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Theory of Computing Systems

The Clique Problem in Intersection Graphs of Ellipses and Triangles*

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Abstract. Intersection graphs of disks and of line segments, respectively, have been well studied, because of both practical applications and theoretically interesting properties of these graphs. Despite partial results, the complexity status of the CLIQUE problem for these two graph classes is still open. Here, we consider the CLIQUE problem for intersection graphs of ellipses, which, in a sense, interpolate between disks and line segments, and show that the problem is $\mathcal{APX}$-hard in that case. Moreover, this holds even if for all ellipses, the ratio of the larger over the smaller radius is some prescribed number. Furthermore, the reduction immediately carries over to intersection graphs of triangles. To our knowledge, this is the first hardness result for the CLIQUE problem in intersection graphs of convex objects with finite description complexity. We also describe a simple approximation algorithm for the case of ellipses for which the ratio of radii is bounded.

1. Introduction

Let $\mathcal{M}$ be a collection of sets. The *intersection graph* of $\mathcal{M}$ is the abstract graph G whose vertices are the sets in $\mathcal{M}$, and two vertices are connected by an edge if the corresponding

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sets intersect; formally,

$$V(G) = \mathcal{M} \quad \text{and} \quad E(G) = \{\{M, N\} \subseteq \mathcal{M} : M \cap N \neq \emptyset\}.$$

The family $\mathcal{M}$ is called a *representation* of the graph G.

Intersection graphs of various classes of geometric objects have been studied, because of both practical applications and interesting structural properties of the graphs in question. Two prominent examples that have received a lot of attention are intersection graphs of disks (see [17] and [6]) and of line segments (see [15] and [11]), respectively.

For instance, intersection graphs of disks, *disk graphs* for short, arise naturally when studying interference in networks of radio or mobile phone transmitters [1]. Many of these graphs are hard to recognize. For example, recognizing unit disk graphs and general disk graphs is NP-hard [7], [12]. However, it is not known whether these problems are actually in NP. Only PSP ACE-membership is known [7]. On the other hand, disk contact graphs can be recognized in linear time, since this class coincides with the class of planar graphs [13].

One reason to study intersection graphs is the hope that they provide classes of graphs for which optimization problems which are hard for general graphs become tractable. As an example, CLIQUE is polynomially solvable in unit disk graphs [8]. Since recognition is hard for many of these classes, usually the geometric representation has to be provided in the input.

Even if a problem remains NP-hard in a certain graph class, using its structure might lead to better approximation algorithms or even allow a PTAS, such as for INDEPENDENT SET and VERTEX COVER in the case of disk graphs [10].

In this article we consider the CLIQUE problem, i.e., the problem of finding a maximal complete subgraph. Its complexity status is unknown for both disk graphs and intersection graphs of line segments.

The graphs consider are intersection graphs of ellipses (which contain both of the above classes) and show that the CLIQUE problem for these graphs is $\mathcal{APX}$-hard. That is, unless $\mathcal{P} = \mathcal{NP}$, there is a constant c such that there is no approximation algorithm with ratio better than c. Hence, there is no PTAS. What is more, this remains true even if all the ellipses are required to be arbitrarily "round" (or circle-like) or arbitrarily "stretched" (or segment-like). More precisely, given $1 < \rho < \infty$, let ELLIPSE$_\rho$CLIQUE, respectively ELLIPSE$_{\leq\rho}$CLIQUE, be the CLIQUE problem in intersection graphs of ellipses for which the ratio of the larger over the smaller radius is exactly ρ, respectively at most ρ. We stress that in this definition, we mean intersection graphs of ellipses *without interior* (i.e., if one ellipse is completely contained in the interior of another one, they are not considered to intersect). When considering ellipses with their interior, we call them *filled* and denote the corresponding problem by FILLEDELLIPSE$_\rho$CLIQUE. Note that FILLEDELLIPSE$_\rho$CLIQUE is at least as hard as ELLIPSE$_\rho$CLIQUE, since intersection graphs of ellipses without interior are also intersection graphs of filled ellipses.

Theorem 1. *For every* $\rho > 1$, *the problem* ELLIPSE$_\rho$CLIQUE *is* $\mathcal{APX}$*-hard.*

This theorem is proved in Section 2 by a reduction from MAX-B-OCC-2SAT, which is the following optimization problem: given a Boolean formula φ in conjunctive normal form with at most two literals per clause and at most B occurrences of every variable, find an assignment of truth values to the variables that satisfies the maximum number of clauses. MAX-B-OCC-2SAT is known to be $\mathcal{APX}$-hard for $B \geq 3$ [5].

We would like to stress that the inapproximability ratio in Theorem 1 is independent of the parameter ρ, so it does not matter how close our ellipses are to the "limit cases" $\rho = 1$ (corresponding to circles) or $\rho = \infty$ (corresponding to segments, or to parabolas).

Furthermore, the reduction immediately carries over to intersection graphs of triangles (they can even be made isosceles if desired).

Theorem 2. *The problem* TRIANGLECLIQUE *is* $\mathcal{APX}$*-hard.*

We note that Theorems 1 and 2 strengthen a result of Kratochvíl and Kuběna [14], who proved that the CLIQUE problem is $\mathcal{NP}$-complete for intersection graphs of general (compact) convex subsets of the plane. (In fact, they proved a stronger result, namely that every co-planar graph has an (efficiently computable) representation as the intersection graph of some family of convex sets in the plane.) The interesting aspect here is that the proof of Kratochvíl and Kuběna relies in an essential way on the fact that the boundary of convex sets has non-constant description complexity—in technical terms, that convex sets have infinite *VC dimension* [16]. Ellipses and triangles, by contrast, have finite VC dimension.

Moreover, if the ratio of radii is bounded, filled ellipses also have a finite *transversal number*. That is, for every $\rho \geq 1$, there is a number $\tau(\rho) \in \mathbf{N}$ such that, for every family $\mathcal{C}$ of pairwise intersecting filled ellipses with ratio of radii at most ρ, there is some set S of at most $\tau(\rho)$ points which *pierce* $\mathcal{C}$ in the sense that every $L \in \mathcal{C}$ contains some point $p \in S$. In Section 3 we exploit this to give an approximation algorithm for FILLEDELLIPSE$_{\leq\rho}$CLIQUE.

Theorem 3. *For every* $1 < \rho < \infty$, *the problem* FILLEDELLIPSE$_{\leq\rho}$CLIQUE *can be approximated within a factor of* $\min\{9\rho^2, \tau(\rho)/2\}$ (*this also applies when we consider ellipses with their interiors*). *For* DISKCLIQUE, *the approximation factor can be improved to* 2.

2. Reduction from MAX-*B*-OCC-2SAT to ELLIPSE$_\rho$CLIQUE

We first recall some facts about ellipses. An ellipse is an affine transformation of the unit circle. That is,

$$E = f(K), \qquad f(x) = R \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} x + a, \tag{1}$$

where K is the unit circle centered at the origin in $\mathbf{R}^2$, R is an orthogonal 2×2 matrix, r, s are positive real numbers, and $a \in \mathbf{R}^2$ (the *center* of E). Then E can also be written as the zero set of a quadratic bivariate polynomial,

$$E = E(A, a) = \{x \in \mathbf{R}^2 : (x - a)^{\mathrm{T}} A (x - a) = 1\}, \tag{2}$$

where "$\cdot^{\mathrm{T}}$" denotes the transpose, and

$$A = R \begin{bmatrix} 1/r^2 & 0 \\ 0 & 1/s^2 \end{bmatrix} R^{\mathrm{T}}$$

is a positive definite symmetric 2×2 matrix (observe that $R^{\mathrm{T}} = R^{-1}$). Thus, A has positive real eigenvalues $\lambda = 1/r^2$, $\mu = 1/s^2$; in other words, $1/\sqrt{\lambda}$ and $1/\sqrt{\mu}$ are the radii of E. Similarly, a filled ellipse is an affine transformation of the unit disk and can be written as $\{x \in \mathbf{R}^2 : (x-a)^{\mathrm{T}} A (x-a) \le 1\}$.

For computational purposes, we assume that in instances of ELLIPSE$_\rho$CLIQUE, the ellipses are specified as in (2) with rational coefficients $a \in \mathbf{Q}^2$ and $A \in \mathbf{Q}^{2\times 2}$.

For the reduction to be polynomial, we also need to ensure that the numbers involved stay polynomial in size. In fact, we describe a construction involving small algebraic numbers. To complete the reduction, we invoke certain perturbation arguments, which we sketch at the end of this section.

We now start with the description of the reduction. Fix $\rho > 1$ and suppose we are given a formula φ in the variables $x_1, \dots, x_n$. We begin by introducing ellipses representing the variables and their negations, respectively, in Section 2.1. In Section 2.2 we prove the existence of suitable ellipses which will represent the clauses. In Section 2.3 we combine these building blocks to prove Theorem 1.

2.1. *Ellipses Representing the Literals*

We introduce ellipses representing the variables and their negations, respectively. We start out with two auxiliary concentric circles C_1 and C_0 of radius r (to be chosen later) and 1, respectively, with common center c.

Let L be an ellipse with radii $r-1$ and $\rho(r-1)$. We place congruent copies $L_1, \dots, L_{2n}$ of L along the outer circle C_1 such that their centers lie on C_1 and form the vertices of a regular $2n$-gon (with numbering in counterclockwise order), and such that, for each L_i, the main axis corresponding to the radius $r-1$ is perpendicular to the circle C_1. Thus, each ellipse L_i touches the inner auxiliary circle C_0 in a point p_i.

By choosing r sufficiently large, we may achieve that these ellipses pairwise intersect, except for pairs L_i, L_{i+n} of antipodal ellipses, which are disjoint (see Figure 1 for an example with $n = 4$).

One can prove that $r = O(n^2)$ is sufficient as follows. First we can assume $\rho = 1$ here, since the radius needed decreases with growing ρ. The minimum value of r needed is determined by a pair of ellipses L_i and $L_{i+(n-1)}$, $1 \le i \le n$. That is, a pair of ellipses that are almost opposite of each other in the rosette. Let the midpoints of L_i and $L_{i+(n-1)}$ be M_1 and M_2 and let α be the angle $\angle M_1 C_0 M_2$. The value of α then is $\pi - \pi/n$. For the two ellipses to intersect, r must be large enough such that

$$\sin\left(\frac{\alpha}{2}\right) \le \frac{r-1}{r}.$$

Solving for r, we obtain

$$r \ge \frac{1}{1-\cos(\pi/2n)} = \frac{8}{\pi^2}n^2 + \frac{1}{6} + O\left(\frac{1}{n^2}\right).$$

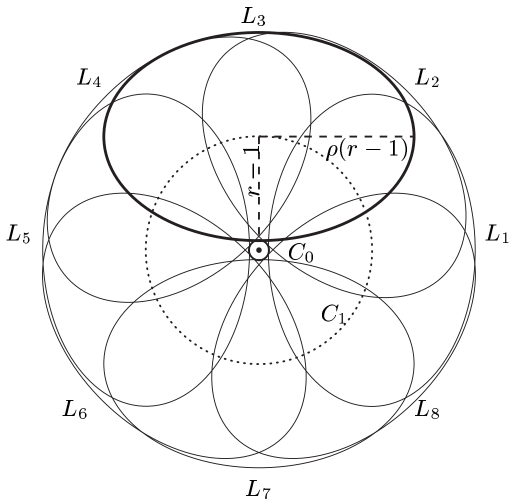

Fig. 1. Rosette of literalipses.

For a literal ξ, let $L(\xi)$ be the ellipse L_i, if ξ is a variable x_i, and L_{i+n} if ξ is a negated variable $\neg x_i$. These ellipses will be called the *literalipses*.

2.2. *Ellipses Representing the Clauses*

The second building block of our reduction are ellipses which avoid two prescribed literalipses but intersect all others. These are used to represent the clauses of φ, as will be described in Section 2.3.

Lemma 1. *Let $\rho > 1$. For any two literalipses $L(\xi)$ and $L(\omega)$, there is a clause ellipse $E = E(\xi, \omega)$ whose ratio of radii is ρ and which intersects all literalipses except $L(\xi)$ and $L(\omega)$. Moreover, all these clause ellipses intersect one another.*

Note that the lemma also holds if only one literalipse needs to be avoided. The proof of Lemma 1 is based on the upcoming, rather technical, Lemma 2. We begin by introducing some notation.

Consider an ellipse L and two points a, b on L. By $(a, b)_L$, we denote the open arc of L that lies to the right of the oriented line $\overrightarrow{ab}$ through a and b. (See Figure 2.)

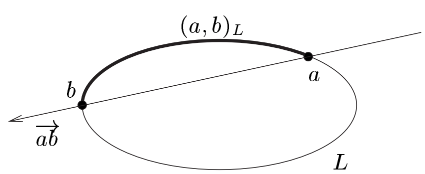

Fig. 2

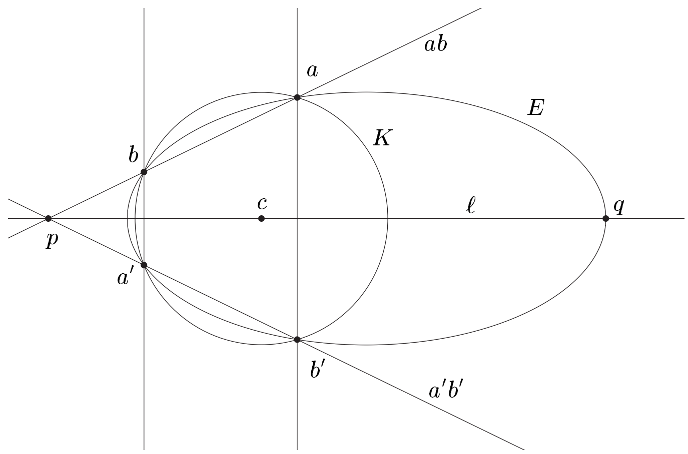

Fig. 3

Lemma 2. *Consider a circle K with center c, and four points a, b, a', b' in counter-clockwise order on K such that the arcs $(a, b)_K$ and $(a', b')_K$ are of the same length and disjoint. Let p be the point where the lines ab and $a'b'$ intersect, and let ℓ be the line through p and c (if ab and $a'b'$ are parallel, we take ℓ to be that line through c which is parallel to both of them). (See Figure 3).*

Then, if q is any point on ℓ such that the segment $[p, q]$ intersects K twice, there is a unique ellipse E through the five points a, b, a', b', and q. Moreover, if we move q away from p towards infinity on ℓ, the ratio of radii of E grows monotonically and tends to ∞.

Furthermore, the arcs $(a, b)_E$ and $(a', b')_E$ are completely contained in the interior of K, and the arcs $(b, a')_E$ and $(b', a)_E$ are contained in the intersection of the open half-planes to the left of $\overrightarrow{ab}$ and to the left of $\overrightarrow{a'b'}$. On the other hand, the arcs $(b, a')_K$ and $(b', a)_K$ of K are contained in the interior of E.

Proof. Without loss of generality, K is the unit circle centered at $c = 0$, ℓ is the x-axis, and, for suitable real parameters r, s, t, $a = (s, \sqrt{1 - s^2})$, $b' = (s, -\sqrt{1 - s^2})$, $b = (r, \sqrt{1 - r^2})$, $a' = (r, -\sqrt{1 - r^2})$, and $q = (t, 0)$.

There is a unique conic E through the five points a, b, a', b', q. Moreover, by the symmetry of these points with respect to the x-axis, E is of the form

$$E = \{(x, y) : \lambda(x - m)^2 + \mu y^2 = 1\} \tag{3}$$

for suitable real parameters λ, μ, and m. Here, m is the x-coordinate of the center of E. Moreover, E is an ellipse if λ and μ have the same sign. Then the radii of E are $1/\sqrt{|\lambda|}$ and $1/\sqrt{|\mu|}$, respectively.

By assumption, $a, b, q \in E$, which yields the three equations

$$\lambda(r-m)^2+\mu(1-r^2) = 1, \qquad \lambda(s-m)^2+\mu(1-s^2) = 1, \qquad \lambda\,(t-m)^2 = 1. \tag{4}$$

Using Maple™, solving these for λ, μ, and m yields

$$m = m(r,s,t) = -\frac{1}{2}\frac{-s + s\,t^2 - r + rt^2}{rs - st + 1 - rt} =: -\frac{1}{2}\frac{f(r,s,t)}{g(r,s,t)},$$

$$\lambda = \lambda(r,s,t) = 4\,\frac{(rs - st + 1 - rt)^2}{(2trs - st^2 + 2t - rt^2 - s - r)^2} =: 4\frac{g(r,s,t)^2}{h(r,s,t)^2},$$

$$\mu = \mu(r,s,t)$$

$$= 4\frac{r^2t^2 - 2r^2st + r^2s^2 + 3st^2r - rt - rt^3 + rs - 2rs^2t + s^2t^2 - st^3 - st + t^2}{(2trs - st^2 + 2t - rt^2 - s - r)^2}$$

$$=: 4\frac{i(r,s,t)}{h(r,s,t)^2}.$$

Let us consider the zeros and singularities of these functions: For given r and s,

$$f(r,s,t) = 0 \Leftrightarrow t \in \{-1, +1\},$$

$$g(r,s,t) = 0 \Leftrightarrow t = \frac{rs+1}{r+s},$$

$$h(r,s,t) = 0 \Leftrightarrow t = \frac{1}{2}\frac{2 + 2rs + 2\sqrt{(1-r^2)(1-s^2)}}{r+s} \qquad \text{(double zero)},$$

$$i(r,s,t) = 0 \Leftrightarrow t \in \left\{r, s, \frac{rs+1}{r+s}\right\}.$$

By symmetry, we may assume that the point p lies to the left of K, i.e., that $r \leq -|s|$. Fix such r and s. Straightforward calculations show that $t = s$ is largest among the roots of $g(r,s,t)$, $h(r,s,t)$, and $i(r,s,t)$. Therefore, on the interval $s < t < \infty$, $\lambda(r,s,t)$ and $\mu(r,s,t)$ are continuous functions of t that do not change signs. Since $\mu(r,s,1) = \lambda(r,s,1) = 1$, we see that $\lambda(r,s,t), \mu(r,s,t) > 0$ for $t \in (s,\infty)$, hence E is an ellipse in that range of t. Note also that the ratio $\mu(r,s,t)/\lambda(r,s,t) \to \infty$ as $t \to \infty$, and that it grows monotonically for

$$t > \max\{(rs+1)/(r+s), (2 + 2rs + 2\sqrt{(1-r^2)(1-s^2)})/(2(r+s))\},$$

in particular for $t \geq 1$.

For the claimed containment properties, it suffices to observe that K and E have no points of intersection except a, b, a', b', and that there are no points of intersection of E and ab except a and b, and analogously for $a'b'$. □

Proof of Lemma 1 (*using Lemma* 2). Suppose $L(\xi) = L_i$ and $L(\omega) = L_j$. Let p_{i-1}, p_i, p_{i+1}, p_{j-1}, p_j, and p_{j+1} be the points at which L_{i-1}, L_i, L_{i+1}, L_{j-1}, L_j, and L_{j+1}, respectively, touch the inner circle C_0. Let a_i be the midpoint of the arc $(p_{i-1}, p_i)_{C_0}$ and let b_i be the midpoint of the arc $(p_i, p_{i+1})_{C_0}$. The points a_j and b_j are defined analogously. Finally, let q' be the midpoint of the arc $(b_j, a_i)_{C_0}$. (See Figure 4.) Consider the point $q = q' + t(q' - c)$ for a parameter $t \geq 0$. Let E be the ellipse through a_i, b_i, a_j, b_j, and q whose existence is guaranteed by the preceding lemma. By the containment properties

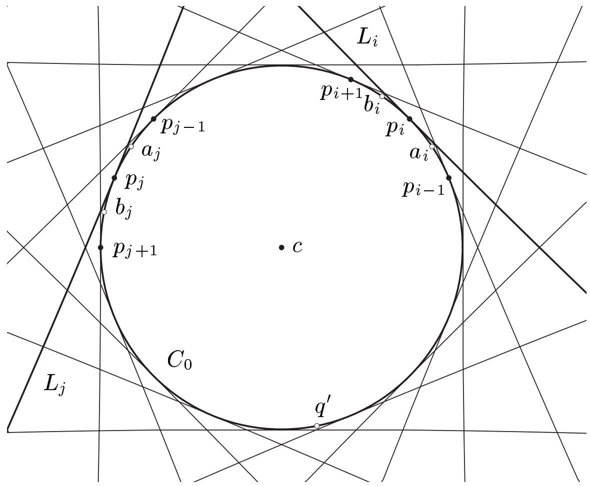

Fig. 4. Detail of the rosette.

asserted above, E avoids L_i and L_j but intersects all L_l, $l \neq i, j$. Moreover, the ratio of radii of E depends continuously on the parameter t and tends to infinity with t. Thus, since we have ratio 1 for $t = 0$, we can achieve any prescribed ratio. Therefore, E is as advertised. It is easy to see that all clause ellipses intersect each other since all of them contain point c. □

2.3. *The Reduction*

We are now ready to complete the reduction: Fix $\rho > 1$. Additionally, fix $B \geq 3$ and let $Q := \lfloor 3B/2 \rfloor$. Given a MAX-$B$-OCC-2SAT formula φ with n variables and m clauses, we construct a collection $\mathcal{L} = \mathcal{L}(\varphi)$ of $2Qn + 3m$ ellipses, as follows:

1. For each variable x that occurs in φ, we take Q copies of the ellipse $L(x)$ and Q copies of the ellipse $L(\neg x)$. These literalipses are arranged into a rosette as described in Section 2.1. We stress that the auxiliary circles C_0 and C_1 are not part of $\mathcal{L}$.
2. For each clause $\kappa = \xi \vee \omega$ of φ, we take the three ellipses $E(\neg\xi, \neg\omega)$, $E(\neg\xi, \omega)$, and $E(\xi, \neg\omega)$ according to Lemma 1. If a clause contains only a single literal ξ, take clause ellipse $E(\neg\xi)$. If there are several clauses $\kappa_1, \ldots, \kappa_l$ that require an ellipse E in this fashion, we take the corresponding number of copies $E_{\kappa_1}, \ldots, E_{\kappa_l}$ of E.

It remains to verify that we have indeed reduced MAX-B-OCC-2SAT to the problem ELLIPSE$_\rho$CLIQUE. This is established by the following:

Lemma 3. *Let φ be an instance of* MAX-B-OCC-2SAT *with n variables and m clauses, and let $\mathcal{L}$ be the corresponding* ELLIPSE$_\rho$CLIQUE *instance just defined. $\mathcal{L}$ contains a clique of size $Qn + k$ if and only if there is an assignment of truth values to the variables of φ that satisfies k clauses of φ.*

Proof. We first show how to find a corresponding clique for a given assignment. Fix an assignment $\mathcal{A}$ of truth values. For each literal ξ that is made TRUE by $\mathcal{A}$, take all Q copies of $L(\xi)$. These form a clique. Moreover, a clause $\xi \vee \omega$ of φ is satisfied by the assignment if and only if one of the following three cases occurs:

1. $\xi =$ TRUE and $\omega =$ TRUE.
2. $\xi =$ TRUE and $\omega =$ FALSE.
3. $\xi =$ FALSE and $\omega =$ TRUE.

In the first case we have already taken Q copies of $L(\xi)$ and of $L(\omega)$, respectively. Thus, we can enlarge our clique by one element by adding the ellipse $E(\neg\xi, \neg\omega)$ (to be more precise: by adding that copy of it which we have taken into $\mathcal{L}$ on account of the clause $\xi \vee \omega$). We cannot, however, add either of the ellipses $E(\xi, \neg\omega)$ or $E(\neg\xi, \omega)$, which avoid ξ and ω, respectively. The other two cases are treated analogously. Altogether, the clique thus constructed contains $Q \cdot n$ literalipses (Q for each satisfied literal) and k clause ellipses (one for each satisfied clause).

Conversely, let $\mathcal{C}$ be a clique of size $Qn + k$, $k \geq 0$. We may assume that for every variable x, $\mathcal{C}$ contains Q copies of $L(x)$ or Q copies $L(\neg x)$. For suppose there is a variable x such that $\mathcal{C}$ does not contain a copy of either $L(x)$ or $L(\neg x)$. Let $\kappa_1^+, \ldots, \kappa_a^+$ and $\kappa_1^-, \ldots, \kappa_b^-$ be the clauses of φ in which x, respectively $\neg x$, occur. We have $a + b \leq B$. Each κ_i^+ yields two clause ellipses in $\mathcal{L}$ that avoid $L(x)$, and one which avoids $L(\neg x)$. Similarly, each κ_j^- yields two ellipses which avoid $L(x)$, and one which avoids $L(\neg x)$. Therefore, $\mathcal{C}(\subseteq \mathcal{L})$ contains at most $3(a + b) \leq 3B$ clause ellipses which avoid either $L(x)$ or $L(\neg x)$. Thus, for some $\xi \in \{x, \neg x\}$, $L(\xi)$ is avoided by at most $Q = \lfloor 3B/2 \rfloor$ ellipses from $\mathcal{C}$. However, then, if we remove these ellipses from $\mathcal{C}$ and replace them by the Q copies of $L(\xi)$, we do not decrease $|\mathcal{C}|$.

Therefore, without loss of generality, $\mathcal{C}$ contains Qn literalipses. Then $\mathcal{C}$ induces a truth value assignment in the obvious fashion: set variable x to TRUE if $\mathcal{C}$ contains (all Q copies of) $L(x)$, and to FALSE otherwise.

The remaining k elements of $\mathcal{C}$ are clause ellipses. Consider such an ellipse E. There must be a clause κ that caused $E = E_\kappa(\xi, \omega)$ to be included in the ELLIPSE$_\rho$CLIQUE instance. Call κ the *witness clause* of E (κ could be $\neg\xi \vee \neg\omega$, $\neg\xi \vee \omega$, or $\xi \vee \neg\omega$). Now, E avoids $L(\xi)$ and $L(\omega)$, hence $\mathcal{C}$ must contain all copies of $L(\neg\xi)$ and all copies of $L(\neg\omega)$. Therefore, the assignment induced by $\mathcal{C}$ satisfies the witness clause κ of E. Since this holds for all clause ellipses in $\mathcal{C}$, the assignment satisfies at least k clauses of φ (one for each clause ellipse contained in $\mathcal{C}$). □

From the above lemma, it is easy to obtain $\mathcal{APX}$-hardness.

Corollary 1. *Let φ be an instance of* MAX-B-OCC-2SAT *consisting of n variables, m clauses and let $\mathcal{L}$ be the corresponding instance of* ELLIPSE$_\rho$CLIQUE. *Let OPT be the maximum number of satisfied clauses of φ by any assignment of the variables and let OPT′ be the size of a maximum clique in $\mathcal{L}$, and let $\varepsilon > 0$ and $\gamma > 0$ be constants. Then*

$$OPT \geq (1 - \varepsilon)m \Longrightarrow OPT' \geq Qn + (1 - \varepsilon)m,$$
$$OPT < (1 - \varepsilon - \gamma)m \Longrightarrow OPT' < Qn + (1 - \varepsilon - \gamma)m.$$

Proof. This follows immediately from Lemma 3. We just have to replace k by $(1-\varepsilon)m$ or $(1-\varepsilon-\gamma)m$, respectively. □

In a promise problem of MAX-B-OCC-2SAT, we are promised that either at least $(1-\varepsilon)m$ clauses or at most $(1-\varepsilon-\gamma)m$ clauses are satisfiable, and we are to find out which of the two cases holds. This problem is *NP*-hard for sufficiently small values of $\varepsilon > 0$ and $\gamma > 0$ (see [4]). Therefore, Lemma 1 implies that the promise problem for ELLIPSE$_\rho$CLIQUE, where we are promised that the maximum clique is either of size at least $Qn + (1-\varepsilon)m$ or at most $Qn + (1-\varepsilon-\gamma)m$, is *NP*-hard as well, for sufficiently small values of $\varepsilon > 0$ and $\gamma > 0$. Thus, ELLIPSE$_\rho$CLIQUE cannot be approximated with a ratio of

$$\frac{Qn + (1-\varepsilon)m}{Qn + (1-\varepsilon-\gamma)m} \geq 1 + \frac{\gamma m}{Qn + (1-\varepsilon-\gamma)m}$$
$$\geq 1 + \frac{(n/2)\gamma}{Qn + Bn(1-\varepsilon-\gamma)} = 1 + \frac{\gamma}{2Q + 2B(1-\varepsilon-\gamma)},$$

where we have used that $m/B \leq n \leq 2m$. We let $\delta := \gamma/(2Q + 2B(1-\varepsilon-\gamma))$. Since $\delta > 0$, we have shown that ELLIPSE$_\rho$CLIQUE cannot be approximated by any polynomial-time approximation algorithm with an approximation ratio of $1 + \delta$. This proves Theorem 1.

The construction of the rosette and Lemma 1 immediately carry over to intersection graphs of triangles (they can even be made isosceles if desired) and therefore prove Theorem 2. Figure 5 sketches a rosette of literal triangles and a triangle $T = T(T_i, T^j)$ avoiding two given literal triangles T_i and T_j but intersecting all others.

2.4. *Perturbations*

As already mentioned above, we need to ensure that the numbers involved in the reduction stay polynomial in size.

The reduction produces two kinds of ellipses. The ellipses representing the variables are defined by (1), where the entries of a and R are of the form $k \cdot \sin(2\pi/n \cdot i)$ and $k \cdot \cos(2\pi/n \cdot i)$ for $i \in \{0, \ldots, n-1\}$ and $k > 1$ integer. Furthermore, we have $r = \rho s$ with integer s.

The literalipses are defined by five points. Four of them are described by trigonometric expressions similar to the entries of R and one point is of the form $(t, 0)$ where t is the root of a polynomial in ρ, t, and n.

Note that the reduction is stable in the sense that ellipses which intersect do this in such a way that some circle of radius polynomial in $1/n$ fits into the intersection. Conversely, if a pair of ellipses is not to intersect, their distance from each other is of the same form. Note that any dependence on ρ is allowed, since ρ is considered to be a constant.

Thus one can argue that one can approximate all numbers involved by polynomial precision without changing the intersection pattern of the ellipses.

We also note that we actually construct a multiset of ellipses (in other words, the ellipses have non-negative integer weights). In order to obtain a set of ellipses in which no element occurs more than once, we have to invoke perturbation arguments as above a second time.

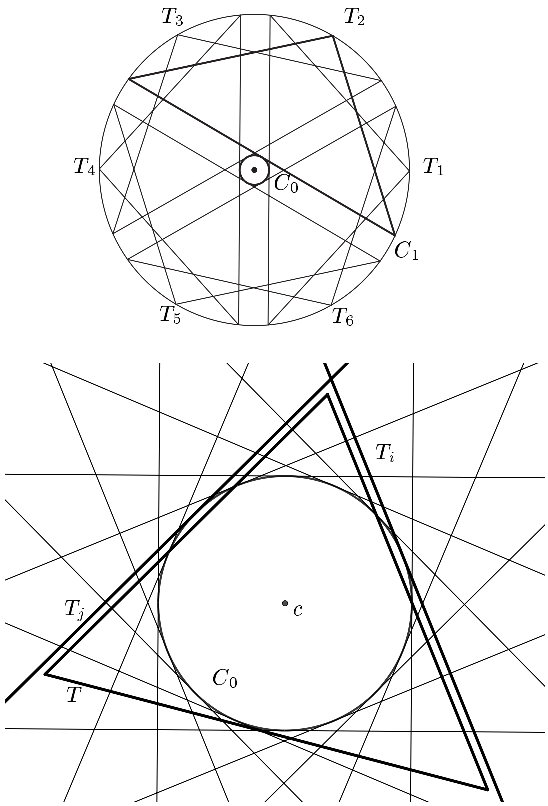

Fig. 5. A rosette of triangles and an avoiding triangle.

2.5. *A More General View on the Reduction*

Looking back, the reduction described in Section 2.3 in fact proves $\mathcal{APX}$-hardness for the CLIQUE problem in a more general context.

Let $B \geq 3$ and $Q = \lfloor 3B/2 \rfloor$. For $n \in \mathbf{N}$, consider the following graph G_n: $V(G_n)$ contains Q vertices $v_i^1, \ldots, v_i^Q$ for every integer $i \in \{1, \ldots, n\} \dot{\cup} \{-1, \ldots, -n\}$ and B vertices $w_{i,j}^1, \ldots, w_{i,j}^5$ for every pair of such integers. Furthermore, all edges are present in $E(G_n)$ except for those connecting vertices v_i^a and v_{-i}^b, and except for those edges connecting $w_{i,j}^c$ to v_i^a and v_j^b, respectively, $1 \leq a, b \leq Q$, $1 \leq c \leq B$.

By taking suitable induced subgraphs of G_n (depending on the formula φ), our reduction immediately yields the following generalization of Theorem 1.

Theorem 4. *Let $\mathcal{K}$ be a class of sets such that for every n, the graph G_n has a representation* (*of description size polynomial in n*) *as a $\mathcal{K}$-intersection graph* (*i.e., as an intersection graph of some subset of $\mathcal{K}$*). *Then the* CLIQUE *problem is $\mathcal{APX}$-hard in $\mathcal{K}$-intersection graphs.*

3. An Approximation Algorithm for Ellipses of Bounded Ratio

In this section we consider ellipses with their interiors (the resulting intersection graphs are slightly more general than those of ellipses without interiors). Suppose $\rho \geq 1$, and let FILLEDELLIPSE$_{\leq\rho}$CLIQUE be the CLIQUE problem for intersection graphs of $(\leq \rho)$-ellipses with interiors. We outline an approximation algorithm for this problem, with the approximation ratio depending on ρ:

Lemma 4. *Let $\mathcal{C}$ be a clique of $(\leq \rho)$-ellipses. Then there is a point p that is contained in at least $|\mathcal{C}|/(9\rho^2)$ ellipses from $\mathcal{C}$.*

Proof. This is an adaptation of the proof of Lemma 4.1 of [3]. Let r be the smallest radius of all ellipses in $\mathcal{C}$, and pick $L \in \mathcal{C}$ which has r as its smaller radius. Furthermore, consider the ellipse $3L$ obtained from L by scaling by a factor of 3. We claim that, for every ellipse $E \in \mathcal{C}$,

$$\text{area}(E \cap 3L) \geq \frac{1}{9\rho^2}\text{area}(3L). \tag{5}$$

To see why (5) holds, consider an ellipse $E \in \mathcal{C}$ and an arbitrary point p in the intersection of L and the boundary of E. Out of E, we construct an ellipse F by applying a dilation at point p such that the largest radius of F has length r. that since E has ratio at most ρ, the

Since the largest radius of E has as least length r, F is smaller than E. Because E is convex and $p \in E$, F is even contained in E.

Note that F is also contained in $3L$ since all points within distance at most $2r$ from L are contained in $3L$ and, therefore, every cycle with radius r which touches L is completely contained in $3L$. Hence, F is contained in $3L$.

We therefore obtain (5) by

$$\text{area}(E \cap 3L) \geq \text{area}(F) \geq \frac{1}{\rho^2}\text{area}(L) \geq \frac{1}{9\rho^2}\text{area}(3L).$$

The second inequality holds because the largest radius of L is at most ρr and the shortest radius of F is at least r/ρ.

Using (5), we conclude that on average, a point $p \in 3L$ is contained in

$$\frac{\sum_{E\in\mathcal{C}} \text{area}(E \cap 3L)}{\text{area}(3L)} \geq \frac{|\mathcal{C}|}{9\rho^2}$$

ellipses. Thus, there is a point that is contained in at least as many ellipses. □

Having Lemma 4 at our disposal, there is as easy $9\rho^2$-approximation algorithm for FILLEDELLIPSE$_{\leq\rho}$CLIQUE:

Algorithm 1. Given ℓ, compute the *arrangement* $\mathcal{A}$ induced by $\mathcal{L}$ and for every cell c, compute the number n_c of ellipses which contain c. (For a family of n ellipses, the

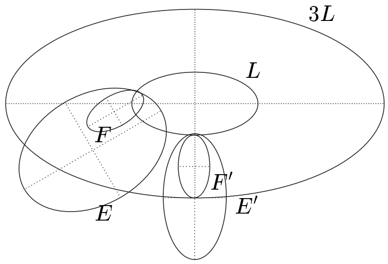

Fig. 6. An area argument for ellipses of bounded ratio.

arrangement can be computed, for instance, by a randomized incremental algorithm with expected runtime of $O(n \log n + v)$, where $v = O(n^2)$ is the number of vertices of the arrangement, or deterministically with a slightly super-quadratic runtime, see [18].) Output the maximum $\max_c n_c$.

Here is an approach for further improvement of the approximation ratio: The proof of Lemma 4 shows that every family $\mathcal{C}$ of pairwise intersecting ellipses has the following property: every subfamily $\mathcal{L} \subseteq \mathcal{C}$ of cardinality greater than $27\rho^2$, contains three distinct ellipses L_1, L_2, L_3 whose intersection $L_1 \cap L_2 \cap L_3$ is non-empty (by (5), there is an ellipse $L \in \mathcal{L}$ such that some point $p \in 3L$ is covered by at least three ellipses from $\mathcal{L}$). By the (p, q)-*Theorem* [2], for every ρ, there is some finite number $\tau(\rho)$, called the *transversal number*, such that every clique $\mathcal{C}$ of $(\leq \rho)$-ellipses can be *pierced* by some set of at most $\tau(\rho)$ points (i.e., every $L \in \mathcal{C}$ contains at least one of the points). This suggests the following variant of Algorithm 1.

Algorithm 2. Compute the arrangement induced by $\mathcal{L}$ as above. For every pair $\{c, c'\}$ of cells (there are at most $O(n^4)$), let $\mathcal{L}_{\{c,c'\}}$ be the set of ellipses in $\mathcal{L}$ which contain c, or c', or both. The intersection graph of $\mathcal{L}_{\{c,c'\}}$ is the complement of a bipartite graph on at most n nodes, so we can find a maximum clique in time $O(n^{2.5})$. Output the maximum for all pairs.

The approximation ratio of this algorithm is at least as good as that of the first one, and it is also at most $\tau(\rho)/2$. In general, the bounds for $\tau(\rho)$ implied by the (p, q)-Theorem, are quite large, but in some cases, better bounds are known. For instance, for disks, the transversal number is $\tau(1) = 4$ (see [9]), so we have a 2-approximation in that case (then again, we do not know whether the problem is hard for disks).

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