag{22}$$

Here h^V is the dual Coxeter number of the Lie algebra $\mathfrak{g}$.

Acting with J_1^b, and J_2^b on (18), we obtain

$$\left(\left(\tau k - \tau h^V - 2 \right) J_{-1}^b + \varkappa J_0^b L_{-1} + i\tau \sum_{a,c} f_{abc} J_0^a J_{-1}^c \right) \phi_\Lambda = 0, \tag{23}$$

$$\left(\varkappa + \tau h^V - 4 \right) J_0^b \phi_\Lambda = 0. \tag{24}$$

Here we used that $\sum_{a,c} f_{bac} f_{dac} = 2h^V \delta_{bd}$. Equations (23)–(24) define a necessary and sufficient conditions for ψ given by (18) to be a null vector. Equation (24) implies that (here we assume $\Lambda \neq 0$)

$$\varkappa + \tau h^V = 4. \tag{25}$$

Equation (23) is more involved. However, acting on it with L_1, we derive the following (simpler) necessary condition:

$$(2\varkappa h_\Lambda + \tau k - 2)\, J_0^b \phi_\Lambda = 0. \tag{26}$$

Another necessary condition can be obtained by requiring $L_2 \psi = 0$, which yields

$$\left(3\varkappa h_\Lambda + \frac{1}{2} \tau c (k + h^V) - 8 h_\Lambda - c \right) \phi_\Lambda = 0. \tag{27}$$

In (26) and (27), it was used that $L_0 \phi_\Lambda = h_\Lambda \phi_\Lambda$. Recall that the conformal dimension h_Λ is given by

$$h_\Lambda = \frac{C_\Lambda}{2(k + h^V)}, \tag{28}$$

where C_Λ is the value of the Casimir operator $C = \sum_a t^a t^a$ in the irreducible representation of a highest weight Λ.

For $k \neq 2 h_\Lambda h^V$, Equations (25)–(26) imply that

$$\varkappa = \frac{2(h^V - 2k)}{2 h_\Lambda h^V - k}, \qquad \tau = \frac{8 h_\Lambda - 2}{2 h_\Lambda h^V - k}. \tag{29}$$

Substituting (22) and (28)–(29) in (27), we arrive at the following condition

$$\begin{aligned} &\left(h^V \dim \mathfrak{g} + 2 C_\Lambda (1 - \dim \mathfrak{g}) \right) k^2 \\ &\quad + \left(h^V \dim \mathfrak{g} - C_\Lambda (1 + \dim \mathfrak{g}) \right) h^V k + 4 C_\Lambda^2 h^V - 3 C_\Lambda (h^V)^2 = 0. \end{aligned} \tag{30}$$

Let us analyse Equations (25)–(27) and (30) in some particular cases.

0. For $k=2h_\Lambda h^V$, formulae (29) do not apply. In this case, Equations (25) and (26) are linearly dependent and they have a solution only if

$$k=\frac{1}{2}h^V, \qquad h_\Lambda=\frac{1}{4}, \qquad C_\Lambda=\frac{3}{4}h^V. \tag{31}$$

Note that the condition $C_\Lambda=\frac{3}{4}h^V$ cannot hold for $\mathfrak{g}=\mathfrak{su}(n), n>2$. Under conditions (31), Equation (27) is equivalent to

$$(3\varkappa-8)(\dim\mathfrak{g}-3)=0. \tag{32}$$

For $\mathfrak{g}=\mathfrak{su}(2)$, this relation holds for any $\varkappa$, and the condition $C_\Lambda=\frac{3}{4}h^V$ implies that Λ is the representation of spin 1/2. Thus, the case $\mathfrak{g}=\mathfrak{su}(2), k=1$, Λ being the fundamental representation is very degenerate, the parameter $\varkappa$ is not fixed and the only relation imposed on $\varkappa$ and τ is Equation (25).

1. $\mathfrak{g}=\mathfrak{su}(2), h^V=2$. Let Λ be a representation of spin j. We have $h_\Lambda=j(j+1)/(k+2)$, and then (30) is equivalent to the condition $(2j-1)(2j+3)(2j-k)(k+2j+2)=0$. That is, either $j=1/2$ or $k=2j$. The latter possibility is actually excluded since it corresponds to the case of $k=2h_\Lambda h^V$. Thus, Λ must be the fundamental representation, i.e. of spin 1/2. Then, for $k\neq 1$, Equation (29) yield

$$\varkappa=\frac{4(k+2)}{k+3}, \qquad \tau=\frac{2}{k+3}. \tag{33}$$

Note that $\mathfrak{g}=\mathfrak{su}(2)$ is the only case, when, for a given C_Λ, condition (30) can hold for more than two different values of k (and thus for any k). Indeed, the polynomial in k given by the l.h.s. of (30) is identically zero only if

$$h^V\dim\mathfrak{g}=2C_\Lambda(\dim\mathfrak{g}-1), \qquad \text{and} \qquad \dim\mathfrak{g}=3. \tag{34}$$

For a simple Lie algebra, the second condition implies that $\mathfrak{g}=\mathfrak{su}(2)$. Then, the first condition implies that Λ is the representation of spin $j=1/2$.

2a. $\mathfrak{g}=\mathfrak{su}(n), h^V=n>2$. Let Λ be the fundamental representation. We have $C_\Lambda=(n^2-1)/n$, and (30) is equivalent to the condition $(k^2-1)(n^2-1)(n^2-4)=0$. Whence, for $n>2$, the only possibility is $k=1$. In this case, (29) yields

$$\varkappa=2, \qquad \tau=\frac{2}{n}. \tag{35}$$

2b. $\mathfrak{g}=\mathfrak{su}(n), h^V=n>2$. Consider the case of $k=1$. Then, (30) is satisfied either if $C_\Lambda=(n^2-1)/n$ (and we recover the case 2a), or if

$$C_\Lambda=\frac{n(n+1)}{4}. \tag{36}$$

This condition holds for self-conjugate representations whose Dynkin labels are $\Lambda \sim (0,1,0), \Lambda \sim (0,0,1,0,0)$, etc. Here n is required to be even. In this case, we have $h_\Lambda = n/8$, and (29) yields

$$\varkappa = \frac{8}{n+2}, \qquad \tau = \frac{4}{n+2}. \tag{37}$$

It is interesting that the set of values of $\varkappa$ in Equation (37) does not meet the set of values of $\varkappa$ in Equation (33). Moreover, $\varkappa$'s of Equation (37) are not contained in the set corresponding to the minimal model $M(p,p')$. Indeed, (37) matches $\varkappa = 4p/p'$ for $p=2$ and $p' = n+2$, but p' must be co-prime with p, which is not the case when n is even.

2c. $\mathfrak{g} = \mathfrak{su}(n), h^\vee = n > 2$. If Λ is the adjoint representation, $\Lambda \sim (1,0,\ldots,0,1)$, then $C_\Lambda = 2n$. In this case, (30) is equivalent to the condition $(3n^2-7)k^2 + n(n^2+1)k - 10n^2 = 0$. For $n>2$, the only positive integer solution is $n=7, k=1$. However, for $k=1$ the adjoint representation does not satisfy the integrability constraint, $|\Lambda| \le k = 1$.

3. KZ Equations and Sufficient Conditions

In this section we obtain and study sufficient conditions for a correlation function to be an SLE martingale.

Below we will use the notation $\nu \equiv 1/(k+h^\vee)$. Recall that $T_{ij} \equiv \sum_a t_i^a t_j^a$. Using that $T_{ii}\phi_{\lambda_i} = (2h_i/\nu)\phi_{\lambda_i}$, we can rewrite (15) as follows:

$$\Theta = \left(\sum_{i=1}^{2N} \left(\frac{2}{w_i} \partial_{w_i} - \frac{2H_i}{w_i^2} \right) + \frac{\varkappa}{2} \sum_{i,j=1}^{2N} \partial_{w_i} \partial_{w_j} + \tau \sum_{i<j}^{2N} \frac{T_{ij}}{w_i w_j} \right) \tag{38}$$

where

$$H_i = h_i \left(1 - \frac{\tau}{2\nu} \right) \tag{39}$$

are renormalized conformal weights. Note that the renormalization of conformal weights is similar to the redistribution of terms in the operator $\mathcal{G}_i$ in Equation (12).

Correlation functions of the WZW model satisfy the Knizhnik–Zamolodchikov (KZ) equations [9]. In our case, they read

$$\partial_{w_i} \left\langle \phi_{\{\lambda\}}\{w_i\} \right\rangle_{\Lambda,0}^{\Lambda^*,\infty} = \nu \left(\frac{T_{0i}}{w_i} + \sum_{j \ne i}^{2N} \frac{T_{ij}}{w_i - w_j} \right) \left\langle \phi_{\{\lambda\}}\{w_i\} \right\rangle_{\Lambda,0}^{\Lambda^*,\infty}. \tag{40}$$

Hence, we have

$$\sum_{i,j=1}^{2N} \partial_{w_i}\partial_{w_j} \left\langle \phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty} = \left(\nu^2 \sum_{i,j=1}^{2N} \frac{T_{0i}T_{0j}}{w_i w_j} - \nu \sum_{i=1}^{2N} \frac{T_{0i}}{w_i^2} \right) \left\langle \phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}, \tag{41}$$

$$\sum_{i=1}^{2N} \frac{1}{w_i}\partial_{w_i} \left\langle \phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty} = \nu \left(\sum_{i=1}^{2N} \frac{T_{0i}}{w_i^2} - \sum_{i<j}^{2N} \frac{T_{ij}}{w_i w_j} \right) \left\langle \phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty}. \tag{42}$$

Applying the operator (38) to the correlation function and using these identities, we rewrite the martingale condition (14) as an algebraic equation:

$$\mathcal{M}(\{w_i\}) \left\langle \phi_{\{\lambda\}}\{w_i\}\right\rangle_{\Lambda,0}^{\Lambda^*,\infty} = 0, \tag{43}$$

where

$$\mathcal{M}(\{w_i\}) = \sum_{i=1}^{2N} \frac{A_i}{w_i^2} + \sum_{i<j}^{2N} \frac{B_{ij}}{w_i w_j}, \tag{44}$$

$$A_i = (4-\varkappa)\, T_{0i} + \varkappa\nu\, (T_{0i})^2 + \frac{1}{\nu} h_i \left(\frac{2}{\nu}\tau - 4 \right), \tag{45}$$

$$B_{ij} = \left(\frac{2}{\nu}\tau - 4 \right) T_{ij} + \varkappa\nu \left(T_{0i}T_{0j} + T_{0j}T_{0i} \right). \tag{46}$$

Thus, the boundary correlation function in question is a martingale if it is in the kernel of the matrix $\mathcal{M}(\{w_i\})$.

Recall that, for $\mathfrak{g} = \mathfrak{su}(2)$, the necessary conditions require Λ to be the representation of spin 1/2 and, for $k \neq 1$,

$$\varkappa = \frac{4}{\nu+1}, \qquad \tau = \frac{2\nu}{\nu+1}, \tag{47}$$

where $\nu = 1/(k+2)$. The properly normalized generators of Λ are $t_0^a = \frac{1}{\sqrt{2}}\sigma^a$, where σ^a are the Pauli matrices. Using their properties, we readily derive the following identities:

$$(T_{0i})^2 = \frac{1}{2}(\sigma^a t_i^a)(\sigma^b t_i^b) = \frac{1}{2}(t_i^a t_i^a - \sqrt{2}\sigma^a t_i^a) = \frac{1}{\nu} h_i - T_{0i}, \tag{48}$$

$$T_{0i}T_{0j} + T_{0j}T_{0i} = \frac{1}{2}(\sigma^a\sigma^b + \sigma^b\sigma^a)\, t_i^a t_j^b = T_{ij}. \tag{49}$$

Substituting (47)–(49) in (45) and (46), we obtain

$$A_i = 0, \qquad B_{ij} = 0. \tag{50}$$

Hence, in this case, $\mathcal{M}(\{w_i\})$ vanishes identically. In conclusion, we have proved that for $\mathfrak{g} = \mathfrak{su}(2)$ and $k \neq 1$ a boundary correlation function is an SLE$_\varkappa$ martingale if and only if the boundary field is in the fundamental representation, and $\varkappa$

and τ are given by (47). We will show below that in the special case of $k=1$ it is possible that $\mathcal{M}(\{w_i\})$ does not vanish but has a non-empty kernel and a certain conformal block lies in the kernel.

Relations (50) are sufficient but not necessary conditions for the martingale property. In fact, their weaker form, $A_i^{\mathfrak{g}}=0$ and $B_{ij}^{\mathfrak{g}}=0$, is also a sufficient condition. Here the superscript $\mathfrak{g}$ denotes the projection onto the $\mathfrak{g}$-invariant subspace of $V_\Lambda \otimes V_{\lambda_1} \otimes \ldots \otimes V_{\Lambda^*}$. However, even this form produces a very restrictive condition. In particular, $A_i^{\mathfrak{g}}=0$ implies that $T_{0i}^{\mathfrak{g}}$ has at most two distinct eigenvalues. In the $\mathfrak{su}(2)$ case, this is true since $V_{1/2} \otimes V_j = V_{j-1/2} \oplus V_{j+1/2}$.

Consider the case $\mathfrak{g}=\mathfrak{su}(n), n>2$, Λ being the fundamental representation. If $\lambda_i=\Lambda$ or $\lambda_i=\Lambda^*$, then T_{0i} has exactly two distinct eigenvalues (cf. Section 4.1): $\left(\frac{n-1}{n}, \frac{-n-1}{n}\right)$ and $\left(\frac{1}{n}, \frac{1-n^2}{n}\right)$, respectively. On the other hand, for Λ being the fundamental representation in the $n>2$ case, the necessary conditions imply $k=1$, and $\varkappa$ and τ must be given by Equation (35). For these data, equation $A_i^{\mathfrak{g}}=0$ implies that the eigenvalues of $T_{0i}^{\mathfrak{g}}$ are equal to $\left(\frac{1-n^2}{n}, \frac{-n-1}{n}\right)$. Thus, $\mathcal{M}(\{w_i\})$ does not vanish. However, similarly to the $\mathfrak{su}(2)$ case, we will show that $\mathcal{M}(\{w_i\})$ may have a non-empty kernel containing a certain conformal block.

4. One-Point Boundary Correlators for $k=1$

In this section, we consider explicit expressions for boundary one-point correlation functions (that is, the case of $N=1$). The normalized correlator is of the form,

$$\langle \phi_\lambda(w) \rangle_{\Lambda,0}^{\Lambda^*,\infty} \equiv \frac{\langle \phi_\Lambda(0)\phi_\lambda(w)\phi_{\lambda^*}(\bar{w})\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}{\langle \phi_\Lambda(0)\phi_{\Lambda^*}(\infty)\rangle^{\mathfrak{g}}}. \tag{51}$$

Recall that the $SL(2,\mathbb{C})$ invariance implies that (see, e.g. [7])

$$\langle \phi_\lambda(w) \rangle_{\Lambda,0}^{\Lambda^*,\infty} = (\bar{w})^{-2h_\lambda} \mathcal{F}(x), \qquad x = w/\bar{w}, \tag{52}$$

where $\mathcal{F}(x)$ belongs to the $\mathfrak{g}$-invariant submodule $W^{\mathfrak{g}}$ of $V_\Lambda \otimes V_\lambda \otimes V_{\lambda^*} \otimes V_{\Lambda^*}$. Substituting this expression in the KZ equations (40) (the variables $w_1=w$ and $w_2=\bar{w}$ are regarded as independent) yields

$$\left(\frac{1}{\nu}\partial_x - \frac{T_{01}}{x} - \frac{T_{12}}{x-1}\right)\mathcal{F}(x)=0, \tag{53}$$

$$\left(\frac{1}{\nu}x\partial_x + \frac{2h_\lambda}{\nu} + T_{02} - \frac{T_{12}}{x-1}\right)\mathcal{F}(x)=0. \tag{54}$$

For $N=1$, it can be derived from (4) that

$$\left(T_{01}+T_{02}+T_{12}+\frac{2}{\nu}h_\lambda\right)\langle \phi_\lambda(w) \rangle_{\Lambda,0}^{\Lambda^*,\infty}=0. \tag{55}$$

Therefore, Equations (53) and (54) are equivalent and it suffices to consider only the first of them.

For $N = 1$, the martingale condition (43) reads:

$$\mathcal{M}(x)\,\mathcal{F}(x) = 0, \qquad \mathcal{M}(x) = A_1 + x^2 A_2 + x B_{12}, \tag{56}$$

where A_1, A_2, and B_{12} are given by (45) and (46). We will analyse condition (56) for the weights Λ allowed by the necessary conditions (see the cases 0–2b) in Section 2), and for the weights λ such that the submodule $W^{\mathfrak{g}}$ be two-dimensional. Recall (see Section 2) that $k > 1$ implies $\mathfrak{g} = \mathfrak{su}(2)$, and this case was analysed in detail in Section 3. For this reason, we will restrict our consideration to the case of $k = 1$. Although $W^{\mathfrak{g}}$ is assumed to be two-dimensional, the space of conformal blocks for $k = 1$ is one-dimensional due to the fusion rules (for a recent account, see [12]).

4.1. $\mathfrak{g} = \mathfrak{su}(n)$, n-DIMENSIONAL REPRESENTATION

Let $\mathfrak{g} = \mathfrak{su}(n), n \geq 2$, and let Λ be the n-dimensional defining representation. If λ coincides with Λ or Λ^*, then $W^{\mathfrak{g}}$ is two-dimensional. For definiteness, we take $\lambda = \Lambda^*$.

If a pair of vectors, v_1 and v_2, forms a basis of $W^{\mathfrak{g}}$, then a solution to the KZ equation (53) has the form $\mathcal{F}(x) = F_1(x)v_1 + F_2(x)v_2$, where $F_1(x)$ and $F_2(x)$ are scalar functions. We take $v_1 = \varepsilon_{12}\varepsilon_{03}$, where ε is the normalised basis vector of the trivial representation appearing in the decomposition of $V_\Lambda \otimes V_{\Lambda^*}$. We choose such v_2 that the pair v_1, v_2 forms an orthonormal basis. Recall that $T_{ij} = T_{ij}^*$. Therefore, $T_{12}^{\mathfrak{g}}$ is represented by a diagonal matrix, and $T_{01}^{\mathfrak{g}}, T_{02}^{\mathfrak{g}}$ are represented by symmetric matrices. Their eigenvalues can be found using the following formula:

$$T_{ij} = \frac{1}{2}\left(C_{ij} - C_i - C_j\right). \tag{57}$$

Since $V_\Lambda \otimes V_\Lambda = V(2, 0, \ldots) \oplus V(0, 1, 0, \ldots)$, the eigenvalues of $T_{02}^{\mathfrak{g}}$ are $\left(\frac{n-1}{n}, \frac{-n-1}{n}\right)$. Since $V_\Lambda \otimes V_{\Lambda^*} = V(0, \ldots, 0) \oplus V(1, 0, \ldots, 0, 1)$, the eigenvalues of $T_{01}^{\mathfrak{g}}$ and $T_{12}^{\mathfrak{g}}$ are $\left(\frac{1-n^2}{n}, \frac{1}{n}\right)$. This, along with relation (55), determines entries of the sought matrices uniquely (up to the sign of the off-diagonal entries of $T_{01}^{\mathfrak{g}}, T_{02}^{\mathfrak{g}}$ which can be reverted by changing $v_2 \to -v_2$):

$$T_{01} = -\frac{1}{n}\begin{pmatrix} 0 & \sqrt{n^2-1} \\ \sqrt{n^2-1} & n^2-2 \end{pmatrix}, \qquad T_{02} = \frac{1}{n}\begin{pmatrix} 0 & \sqrt{n^2-1} \\ \sqrt{n^2-1} & -2 \end{pmatrix},$$
$$T_{12} = \frac{1}{n}\begin{pmatrix} 1-n^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{58}$$

Here, we use the identification $v_1 \sim \binom{1}{0}, v_2 \sim \binom{0}{1}$.

4.1.1. $n=2$

Substituting (58) for $n=2$ in (56) and setting $\tau=2-\varkappa/2$ according to (25), we obtain the following matrix $\mathcal{M}(x)$ for $N=1, k=1$:

$$\mathcal{M}(x)=(3-\varkappa)\begin{pmatrix} 2(x-1)^2 & \frac{2}{\sqrt{3}}(x^2-1) \\ \frac{2}{\sqrt{3}}(x^2-1) & \frac{2}{3}(x+1)^2 \end{pmatrix}. \tag{59}$$

For $\varkappa\neq 3$ and generic x, the rank of $\mathcal{M}(x)$ is one. The eigenvector corresponding to the zero eigenvalue is

$$\mathcal{F}_0(x)=(x+1)v_1+\sqrt{3}\,(1-x)v_2. \tag{60}$$

Using again (58), it is straightforward to check that

$$\mathcal{F}(x)=(x(1-x))^{-\frac{1}{2}}\,\mathcal{F}_0(x) \tag{61}$$

satisfies the KZ equation (53). Note that $-\frac{1}{2}=-2h_\lambda$ is consistent with (52). Thus, we conclude that, in the $\mathfrak{su}(2)_1$ case, the boundary one-point correlator is an SLE$_\varkappa$ martingale for any $\varkappa$.

4.1.2. $n>2$

Substituting (58) in (56) and setting $\varkappa=2, \tau=2/n$ according to (35), we obtain the following matrix $\mathcal{M}(x)$ for $N=1, k=1$:

$$\mathcal{M}(x)=\frac{2(n^2+n-2)}{n^2}\begin{pmatrix} (x-1)^2 & \frac{(x-1)(nx-x+1)}{\sqrt{n^2-1}} \\ \frac{(x-1)(nx-x+1)}{\sqrt{n^2-1}} & \frac{(nx-x+1)^2}{n^2-1} \end{pmatrix}. \tag{62}$$

For $n>1$ and generic x, the rank of $\mathcal{M}(x)$ is one. The eigenvector corresponding to the zero eigenvalue is

$$\mathcal{F}_0(x)=(nx-x+1)v_1+\sqrt{n^2-1}\,(1-x)v_2. \tag{63}$$

Using (58), it is straightforward to check that

$$\mathcal{F}(x)=(x(1-x))^{\frac{1}{n}-1}\,\mathcal{F}_0(x) \tag{64}$$

satisfies the KZ equation (53). Note that $\frac{1}{n}-1=-2h_\lambda$ is consistent with (52). We conclude that the boundary one-point correlator is an SLE$_2$ martingale (recall that the space of conformal blocks is one-dimensional, cf. Theorem 4.7 in [12]).

4.2. $\mathfrak{g}=\mathfrak{su}(n)$, SELF-ADJOINT REPRESENTATION

Let $\mathfrak{g}=\mathfrak{su}(n), n\geq 2$ is even, and let $\Lambda=\Lambda^*=\omega_{n/2}$ (ω_i denotes the ith fundamental weight, $\omega_i^*=\omega_{n-i}$). If λ or λ^* is equal to ω_1 (the highest weight of the defining

n-dimensional representation), then $W^{\mathfrak{g}}$ is two-dimensional. Indeed, we have

$$\begin{aligned} &V\left(\omega_{\frac{n}{2}}\right)\otimes V(\omega_1)=V\left(\omega_{\frac{n}{2}+1}\right)\oplus V\left(\omega_1+\omega_{\frac{n}{2}}\right),\\ &V(\omega_{n-1})\otimes V\left(\omega_{\frac{n}{2}}\right)=V\left(\omega_{\frac{n}{2}-1}\right)\oplus V\left(\omega_{\frac{n}{2}}+\omega_{n-1}\right), \end{aligned} \tag{65}$$

where $V(\omega)$ is the irreducible representation of highest weight ω. In the decomposition of the tensor product of the l.h.s., the trivial representation appears once in the product of the first terms and once in the product of the second terms on the r.h.s. For definiteness, we take λ to be the fundamental representation.

As a basis of $W^{\mathfrak{g}}$ we take $v_1=\varepsilon_{12}\varepsilon'_{03}$ (ε and ε' are the normalised basis vectors of the trivial representation appearing in the decomposition of $V_\lambda\otimes V_{\lambda^*}$ and $V_\Lambda\otimes V_\Lambda$, respectively) and such v_2 that the basis be orthonormal. Then, $T^{\mathfrak{g}}_{12}$ is represented by a diagonal matrix, and $T^{\mathfrak{g}}_{01}, T^{\mathfrak{g}}_{02}$ are represented by symmetric matrices. Their eigenvalues are found from (57) and (65), and they are equal to $\left(\frac{1-n^2}{n},\frac{1}{n}\right)$ for $T^{\mathfrak{g}}_{12}$, and $\left(\frac{-n-1}{2},\frac{1}{2}\right)$ for $T^{\mathfrak{g}}_{01}$ and $T^{\mathfrak{g}}_{02}$. This, along with relation (55), determines entries of the sought matrices uniquely (up to the sign of the off-diagonal entries of $T^{\mathfrak{g}}_{01}, T^{\mathfrak{g}}_{02}$ which can be reverted by changing $v_2\to -v_2$):

$$\begin{aligned} &T_{01}=-\frac{1}{2}\begin{pmatrix}0 & \sqrt{n-1}\\ \sqrt{n-1} & n\end{pmatrix},\quad T_{02}=\frac{1}{2}\begin{pmatrix}0 & \sqrt{n-1}\\ \sqrt{n-1} & -n\end{pmatrix},\\ &T_{12}=\frac{1}{n}\begin{pmatrix}1-n^2 & 0\\ 0 & 1\end{pmatrix}. \end{aligned} \tag{66}$$

Substituting (66) in (56) and setting $\varkappa=8/(n+2), \tau=4/(n+2)$ according to (37), we obtain the following matrix $\mathcal{M}(x)$ for $N=1, k=1$:

$$\mathcal{M}(x)=\frac{2n^2}{n+2}\begin{pmatrix}(x-1)^2 & \frac{x^2-1}{\sqrt{n+1}}\\ \frac{x^2-1}{\sqrt{n+1}} & \frac{(x+1)^2}{n+1}\end{pmatrix}. \tag{67}$$

For generic x, the rank of $\mathcal{M}(x)$ is one. The eigenvector corresponding to the zero eigenvalue is

$$\mathcal{F}_0(x)=(x+1)v_1+\sqrt{n+1}\,(1-x)v_2. \tag{68}$$

Using (66), it is straightforward to check that

$$\mathcal{F}(x)=(1-x)^{\frac{1}{n}-1}x^{-\frac{1}{2}}\mathcal{F}_0(x) \tag{69}$$

satisfies the KZ equation (53). Note that $\frac{1}{n}-1=-2h_\lambda$ is consistent with (52). We conclude that the boundary one-point correlator is an SLE$_\varkappa$ martingale for $\varkappa=8/(n+2)$.

5. $\mathrm{SLE}_{\varkappa,\rho}$ Version

The $\mathrm{SLE}_{\varkappa,\rho}$ process was introduced in [10] as a generalization of the SLE process. More specifically, consider a random curve γ_t which starts at the origin of the upper half plane $\mathbb{H}$, and chose a point $r\in\mathbb{R}$. In the $\mathrm{SLE}_{\varkappa,\rho}$ process, the dynamics of the tip z_t of the curve is given by the law $z_t=g_t^{-1}(x_t)$, where $g_t(z)$ is the uniformizing conformal map which maps the slit domain $\mathbb{H}/\gamma_t$ back to $\mathbb{H}$ and which satisfies the following stochastic differential equation:

$$\mathrm{d}g_t(z)=\frac{2\mathrm{d}t}{g_t(z)-x_t}, \quad g_0(z)=z, \tag{70}$$

where $x_t=g_t(z_t)\in\mathbb{R}$ in turn satisfies

$$\mathrm{d}x_t=\sqrt{\varkappa}\,\mathrm{d}\xi_t+\frac{\rho\,\mathrm{d}t}{x_t-g_t(r)}, \quad x_0=0. \tag{71}$$

Here ξ_t is the normalized Brownian process on $\mathbb{R}$ starting at the origin, $\varkappa$ is the diffusion coefficient, and $\rho\in\mathbb{R}$ is the drift parameter.

This setting can be used to study martingale properties of boundary correlation functions in the case when the second boundary changing operator is inserted at the finite distance from the origin (instead of infinity). For the $U(1)$ CFT, this approach was used in [6]. In the WZW case, we consider the boundary correlator

$$\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,r}\equiv\frac{\langle\phi_\Lambda(z_t)\phi_{\lambda_1}(z_1)\ldots\phi_{\lambda_N}(z_N)\,\phi_{\lambda_1^*}(\bar z_1)\ldots\phi_{\lambda_N^*}(\bar z_N)\phi_{\Lambda^*}(r)\rangle^{\mathfrak{g}}}{\langle\phi_\Lambda(z_t)\phi_{\Lambda^*}(r)\rangle^{\mathfrak{g}}}. \tag{72}$$

Conformal covariance implies that $\langle\phi_\Lambda(z_t)\phi_{\Lambda^*}(r)\rangle^{\mathfrak{g}}=\mathrm{const}\;|z_t-r|^{-2h_\Lambda}$.

It is convenient to introduce a conformal map $w_t(z)=g_t(z)-x_t$, so that the dynamics of the tip of the trace is given by $z_t=w_t^{-1}(0)$. The map $w_t(z)$ maps the initial configuration of fields on the slit domain $\mathbb{H}/\gamma_t$ into a configuration on the upper half plane $\mathbb{H}$. The boundary condition changing operators ϕ_Λ and ϕ_{Λ^*} are now inserted at the points $w=0$ and $y\equiv w_t(r)=g_t(r)-x_t$, and the bulk primary fields ϕ_{λ_i} are positioned at the points $w_i\equiv w_t(z_i)$. It follows from (70) to (71) that

$$\mathrm{d}w_i=\left(\frac{2}{w_i}+\frac{\rho}{y}\right)\mathrm{d}t-\sqrt{\varkappa}\,\mathrm{d}\xi_t, \quad w_i\Big|_{t=0}=z_i, \tag{73}$$

where $i=1,\ldots,2N+1$ with understanding that $w_{i+N}=\bar w_i$ and $w_{2N+1}\equiv y$. Note that as t increases the boundary field ϕ_{Λ^*} moves along the boundary.

Using the conformal map $w_t(z)$, we rewrite correlator (72) in the new coordinates:

$$\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,r}=\left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)y^{2h_\Lambda}\,\langle\{w_i\},y\rangle\,, \tag{74}$$

where

$$\langle\{w_i\},y\rangle \equiv \langle\phi_\Lambda(0)\phi_{\lambda_1}(w_1)\ldots\ldots\phi_{\lambda_N^*}(w_{2N})\phi_{\Lambda^*}(y)\rangle^{\mathfrak{g}}. \tag{75}$$

When t is increased by dt, the fields at $w_i, i=1,\ldots,2N+1$ are changed as follows

$$\mathrm{d}\phi_{\lambda_i}(w_i) = \mathcal{G}_i\phi_{\lambda_i}(w_i), \tag{76}$$

where

$$\mathcal{G}_i = \left(\left(\frac{2}{w_i}+\frac{\rho}{y}\right)\mathrm{d}t - \sqrt{\varkappa}\,\mathrm{d}\xi_t\right)\partial_{w_i} + \sqrt{\tau}\left(\frac{1}{w_i}-\frac{1}{y}\right)\sum_{a=1}^{\dim\mathfrak{g}}\left(\mathrm{d}\theta^a\, t_i^a\right). \tag{77}$$

Here the first term is due to (73) while the coordinate dependent coefficient in the second term is obtained from the analogous coefficient in (9) by the inverse to the Möbius map $\tilde{w} = yw/(w+y)$ (which maps the infinity to y while preserving the origin).

A computation similar to (15) shows that

$$\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\phi_{\{\lambda\}}\{z_i\}\right\rangle_{\Lambda,z_t}^{\Lambda^*,\infty}\right] = \left(y^{2h_\Lambda}\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\tilde{\Theta}\,\langle\{w_i\},y\rangle\,,$$

where

$$\begin{aligned}\tilde{\Theta} = \Bigg(&\frac{h_\Lambda}{y^2}(6+2\rho+\varkappa(2h_\Lambda-1)) + \sum_{i=1}^{2N+1}\left(\frac{2}{w_i}\partial_{w_i} - \frac{2h_i}{w_i^2} + \frac{\varkappa}{2}\partial_{w_i}\partial_{w_j}\right)\\ &+(\rho+2\varkappa h_\Lambda)\frac{1}{y}\sum_{i=1}^{2N+1}\partial_{w_i} + \frac{\tau}{2}\sum_{i,j=1}^{2N+1}\left(\frac{T_{ij}}{w_i\,w_j} - \frac{2T_{ij}}{y\,w_i} + \frac{T_{ij}}{y^2}\right)\Bigg).\end{aligned} \tag{78}$$

Thus, the correlator (72) is an SLE$_{\varkappa,\rho}$ martingale if and only if $\langle\{w_i\},y\rangle$ is annihilated by $\tilde{\Theta}$. Using (16) and the Sugawara relations, $L_0 = \frac{\nu}{2}\sum_a J_0^a J_0^a$, $L_{-1} = \nu\sum_a J_{-1}^a J_0^a$, we can rewrite this condition as follows

$$\begin{aligned}\Bigg(&-2L_{-2} + \frac{\varkappa}{2}L_{-1}^2 + \frac{\tau}{2}\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \left(\rho + 2\varkappa h_\Lambda + \frac{\tau}{\nu}\right)\frac{1}{y}L_{-1}\\ &+\frac{h_\Lambda}{y^2}\left(6+2\rho+\varkappa(2h_\Lambda-1)+\frac{\tau}{\nu}\right)\Bigg)\langle\{w_i\},y\rangle = 0.\end{aligned}$$

Here L's and J's act on $\phi_\Lambda(0)$.

Note that the terms involving ρ and y are of level 0 and -1. Therefore, applying level two operators $J_2^b, L_1 J_1^b$, and L_2 to (79) yields the same relations (25), (26), and (27). A combination of the first two of them leads to the following constraint:

$$2\varkappa h_\Lambda + \varkappa + \frac{\tau}{\nu} = 6. \tag{79}$$

Furthermore, by applying L_1 to (79) we obtain

$$\frac{2}{y}h_\Lambda\left(\rho+2\varkappa h_\Lambda+\frac{\tau}{\nu}\right)\phi_\Lambda=\left(\varkappa(2h_\Lambda+1)+\frac{\tau}{\nu}-6\right)L_{-1}\phi_\Lambda. \tag{80}$$

Assuming that $L_{-1}\phi_\Lambda\neq 0$, we conclude from (79) and (80) that ρ is uniquely determined:

$$\rho=\varkappa-6. \tag{81}$$

It is well known [15] that, up to a time change, the $\mathrm{SLE}_{\varkappa,\varkappa-6}$ process is an image of the $\mathrm{SLE}_\varkappa$ process under the Möbius map preserving zero and mapping ∞ to a finite point y. As a consequence, $\mathrm{SLE}_{\varkappa,\varkappa-6}$ describes a random curve which starts at the origin and aims at the point y on the real axis. Choose a Möbius map preserving the singularity at 0 (that is, $\mathrm{d}\tilde{w}/\mathrm{d}w|_{w=0}=1$) and, hence, preserving the parametrization of the Loewner chain. Then, the coefficient $\frac{1}{w_i}-\frac{1}{y}$ in front of the second term of (77) is exactly the push-forward of its counterpart in (9), and the whole $\mathrm{SLE}_{\varkappa,\rho}$ picture is a Möbius image of the $\mathrm{SLE}_\varkappa$ one.

Taking constraints (79) and (81) into account, we see that the martingale condition (79) reduces to

$$\left(-2L_{-2}+\frac{\varkappa}{2}L_{-1}^2+\frac{\tau}{2}\sum_{a=1}^{\dim\mathfrak{g}}J_{-1}^aJ_{-1}^a\right)\langle\{w_i\},y\rangle=0. \tag{82}$$

Thus, a sufficient condition for the boundary correlator (72) to be a covariant $\mathrm{SLE}_{\varkappa,\varkappa-6}$ martingale is the same as in the $\mathrm{SLE}_\varkappa$ case, i.e. that the operator ψ given by (18) be a level two null vector. Therefore, all the results of Section 2 apply here as well.

A counterpart of the necessary condition (43) can be obtained as follows. The correlator $\langle\{w_i\},y\rangle$ satisfies the KZ equation (40), where $2N$ is replaced by $2N+1$ (recall that $w_{2N+1}\equiv y$). Using the corresponding versions of Equations (41)–(42) along with the following relations

$$\left(T_{0i}+T_{i,2N+1}+\sum_{j=1}^{2N}T_{ij}\right)^{\mathfrak{g}}=0,\quad \left(\sum_{1\le i<j}^{2N}T_{ij}+\frac{1}{\nu}\sum_{i=1}^{2N}h_i-\frac{2}{\nu}h_\Lambda-T_{0,2N+1}\right)^{\mathfrak{g}}=0 \tag{83}$$

which are consequences of the $\mathfrak{g}$-invariance, cf. (4), we repeat the computations of Section 3. As a result, we find that the condition $\tilde{\Theta}\langle\{w_i\},y\rangle=0$ is equivalent to the condition that $\langle\{w_i\},y\rangle$ belongs to the kernel of a certain matrix:

$$\tilde{\mathcal{M}}(\{w_i\})\,\langle\{w_i\},y\rangle=0, \tag{84}$$

where

$$\tilde{\mathcal{M}}(\{w_i\})=\sum_{i=1}^{2N+1}\frac{A_i}{w_i^2}+\sum_{1\le i<j}^{2N+1}\frac{B_{ij}}{w_iw_j}, \tag{85}$$

with A_i and B_{ij} given by the same formulae (45) and (46), respectively. The difference with the SLE$_\varkappa$ case is that $\tilde{\mathcal{M}}(\{w_i\})$ contains more terms. Equation (50) imply that $\tilde{\mathcal{M}}(\{w_i\})$ vanishes identically in the case $\mathfrak{g} = \mathfrak{su}(2), \Lambda$ being the fundamental representation.

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