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DIOPHANTINE APPROXIMATIONS ON FRACTALS

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Abstract. We exploit dynamical properties of diagonal actions to derive results in Diophantine approximations. In particular, we prove that the continued fraction expansion of almost any point on the middle third Cantor set (with respect to the natural measure) contains all finite patterns (hence is well approximable). Similarly, we show that for a variety of fractals in $[0,1]^2$, possessing some symmetry, almost any point is not Dirichlet improvable (hence is well approximable) and has property C (after Cassels). We then settle by similar methods a conjecture of M. Boshernitzan saying that there are no irrational numbers x in the unit interval such that the continued fraction expansions of $\{nx \bmod 1\}_{n\in\mathbb{N}}$ are uniformly eventually bounded.

1 Introduction

1.1 Preface. In the theory of metric Diophantine approximations, one wishes to understand how well vectors in $\mathbb{R}^d$ can be approximated by rational vectors. The quality of approximation can be measured in various forms leading to numerous Diophantine classes of vectors such as *WA (well approximable), VWA (very well approximable), DI (Dirichlet improvable)* and so forth. Usually such a class is either a null set or generic (i.e. its complement is a null set) and often one encounters the phenomena of the class being null but of full dimension. Given a closed subset $M \subset \mathbb{R}^d$ supporting a natural measure (for example a lower-dimensional submanifold with the volume measure or a fractal with the Hausdorff measure), it is natural to investigate the intersection of M with the various Diophantine classes. It is natural to expect that unless there are obvious obstacles, the various Diophantine classes will intersect M in a set which will inherit the characteristics of the class, i.e. if the class is null, generic or of full dimension in $\mathbb{R}^d$, then its intersection with M would be generic, null or of full dimension in M as well.

Let us demonstrate this with two examples in the real line. We consider the intersection of the middle third Cantor set, C, in the unit interval, with two classes: BA and VWA. The class BA of badly approximable numbers consists of real numbers whose coefficients in their continued fraction expansion are bounded and the class

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WA is its complement. A classical result of Borel says that BA is null. Nevertheless, Schmidt showed in [S] that it is of dimension 1. It was shown independently in [KW1] and [KrTV], that the dimension of $C \cap \mathrm{BA}$ is full, i.e. equals $\log 2/\log 3$ (see [F] for some recent developments). One of the motivating questions for this paper, answered affirmatively in Corollary 1.10, was to decide whether $C \cap \mathrm{BA}$ is null with respect to the Hausdorff measure on C.

The class VWA (in the real line) is a subclass of WA and consists of numbers x for which there exists $\delta > 0$ such that one can find infinitely many solutions over $\mathbb{Z}$ to the inequality $|qx - p| < q^{-(1+\delta)}$. It is null and of full dimension in $\mathbb{R}$. It was shown in [W] that $C \cap \mathrm{VWA}$ is null with respect to the Hausdorff Measure on C and in [LSV] a lower bound for the dimension of this intersection is given. As far as we know it is not known if the dimension equals $\dim C$.

The intersection of the class of VWA vectors with submanifolds and fractals in $\mathbb{R}^d$ has attracted much attention. As this class will not concern us in this paper we refer the reader to the following papers for further discussions: [KLW], [KM98], [W], [K], [PV], and [LSV].

In this paper we will be concerned with inheritance of genericity to certain fractals in $\mathbb{R}$ and $\mathbb{R}^2$, with respect to three Diophantine classes WA, DI and C (see Definitions 1.1, 1.2, 1.3). In Theorems 1.5 through 1.8 we prove that the above Diophantine classes remain generic or null when additional assumptions on the fractal (and the measure supported on it) are imposed. These involve positivity of dimension and invariance under an appropriate map. The reader is referred to Remarks 1.12 for a discussion about the necessity of these additional assumptions as well as the restriction to dimensions 1 and 2.

Our arguments rely on the measure classification results obtained by E. Lindenstrauss in [Li2] and by M. Einsiedler, E. Lindenstrauss and A. Katok in [EKL].

1.2 Diophantine classes. Vectors in $\mathbb{R}^d$ will be thought of as column vectors and the action of matrices on them will be from the left. We now define the Diophantine classes we will consider.

DEFINITION 1.1. *A vector $v \in \mathbb{R}^d$ is said to be well approximable (WA), if for any $\epsilon > 0$ one can find $\vec{m} \in \mathbb{Z}^d$, $n \in \mathbb{N}$ such that*

$$|nv - \vec{m}|_\infty < \frac{\epsilon}{n^{1/d}}\,. \tag{1.1}$$

We denote WA $= \{v \in \mathbb{R}^d : v$ is WA$\}$.

It is well known that WA is a generic class.

DEFINITION 1.2. *A vector $v \in \mathbb{R}^d$ is said to be Dirichlet improvable if there exists $0 < \mu < 1$, such that for all sufficiently large N the following statement holds:*

There exists $\vec{m} \in \mathbb{Z}^d,\ n \in \mathbb{N}$ such that $0 < n \le N^d$ and $|nv - \vec{m}| < \mu N^{-1}$.

We denote DI $= \{v \in \mathbb{R}^2 : v$ is Dirichlet improvable$\}$. We say that v is not DI if $v \notin DI$.

In [DaS2] Davenport and Schmidt introduced the notion of Dirichlet improvable vectors and showed amongst other things that the class, BA, of badly approximable vectors (which is the complement of WA) is contained in the class DI. Moreover, they showed that in dimension 1 the two classes are equal (modulo the rationals). In [DaS1] it is shown that DI is a null set. Recently N. Shah, motivated by the work of Kleinbock and Weiss [KW2], showed in [Sh] that the intersection of DI with any non-degenerate analytic curve in $\mathbb{R}^d$ is null as well. In the following definition we use the notation $\langle\gamma\rangle$ for the distance of a real number γ to the integers.

Definition 1.3. *A vector $v \in \mathbb{R}^d$ is said to have property C (after Cassels) of the first type if the following statement holds:*

$$\text{For all } \vec{\gamma} \in \mathbb{R}^d \quad \liminf_{|n|\to\infty} |n| \prod_1^d \langle nv_i - \gamma_i \rangle = 0\,.$$

It is said to have property C of the second type if the following statement holds:

$$\text{For all } \gamma \in \mathbb{R} \quad \liminf_{\vec{n}\in\mathbb{Z}^d,\, \prod|n_i|\to\infty} \left(\prod_1^d |n_i|\right) \left\langle \sum_1^d n_i v_i - \gamma \right\rangle = 0\,.$$

We denote $C = \{v \in \mathbb{R}^d : v \text{ has property C of the first and the second type}\}$. We say that v has property C if $v \in C$.

For $d = 1$, it is shown in [Da] that there are no real numbers with property C (of the first or second type which coincide in this case). In [Sha2] the third named author showed that for $d \geq 2$ the class C is generic. We remark that if a vector $(\alpha,\beta)^t \in \mathbb{R}^2$ (t stands for transpose) has property C of the first type, then in particular, α,β satisfy the well-known Littlewood conjecture, i.e.

$$\liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle = 0\,.$$

Note that WA, DI and C are invariant under translations by integer vectors, hence define subsets of the d-torus $\mathbb{R}^d/\mathbb{Z}^d$. We use the same notation for the corresponding subsets of the d-torus.

1.3 Statements of results. Before stating the main results which we prove in this paper we need to recall the notion of *dimension of a measure*. Let K be a compact metric space and μ a Borel probability measure on K. The upper and lower local dimension functions of μ are defined to be

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log\left(\mu\left(B_r(x)\right)\right)}{\log(r)}\,, \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log\left(\mu\left(B_r(x)\right)\right)}{\log(r)}\,. \tag{1.2}$$

We say that μ has *exact dimension* if there exists a number d such that $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for μ-almost any x. In this case we sometimes simply say that μ is of dimension d.

Remark 1.4. In the following theorems the fact that measures have exact dimension follows from the other assumptions; it follows from [BK] that both the lower and upper dimension functions are equal almost surely to a positive multiple of the entropy of the system.

Theorem 1.5. *Let $n \in \mathbb{N}$ and let μ be a probability measure on the unit interval which is invariant and ergodic under $\times n$ modulo 1 (i.e. under multiplication by n modulo 1), and has positive dimension. Then μ almost any $x \in [0, 1]$ is WA.*

Theorem 1.6. *Let $\gamma : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be a hyperbolic automorphism, induced by the linear action of a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and let μ be a probability measure which is invariant and ergodic with respect to γ, and has positive dimension. Then μ almost any $v \in \mathbb{R}^2/\mathbb{Z}^2$ is WA.*

One way to construct examples of measures μ on the unit interval or on the 2-torus satisfying the assumptions of Theorems 1.5 and 1.6 respectively is to choose an appropriate partition of the underlying space for which the resulting factor map to the symbolic system is an isomorphism of measurable dynamical systems. In the case of the unit interval, one chooses the partition into n intervals of equal length, and in the case of the 2-torus, a Markov partition corresponding to γ (see [AW]). Then, one takes a (topologically transitive) subshift of finite type of the symbolic system and the unique maximal entropy probability measure supported on it and translates this measure to the original space.

Before stating further results we briefly introduce some notation (see section 2.2 for a more thorough account). For any positive integer d ($d = 1$ or 2 in our discussions), let $\{a_t\}_{t\in\mathbb{R}}, \{u_v\}_{v\in\mathbb{R}^d} < \mathrm{PGL}_d(\mathbb{R})$ be the subgroups given by

$$a_t = \mathrm{diag}(e^t, \dots, e^t, e^{-dt}), \quad u_v = \begin{pmatrix} I_d & -v \\ 0 & 1 \end{pmatrix}, \tag{1.3}$$

where I_d is the $d\times d$ identity matrix. Our arguments rely on the natural identification of the d-torus with the periodic orbit $\{u_v \mathrm{PGL}_{d+1}(\mathbb{Z}) : v \in \mathbb{R}^d\}$ in the homogeneous space $\mathrm{PGL}_{d+1}(\mathbb{R})/\mathrm{PGL}_{d+1}(\mathbb{Z})$ (see (2.2)). This enables us to view measures supported on the d-torus as measures supported in the space $\mathrm{PGL}_{d+1}(\mathbb{R})/\mathrm{PGL}_{d+1}(\mathbb{Z})$.

In the following two theorems we are able to obtain stronger results than in Theorems 1.5, 1.6. The price is reflected in the stronger assumptions which are automatically satisfied in many applications (see Remark 1.9).

Theorem 1.7. *Let n and μ be as in Theorem 1.5. Viewing μ as a probability measure on $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$, if we further assume that any weak* limit of $\frac{1}{T}\int_0^T (a_t)_*\mu dt$ is a probability measure on $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$ (i.e. there is no escape of mass on average), then for μ almost any $s \in \mathbb{R}/\mathbb{Z}$*

$$\overline{\{a_t u_s \mathrm{PGL}_2(\mathbb{Z})\}_{t\geq 0}} = \mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z}). \tag{1.4}$$

Furthermore, if for a given s (1.4) holds, then the continued fraction expansion of s contains all patterns.

Theorem 1.8. *Let γ, μ be as in Theorem 1.6. Viewing μ as a probability measure on $\mathrm{PGL}_3(\mathbb{R})/\mathrm{PGL}_3(\mathbb{Z})$, if we further assume that any weak* limit of $\frac{1}{T}\int_0^T (a_t)_*\mu dt$ is a probability measure on $\mathrm{PGL}_3(\mathbb{R})/\mathrm{PGL}_3(\mathbb{Z})$ (i.e. there is no escape of mass on average), then for μ almost any $v \in \mathbb{R}^2/\mathbb{Z}^2$*

$$\overline{\{a_t u_v \mathrm{PGL}_3(\mathbb{Z})\}_{t\geq 0}} = \mathrm{PGL}_3(\mathbb{R})/\mathrm{PGL}_3(\mathbb{Z}).$$

In particular v is not DI (hence is WA) and has property C.

Remark 1.9. In [KLW], Kleinbock, Lindenstrauss, and Weiss showed that if μ is *friendly* (see section 2 of [KLW] for the definition), then there is no escape of mass on average and so the further assumptions in Theorems 1.7, 1.8 are satisfied automatically. For a detailed proof of this statement the reader is further referred to [Shi, Cor. 3.2].

In section 2 of the paper [KLW] it is shown that if $F \subset [0,1]$ is a fractal defined as the attractor of an irreducible system of contracting self-similar maps satisfying the open set condition, then the Hausdorff measure on F is of positive dimension and is friendly. In many examples, the fractal F is invariant and ergodic (with respect to the Hausdorff measure) under $\times n$ (for a suitable choice of n), hence Theorem 1.7 applies by the above remark. In particular we have the following corollary which served as one of the motivating questions for this work.

Corollary 1.10. *Almost any point in the middle third Cantor set (with respect to the natural measure) is WA and moreover its continued fraction expansion contains all patterns.*

Our last theorem is of a different nature as it is an everywhere statement. It was conjectured to hold by M. Boshernitzan and communicated by the second named author.

Theorem 1.11. *If we denote for $x \in [0,1]$, $c(x) = \limsup a_n(x)$ where $a_n(x)$ are the coefficients in the continued fraction expansion of x, then for any irrational $x \in [0,1]$, $\sup_n c(nx) = \infty$, where nx is calculated modulo 1.*

The proofs of all the above theorems are similar in nature. We shall first prove Theorems 1.6, 1.8 which are somewhat simpler but contain the ideas. We then prove Theorems 1.5, 1.7 which involves S-arithmetic arguments and finally prove Theorem 1.11 which involves adelic arguments.

Remarks 1.12. (1) We note that there are fractals of positive dimension which intersect the various generic Diophantine classes trivially. For example it is not hard to construct a closed set of positive dimension in the unit interval which is contained in the class BA of badly approximable numbers. Hence in order to obtain results as above one must impose some further assumptions, which in our case, are reflected in the symmetry of the fractal given by the invariance under the appropriate map.

(2) One can build examples of probability measures μ of positive dimension on the d-torus which do have escape of mass on average (in the context of Theorems 1.7, 1.8). The constructions we suggest depend on the dimension.

For $d = 1$ consider the set $D \subset [0,1)$ consisting of numbers with diverging c.f.e. coefficients. Any probability measures supported in D will produce an example of a measure with full escape of mass. As the dimension of D is $1/2$ (see [G] or [C]), we conclude that such measures with positive dimension exist.

For $d = 2$ one can take, in a similar manner, a probability measure of positive dimension supported on the set of singular vectors (recall that a vector $v \in \mathbb{R}^d$ is said to be singular if the orbit $a_t u_v \operatorname{PGL}_{d+1}(\mathbb{Z})$ goes to infinity as $t \to \infty$). In [C] the dimension of this set was calculated. For such measures there will be full escape of mass.

We do not know however if the further invariance assumption on the measure μ which appears in the statements of Theorems 1.7, 1.8 actually excludes the possibility of escape of mass on average or even of escape of mass.

(3) We expect that the analogues for Theorems 1.6, 1.8 for higher-dimensional torus still hold with some assumptions on the automorphism γ (or even if γ is an epimorphism). Using the high entropy method developed in [EK], it can be proved (and will be done elsewhere), that for any $d \geq 3$, if γ is an automorphism of the d-torus with characteristic polynomial having only real roots which are distinct in absolute value, and μ is a γ-invariant and ergodic measure of dimension greater than 1, then μ almost any point is WA. Moreover if μ is friendly, then μ almost any point is not DI and has property C.

(4) Theorems 1.7, 1.8 seem to have many applications to Diophantine approximations and the list of properties in their statements is not complete. For example, the third named author proved that for $s \in \mathbb{R}$, if (1.4) holds, then the 2-dimensional lattice $u_s \operatorname{PGL}_2(\mathbb{Z})$ (see subsection 2.2) satisfies the generalized Littlewood conjecture. For a proof of this statement and the discussion on the generalized Littlewood conjecture, the reader is referred to [Sha1].

(5) Boshernitzan reported to us that a stronger version of Theorem 1.11 holds for the special case of quadratic irrationals.

(6) B. de Mathan and O. Teulié have conjectured in [MT] that for any prime p and for any irrational number $x \in [0,1]$, if we denote by $\tilde{c}(x) = \sup_n a_n(x)$ (where $a_n(x)$ are the coefficients in the continued fraction expansion of x), one has $\sup_\ell \tilde{c}(p^\ell x) = \infty$. In [EKl] it was shown that the set of exceptions to de Mathan–Teulié's conjecture is of Hausdorff dimension zero. Although in Theorem 1.11 we allow to multiply x by a much bigger set of integers than powers of a single prime, our result does not follow from de Mathan–Teulié's conjecture because of the fundamental difference between the definitions of $c(x)$ using $\limsup$ and $\tilde{c}(x)$ using sup.

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2 Preliminaries

Most of the arguments appearing in our proofs are dynamical. In this section we present the dynamical systems in which our discussion takes place and give the necessary preliminaries needed to understand the proofs of the results stated in subsection 1.3.

2.1 Homogeneous spaces. Let G be a second countable locally compact topological group and $\Gamma < G$ a discrete subgroup. The space G/Γ is called a homogeneous space as G acts transitively on G/Γ by left translation. The topology we take on G/Γ is the quotient topology which then makes the natural projection $G \to G/\Gamma$ a covering map. When G/Γ supports a G-invariant probability measure we say that Γ is a lattice in G. In this case, this probability measure is unique and is denoted by μ_G. If $\Gamma < G$ is a lattice, then the support of μ_G equals of course G/Γ. This simple fact is used without reference in our arguments. In this paper we will be interested in a very restrictive family of examples. We now describe the most important one.

2.2 The space of lattices. Fix $d \geq 1$ and let $X = \mathrm{PGL}_{d+1}(\mathbb{R})/\mathrm{PGL}_{d+1}(\mathbb{Z})$. It is well known that $\mathrm{PGL}_{d+1}(\mathbb{Z}) < \mathrm{PGL}_{d+1}(\mathbb{R})$ is a lattice. The space X can be identified with the space of unimodular lattices in $\mathbb{R}^{d+1}$ (i.e. of covolume 1) in the following manner: Given a coset $g\,\mathrm{PGL}_{d+1}(\mathbb{Z})$ we choose a matrix in $\mathrm{GL}_{d+1}(\mathbb{R})$ representing g and denote it also by g. We then take the lattice spanned by the columns of g and normalize it to have covolume 1. The reader should check that this defines a bijection between X and the space of unimodular lattices in $\mathbb{R}^{d+1}$. The group $\mathrm{SL}_{d+1}(\mathbb{R})$ is mapped in a natural way into $\mathrm{PGL}_{d+1}(\mathbb{R})$ and hence acts on X by left translation. When we think of points of X as lattices in $\mathbb{R}^{d+1}$, this action translates to the linear action of $\mathrm{SL}_{d+1}(\mathbb{R})$ on $\mathbb{R}^{d+1}$. The following is known as Mahler's compactness criterion. It gives a geometric criterion for divergence in X and in particular, shows that X is not compact:

Theorem 2.1 (Mahler's compactness criterion)**.** *A subset $C \subset X$ is bounded (i.e. its closure is compact) if and only if there exists $\epsilon > 0$ such that for any lattice $\Lambda \in C$, $\Lambda \cap B_\epsilon(0) = \{0\}$ i.e. if and only if there exists a uniform lower bound for the lengths of nonzero vectors belongings to points in C.*

We denote for $t \in \mathbb{R}$ and $v \in \mathbb{R}^d$,

$$a_t = \mathrm{diag}(e^t, \ldots, e^t, e^{-dt}), \quad u_v = \begin{pmatrix} I_d & -v \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_{d+1}(\mathbb{R}), \tag{2.1}$$

where I_d is the $d \times d$ identity matrix. The mysterious minus sign in front of v in (2.1) is explained in the discussion in Appendix 5. Note that $\{a_t\}_{t\in\mathbb{R}}, \{u_v\}_{v\in\mathbb{R}^d}$ are subgroups of $\mathrm{PGL}_{d+1}(\mathbb{R})$. In the base of our arguments lies the identification of the d-torus $\mathbb{R}^d/\mathbb{Z}^d$ with the periodic orbit of the group $\{u_v\}_{v\in\mathbb{R}^d}$ through the identity coset,

$$\text{for all } v \in \mathbb{R}^d, \quad v + \mathbb{Z}^d \leftrightarrow u_v\,\mathrm{PGL}_{d+1}(\mathbb{Z}). \tag{2.2}$$

Using this identification, many of the Diophantine properties of a vector $v \in \mathbb{R}^d$, correspond to dynamical properties of the orbit $\{a_t u_v \operatorname{PGL}_{d+1}(\mathbb{Z})\}_{t>0}$. This is the content of Lemmas 2.2–2.6. Although these are probably well known, the proofs of Lemmas 2.2, 2.5 and 2.6 appear in Appendix 5 for completeness of the exposition. The following lemma is essentially contained in Theorem 2.20 in [D]:

LEMMA 2.2. *For any $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \operatorname{PGL}_{d+1}(\mathbb{R})/\operatorname{PGL}_{d+1}(\mathbb{Z})$ such that for any $v \in \mathbb{R}^d$, if the inequality $\left|v - \frac{\vec{m}}{n}\right|_\infty < \frac{\epsilon}{n^{1+1/d}}$ has only finitely many solutions $\vec{m} \in \mathbb{Z}^d$, $n \in \mathbb{N}$, then for large enough T, $a_t u_v \operatorname{PGL}_{d+1}(\mathbb{Z}) \in K_\epsilon$ for $t > T$. In particular if the vector $v \in \mathbb{R}^d$ is not WA then the orbit $\{a_t u_v \operatorname{PGL}_{d+1}(\mathbb{Z})\}_{t\geq 0}$ is bounded.*

LEMMA 2.3. *Let $d \geq 2$. If $v \in \mathbb{R}^d$ is such that*

$$\overline{\{a_t u_v \operatorname{PGL}_{d+1}(\mathbb{Z})\}_{t>0}} = \operatorname{PGL}_{d+1}(\mathbb{R})/\operatorname{PGL}_{d+1}(\mathbb{Z}),$$

then v is not DI and has property C.

Proof. The proof of this lemma follows from Corollaries 4.6, 4.8 in [Sha2] and Proposition 2.1 in [KW2]. □

The following lemma is left as an exercise.

LEMMA 2.4. *The class of WA points in the d-torus is invariant under the natural action of $M_d(\mathbb{Z}) \cap \operatorname{GL}_d(\mathbb{Q})$.*

2.3 Connection with continued fraction expansion. We identify the circle $\mathbb{R}/\mathbb{Z}$ with the interval $[0,1)$. For each irrational $s \in [0,1)$, there exists a unique infinite sequence of positive integers $a_n(s) = a_n$ such that the sequence

$$[a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \frac{1}{a_n}}}} \tag{2.3}$$

converges to s. This correspondence is a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and the irrational points on the circle. We then denote $s = [a_1, a_2, \ldots]$ and refer to the sequence $a_n(s)$ as the continued fraction expansion (abbreviated c.f.e.) of s. We denote, as in Theorem 1.11, $c(s) = \limsup a_n(s)$.

LEMMA 2.5. *For any $N \in \mathbb{N}$, there exists a compact set $K_N \subset \operatorname{PGL}_2(\mathbb{R})/\operatorname{PGL}_2(\mathbb{Z})$ such that if $s \in \mathbb{R}/\mathbb{Z}$ is irrational and $c(s) < N$ then the orbit $\{a_t u_s \operatorname{PSL}_2(\mathbb{Z})\}_{t\geq T}$ is contained in K_N for large enough T (which depends of course on s).*

We say that the c.f.e. of an irrational $s \in \mathbb{R}/\mathbb{Z}$ contains all patterns if given a finite sequence of integers $b_1, \ldots, b_n$, there exists k such that $a_{k+i}(s) = b_i$ for any $1 \leq i \leq n$.

LEMMA 2.6. *If $s \in \mathbb{R}/\mathbb{Z}$ is such that $\overline{\{a_t u_s \operatorname{PGL}_2(\mathbb{Z})\}_{t>0}} = \operatorname{PGL}_2(\mathbb{R})/\operatorname{PGL}_2(\mathbb{Z})$, then the c.f.e. of s contains all patterns.*

2.4 Escape of mass. Given a probability measure μ on $\mathbb{R}^d/\mathbb{Z}^d$, we may think of it (see (2.2)) as a measure supported on the periodic orbit

$$\left\{u_v \operatorname{PGL}_{d+1}(\mathbb{Z})\right\}_{v\in\mathbb{R}^d} \subset \operatorname{PGL}_{d+1}(\mathbb{R})/\operatorname{PGL}_{d+1}(\mathbb{Z}).$$

This enables us to define

Definition 2.7. *We say that μ has no escape of mass on average with respect to $\{a_t\}_{t\geq 0}$ if any weak* limit of $\frac{1}{T}\int_0^T (a_t)_*\mu dt$ is a probability measure on $\operatorname{PGL}_{d+1}(\mathbb{R})/\operatorname{PGL}_{d+1}(\mathbb{Z})$.*

We can now state Theorem 5.3 from [Shi] which will be needed to prove Theorems 1.5, 1.7.

Theorem 2.8 (Theorem 5.3 from [Shi])**.** *Let μ be a probability measure on $\mathbb{R}^d/\mathbb{Z}^d$ of dimension κ such that μ has no escape of mass on average with respect to $\{a_t\}_{t\geq 0}$. Then any weak* limit, ν, of $\frac{1}{T}\int_0^T (a_t)_*\mu dt$ satisfies $h_\nu(a_1) \geq (d+1)\kappa$. In particular, if $\kappa > 0$ then $h_\nu(a_1) > 0$.*

2.5 Group action on measures. Let $X = G/\Gamma$ be a homogeneous space (G a locally compact group and Γ a discrete subgroup of G). G acts on X by left translations. This action induces an action of G on the space of Borel probability measures on X. Given a probability measure μ on X and $g \in G$, we denote by $g_*\mu$ the probability measure defined by the equation

$$\int_X f(x) dg_*\mu(x) = \int_X f(gx) d\mu(x) \tag{2.4}$$

for any $f \in C_c(X)$. μ is said to be g-invariant if $g_*\mu = \mu$. Given a subgroup $H < G$, the set of H-invariant probability measures will be denoted by $\mathcal{M}_X(H)$.

Let $H < G$ be a commutative closed group and let $\mu \in \mathcal{M}_X(H)$. The ergodic decomposition of μ with respect to H is the unique Borel probability measure θ_H concentrated on the extreme points of $\mathcal{M}_X(H)$ (i.e. the extreme points have θ_H-measure 1) and having μ as its center of mass. Existence and uniqueness of the ergodic decomposition follow from Choquet's theorem. We say that an ergodic H-invariant measure, μ_0, appears as a component with positive weight in the ergodic decomposition of μ with respect to H, if $\theta_H(\{\mu_0\}) > 0$. An equivalent (and perhaps simpler) condition is the existence of a constant $c > 0$, such that for any nonnegative function $f \in C_c(X)$ one has $\int_X f d\mu \geq c \int_X f d\mu_0$.

Let $H' < H$ be a closed subgroup. If μ_0 is ergodic with respect to H' (and hence with respect to H), then it appears with positive weight in the ergodic decomposition of μ with respect to H, if and only if it appears as a component with positive weight in the ergodic decomposition with respect to H'.

H acts on $\mathcal{M}_X(H')$ and as H' acts trivially, this action induces an action of the quotient H/H' on $\mathcal{M}_X(H')$. Denote the natural projection from H to H/H' by $g \mapsto \hat{g}$. Let $\mu \in \mathcal{M}_X(H')$. If the quotient H/H' is compact, one can define an H-invariant probability measure

$$\tilde{\mu} = \int_{H/H'} \hat{g}_*\mu d\hat{g},$$

where $d\hat{g}$ is the Haar probability measure on H/H'. The meaning of this equation is that

$$\int_X f(x)d\tilde{\mu} = \int_{H/H'} \left(\int_X f(x)d\hat{g}_*\mu \right) d\hat{g}$$

for any $f \in C_c(X)$. For $b \in H'$ and $g \in H$, the entropies $h_\mu(b), h_{g_*\mu}(b)$ are equal. This implies that $h_{\tilde{\mu}(b)} = h_\mu(b)$ too. We shall need the following theorem about entropy (see [EL] for the proof).

Theorem 2.9 (Upper semi-continuity of entropy). *Let $X = G/\Gamma$ be as above and let $b \in G$. Let μ_n be a sequence of b-invariant probability measures converging in the weak* topology to a probability measure μ (which is automatically b-invariant). Then $h_\mu(b) \geq \limsup h_{\mu_n}(b)$.*

3 Proofs of Theorems 1.6, 1.8

In this section $G = \mathrm{PGL}_3(\mathbb{R})$, $\Gamma = \mathrm{PGL}_3(\mathbb{Z})$ and $X = G/\Gamma$. The identity coset in X will be denoted by $\bar{e}$. We use the notation of (2.1) and the identification of (2.2). Hence the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ is identified with the periodic orbit $\left\{u_v\bar{e} : v \in \mathbb{R}^2\right\}$ of the two-dimensional unipotent group $\{u_v\}_{v\in\mathbb{R}^2} < G$. This enables us to view the measure μ from the statement of Theorems 1.6, 1.8, as a measure supported on this periodic orbit. Let γ be as in the statement of Theorem 1.6. Under this identification, the action of γ translates to the action from the left of

$$\gamma' = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma\,. \tag{3.1}$$

Hence, the assumptions of Theorem 1.6 translate to μ being of positive dimension, γ'-invariant, and ergodic.

Proof of Theorem 1.6. Assume that the statement of the theorem is false. As the set of WA points on the torus is γ'-invariant (see Lemma 2.4), it follows from ergodicity and Lemma 2.2 that, for μ-almost any $x \in X$, the orbit $\{a_tx\}_{t\geq0}$ is bounded. Let K_i be an increasing sequence of compact subsets exhausting X. We shall build an invariant measure on X having the Haar measure appearing as a component with positive weight in its ergodic decomposition, while at the same time this measure will be the sum of invariant measures supported on the sets K_i. This contradicts the uniqueness of the ergodic decomposition.

To this end we define

$$E_i = \left\{x \in \mathrm{supp}(\mu) : \{a_tx\}_{t\geq0} \text{ is contained in } K_i \text{ but not in } K_{i-1}\right\}.$$

Hence, E_i form a partition (up to a null set) of the support of μ. Denote by μ_i the restriction of μ to E_i. Hence $\mu = \sum \mu_i$. We denote $\mu_i^T = \frac{1}{T}\int_0^T (a_t)_*\mu_i dt$ and $\mu^T = \sum \mu_i^T$. Let $T_j \to \infty$ be chosen so that the sequences $\mu_i^{T_j}, \mu^{T_j}$ converge weak* to some measures ν_i, ν respectively. Since for any $t \geq 0$, $a_t(E_i) \subset K_i$, ν_i is supported in K_i and there could be no escape of mass and ν is a probability measure. ν and the ν_i's are a_t-invariant and $\nu = \sum \nu_i$. In particular, the ergodic decomposition of

ν with respect to $\{a_t\}_{t\in\mathbb{R}}$ is the sum of the ergodic decompositions of the ν_i's. As ν_i is supported in K_i, we deduce that the G-invariant probability measure, μ_G, cannot appear as a component with positive weight in the ergodic decomposition of ν with respect to $\{a_t\}_{t\in\mathbb{R}}$. Since the action of γ' commutes with a_t, μ^{T_j} is γ'-invariant for any j and as a consequence ν is γ'-invariant too. Note also that for any T we have the following equality of entropies: $h_\mu(\gamma') = h_{\mu^T}(\gamma')$. Hence it follows from Remark 1.4 and Theorem 2.9 that $h_\nu(\gamma') > 0$. From our assumption on the hyperbolicity of γ (which in this case implies $\mathbb{R}$-diagonability), it follows that the group, H, generated by $\{a_t\}_{t\in\mathbb{R}}$ and γ', is cocompact in a maximal $\mathbb{R}$-split torus T in G. The desired contradiction now follows from Corollary 3.2 below, which in turn follows from the following theorem from [EKL]. □

Theorem 3.1 (Theorem 1.3 from [EKL])**.** *Let ν be a Borel probability measure on $X = \mathrm{PGL}_3(\mathbb{R})/\mathrm{PGL}_3(\mathbb{Z})$ which is invariant under the action of a maximal $\mathbb{R}$-split torus $T < G = \mathrm{PGL}_3(\mathbb{R})$. If there exists $b \in T$ which acts with positive entropy with respect to ν, then the G-invariant probability measure μ_G, appears as a component with positive weight in the ergodic decomposition of ν with respect to T.*

Corollary 3.2. *Let ν be a Borel probability measure on X which is invariant under the action of a group H which is cocompact in a maximal $\mathbb{R}$-split torus $T < G$. If there exists $b \in H$ which acts with positive entropy with respect to ν, then the G-invariant probability measure μ_G, appears as a component with positive weight in the ergodic decomposition of ν with respect to H.*

Proof. Denote the natural projection $T \to T/H$ by $g \mapsto \hat{g}$. Define

$$\lambda = \int_{T/H} \hat{g}_* \nu d\hat{g}\,,$$

where $d\hat{g}$ is the Haar measure in T/H (recall the discussion of subsection 2.5). λ is a T-invariant measure on X and $h_\lambda(b) = h_\nu(b)$, hence Theorem 3.1 implies that μ_G appears with positive weight in the ergodic decomposition of λ with respect to T. By the Howe–Moore theorem μ_G is H-ergodic, hence we conclude that μ_G appears with positive weight in the ergodic decomposition of λ with respect to H. The ergodic decomposition of ν with respect to H is a probability measure θ, supported on the extreme points of $\mathcal{M}_X(H)$, which is the set of H-invariant probability measures on Y, having ν as its center of mass. The ergodic decomposition of λ with respect to H is $\theta' = \int_{T/H} \hat{g}\theta d\hat{g}$. This equation means that θ' is the probability measure on $\mathcal{M}_X(H)$, characterized by the following equation:

$$\int_{\mathcal{M}_X(H)} F(\varphi) d\theta'(\varphi) = \int_{T/H} \int_{\mathcal{M}_X(H)} F(\hat{g}_*\varphi) d\theta(\varphi) d\hat{g} \tag{3.2}$$

for any $F \in C(\mathcal{M}_X(H))$. In order to show that $\theta'(\{\mu_G\}) > 0$ and conclude the proof, we need to show that for any open neighborhood $\mu_G \in V \subset \mathcal{M}_X(H)$, $\theta'(V) > \alpha$ for some positive constant α. Let V be such an open neighborhood. Let $U \subset V$ be another open neighborhood of μ_G such that there exists a bump function F which equals 1 on U and vanishes outside V. Let $U' \subset U$ be a smaller neighborhood of μ_G, such that

$$U' \subset \cap_{\hat{g} \in T/H} \hat{g}_*(U) \,. \tag{3.3}$$

The existence of U' follows from the compactness of T/H and the G-invariance of μ_G. Then

$$\theta'(V) \geq \int_{\mathcal{M}_X(H)} F d\theta' = \int_{T/H} \int_{\mathcal{M}_X(H)} F(\hat{g}_*\varphi) d\theta(\varphi) \hat{g} \,. \tag{3.4}$$

By construction, for any $\hat{g} \in T/H$,

$$\int_{\mathcal{M}_X(H)} F(\hat{g}_*\varphi) d\theta(\varphi) \geq \theta(U') \geq \alpha \,, \tag{3.5}$$

where $\alpha = \theta(\{\mu_G\})$ is positive by assumption. □

Proof of Theorem 1.8. As γ' and a_t commute, the set $F = \{x \in X : \overline{\{a_t x\}_{t>0}} \neq X\}$ is γ'-invariant. Assume to get a contradiction that $\mu(F) > 0$. It follows from the ergodicity that $\mu(F) = 1$. Let $\{U_i\}$ be a countable base for the topology of X. Define recursively

$$E_1 = \left\{x \in \operatorname{supp}(\mu) : \{a_t x\}_{t>0} \cap U_1 = \emptyset\right\} \,, \quad \text{and}$$
$$E_n = \left\{x \in \operatorname{supp}(\mu) : \{a_t x\}_{t>0} \cap U_n = \emptyset\right\} \setminus E_{n-1} \,,$$

for any $n > 1$. Hence, $\{E_i\}$ form a partition up to a null set of the support of μ. We continue as in the proof of Theorem 1.6 using the same notation as there. We now highlight the differences between the arguments: In the proof of Theorem 1.6 we used the fact that μ_i^T is compactly supported in order to pass to a weak* limit without losing mass. Here we do not know that μ_i^T is compactly supported, instead we use our further assumption that any weak* limit of μ^T is a probability measure. Another difference is that in the proof of Theorem 1.6, ν_i was supported in a compact set and hence could not have μ_G appear as a component with positive weight in its ergodic decomposition with respect to $\{a_t\}_{t \in \mathbb{R}}$. Here the reason that ν_i cannot have μ_G appearing as a component with positive weight is that $\nu_i(U_i) = 0$.

To end the proof we note that Lemmas 2.2, 2.3 imply that for μ-almost any $v \in \mathbb{R}^2/\mathbb{Z}^2$, v is WA, not DI, and has property C. □

4 Proof of Theorems 1.5, 1.7

4.1 Preparations. Let $\mathbb{G} = \mathrm{PGL}_2$ and $S = \{p_1, \ldots, p_k, \infty\}$, where the p_i's are the primes appearing in the prime decomposition of the number n appearing in the statement of Theorem 1.5. We denote

$$G_\infty = \mathbb{G}(\mathbb{R}) \,, \quad G_f = \prod_1^k \mathbb{G}(\mathbb{Q}_{p_i}) \,, \quad G_S = G_\infty \times G_f \,, \quad K = \prod_1^k \mathbb{G}(\mathbb{Z}_{p_i}) \,. \tag{4.1}$$

Denote $\Gamma_S = \mathbb{G}\big(\mathbb{Z}\big[\frac{1}{p_1} \ldots \frac{1}{p_k}\big]\big)$ and $\Gamma_\infty = \mathbb{G}(\mathbb{Z})$. We shall abuse notation (as usual) and identify Γ_S with its various diagonal embeddings in G_S, G_f, etc. The meaning should be clear from the context. Γ_S, Γ_∞ are lattices in G_S, G_∞ respectively.

Nonetheless, Γ_S is dense in G_f. Let $X = G_\infty/\Gamma_\infty$ and $Y = G_S/\Gamma_S$. We denote the identity cosets in both spaces by $\bar{e}$. The elements of G_S will be denoted by (g_∞, g_f) where $g_\infty \in G_\infty$ and $g_f \in G_f$. Denote by

$$\pi : Y \to K\backslash Y = K\backslash G_S/\Gamma_S\,, \tag{4.2}$$

the natural projection. The double coset space $K\backslash G_S/\Gamma_S$ can be identified with X in the following manner: Given a double coset $K(g_\infty, g_f)\Gamma_S$ as K is an open subgroup of G_f and $\Gamma_S < G_f$ is dense, there exists $\gamma \in \Gamma_S$ such that $g_f\gamma \in K$. We then identify $K(g_\infty, g_f)\Gamma_S$ with $g_\infty\gamma\bar{e} \in X$. The reader should check that this map is indeed well defined, a bijection, and respects the topologies. In other words the map $\pi : Y \to X$ is defined by

$$\pi\big((g_\infty, g_f)\bar{e}\big) = g_\infty\bar{e} \quad \text{if } g_f \in K\,.$$

G_S, G_∞ act on Y, X by left translation respectively. The action of G_∞ on X is via π a factor of the action of $G_\infty \times \{e_f\}$ on Y.

4.2 Proofs.

Proof of Theorem 1.5. We identify $\mathbb{R}/\mathbb{Z}$ with the periodic orbit of the horocycle flow $\{u_t\}_{t\in\mathbb{R}}$ through $\bar{e} \in X$ (see (2.1), (2.2)). Under this identification, the map $\times n$ becomes the map $u_s\bar{e} \mapsto u_{ns}\bar{e}$. This identification enables us to view the measure μ from Theorem 1.5 as a probability measure supported on this periodic orbit. The next thing we wish to do is to lift this measure to a measure on Y. We do so by pushing it with the map $u_t\bar{e} \mapsto (u_t, e_f)\bar{e}$ defined for $t \in [0, 1)$. We denote the resulting measure on Y by ν_1. It is obvious that $\pi_*(\nu_1) = \mu$. We let $b = \operatorname{diag}(n, 1) \in \Gamma_S$ and note that the action of b on Y, when restricted to $\{(u_s, e_f)\bar{e} : s \in \mathbb{R}\}$, factors via π to the map $\times n$ on the circle; i.e. the following diagram commutes:

$$\begin{array}{ccc} (u_s, e_f)\bar{e} & \xrightarrow{\;b\;} & (u_{ns}, e_f)\bar{e} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ u_s\bar{e} & \xrightarrow{\;\times n\;} & u_{ns}\bar{e} \end{array}$$

Although μ is $\times n$-invariant, ν_1 is not invariant under the action of b on Y. We replace it by a different measure which is invariant under b and projects to μ by the following procedure: We denote $\nu_N = \frac{1}{N}\sum_{i=0}^{N-1} b_*^i(\nu_1)$. Note that for any N, $\pi_*(\nu_N) = \mu$. Let ν be a weak* limit of the sequence ν_N. It follows that $\pi_*(\nu) = \mu$ and in particular, that ν is a probability measure (note that here we used the fact that the fibers of π are compact). One could modify the above construction and first lift the measure μ from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R} \times \mathbb{Q}_S/\mathbb{Z}_S$ and then average the lift to get invariance under the (invertible extension of) $\times n$ and only then identify the resulting measure ν with a measure on Y which projects to μ. To summarize what we established so far, we constructed a b-invariant probability measure, ν, on Y such that $\pi : (Y, \nu, b) \to (X, \mu, \times n)$, is a factor map. Assume that the statement of Theorem 1.5 is false. It follows from Lemmas 2.4 and 2.2 that for μ-almost any $x \in X$, $\{a_t x : t \geq 0\}$ is bounded. Let K_i be an increasing sequence of compact subsets exhausting X. Let

$$E_i = \left\{ x \in \operatorname{supp}(\mu) : \{a_t x\}_{t\geq 0} \text{ is contained in } K_i, \text{ but not in } K_{i-1} \right\}.$$

Thus E_i form a partition (up to a null set) of the support of μ. Denote by μ_i the restriction of μ to E_i, hence $\mu = \sum_i \mu_i$. We denote for $T > 0$, $\nu^T = \frac{1}{T}\int_0^T (a_t, e_f)_*(\nu)dt$. Then

$$\pi_*(\nu^T) = \frac{1}{T}\int_0^T (a_t)_*(\mu)dt = \sum_i \frac{1}{T}\int_0^T (a_t)_*(\mu_i)dt\,. \tag{4.3}$$

Denote $\mu_i^T = \frac{1}{T}\int_0^T (a_t)_*(\mu_i)dt$ and $\mu^T = \sum_i \mu_i^T$. Thus, (4.3) becomes $\pi_*(\nu^T) = \mu^T = \sum \mu_i^T$. Let $T_j \to \infty$ be chosen such that all the following sequences converge in the weak* topology: $\nu^{T_j}, \mu_i^{T_j}, \mu^{T_j}$. Denote their corresponding limits by $\tilde{\nu}, \tilde{\mu}_i, \tilde{\mu}$ respectively. It is evident that $\tilde{\nu}$ is (a_t, e_f)-invariant, while $\tilde{\mu}, \tilde{\mu}_i$ are a_t-invariant. As the fibers of π are compact, we can deduce that $\pi_*(\tilde{\nu}) = \tilde{\mu} = \sum_i \tilde{\mu}_i$. Moreover since $\tilde{\mu}_i$ is supported in K_i, there is no escape of mass and $\tilde{\nu}, \tilde{\mu}$ are probability measures. We will derive the desired contradiction by using the following lemma:

Lemma 4.1. *In the ergodic decomposition of $\tilde{\mu}$ with respect to $\{a_t\}_{t\in\mathbb{R}}$, the G_∞-invariant measure μ_{G_∞} has positive weight.*

To finish the proof of the theorem, note that since for each i, $\tilde{\mu}_i$ is a_t-invariant, the ergodic decomposition of $\tilde{\mu}$ with respect to the action of $\{a_t\}_{t\in\mathbb{R}}$ is the sum of the corresponding ergodic decompositions of the $\tilde{\mu}_i$'s which are supported in K_i and hence cannot have μ_{G_∞} appearing with positive weight in their ergodic decomposition. □

In the proof of Lemma 4.1 we will use the following simplification of Theorem 1.1 from [Li2]. To state it we use the notation from the beginning of this subsection and we denote by $\mathbb{T}$ the subgroup of $\mathbb{G}$ consisting of diagonal matrices.

Theorem 4.2. *Let $\tilde{\nu}$ be a probability measure on Y which is invariant under the action of $\mathbb{T}(\mathbb{R})$, has positive entropy with respect some (hence any) element in $\mathbb{T}(\mathbb{R})$ and is invariant under the action of of a noncompact subgroup of G_f. Then in the ergodic decomposition of $\pi_*(\tilde{\nu})$ with respect to $\mathbb{T}(\mathbb{R})$, the G_∞-invariant measure μ_{G_∞} appears as a component with positive weight.*

Proof of Lemma 4.1. By construction, the measure $\tilde{\nu}$ is invariant under the group generated by $\mathbb{T}(\mathbb{R}) = \{(a_t, e)\}_{t\in\mathbb{R}}$ and (b, b) (here we use the fact that (a_t, e) and (b, b) commute). In particular $\tilde{\nu}$ is invariant under a noncompact subgroup of G_f. It follows from the positivity of the dimension of μ and Theorem 2.8, that $h_{\tilde{\mu}}(a_1) > 0$. Then, since $(X, \tilde{\mu}, a_t)$ is a factor of $(Y, \tilde{\nu}, (a_t, e_f))$, we must have $h_{\tilde{\nu}}((a_1, e_f)) > 0$. We see that the conditions of Theorem 4.2 are satisfied and as a consequence that $\tilde{\nu} = \pi_*(\tilde{\nu})$ has μ_{G_∞} appearing as a component with positive weight in the ergodic decomposition of it with respect to the action of $\{a_t\}_{t\in\mathbb{R}}$ as desired. □

In order to complete the proof of Theorem 1.7 we shall need the following lemma:

Lemma 4.3. *The set $F = \{s \in \mathbb{R}/\mathbb{Z} : \overline{\{a_t u_s \Gamma_\infty\}_{t\geq 0}} \neq X\}$ is $\times n$-invariant.*

Proof of Theorem 1.7. Let F be as in Lemma 4.3. Assume to get a contradiction that $\mu(F) > 0$. It follows from the ergodicity that $\mu(F) = 1$. Let $\{U_i\}$ be a countable base for the topology of X. Define recursively

$$E_1 = \left\{x \in \operatorname{supp}(\mu) : \overline{\{a_t x\}_{t>0}} \cap U_1 = \emptyset\right\}, \quad \text{and}$$
$$E_n = \left\{x \in \operatorname{supp}(\mu) : \overline{\{a_t x\}_{t>0}} \cap U_n = \emptyset\right\} \setminus E_{n-1},$$

for any $n > 1$. Hence, $\{E_i\}$ form a partition up to a null set of the support of μ. Denote by μ_i, the restriction of μ to E_i. We continue as in the proof of Theorem 1.5 using the same notation as there. We now highlight the differences between the arguments: In the proof of Theorem 1.5 we used the fact that μ_i^T is compactly supported in order to pass to a weak* limit without losing mass. Here we do not know that μ_i^T is compactly supported, instead we use our further assumption that any weak* limit of μ^T is a probability measure. In particular $\tilde{\mu}(X) = 1$. This in turn implies that $\tilde{\nu}(Y) = 1$. Another difference is that in the proof of Theorem 1.5, μ_i was supported in a compact set and hence could not have μ_{G_∞} appear as a component with positive weight in its ergodic decomposition with respect to $\{a_t\}_{t\in\mathbb{R}}$. Here the reason that μ_i cannot have μ_{G_∞} appearing as a component with positive weight is that $\mu_i(U_i) = 0$.

After establishing the density of $\{a_t u_s \Gamma_\infty\}_{t\geq 0}$ for μ-almost any $s \in \mathbb{R}/\mathbb{Z}$, Lemma 2.6 implies that the continued fraction expansion of any such s contains any given pattern. □

In order to prove Lemma 4.3 we shall need the following lemma which follows immediately from ergodicity of the a_t action on X:

Lemma 4.4. *Let $C \subset X$ be closed and $\{a_t\}_{t\in\mathbb{R}}$-invariant. Then either $C = X$ or C has empty interior.*

Proof of Lemma 4.3. Let us change notation and set

$$G = \mathrm{PGL}_2(\mathbb{R}),\ \Gamma_1 = \mathrm{PGL}_2(\mathbb{Z}),\ \Gamma_2 = \operatorname{diag}(n^{-1},1)\Gamma_1 \operatorname{diag}(n,1),\ \text{and } \Gamma = \Gamma_1 \cap \Gamma_2.$$

Note that Γ is of finite index in both of the Γ_i's. It means that the natural projections $p_i : G/\Gamma \to G/\Gamma_i$ are finite covers. As such, they satisfy:

$$\text{for any } M \subset G/\Gamma, \quad p_i(\overline{M}) = \overline{p_i(M)}. \tag{4.4}$$

Now let $s \in \mathbb{R}/\mathbb{Z}$ be such that $ns \notin F$, i.e. such that $\overline{\{a_t u_{ns}\Gamma_1\}}_{t>0} = G/\Gamma_1$. We need to show that $s \notin F$, i.e. that the same holds for s instead of ns. Assume first that

$$\overline{\{a_t u_s \Gamma_2\}}_{t>0} = G/\Gamma_2. \tag{4.5}$$

It follows from (4.4) that $p_2(\overline{\{a_t u_s\Gamma\}}_{t>0}) = G/\Gamma_2$ so $\overline{\{a_t u_s\Gamma\}}_{t>0}$ must have nonempty interior in G/Γ (by Baire's category theorem for example) and in turn $p_1(\overline{\{a_t u_s\Gamma\}}_{t>0}) = \overline{\{a_t u_s\Gamma_1\}}_{t>0}$ has nonempty interior in G/Γ_1. Lemma 4.4 now implies that $\overline{\{a_t u_s\Gamma_1\}}_{t>0} = G/\Gamma_1$ as desired. We now argue the validity of (4.5). The fact that $\overline{\{a_t u_{ns}\Gamma_1\}}_{t>0} = G/\Gamma_1$ is equivalent to the set

$$\left\{a_t \operatorname{diag}(n,1) u_s \operatorname{diag}(n^{-1},1)\gamma : t > 0, \gamma \in \Gamma_1\right\}$$

being dense in G. As a_t and $\mathrm{diag}(n,1)$ commute, this is the same as to say that the set

$$\left\{a_t u_s \,\mathrm{diag}(n^{-1},1)\gamma\,\mathrm{diag}(n,1) : t>0, \gamma\in\Gamma_1\right\} = \{a_t u_s \gamma : t>0, \gamma\in\Gamma_2\}$$

is dense in G, which is exactly (4.5). □

5 Proof of Theorem 1.11

In this section we use the following notation. Let $\mathbb{A}$ denote the ring of adeles, $\mathbb{G}=\mathrm{PGL}_2$, $G=\mathbb{G}(\mathbb{R})$, $G'=\mathbb{G}(\mathbb{A})$, $\Gamma=\mathbb{G}(\mathbb{Z})$ and $\Gamma'=\mathbb{G}(\mathbb{Q})$. Γ' is a lattice in G' when embedded diagonally. We denote elements of G' as $(g_\infty, g_2, g_3, g_5\dots)$ and will abbreviate and denote them simply as (g_∞, g_f), where $g_f=(g_2,g_3\dots)$. Let $\mathbb{T}<\mathbb{G}$ be the the subgroup consisting of (classes of) diagonal matrices and denote $T'=\mathbb{T}(\mathbb{A})$, $T=\mathbb{T}(\mathbb{R})$. We denote $X=G/\Gamma$ and $Y=G'/\Gamma'$. $\bar{e}$ will denote the identity coset in both spaces. Define $\pi: Y\to X$ in the following way: For a point $y\in Y$, we choose a representative $(g_\infty,g_f)\in G'$, for which $g_f\in\mathbb{G}(\prod_p \mathbb{Z}_p)$, and define $\pi(y)=g_\infty\bar{e}$. π is well defined, continuous and has compact fibers. We use the notation and identification of (2.1), (2.2) and identify $\mathbb{R}/\mathbb{Z}$ with the periodic orbit of the horocycle u_t, through the identity coset $\bar{e}\in X$. We shall also need the following theorem of E. Lindenstrauss and its corollary.

Theorem 5.1 (Theorem 1.5 of [Li1]). *The action of the group, T', of adelic points of the torus $\mathbb{T}=\{\mathrm{diag}(*,*)\}<\mathrm{PGL}_2$ on $Y=\mathrm{PGL}_2(\mathbb{A})/\mathrm{PGL}_2(\mathbb{Q})$ is uniquely ergodic.*

COROLLARY 5.2. *Let $H<T'$ be a cocompact subgroup. Then there are no compactly supported H-invariant measures on Y.*

Proof. Assume by way of contradiction that ν is a compactly supported H-invariant measure on Y. Define

$$\tilde{\nu} = \int_{T'/H} \hat{g}_*\nu d\hat{g}\,.$$

Then $\tilde{\nu}$ is T'-invariant and compactly supported (because H is cocompact in T'). This contradicts Theorem 5.1. □

Proof of Theorem 1.11. For any prime p, let $b_p=\mathrm{diag}(p,1)\in\Gamma'$. We denote the diagonal embedding of b_p in G' by the same letter. Note that for any $s\in\mathbb{R}$, $b_p(u_s,e_f)\bar{e}=(u_{ps},e_f)\bar{e}$ and in particular, if $n=p_1\dots p_k$, then $\pi\left(b_{p_1}\dots b_{p_k}(u_s,e_f)\bar{e}\right)$ $=u_{ns}\bar{e}$. Assume that the statement of the theorem is false. Thus, by Lemma 2.5 there exists a compact set $K\subset X$ and an irrational $s\in[0,1)$ such that for any $n=p_1\dots p_k$, for large enough t,

$$K\ni a_t u_{ns}\bar{e}=\pi\big((a_t,e_f)b_{p_1}\dots b_{p_k}(u_s,e_f)\bar{e}\big)\,.$$

Hence, if we denote $K'=\pi^{-1}(K)\subset Y$ then for fixed $b_{p_1},\dots,b_{p_k}$ and all sufficiently large t

$$(a_t,e_f)b_{p_1}\dots b_{p_k}(u_s,e_f)\bar{e}\in K'. \tag{5.1}$$

Let $C < T'$ be the semigroup generated by (a_1, e_f) and the b_p's and let H be the group generated by C. H is cocompact in T'. To see this note that the compact set

$$\left\{\mathbf{a} = (\operatorname{diag}(e^t, e^{-t}), a_2, a_3 \dots) : t \in [0,1]\,,\ a_p \in \mathbb{T}(\mathbb{Z}_p)\right\}\,,$$

contains a fundamental domain for H in T'; this follows from the fact that for any element $\mathbf{a} = (a_\infty, a_2, a_3, \dots) \in T'$, for almost all primes p, the matrix a_p has entries in $\mathbb{Z}_p$ (see Remark 5.3).

Let F_n be a Følner sequence for C and define

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g_* \delta_{(u_s, e_f)\bar{e}}\,, \tag{5.2}$$

where $\delta_{(u_s,e_f)\bar{e}}$ is the Dirac measure centered at the point $(u_s, e_f)\bar{e}$. Let μ be a weak* limit of μ_n. It is H-invariant. On the other hand, we claim that if the Følner sequence is chosen appropriately then by (5.1), it is a probability measure supported in K'. This contradicts Corollary 5.2. We define F_n inductively in the following manner: We first choose a Følner sequence, F'_n, for the semigroup C' generated only by the b_p's. Then for a fixed n, there is some T_n such that, for any $g \in F'_n$ and for any $t > T_n$, $(a_t, e_f) g (u_s, e_f)\bar{e} \in K'$. It follows that there exists an integer $m_n > T_n$, such that if we define

$$F_n = F'_n \cup \left\{(a_1, e_f)^k\right\}_1^{m_n}, \tag{5.3}$$

then the weight that μ_n from (5.2) gives to K' is greater than $1 - 1/n$. □

Remark 5.3. It is tempting to replace in the above argument the group H by the group generated by (a_1, e_f) and the elements b_p^k for a fixed positive integer k. This would have implied the same statement of Theorem 1.11 with the sequence ns replaced by $n^k s$. Unfortunately the argument fails for any $k \geq 2$, as then H is no longer cocompact in T' (due to the fact that the topology on T' is not the product topology but the restricted one). Nonetheless, using a version of Theorem 5.1 for the group SL_2 (which is not available in the literature) and the choice $b_p = \operatorname{diag}(p, p^{-1})$, leads to a proof of the validity of statement of Theorem 1.11 for the sequence $n^2 s$. It seems plausible that a better understanding of the proof of Theorem 5.1 could lead to a proof of the validity of the statement for $n^k s$ for general k as well.

Remark 5.4. It is worth noting that a slight variant of the above argument actually yields a stronger uniform version of Theorem 1.11 namely

Theorem 5.5. *For any $M > 0$ there exists a number N such that for any irrational $s \in [0,1]$, there exists some $1 \leq n \leq N$ for which $c(np) \geq M$.*

We end this section with two natural questions which emerge from the proof of Theorem 1.11. We use the notation presented in that proof. In the argument yielding the proof of Theorem 1.11 we used the assumption that the sequence $c(ns)$ is bounded to guarantee that the sequence of measures μ_n constructed in (5.2) has no escape of mass. It seems plausible that if the number s is assumed to be badly approximable, then the non-escape of mass might be automatic for certain constructions of μ_n. More precisely:

QUESTION 5.6. Is it true that for any badly approximable number $s \in [0,1]$, one can choose the Følner sequence F'_n of C', such that if F_n is defined as in (5.3), with m_n arbitrarily large, then the sequence of probability measures μ_n defined in (5.2) has no escape of mass?

We note that by applying the results from [AkS] one can give a positive answer to Question 5.6 for quadratic irrationals which are of course badly approximable. It is not hard to see by applying Theorem 5.1, that a positive answer for Question 5.6 leads to a positive answer to the following question:

QUESTION 5.7. Is it true that for any badly approximable number $s \in [0,1]$, and for any finite pattern $\mathbf{w} = (w_1, \dots, w_\ell)$ of natural numbers, there exists $n \in \mathbb{N}$ such that the continued fraction expansion of ns contains the pattern $\mathbf{w}$ infinitely many times?

Appendix

Proofs of Several Lemmas

In this section we give proofs for some of the lemmas appearing in section 2.

Proof of Lemma 2.2. We think of points in $\mathrm{PGL}_{d+1}(\mathbb{R})/\mathrm{PGL}_{d+1}(\mathbb{Z})$ as unimodular lattices in $\mathbb{R}^{d+1}$ as in subsection 2.2. For $v \in \mathbb{R}^d$, the general form of a vector w in the lattice $a_t u_v \mathrm{PGL}_{d+1}(\mathbb{Z})$ is given by

$$w = \sum_1^d e^t(nv_i + m_i) + e^{-dt} n e_{d+1}\,, \tag{A.1}$$

where e_i denotes the standard basis of $\mathbb{R}^{d+1}$, v_i denotes the i-th coordinate of v and $m_i, n \in \mathbb{Z}$. Assume that $\epsilon > 0$ is given so that the inequality

$$|nv + \vec{m}|_\infty < \frac{\epsilon}{n^{1/d}} \tag{A.2}$$

has only finitely many solutions $\vec{m} \in \mathbb{Z}^d$, $n \in \mathbb{Z} \setminus \{0\}$. We will show that for $w \neq 0$ as in (A.1) $|w|_\infty > \epsilon$ for large enough t's. Theorem 2.1 then implies the validity of the lemma.

Let N_0 be given so that for $|n| \geq N_0$, there are no solutions to (A.2). For each n with $0 < |n| < N_0$, set $\delta_{v,n} = \min_{\vec{m}\in\mathbb{Z}^d} |nv + \vec{m}|_\infty$, and for $n = 0$ set $\delta_{v,0} = 1$. Note that as v is irrational (otherwise there would have been infinitely many solutions to (A.2)), we have for all $0 \leq |n| < N_0$ that $\delta_{v,n} > 0$. We denote $\min_{0\leq|n|<N_0} \delta_{v,n} = \delta$. Let $T > 0$ be such that for $t > T$, $e^t\delta > 1$. Let $t > T$ be given. We now estimate the norm of $w \neq 0$ in (A.1). There are two possibilities. If $0 \leq |n| < N_0$ then by construction, one of the first d coordinates of w is greater in absolute value than $e^t\delta > 1$. If $|n| \geq N_0$ then by the choice of N_0, (A.2) is violated and there exists $1 \leq i \leq d$ with $|nv_i + m_i| > \epsilon/n^{1/d}$. This means that the product of the i-th coordinate of w to the power of d, times the $(d+1)$-th coordinate satisfies $|(e^t(nv_i + m_i))^d(e^{-dt}n)| > \epsilon$. This shows (assuming $\epsilon < 1$) that one of the coordinates of w must be of absolute value greater than ϵ, as desired. □

Proof of Lemma 2.5. In this proof we use some basic facts about continued fractions. The reader is referred to [EW] and to [Po]. Let $s \in [0,1)$ be irrational with c.f.e. $s = [a_1, a_2 \dots]$. For $n \in \mathbb{N}$, let $p_n(s) = p_n$, $q_n(s) = q_n \in \mathbb{N}$ be the co-prime positive integers defined by the equation $p_n/q_n = [a_1, \dots a_n]$ (see (2.3)). p_n/q_n is called the n-th convergent of s. The following two identities are well known for all $n > 0$:

$$q_{n+1} = a_{n+1}q_n + q_{n-1}\,, \tag{A.3}$$

$$s - \frac{p_n}{q_n} = \sum_{k \geq n} (-1)^k \frac{1}{q_k q_{k+1}}\,.$$

It follows that $q_n \nearrow \infty$ and hence the above series is a Leibniz series and therefore we have

$$\left| s - \frac{p_n}{q_n} \right| \geq \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}}\,, \tag{A.4}$$

where the last equality follows from (A.3). By applying (A.3) twice, we have $q_{n+2} < (a_{n+2}+1)(a_{n+1}+1)q_n$. This together with (A.4) yields

$$|q_n s - p_n| \geq \frac{a_{n+2}}{(a_{n+2}+1)(a_{n+1}+1)q_n}\,. \tag{A.5}$$

It is also well known that the convergents give the best possible approximations to s in the following sense: For any rational a/b with $0 < b \leq q_n$ one has $|q_n s - p_n| \leq |bs - a|$. It follows that if $c(s) = \limsup a_n$ satisfies $c(s) < N$ for some $N \in \mathbb{N}$, then there are only finitely many solutions $a, b \in \mathbb{Z}$, $b \neq 0$, to the inequality

$$|bs + a| < \frac{(N+2)^{-2}}{b}\,.$$

Lemma 2.2 now gives us the desired result. □

For the proof of Lemma 2.6 we need some theory which we now survey. This theory dates back to the work of E. Artin (see [Se]). For a thorough discussion we refer the reader to [EW]. We first note that $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z}) \simeq \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSL}_2(\mathbb{Z})$. So we might as well carry on our analysis in the latter space. We let $\mathbb{H}$ denote the upper half plane. On $\mathbb{H}$ we take the Riemannian metric defined as usual by taking at the tangent space to the point $z = x + iy \in \mathbb{H}$, the inner product given by the usual Euclidean one, multiplied by $1/y^2$. With this metric, the right action of $G = \mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$, given by

$$z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{dz - b}{-cz + a}\,, \tag{A.6}$$

becomes an action by isometries. Hence, this action induces an action on the unit tangent bundle $T^1(\mathbb{H})$. One can easily check that this action is transitive and free, hence, once we choose a base point of $T^1(\mathbb{H})$, the orbit map gives a diffeomorphism between G and $T^1(\mathbb{H})$. We make the common choice for the base point and choose the point $i^{\uparrow}$ which denotes the unit vector pointing upwards in the tangent space to $i \in \mathbb{H}$. Fixing this identification of $T^1(\mathbb{H})$ and G once and for all, we are able to talk about the geodesic flow on G. It is an easy exercise to show that the geodesic flow is given by the action from the left of the diagonal group in G. More precisely, given $g \in G$ the point $a_{-t/2}g$ corresponds to the time t flow starting at g. Hence the action of the group a_t is then the *backwards geodesic flow in double speed*. We define for each $g \in G$ the starting (resp. end) point of the geodesic through g, $e_-(g)$ (resp. $e_+(g)$), to be the intersection of the path $\{a_t g\}_{t>0}$, projected to $\mathbb{H}$ (resp. $\{a_t g\}_{t<0}$) with the boundary of $\mathbb{H}$ in $\mathbb{C} \cup \{\infty\}$, namely with $\mathbb{R} \cup \{\infty\}$. In other words, in the notation of (A.6) we have

$$e_-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{-b}{a}, \quad e_+\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{-d}{c}. \tag{A.7}$$

We see that the starting point of u_s is s. We now wish to connect the continued fraction expansion (c.f.e.) of s with the geodesic ray $\{a_t u_s\}_{t>0}$ which starts at s. We denote the projection from G to $G/\operatorname{PSL}_2(\mathbb{Z})$ by π. We will need the following three subsets of G:

$$\begin{aligned} C^+ &= \left\{g \in G : g \text{ lies on the } y \text{ axis, and } e_-(g) \in [0,1], e_+(g) < -1\right\}, \\ C^- &= \left\{g \in G : g \text{ lies on the } y \text{ axis, and } e_-(g) \in [-1,0], e_+(g) > 1\right\}, \\ C &= C^+ \cup C^-. \end{aligned}$$

The reader could prove the following theorem by simple geometric arguments (see [EW]).

Theorem A.1. *The submanifold $C \subset G$ has the following properties:*

(1) *$\pi : C \to \pi(C)$ is injective. Hence we have a canonical way of defining the starting (resp. end) point $e_-(x)$ (resp. $e_+(x)$) of $x \in \pi(C)$.*
(2) *$\pi(C)$ is a cross section for the geodesic flow. We denote the first return map (with respect to the a_t-action) by $\rho : \pi(C) \to \pi(C)$. A point $x \in \pi(C)$ returns to $\pi(C)$ infinitely often (i.e. $\rho^n(x)$ is defined for all $n > 0$) if and only if $e_-(x)$ is irrational. In this case, its visits to $\pi(C)$ alternate between $\pi(C^+)$ and $\pi(C^-)$.*
(3) *The map $x \mapsto |e_-(x)|$ from $\pi(C)$ to $[0,1]$ is a factor map connecting the first return map ρ and the Gauss map on the unit interval (which is the shift on the c.f.e.).*

The last bit of information we need in order to argue the proof of Lemma 2.6, is that if $s_1, s_2 \in [0,1] \setminus \mathbb{Q}$ satisfy $s_1 = s_2\gamma$ for some $\gamma \in \operatorname{PSL}_2(\mathbb{Z})$ (the action given in (A.6)), then the continued fraction expansions of s_1 and s_2 only differ at their beginnings.

Proof of Lemma 2.6. Let $s \in [0,1]$ be such that $\{a_t u_s \operatorname{PSL}_2(\mathbb{Z})\}_{t>0}$ is dense in $G/\operatorname{PSL}_2(\mathbb{Z})$. In particular, s is irrational. Given a pattern $(b_1, \dots, b_k) \in \mathbb{N}^k$, the set

$$P = \left\{s \in [0,1) \setminus \mathbb{Q} : a_i(s) = b_i \text{ for } 1 \le i \le k\right\}$$

is an open set in $[0,1) \setminus \mathbb{Q}$. It follows from (3) of Theorem A.1 that there is an open set $\tilde{P} \subset \pi(C)$, so that for any point $x \in \tilde{P}$, the starting point $e_-(x)$, if irrational, is in P. The density assumption gives us that there exists a sequence of times $t_i \nearrow \infty$ such that $x_i = a_{t_i} u_s \operatorname{PGL}_2(\mathbb{Z}) \in \tilde{P}$ and moreover by (2) of Theorem A.1 we may assume that $x_i \in C^+$, hence $e_-(x_i) \in [0,1]$. Now the c.f.e. of $s = e_-(u_s)$ differs from that of $e_-(x_1)$ only in their beginnings (by the paragraph preceding this proof) but by (3) of Theorem A.1, the c.f.e. of $e_-(x_1)$ must contain the pattern $b_1 \dots b_k$ infinitely many times (as the c.f.e. of $e_-(x_i)$ starts with this pattern and is a shift of the c.f.e. of $e_-(x_1)$), hence so does the c.f.e. of s as desired. □

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