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Boundary-layer similarity flows driven by a power-law shear over a permeable plane surface

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Summary. The boundary-layer similarity flow driven over a semi-infinite permeable flat plate by a power-law shear with asymptotic velocity profile $U_\infty(y) = \beta y^\alpha (y \to \infty, \beta > 0)$ is considered in the presence of lateral suction or injection of the fluid (y denotes the coordinate normal to the plate). The analytically tractable cases $\alpha = -2/3$ and $\alpha = -1/2$ are examined in detail. It is shown that while for $\alpha = -2/3$ the adjustment of the flow over an impermeable plate to the power-law shear is not possible, in the permeable cases the presence of suction allows for a family of boundary-layer solutions with the proper algebraic decay. The value of the skin friction corresponding to this family of solutions is given by the parameter $s = 9\beta^3/(4f_w)$, where f_w denotes the suction parameter. In the limiting case of a vanishing suction and a properly vanishing value of the parameter β (such that $s =$ finite), this family of algebraically decaying solutions goes over into the (exponentially decaying) Glauert-jet. In the case $\alpha = -1/2$, solutions showing the proper algebraic decay were found both for suction ($f_w > 0$) and injection ($f_w < 0$) in the whole range $-\infty < f_w < +\infty$. In this case the skin friction parameters $s = 2\beta^2/3$ is independent of the suction/injection parameter f_w.

1 Introduction and basic equations

The shear driven flows, like the wall-driven Couette-flow, the wind-driven Ekman-flow, the Lock-type flows near to the interface of two parallel streams etc. belong to the classical topics of fluid mechanics. Due to their wide technical and environmental applications, the general research interest in the shear driven flows is still present in our days.

Recently, the adjustment of a zero pressure gradient laminar flow near a flat impermeable boundary to an exterior power-law velocity profile of the form

$$U_\infty(y) = \beta y^\alpha \quad (y \to \infty, \ \beta > 0) \tag{1}$$

has been investigated for a wide range of values of the exponent α by Weidman et al. [1]. These authors have shown that by the similarity transformation

$$\psi(x,y) = x^{\frac{\alpha+1}{\alpha+2}} \cdot f(\eta), \quad \eta = x^{-\frac{1}{\alpha+2}} \cdot y \tag{2}$$

where the stream function $\psi(x,y)$ is subject to impermeability, the no-slip and the asymptotic conditions

$$\psi(x,0) = 0, \ \frac{\partial \psi}{\partial y}(x,0) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial y} \to \beta y^\alpha \ (y \to \infty) \tag{3}$$

Prandtl's steady boundary-layer equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3} \tag{4}$$

reduces to the ordinary differential equation

$$(\alpha+2)f'''+(\alpha+1)ff''-\alpha f'^2=0 \tag{5}$$

along with boundary conditions

$$f(0)=0,\ f'(0)=0,\ f'(\eta)\to\beta\eta^\alpha \text{ as } \quad \eta\to\infty. \tag{6}$$

In the above equations all the quantities ψ, x, y etc. are nondimensional. The stream function is defined by $u(x,y)=\partial\psi/\partial y,\ v(x,y)=-\partial\psi/\partial x,\ x$ and y are the streamwise and the plate normal coordinates, u and v the corresponding (nondimensional) velocity components given by

$$u(x,y)=x^{\frac{\alpha}{\alpha+2}}\cdot f'(\eta),$$
$$v(x,y)=-x^{-\frac{1}{\alpha+2}}\cdot\left[\frac{\alpha+1}{\alpha+2}f(\eta)-\frac{\eta}{\alpha+2}f'(\eta)\right], \tag{7}$$

and the primes denote derivatives with respect to η.

The outstanding feature of the problem considered by Weidman et al. [1] consists of the algebraic asymptotic behavior of the streamwise velocity profile u as required by Eq. (1). As proven in [1], such solutions of the boundary value problem (5), (6) only exists in the range $\alpha>-2/3$, but not for $\alpha=-2/3$ and below it .

The aim of the present paper is to examine the effect of a lateral suction and injection of the fluid, i.e. the existence of similarity solutions of Eq. (5) which satisfy the boundary conditions

$$f(0)=f_w,\ f'(0)=0,\ f'(\eta)\to\beta\eta^\alpha \quad (\eta\to\infty) \tag{8.1–3}$$

with $f_w\neq 0$. Our investigation is restricted to the values $\alpha=-2/3$ and $\alpha=-1/2$ only, where exact analytic solutions are available.

It is worth underlying at this place that the problem of the shear driven boundary-layer flows considered in the present paper is structurally quite different from the more familiar problem of the pressure gradient driven Falkner-Skan flows with an inviscid and irrotational free stream. Here we are faced with strictly zero pressure gradient flows, exhibiting a rotational free stream velocity of the power-law form, $U_\infty(y)=\beta y^\alpha$, i.e. shear.

2 The case $\alpha=-1/2$

We first notice that our basic equation (5) can be re-written in the form

$$\frac{d^2}{d\eta^2}\left[(\alpha+2)f'+\frac{\alpha+1}{2}f^2\right]-(2\alpha+1)f'^2=0. \tag{9}$$

Thus, for $\alpha=-1/2$ we obtain after two elementary integrations the Riccati equation

$$f'+\frac{1}{6}f^2=C_1\eta+C_2, \tag{10}$$

where C_1 and C_2 are constants of integration. Using the first two boundary conditions (8) yields

$$C_1=f''(0)\equiv s,\ \ C_2=\frac{1}{6}f_w^2. \tag{11}$$

Now, letting in Eq. (10) $\eta \to \infty$ and having in mind the boundary condition (8.3), we obtain for the wall shear stress (skin friction) parameter s the result

$$s = \frac{2}{3}\beta^2. \tag{12}$$

Surprisingly, this result is independent of the suction/injection parameter f_w, and thus it coincides with the expression of s found by Weidman et al. [1] for the impermeable plate ($f_w = 0$). Furthermore, with the aid of the transformation

$$\begin{aligned} f(\eta) &= (36s)^{1/3}\frac{d}{dz}[\ln W(z)], \\ z &= \left(\frac{s}{6}\right)^{1/3}\left(\eta + \frac{f_w^2}{6s}\right) \end{aligned} \tag{13}$$

we obtain for the new dependant variable $W = W(z)$ the Airy equation $W'' = zW$ (where primes denote now differentiation with respect to the new independent variable z). In this way, the general solution of Eq. (10) for C_1 and C_2 given by Eqs. (11) is

$$f(\eta) = (36s)^{1/3}\frac{a \cdot Ai'(z) + b \cdot Bi'(z)}{a \cdot Ai(z) + b \cdot Bi(z)}, \tag{14}$$

where $Ai(z)$ and $Bi(z)$ denote the Airy functions [3], and a and b are integration constants. Bearing in mind the asymptotic behavior of the Airy functions it can be shown that the function (14) satisfies the asymptotic condition (8.3) for any nonzero value of the integration constant b and any (vanishing or non-vanishing) value of a. In the case $a = 0$ and $b \neq 0$, the corresponding solution

$$f(\eta) = (36s)^{1/3}\frac{Bi'(z)}{Bi(z)} \tag{15}$$

satisfies the boundary condition (8.1) for the value of f_w which satisfies transcendental equation

$$(36s)^{1/3} \cdot Bi'(z_0) - f_w \cdot Bi(z_0) = 0, \tag{16}$$

where

$$z_0 = z|_{\eta=0} = (36s)^{-2/3} \cdot f_w^2. \tag{17}$$

The (unique) solution of Eq. (16) found numerically for $\beta = 1$ is $f_w = 1.82502$ (for $\beta = 0.2$ and $\beta = 3$, one obtains $f_w = 0.62415$ and $f_w = 3.79619$, respectively).

For non-vanishing values of a and b the condition (8.1) yields

$$a = \frac{(36s)^{1/3} \cdot Bi'(z_0) - f_w \cdot Bi(z_0)}{(36s)^{1/3} \cdot Ai'(z_0) - f_w \cdot Ai(z_0)} b \tag{18}$$

such that the solution (14) becomes again independent of b. In the case of the impermeable plate ($f_w = 0$), Eq. (18) yields $a = \sqrt{3} \cdot b$, and thus we recover the result of Weidman et al. [1]. In Fig. 1, the ratio a/b as given by Eq. (18) is plotted as a function of f_w for $\beta = 0.2, 1$ and 3, respectively. These curves show that Eq. (14) possesses a unique solution of the boundary value problem (5), (8) for any given $\beta > 0$ and any specified value of the suction/injection parameter f_w in the range $-\infty < f_w < +\infty$. The only difference is that the zero of a is shifted to larger values with increasing f_w, but the intersection point of the curve with the a/b-axis of Fig. 1, i.e., $a/b = \sqrt{3}$ is still independent of β. Therefore, the algebraically decaying wall jet with $\alpha = -1/2$ survives also on an impermeable plate, regardless of the value of suction/injection parameter f_w, but, as shown above, its skin friction given by Eq. (12) is independent of f_w. As an illustration,

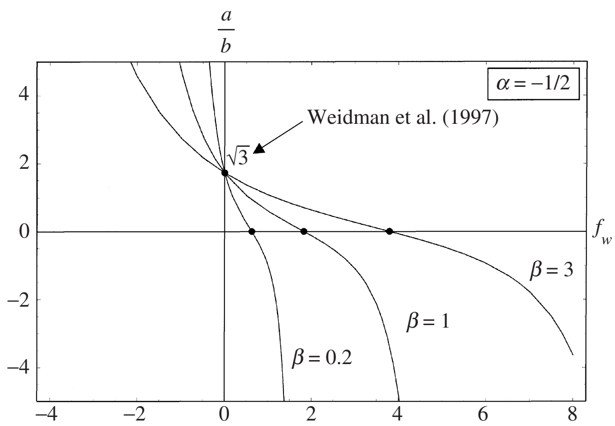

Fig. 1. Case $\alpha = -1/2$: The ratio a/b given by Eq. (18) is plotted as a function of f_w for different values of β

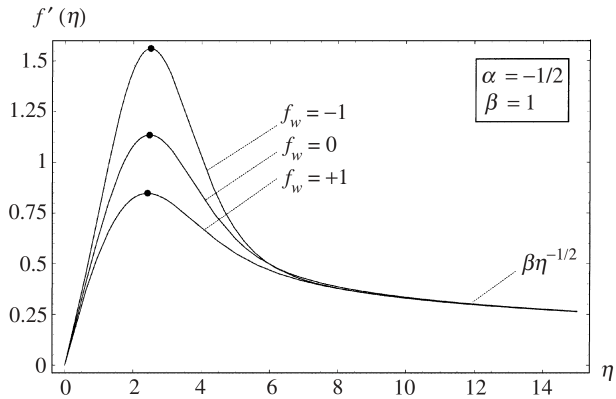

Fig. 2. Case $\alpha = -1/2$: The downstream velocity profiles $f'(\eta)$ corresponding to different values of f_w decay algebraically as $\beta\eta^{-1/2}$

in Fig. 2 the downstream velocity profiles $f'(\eta)$ are plotted versus η for $\beta = 1$ and three different values of f_w (with $f_w > 0$ for suction and $f_w < 0$ for injection). It is seen that the maximum of $f'(\eta)$, i.e., the "jet velocity" $f'_{\max} = f'(\eta_0)$, where η_0 is the solution of equation $f''(\eta_0) = 0$, decreases with increasing f_w monotonically (see Fig. 3). The abscissa η_0 of the jet velocity, on the other hand, moves from 0 to maximum value, $\eta_{0,\max} = 2.50204$ as f_w increases from $-\infty$ to -1 and then returns to zero again as f_w further increases from -1 to $+\infty$ (see Fig. 4). It can be shown that in the present case η_0 and $f'_{\max}$ are related to each other according to the relationship

$$\eta_0 = \frac{f'_{\max}}{s} + \frac{3s}{2f'^2_{\max}} - \frac{f_w^2}{6s}. \tag{19}$$

The values $\eta_0 = 2.47878$ and $f'_{\max} = 1.13517$ found by Weidman et al. [1] for $f_w = 0$ and $\beta = 1$ satisfy accurately Eq. (19).

It is also worth noticing here that, according to Eqs. (5), (8) and (12), the curvature of the velocity profiles $f'(\eta)$ at $\eta = 0$ is given by

$$f'''(0) = -\frac{\alpha+1}{\alpha+2} f_w s = -\frac{2}{9} f_w \beta^2. \tag{20}$$

Hence, at $\eta = 0$ the shape of $f'(\eta)$ is concave for $f_w > 0$ (suction), convex for $f_w < 0$ (injection) and flat (vanishing curvature) for the impermeable plate ($f_w = 0$). These features may indeed be seen by a careful inspection of Fig. 2.

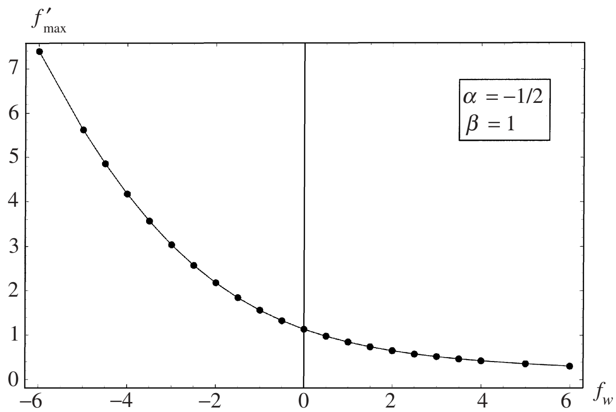

Fig. 3. Case $\alpha = -1/2$: The jet velocity decreases monotonically with increasing f_w

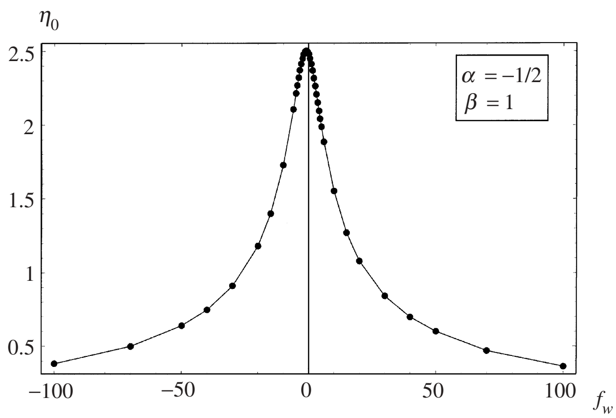

Fig. 4. Case $\alpha = -1/2$: The abscissa η_0 of the jet velocity moves from 0 to a maximum value, $\eta_{0,\max} = 2.50204$ as f_w increases from $-\infty$ to -1 and then returns to zero again as f_w further increases from -1 to $+\infty$

3 The case $\alpha = -2/3$

It is convenient in this case to re-write the basic Eq. (5) into the following form:

$$\frac{1}{f}\frac{d}{d\eta}\left\{f^{3/2}\frac{d}{d\eta}\left[(\alpha+2)f^{-1/2}\cdot f' + \frac{2}{3}(\alpha+1)f^{3/2}\right]\right\} - (3\alpha+2)f'^2 = 0 \tag{21}$$

which for $\alpha = -2/3$ implies

$$f^{3/2}\frac{d}{d\eta}\left(4f^{-1/2}\cdot f' + \frac{2}{3}f^{3/2}\right) = \text{const.} \tag{22}$$

Now, putting in Eq. (22) $\eta = 0$ and taking into account the boundary conditions (8), one obtains for the integration constant the value $4f_w s$ where s denotes again the dimensionless skin friction parameter, $s = f''(0)$. Thus Eq. (22) becomes

$$4ff'' + f^2 f' - 2f'^2 = 4f_w s. \tag{23}$$

Letting in Eq. (23) $\eta \to \infty$ and having in mind the boundary condition (8.3), we obtain for the skin friction parameter s the result

$$s = \frac{9}{4}\frac{\beta^3}{f_w}. \tag{24}$$

On the other hand, integrating Eq. (5) once from 0 to ∞ and having in mind the boundary conditions (8) we easily deduce that for $\alpha = -2/3$ the parameter s is given by

$$s = \frac{1}{4}\int_0^\infty f'^2(\eta)d\eta, \tag{25}$$

which shows explicitly that the skin friction s is a positive quantity. Thus, Eq. (24) requires $f_w > 0$ which means suction. We also see that $s \to \infty$ as $f_w \to 0$, and we thus recover here the singularity found by Weidman et al. [1] as the lower limit of existence of the shear driven boundary layer flows over impermeable plane surfaces. Therefore, in the present case, a lateral suction of the fluid makes possible that over a permeable plate a shear driven steady boundary layer can form. As an illustration, in Fig. 5 the corresponding wall-jet like velocity profiles $f'(\eta)$ are plotted versus η for $\beta = 1$ and four different values of the suction parameter f_w. The jet velocity $f'_{\max}$ decrease now from ∞ to 0 monotonically as f_w increase from 0 to ∞. Its abscissa η_0 shows a similar behavior to that shown in Fig. 4: it moves from 0 to a maximum value $\eta_{0,\max} = 2.15208$ as f_w increases from 0 to 5.7 and then returns to zero again as f_w further increases from 5.7 to $+\infty$.

The divergence of the jet velocity which occurs for any finite value of β as $f_w \to 0$ corresponds precisely to the singularity described by Weidman et al. [1]. If, however, simultaneously with $f_w \to 0$ also β goes to zero in such a way that the skin friction given by Eq. (24) approaches a finite value, it can be expected that our algebraically decaying wall jet goes over into the classical (exponentially decaying) Glauert-jet [2]. This is actually the case as shown recently by Magyari and Keller [4]. Indeed, the Glauert-jet represents the solution of the above Eq. (22) subject to the boundary conditions

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 0. \tag{26}$$

It can be expressed in the implicit analytic form

$$\eta = \frac{4}{f_\infty}\left[\ln\left(\frac{\sqrt{1+W+W^2}}{1-W}\right) + \sqrt{3}\cdot\arctan\left(\frac{\sqrt{3}\cdot W}{2+W}\right)\right], \tag{27}$$

where

$$W = \left[\frac{f(\eta)}{f_\infty}\right]^{1/2} \quad \text{and} \quad f_\infty = \lim_{\eta\to\infty} f(\eta). \tag{28}$$

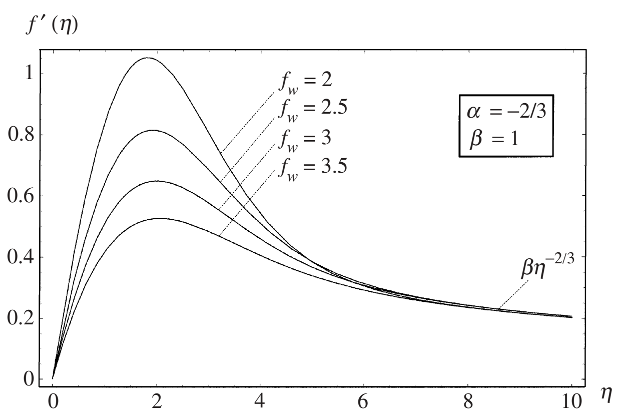

Fig. 5. Case $\alpha = -2/3$: The downstream velocity profiles $f'(\eta)$ corresponding to different values of f_w decay algebraically as $\beta\eta^{-2/3}$

The corresponding downstream velocity profile decays exponentially according to

$$f'(\eta) = \frac{\sqrt{3}}{2} f_\infty^2 \cdot \exp\left(\frac{\pi}{2\sqrt{3}} - \frac{f_\infty}{4}\eta\right) \text{ as } \eta \to \infty. \tag{29}$$

The skin friction of the Glauert-jet is thus given by

$$f''(0) \equiv s_G = \frac{f_\infty^3}{72}. \tag{30}$$

Since our basic equation (5) is invariant under the scaling transformation $\{f \to \lambda \cdot f,\ \eta \to \eta/\lambda\}$ (for any α) and all the boundary conditions (26) are homogeneous, the solution (27) is determined only up to a normalization factor, and thus the value of f_∞ may be fixed arbitrarily. We put in this paper $f_\infty = 4$ (since our present f is 4 times larger than Glauert's original one, [2]). This choice results in $s_G = 8/9$, $f'_{\max} = f'(\eta_0) = 2^{-11/3} \cdot f_\infty^2 = 1.25992$ and $\eta_0 = 2.02854$, respectively. Thus, the skin friction (24) of the algebraically decaying jet equals the skin friction (30) of the Glauert-jet if

$$\beta = \frac{4}{3}\left(\frac{f_w}{6}\right)^{1/3}. \tag{31}$$

Therefore, when $f_w \to 0$ and at the same time $\beta \to 0$ according to Eq. (31), the wall jet decaying according to $f'(\eta) \to \beta\eta^{-2/3}$ as $\eta \to \infty$ must go over gradually into the Glauert-jet (27). This crossover actually takes place as illustrated in Fig. 6.

4 Summary and conclusions

The effect of a lateral suction and injection on the boundary-layer similarity flow driven over a permeable semi-infinite flat plate by a power-law shear with the asymptotic velocity profile $U_\infty(y) = \beta y^\alpha (y \to \infty,\ \beta > 0)$ has been investigated in the analytically tractable particular cases $\alpha = -2/3$ and $\alpha = -1/2$, respectively.

The main results of the paper can be summarized as follows:

(i) While for $\alpha = -2/3$ the adjustment of the flow over an impermeable plate to the power-law shear is not possible [1], we have shown that in permeable case the presence of a (similarity preserving) suction allows for a family of wall jet like solutions with the proper algebraic decay. The value of the skin friction corresponding to this family of solutions is given by $s = 9\beta^3/(4f_w)$ where $f_w > 0$ is the suction parameter.

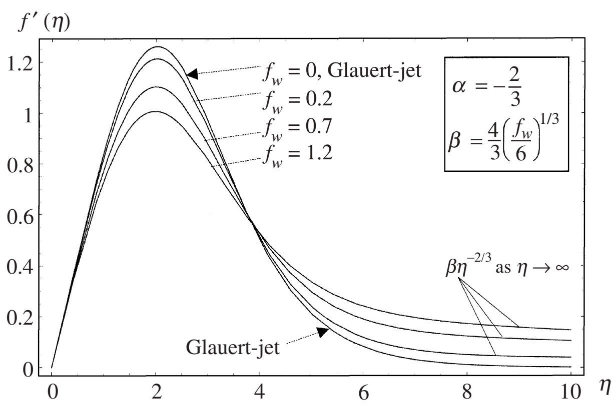

Fig. 6. Case $\alpha = -2/3$: When $f_w \to 0$, and at the same time $\beta \to 0$ according to $\beta = 4(f_w/6)^{1/3}/3$, the family of the algebraically decaying wall jets approaches the exponentially decaying Glauert-jet gradually

(ii) The jet velocity $f'_{\max}$ decreases for $\alpha = -2/3$ from ∞ to 0 monotonically as f_w increases from 0 to ∞. Its abscissa η_0 on the other hand moves from 0 to a maximum value $\eta_{0,\max} = 2.15208$ as f_w increases from 0 to 5.7 and then returns to zero again as f_w further increases from 5.7 to $+\infty$. We notice that this behavior also occurs in the case of (exponentially decaying) boundary-layer flows induced by continuous stretching surfaces, as shown recently by Magyari and Keller [5].

(iii) when $f_w \to 0$ and at the same time $\beta \to 0$ according to $\beta = 4(f_w/6)^{1/3}/3$, the family of the algebraically decaying wall jets with $\alpha = -2/3$ approaches the well-known (exponentially decaying) Glauert-jet[2] gradually.

(iv) In the case of $\alpha = -1/2$, solutions showing the proper algebraic decay were found both for suction ($f_w > 0$) and injection ($f_w < 0$) in the whole range $-\infty < f_w < +\infty$. They can be expressed in terms of Airy functions (as done by Weidman et al. [1] in the case $f_w = 0$, as well as much earlier by Kuiken [6] in the context of boundary-layer flows induced by continuous stretching surfaces). In this case the skin friction $s = 2\beta^2/3$ is independent of the suction/injection parameter f_w. The jet velocity $f'_{\max} = f'(\eta_0)$ decreases with increasing f_w monotonically also for $\alpha = -1/2$. Its abscissa η_0 moves from 0 to a maximum value $\eta_{0,\max} = 2.50204$, as f_w increases from $-\infty$ to -1 and then returns to zero again as f_w further increases from -1 to $+\infty$.

The result of the present paper show explicitly that in the case of the shear driven wall jets the lateral suction or injection of the fluid is able to extend the existence domain of the similarity solutions significantly.

Finally, it is worth mentioning here that the phenomenon of the seemingly "missing solutions", as it has been described recently in [7], can also be encountered in the present context of the shear driven flows. It occurs in the case $\alpha = -1$ when the second term of the left-hand side of Eq. (5) identically vanishes and thus neither the boundary value problems (5), (6), nor (5), (8) admit solutions. Putting however $\psi(x,y) = f(\eta) + \beta \ln x,\ \eta = y/x$, instead of the usual similarity ansatz (2) which for $\alpha = -1$ is much too restrictive, [7], the "missing solutions" can readily be found also in this case.

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