



Hölder Equicontinuity of the Integrated Density of States at Weak Disorder

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Abstract. Hölder continuity, $|N_\lambda(E) - N_\lambda(E')| \leq C|E - E'|^\alpha$, with a constant C independent of the disorder strength λ is proved for the integrated density of states $N_\lambda(E)$ associated to a discrete random operator $H = H_o + \lambda V$ consisting of a translation invariant hopping matrix H_o and i.i.d. single site potentials V with an absolutely continuous distribution, under a regularity assumption for the hopping term.

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1. Introduction

Random operators on $\ell^2(\mathbb{Z}^d)$ of the general form

$$H_\omega = H_o + \lambda V_\omega, \tag{1}$$

play a central role in the theory of disordered materials, where:

- (1) $V_\omega \psi(x) = \omega(x)\psi(x)$ with $\omega(x)$, $x \in \mathbb{Z}^d$, independent identically distributed random variables whose common distribution is $\rho(\omega)d\omega$ with ρ a bounded function. The coupling $\lambda \in \mathbb{R}$ is called the *disorder strength*.
- (2) H_o is a bounded translation invariant operator, i.e., $[S_\xi, H_o] = 0$ for each translation $S_\xi \psi(x) = \psi(x - \xi)$, $\xi \in \mathbb{Z}^d$.

The *density of states measure* for an operator H_ω of the form Equation (1) is the (unique) Borel measure $dN_\lambda(E)$ on the real line defined by

$$\int f(E)dN_\lambda(E) = \lim_{L \rightarrow \infty} \frac{1}{\#\{x \in \mathbb{Z}^d : |x| < L\}} \sum_{x:|x|<L} \langle \delta_x, f(H_\omega)\delta_x \rangle,$$

and the *integrated density of states* $N_\lambda(E)$ is

$$N_\lambda(E) := \int_{(-\infty, E)} dN_\lambda(\varepsilon).$$

It is a well known consequence, e.g., reference [6], of the translation invariance of the distribution of H_ω that the density of states exists and equals

$$N_\lambda(E) = \int_{\Omega} \langle \delta_0, P_{(-\infty, E)}(H_\omega) \delta_0 \rangle d\mathbb{P}(\omega), \quad \text{every } E \in \mathbb{R};$$

for \mathbb{P} almost every ω , where \mathbb{P} is the joint probability distribution for ω and $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ is the probability space.

The density of states measure is an object of fundamental physical interest. For example, the free energy f per unit volume of a system of non-interacting identical Fermions, each governed by a Hamiltonian H_ω of the form Equation (1), is

$$f(\mu, \beta) = -\beta \int \ln(1 + e^{-\beta(E-\mu)}) dN_\lambda(E),$$

where β is the inverse temperature and μ is the chemical potential. Certain other thermodynamic quantities (density, heat capacity, etc.) of the system can also be expressed in terms of N_λ .

Our main result is equicontinuity of the family $\{N_\lambda(\cdot), \lambda > 0\}$ within a class of Hölder continuous functions, that is

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq C_\alpha \delta^\alpha, \quad \text{for all } \lambda > 0, \quad (2)$$

under appropriate hypotheses on H_o . The exponent $\alpha < 1$ depends on H_o as well as the probability density, with $\alpha = \frac{1}{2}$ at generic E for a large class of hopping terms if ρ is compactly supported.

A bound of the form Equation (2) for the integrated density of states associated to a continuum random Schrödinger operator is implicit in Theorem 1.1 of reference [1], although uniformity in λ is not explicitly noted there. The tools of reference [1] carry over easily to the discrete context to give an alternative proof of Equation (2). However the methods employed herein are in fact quite different from those of reference [1], and may be interesting in and of themselves.

The main point of Equation (2) is the uniformity of the bound as $\lambda \rightarrow 0$, since the well known *Wegner estimate* [9], see also [7, Theorem 8.2],

$$\frac{dN_\lambda(E)}{dE} \leq \frac{\|\rho\|_\infty}{\lambda}, \quad (3)$$

implies that $N_\lambda(E)$ is in fact Lipschitz continuous,

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq \frac{\|\rho\|_\infty}{\lambda} 2\delta. \quad (4)$$

However, the Lipschitz constant $\|\rho\|_\infty/\lambda$ in Equation (4) diverges as $\lambda \rightarrow 0$. Such a singularity is inevitable for a bound which makes no reference to the hopping term, since $dN_\lambda(E) = \lambda^{-1} \rho(E/\lambda) dE$ for $H_o = 0$, as may easily be verified. However if the background itself has an absolutely continuous density of states, the Wegner estimate is far from optimal at weak disorder.

The translation invariant operator H_o may be written as a superposition of translations,

$$H_o = \sum_{\xi} \check{\varepsilon}(\xi) S_{\xi},$$

where

$$\check{\varepsilon}(\xi) = \int_{T^d} \varepsilon(\mathbf{q}) e^{-i\xi \cdot \mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^d},$$

is the inverse Fourier transform of a bounded real function ε on the torus $T^d = [0, 2\pi)^d$, called the symbol of H_o . For any bounded measurable function f ,

$$f(H_o) = \sum_{\xi \in \mathbb{Z}^d} \left[\int_{T^d} f(\varepsilon(\mathbf{q})) e^{-i\xi \cdot \mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^d} \right] S_{\xi},$$

from which it follows that the density of states $N_o(E)$ for H_o obeys

$$\int f(E) dN_o(E) = \int_{T^d} f(\varepsilon(\mathbf{q})) \frac{d\mathbf{q}}{(2\pi)^d}.$$

In particular,

$$N_o(E) = \int_{\{\varepsilon(\mathbf{q}) < E\}} \frac{d\mathbf{q}}{(2\pi)^d}.$$

We define a *regular point* for ε to be a point $E \in \mathbb{R}$ at which

$$N_o(E + \delta) - N_o(E - \delta) \leq \Gamma(E) \delta, \quad (5)$$

for some $\Gamma(E) < \infty$. In particular if ε is C^1 and $\nabla \varepsilon$ is non-zero on the level set $\{\varepsilon(\mathbf{q}) = E\}$, then E is a regular point. For example, with H_o the discrete Laplacian on $\ell^2(\mathbb{Z})$,

$$H_o \psi(x) = \psi(x+1) + \psi(x-1),$$

we have the symbol $\varepsilon(q) = 2 \cos(q)$ and every $E \in (-2, 2)$ is a regular point. However at the band edges, $E = \pm 2$, the difference on the left hand side of Equation (5) is only $\mathcal{O}(\delta^{\frac{1}{2}})$, and these points are not regular points. We consider the behavior of $N_{\lambda}(E)$ at such ‘points of order α ’, here $\alpha = 1/2$, in Theorem 3 below.

Our main result involves the density of states of H_{λ} at a regular point:

THEOREM 1. *Suppose $\int |\omega|^q \rho(\omega) d\omega < \infty$ for some $2 < q < \infty$ or that ρ is compactly supported, in which case set $q = \infty$. If E is a regular point for ε , then there is $C_q = C_q(\rho, \Gamma(E)) < \infty$ such that*

$$N_{\lambda}(E + \delta) - N_{\lambda}(E - \delta) \leq \Gamma(E) \delta + C_q \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} \quad (6)$$

for all $\lambda, \delta \geq 0$.

For very small δ , namely

$$\frac{\delta}{\lambda} \lesssim \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})},$$

the Wegner bound Equation (3) is stronger than Equation (6).¹ Thus Theorem 1 is useful only for

$$\delta \gtrsim \lambda^{\frac{2q+1}{q+1}}.$$

Combining the Wegner estimate and Theorem 1 for these separate regions yields the following:

COROLLARY 2. *Under the hypotheses of Theorem 1, there is $C_q < \infty$, with $C_q = C_q(\rho, \Gamma(E))$, such that*

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq C_q \delta^{\frac{1}{2}(1-\frac{1}{2q+1})} \quad (7)$$

for all $\lambda, \delta \geq 0$.

Thus, the integrated density of states is Hölder equi-continuous of order $\frac{1}{2}$ as $\lambda \rightarrow 0$ (if ρ is compactly supported).

The starting point for our analysis of the density of states is a well-known formula relating dN_λ to the resolvent of H_ω ,

$$\frac{dN_\lambda(E)}{dE} = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im} \langle \delta_0, (H_\omega - E - i\eta)^{-1} \delta_0 \rangle d\mathbb{P}(\omega).$$

The general idea of the proof is to express $\text{Im} \langle \delta_0, (H_\omega - E - i\eta)^{-1} \delta_0 \rangle$ using a finite resolvent expansion to second order

$$\begin{aligned} & (H_\omega - E - i\eta)^{-1} \\ &= (H_o - E - i\eta)^{-1} - \lambda (H_o - E - i\eta)^{-1} V_\omega (H_o - E - i\eta)^{-1} + \\ & \quad + \lambda^2 (H_o - E - i\eta)^{-1} V_\omega (H_\omega - E - i\eta)^{-1} V_\omega (H_o - E - i\eta)^{-1}, \end{aligned} \quad (8)$$

and to use the Wegner bound Equation (3) to estimate the last term, with the resulting factor of $1/\lambda$ controlled by the factor λ^2 .

Here is a simplified version of the argument which works if E falls outside the spectrum of H_o and $\psi_E = (H_o - E)^{-1} \delta_0 \in \ell^1(\mathbb{Z}^d)$. The first two terms of Equation (8) are bounded and self-adjoint when $\eta = 0$, so

$$\begin{aligned} & \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im} \langle \delta_0, (H_\omega - E - i\eta)^{-1} \delta_0 \rangle d\mathbb{P}(\omega) \\ &= \lambda^2 \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\Omega} \text{Im} \langle \psi_E, V_\omega (H_\omega - E - i\eta)^{-1} V_\omega \psi_E \rangle d\mathbb{P}(\omega) \\ &\leq \lambda^2 \lim_{\eta \downarrow 0} \sum_{x,y} |\psi_E(x)| |\psi_E(y)| \times \\ & \quad \times \frac{\eta}{\pi} \int_{\Omega} \left| \omega(x) \omega(y) \left\langle \delta_x, ((H_\omega - E)^2 + \eta^2)^{-1} \delta_y \right\rangle \right| d\mathbb{P}(\omega). \end{aligned}$$

¹We thank M. Disertori for this observation.

If ρ is, say, compactly supported, then

$$\begin{aligned} & \lim_{\eta \downarrow 0} \frac{\eta}{\pi} \int_{\Omega} \left| \omega(x)\omega(y) \left\langle \delta_x, ((H_{\omega} - E)^2 + \eta^2)^{-1} \delta_y \right\rangle \right| d\mathbb{P}(\omega) \\ & \lesssim \lim_{\eta \downarrow 0} \frac{\eta}{\pi} \int_{\Omega} \left\langle \delta_x, ((H_{\omega} - E)^2 + \eta^2)^{-1} \delta_y \right\rangle d\mathbb{P}(\omega) \lesssim \frac{1}{\lambda}, \end{aligned}$$

by the Wegner bound, and therefore

$$\frac{dN_{\lambda}(E)}{dE} \lesssim \lambda \|\psi_E\|_1^2, \quad \text{for } E \notin \sigma(H_o). \tag{9}$$

We have used second order perturbation theory to ‘boot-strap’ the Wegner estimate and obtain an estimate of lower order in λ . Unfortunately, as ρ was assumed compactly supported, E is not in the spectrum of H_{λ} for sufficiently small λ , and thus $dN_{\lambda}(E)/dE=0$. So, in practice, Equation (9) is not a useful bound.

Nonetheless, in the cases covered by Theorem 1, H_{λ} can have spectrum in a neighborhood of E , even for small λ , since E may be in the interior of the spectrum of H_o . Although, the above argument does not go through, we shall exploit the translation invariance of the distribution of H_{ω} by introducing a Fourier transform on the Hilbert space of ‘random wave functions’, complex valued functions $\Psi(x, \omega)$ of $(x, \omega) \in \ell^2(\mathbb{Z}^d) \times \Omega$ with

$$\sum_x \int_{\Omega} |\Psi(x, \omega)|^2 d\mathbb{P}(\omega) < \infty.$$

Under this Fourier transform an integral \int_{Ω} of a matrix element of $f(H_{\omega})$ is replaced by an integral \int_{T^d} over the d -torus of a matrix element of $f(\widehat{H}_{\mathbf{k}})$, with $\widehat{H}_{\mathbf{k}}$ a certain family of operators on $L^2(\Omega)$ (see Equation (16)). Off the set $S_{\epsilon} := \{\mathbf{k} \in T^d \mid |\epsilon(\mathbf{k}) - E| > \epsilon\}$ with $\epsilon \gg \delta$, we are able to carry out an argument similar to that which led to Equation (9). To prove Theorem 1, we shall directly estimate

$$N(E + \delta) - N(E - \delta) = \int_{\Omega} \langle \delta_0, P_{\delta}(H_{\omega}) \delta_0 \rangle d\mathbb{P}(\omega),$$

with P_{δ} the characteristic function of the interval $[E - \delta, E + \delta]$, because the integrand on the r.h.s. is bounded by 1. Since E is a regular point, the error in restricting to S_{ϵ} will be bounded by $\Gamma(E)\epsilon$. Choosing ϵ optimally will lead to Theorem 1.

More generally, we say that E is a point of order α for ϵ , if there exists $\Gamma(E; \alpha)$ such that

$$N_o(E + \delta) - N_o(E - \delta) \leq \Gamma(E; \alpha) \delta^{\alpha}.$$

If $E \notin \sigma(H_o)$, we say that E is a point of order ∞ and set $\Gamma(E; \infty) = 0$. For points of order α , we have the following extension of Theorem 1.

THEOREM 3. *Suppose $\int |\omega|^q \rho(\omega) d\omega < \infty$ for some $2 < q < \infty$ or that ρ is compactly supported, in which case set $q = \infty$. If E is a point of order $\alpha \leq \infty$ for ε , then there is $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E; \alpha)) < \infty$ such that*

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq \Gamma(E; \alpha) \delta^\alpha + C_{q,\alpha} \left[\lambda^{1+\frac{2}{q}} \delta^{1-\frac{2}{q}} \right]^{\frac{1}{1+\frac{2}{\alpha}}} \quad (10)$$

for all $\lambda, \delta \geq 0$.

When $\alpha = \infty$ and $q = \infty$, so $E \notin \sigma(H_0)$ and ρ is compactly supported, the result is technically true but uninteresting since $E \notin \sigma(H_\lambda)$ for small λ , as discussed above. However for $q < \infty$, we need not have that ρ is compactly supported, and $E \notin \sigma(H_0)$ may still be in the spectrum of H_λ for arbitrarily small λ . In this case, Equation (10) signifies that

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq C_{q,\infty} \lambda^{1+\frac{2}{q}} \delta^{1-\frac{2}{q}},$$

which in fact improves on the Wegner bound for appropriate λ, δ .

As above, we may use the Wegner bound for δ very small to improve on Equation (10):

COROLLARY 4. *Under the hypotheses of Theorem 3, there is $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E; \alpha)) < \infty$ such that*

$$N_\lambda(E + \delta) - N_\lambda(E - \delta) \leq C_{q;\alpha} \delta^{\frac{\alpha}{\alpha+1}} \left(1 - \frac{1}{\frac{\alpha+1}{\alpha} q + 1} \right)$$

for all $\lambda, \delta \geq 0$.

The inspiration for these results is the (non-rigorous) renormalized perturbation theory for dN_λ which has appeared in the physics literature, e.g., reference [8] and references therein. If $\int \omega \rho(\omega) d\omega = 0$ and $\int \omega^2 \rho(\omega) d\omega = 1$, as can always be achieved by shifting the origin of energy and re-scaling λ , then the central result of that analysis is that

$$\frac{dN_\lambda(E)}{dE} \approx \frac{1}{\pi} \text{Im} \left\langle \delta_0 (H_0 - E - \lambda^2 \Gamma_\lambda(E))^{-1} \delta_0 \right\rangle,$$

where $\Gamma_\lambda(E)$, the so-called ‘self energy’, satisfies $\text{Im} \Gamma_\lambda(E) > 0$ with

$$\lim_{\lambda \rightarrow 0} \text{Im} \Gamma_\lambda(E) \approx \lim_{\eta \rightarrow 0} \text{Im} \langle \delta_0, (H_0 - E - i\eta)^{-1} \delta_0 \rangle = \pi \frac{dN_0(E)}{dE}.$$

Up to a point, the self-energy analysis may be followed rigorously. Specifically, one can show (see Section 2):

PROPOSITION 1.1. *If $\int \omega \rho(\omega) d\omega = 0$ and $\int \omega^2 \rho(\omega) d\omega = 1$, then for each $\lambda > 0$ there is a map Γ_λ from $\{\text{Im } z > 0\}$ to the translation invariant operators with non-negative imaginary part on $\ell^2(\mathbb{Z}^2)$ such that*

$$\int_{\Omega} (H_\omega - z)^{-1} d\mathbb{P}(\omega) = (H_o - z - \lambda^2 \Gamma_\lambda(z))^{-1}, \tag{11}$$

and for fixed $z \in \{\text{Im } z > 0\}$

$$\lim_{\lambda \rightarrow 0} \langle \delta_x, \Gamma_\lambda(z) \delta_y \rangle = \langle \delta_0, (H_o - z)^{-1} \delta_0 \rangle \delta_{x,y}. \tag{12}$$

However there is *a priori* no uniformity in z for the convergence in Equation (12), so for fixed λ we may conclude nothing about

$$\lim_{\eta \downarrow 0} (H_o - E - i\eta - \lambda^2 \Gamma_\lambda(E + i\eta))^{-1}.$$

Still, one is left feeling that Theorem 1 and Corollary 2 are not optimal, and the ‘standard wisdom’ is that something like the following is true.

CONJECTURE 5. Let ρ have moments of all orders, i.e., $\int |\omega|^q \rho(\omega) < \infty$ for all $q \geq 1$. Given $E_o \in \mathbb{R}$, if there is $\delta > 0$ such that on the set $\{\mathbf{q} : |\varepsilon(\mathbf{q}) - E_o| < \delta\}$ the symbol ε is C^1 with $\nabla \varepsilon(\mathbf{q}) \neq 0$, then there is $C_\delta < \infty$ such that

$$\frac{dN_\lambda(E)}{dE} \leq C_\delta$$

for all $\lambda \in \mathbb{R}$ and $E \in [E_o - \frac{1}{2}\delta, E_o + \frac{1}{2}\delta]$.

Remark. The requirement that ρ have moments of all orders is simply the minimal requirement for the infinite perturbation series for $(H_o - z - \lambda V_\omega)^{-1}$ to have finite expectation at each order (for $\text{Im } z > 0$). In fact, this may be superfluous, as suggested by the example of Cauchy randomness, for which the density of states can be explicitly computed, see reference [7]:

$$dN_\lambda(E) = \frac{1}{\pi} \int_{T^d} \frac{\lambda}{(\varepsilon(\mathbf{q}) - E)^2 + \lambda^2} \frac{d\mathbf{q}}{(2\pi)^d}, \quad \text{for } \rho(\omega) = \frac{1}{\pi} \frac{1}{1 + \omega^2},$$

although $\int \rho(\omega) |\omega|^q = \infty$ for every $q \geq 1$.

2. Translation Invariance, Augmented Space, and a Fourier Transform

The joint probability measure $\mathbb{P}(\omega)$ for the random function $\omega: \mathbb{Z}^d \rightarrow \mathbb{R}$ is

$$d\mathbb{P}(\omega) := \prod_{x \in \mathbb{Z}^d} \rho(\omega(x)) d\omega(x)$$

on the probability space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. Clearly, $\mathbb{P}(\omega)$ is invariant under the translations $\tau_\xi: \Omega \rightarrow \Omega$ defined by

$$\tau_\xi \omega(x) = \omega(x - \xi).$$

In particular, since

$$S_\xi H_\omega S_\xi^\dagger = H_\omega + V_{\tau_\xi \omega} = H_{\tau_\xi \omega}, \quad (13)$$

H_ω and $S_\xi H_\omega S_\xi^\dagger$ are identically distributed for any $\xi \in \mathbb{Z}^d$.

To express this invariance in operator theoretic terms, we introduce the fibred action of H_ω on the Hilbert space $L^2(\Omega; \ell^2(\mathbb{Z}^d))$ – the space of ‘random wave functions’ – namely,

$$\Psi(\omega) \mapsto H_\omega \Psi(\omega).$$

We identify $L^2(\Omega; \ell^2(\mathbb{Z}^d))$ with $L^2(\Omega \times \mathbb{Z}^d)$ and denote the action of H_ω on the latter space by \mathbf{H} , so

$$[\mathbf{H}\Psi](\omega, x) = \sum_{\xi} \check{\xi}(\xi) \Psi(\omega, x - \xi) + \lambda \omega(x) \Psi(\omega, x).$$

The following elementary identity relates $\int_{\Omega} f(H_\omega) d\mathbb{P}(\omega)$ to $f(\mathbf{H})$, for any bounded measurable function f ,

$$\int_{\Omega} d\mathbb{P}(\omega) \langle \delta_x, f(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, f(\mathbf{H}) \mathbb{E}^\dagger \delta_y \rangle, \quad (14)$$

where \mathbb{E}^\dagger is the adjoint of the linear expectation map $\mathbb{E}: L^2(\Omega \times \mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ defined by

$$[\mathbb{E}\Psi](x) = \int_{\Omega} \Psi(\omega, x) d\mathbb{P}(\omega).$$

Note that \mathbb{E}^\dagger is an isometry from $\ell^2(\mathbb{Z}^d)$ onto the subspace of functions independent of ω – ‘non-random functions’.

The general fact that averages of certain quantities depending on H_ω can be represented as matrix elements of \mathbf{H} is known, and is sometimes called the ‘augmented space representation’ (e.g., references [3–5]) where ‘augmented space’ refers to the Hilbert space $L^2(\Omega \times \mathbb{Z}^d)$. There are ‘augmented space’ formulae other than Equation (14), such as

$$\int_{\Omega} d\mathbb{P}(\omega) \omega(x) \omega(y) \langle \delta_x, f(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, \mathbf{V} f(\mathbf{H}) \mathbf{V} \mathbb{E}^\dagger \delta_y \rangle, \quad (15)$$

with \mathbf{V} defined below, and

$$\int_{\Omega} d\mathbb{P}(\omega) \langle \delta_x, f(H_\omega) \delta_0 \rangle \langle \delta_0, g(H_\omega) \delta_y \rangle = \langle \mathbb{E}^\dagger \delta_x, f(\mathbf{H}) P_0 g(\mathbf{H}) \mathbb{E}^\dagger \delta_y \rangle,$$

where P_0 denotes the projection $P_0 \Psi(\omega, x) = \Psi(\omega, 0)$ if $x = 0$ and 0 otherwise. The first of these (Equation (15)) will play a roll in the proof of Theorem 1.

There are two natural groups of unitary translations on $L^2(\Omega \times \mathbb{Z}^d)$:

$$S_\xi \Psi(\omega, x) = \Psi(\omega, x - \xi),$$

and

$$T_\xi \Psi(\omega, x) = \Psi(\tau_{-\xi} \omega, x).$$

Note that these groups commute: $[S_\xi, T_{\xi'}] = 0$ for every $\xi, \xi' \in \mathbb{Z}^d$. A key observation is that the distributional invariance of H_ω , Equation (13), results in the *invariance* of \mathbf{H} under the combined translations $T_\xi S_\xi = S_\xi T_\xi$:

$$S_\xi T_\xi \mathbf{H} T_\xi^\dagger S_\xi^\dagger = \mathbf{H}.$$

In fact, let us define

$$\mathbf{H}_o = \sum_{\xi} \check{\xi}(\xi) S_\xi, \quad \mathbf{V} \Psi(\omega, x) = \omega(x) \Psi(\omega, x).$$

Then $\mathbf{H} = \mathbf{H}_o + \lambda \mathbf{V}$ where \mathbf{H}_o commutes with S_ξ and T_ξ while for \mathbf{V} we have

$$\mathbf{V} S_\xi = T_{-\xi} \mathbf{V}.$$

To exploit this translation invariance of \mathbf{H} , we define a Fourier transform which diagonalizes the translations $S_\xi T_\xi$ (and therefore partially diagonalizes \mathbf{H}). The result is a unitary map $\mathcal{F}: L^2(\Omega \times \mathbb{Z}^d) \rightarrow L^2(\Omega \times T^d)$, with T^d the d -torus $[0, 2\pi)^d$. Let us define \mathcal{F} first on functions having finite support in \mathbb{Z}^d by

$$\mathcal{F} \Psi(\omega, \mathbf{k}) = \sum_{\xi} e^{-i\mathbf{k} \cdot \xi} \Psi(\tau_{-\xi} \omega, -\xi).$$

It is easy to verify, using well-known properties of the usual Fourier series mapping $\ell^2(\mathbb{Z}^d) \rightarrow L^2(T^d)$, that \mathcal{F} extends to a unitary isomorphism $L^2(\Omega \times \mathbb{Z}^d) \rightarrow L^2(\Omega \times T^d)$, i.e. that $\mathcal{F} \mathcal{F}^\dagger = 1$ and $\mathcal{F}^\dagger \mathcal{F} = 1$ where \mathcal{F}^\dagger is the adjoint map

$$\mathcal{F}^\dagger \widehat{\Psi}(\omega, x) = \int_{T^d} e^{-i\mathbf{k} \cdot x} \widehat{\Psi}(\tau_{-x} \omega, \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^d}.$$

Another way of looking at \mathcal{F} is to define for each $\mathbf{k} \in T^d$ an operator $\mathcal{F}_\mathbf{k}: L^2(\Omega \times \mathbb{Z}^d) \rightarrow L^2(\Omega)$ by

$$\mathcal{F}_\mathbf{k} \Psi = \lim_{L \rightarrow \infty} \sum_{|\xi| < L} e^{-i\mathbf{k} \cdot \xi} \mathcal{J} S_\xi T_\xi \Psi,$$

where \mathcal{J} is the evaluation map $\mathcal{J} \Psi(\omega) = \Psi(\omega, 0)$. The maps $\mathcal{F}_\mathbf{k}$ are *not* bounded, but are densely defined with $\mathcal{F}_\mathbf{k} \Psi \in L^2(\Omega)$ for almost every \mathbf{k} , and

$$\mathcal{F} \Psi(\omega, \mathbf{k}) = \mathcal{F}_\mathbf{k} \Psi(\omega) \quad \text{a.e. } \omega, \mathbf{k}.$$

If we look at $L^2(\Omega \times T^d)$ as the direct integral $\int^\oplus d\mathbf{k} L^2(\Omega)$, then

$$\mathcal{F} = \int^\oplus d\mathbf{k} \mathcal{F}_\mathbf{k}.$$

This Fourier transform diagonalizes the combined translation $S_\xi T_\xi$,

$$\mathcal{F}_{\mathbf{k}} S_{\xi} T_{\xi} = e^{i\mathbf{k}\cdot\xi} \mathcal{F}_{\mathbf{k}},$$

as follows from the following identities for S and T ,

$$\mathcal{F}_{\mathbf{k}} T_{\xi} = T_{\xi} \mathcal{F}_{\mathbf{k}}, \quad \mathcal{F}_{\mathbf{k}} S_{\xi} = e^{i\mathbf{k}\cdot\xi} T_{-\xi} \mathcal{F}_{\mathbf{k}},$$

where, on the right hand sides, T_{ξ} denotes the operator $T_{\xi}\psi(\omega) = \psi(\tau_{-\xi}\omega)$ on $L^2(\Omega)$. Furthermore, explicit computation shows that

$$\mathcal{F}_{\mathbf{k}} \mathbf{V} = \omega(0) \mathcal{F}_{\mathbf{k}},$$

where $\omega(0)$ denotes the operator of multiplication by the random variable $\omega(0)$, $\psi(\omega) \mapsto \omega(0)\psi(\omega)$. Putting this all together yields:

PROPOSITION 2.1. *Under the natural identification of $L^2(\Omega, T^d)$ with the direct integral $\int^{\oplus} d\mathbf{k} L^2(\Omega)$, the operator $\widehat{\mathbf{H}} = \mathcal{F} \mathbf{H} \mathcal{F}^{\dagger}$ is partially diagonalized, $\widehat{\mathbf{H}} = \int^{\oplus} \widehat{H}_{\mathbf{k}}$, with $\widehat{H}_{\mathbf{k}}$ operators on $L^2(\Omega)$ given by the following formula*

$$\widehat{H}_{\mathbf{k}} = \sum_{\xi} e^{-i\mathbf{k}\cdot\xi} \check{\varepsilon}(-\xi) T_{\xi} + \lambda \omega(0).$$

Let us introduce for each $\mathbf{k} \in T^d$,

$$\widehat{H}_{\mathbf{k}}^o := \sum_{\xi} e^{-i\mathbf{k}\cdot\xi} \check{\varepsilon}(-\xi) T_{\xi} = \sum_{\xi} \left[\int_{T^d} \varepsilon(\mathbf{q} + \mathbf{k}) e^{i\xi\cdot\mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^d} \right] T_{\xi},$$

so $\widehat{H}_{\mathbf{k}} = \widehat{H}_{\mathbf{k}}^o + \lambda \omega(0)$. Note that

$$\widehat{H}_{\mathbf{k}}^o \chi_{\Omega} = \varepsilon(\mathbf{k}) \chi_{\Omega},$$

where $\chi_{\Omega}(\omega) = 1$ for every $\omega \in \Omega$. That is, χ_{Ω} is an eigenvector for $H_{\mathbf{k}}^o$.²

Applying the Fourier transform \mathcal{F} to the right hand side of the ‘‘augmented space’’ formula Equation (14) we obtain the following beautiful identity, central to this work:

$$\int_{\Omega} d\mathbb{P}(\omega) \langle \delta_x, f(H_{\omega}) \delta_y \rangle = \int_{T^d} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(x-y)} \langle \chi_{\Omega}, f(\widehat{H}_{\mathbf{k}}) \chi_{\Omega} \rangle. \quad (16)$$

Similarly, we obtain

$$\begin{aligned} & \int_{\Omega} d\mathbb{P}(\omega) \omega(x) \omega(y) \langle \delta_x, f(H_{\omega}) \delta_y \rangle \\ &= \int_{T^d} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot(x-y)} \langle \omega(0) \chi_{\Omega}, f(\widehat{H}_{\mathbf{k}}) \omega(0) \chi_{\Omega} \rangle \end{aligned} \quad (17)$$

²In fact, if ε is almost everywhere non-constant (so H_o has no eigenvalues) then $\varepsilon(k)$ is the *unique* eigenvalue for $\widehat{H}_{\mathbf{k}}^o$ and the remaining spectrum of $\widehat{H}_{\mathbf{k}}^o$ is infinitely degenerate absolutely continuous spectrum. One way to see this is to let $\phi_n(v)$ be the orthonormal polynomials with respect to the weight $\rho(v)$, and look at the action of $\widehat{H}_{\mathbf{k}}^o$ on the basis for $L^2(\Omega)$ consisting of products of the form $\prod_{x \in \mathbb{Z}^d} \phi_{n(x)}(\omega(x))$ with only finitely many $n(x) \neq 0$.

from Equation (15). Related formulae have been used, for example, to derive the Aubry duality between strong and weak disorder for the almost Mathieu equation, see reference [2] and references therein.

As a first application of Equation (16), let us prove the existence of the self energy (Proposition 1.1) starting from the identity

$$\int_{\Omega} d\mathbb{P}(\omega) \left\langle \delta_0, (H_{\omega} - z)^{-1} \delta_0 \right\rangle = \int_{T^d} \frac{d\mathbf{k}}{(2\pi)^d} \left\langle \chi_{\Omega}, (\widehat{H}_{\mathbf{k}} - z)^{-1} \chi_{\Omega} \right\rangle.$$

Proof of Proposition 1.1. Since χ_{Ω} is an eigenvector of $\widehat{H}_{\mathbf{k}}^o$ and

$$\langle \chi_{\Omega}, \omega(0) \chi_{\Omega} \rangle = \int \omega \rho(\omega) d\omega = 0,$$

the Feschbach mapping implies

$$\left\langle \chi_{\Omega}, (\widehat{H}_{\mathbf{k}} - z)^{-1} \chi_{\Omega} \right\rangle = (\varepsilon(k) - z - \lambda^2 \Gamma_{\lambda}(z; \mathbf{k}))^{-1}, \quad (18)$$

with

$$\Gamma_{\lambda}(z; \mathbf{k}) = \left\langle \omega(0) \chi_{\Omega}, (P^{\perp} \widehat{H}_{\mathbf{k}} P^{\perp} - z)^{-1} \omega(0) \chi_{\Omega} \right\rangle,$$

where P^{\perp} denotes the projection onto the orthogonal complement of χ_{Ω} in $L^2(\Omega)$.

Let the self energy $\Gamma_{\lambda}(z)$ be the translation invariant operator with symbol $\Gamma_{\lambda}(z; \mathbf{k})$, i.e.,

$$\langle \delta_x, \Gamma_{\lambda}(z) \delta_y \rangle = \int_{T^d} e^{i\mathbf{k} \cdot (x-y)} \Gamma_{\lambda}(z; \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^d}.$$

Clearly $\Gamma_{\lambda}(z)$ is bounded with non-negative imaginary part. Furthermore by Equations (16) and (18), the identity Equation (11) holds, namely

$$\int_{\Omega} (H_{\omega} - z)^{-1} d\mathbb{P}(\omega) = (H_o - z - \lambda^2 \Gamma_{\lambda}(z))^{-1}.$$

It is clear that

$$\lim_{\lambda \rightarrow 0} \Gamma_{\lambda}(z; \mathbf{k}) = \left\langle \omega(0) \chi_{\Omega}, (\widehat{H}_{\mathbf{k}}^o - z)^{-1} \omega(0) \chi_{\Omega} \right\rangle,$$

from which Equation (12) follows easily. \square

3. Proofs

We first prove Theorem 1 and then describe modifications of the proof which imply Theorem 3.

3.1. PROOF OF THEOREM 1

Fix a regular point E for ε , and for each $\delta > 0$ let

$$\begin{aligned} f_\delta(t) &= \frac{1}{2} (\chi_{(E-\delta, E+\delta)}(t) + \chi_{[E-\delta, E+\delta]}(t)) \\ &= \begin{cases} 1 & t \in (E-\delta, E+\delta), \\ \frac{1}{2} & t = E \pm \delta, \\ 0 & t \notin [E-\delta, E+\delta]. \end{cases} \end{aligned}$$

Since $N_\lambda(E)$ is continuous (see Equation (4)),

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) = \int_\Omega \langle \delta_0, f_\delta(H_\omega) \delta_0 \rangle d\mathbb{P}(\omega).$$

Thus, in light of Equation (16), our task is to show that

$$\int_{T^d} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}}) \chi_\Omega \rangle \frac{d\mathbf{k}}{(2\pi)^d} \leq \Gamma(E) \delta + C_q \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})}, \quad (19)$$

with a constant C_q independent of δ and λ . Note that for each $\mathbf{k} \in T^d$

$$|\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}}) \chi_\Omega \rangle| \leq 1,$$

so we can afford to neglect a set of Lebesgue measure $\lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})}$ on the left-hand side of Equation (19).

Consider $\mathbf{k} \in T^d$ with $|\varepsilon(\mathbf{k}) - E| > \delta$. Then

$$f_\delta(\widehat{H}_{\mathbf{k}}^o) \chi_\Omega = f_\delta(\varepsilon(\mathbf{k})) \chi_\Omega = 0.$$

Thus

$$\begin{aligned} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}}) \chi_\Omega \rangle &= \langle \chi_\Omega, (f_\delta(\widehat{H}_{\mathbf{k}}) - f_\delta(\widehat{H}_{\mathbf{k}}^o)) \chi_\Omega \rangle \\ &= \lim_{\eta \rightarrow 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \operatorname{Im} \left\langle \chi_\Omega, \left(\frac{1}{\widehat{H}_{\mathbf{k}} - t - i\eta} - \frac{1}{\widehat{H}_{\mathbf{k}}^o - t - i\eta} \right) \chi_\Omega \right\rangle dt \\ &= \lambda \lim_{\eta \rightarrow 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \operatorname{Im} \frac{1}{t + i\eta - \varepsilon(\mathbf{k})} \left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - t - i\eta} \omega(0) \chi_\Omega \right\rangle dt \\ &= \lambda \left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}_{\mathbf{k}}) \omega(0) \chi_\Omega \right\rangle, \end{aligned} \quad (20)$$

since $(t - \varepsilon(\mathbf{k}))^{-1}$ is continuous for $t \in [E-\delta, E+\delta]$. Using again that $f_\delta(\widehat{H}_{\mathbf{k}}^o) \chi_\Omega = 0$, we find that the final term of Equation (20) equals

$$\begin{aligned}
&= \left\langle \left[\frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_{\delta}(\widehat{H}_{\mathbf{k}}) - \frac{1}{\widehat{H}_{\mathbf{k}}^o - \varepsilon(\mathbf{k})} f_{\delta}(\widehat{H}_{\mathbf{k}}^o) \right] \chi_{\Omega}, \omega(0) \chi_{\Omega} \right\rangle \\
&= \lambda \lim_{\eta \rightarrow 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \frac{1}{t - \varepsilon(\mathbf{k})} \operatorname{Im} \frac{1}{t + i\eta - \varepsilon(\mathbf{k})} \times \\
&\quad \times \left\langle \frac{1}{\widehat{H}_{\mathbf{k}} - t - i\eta} \omega(0) \chi_{\Omega}, \omega(0) \chi_{\Omega} \right\rangle dt \\
&= \lambda \left\langle \omega(0) \chi_{\Omega}, \frac{f_{\delta}(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0) \chi_{\Omega} \right\rangle.
\end{aligned} \tag{21}$$

Putting Equations (20) and (21) together yields

$$\begin{aligned}
\langle \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \chi_{\Omega} \rangle &= \lambda^2 \langle \omega(0) \chi_{\Omega}, \frac{f_{\delta}(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0) \chi_{\Omega} \rangle \\
&\leq \lambda^2 \frac{1}{(|\varepsilon(\mathbf{k}) - E| - \delta)^2} \langle \omega(0) \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \omega(0) \chi_{\Omega} \rangle.
\end{aligned}$$

Thus, for any $\epsilon > \delta$,

$$\begin{aligned}
&\int_{\{|\varepsilon(\mathbf{k}) - E| > \epsilon\}} \langle \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \chi_{\Omega} \rangle \\
&\leq \lambda^2 \frac{1}{(\epsilon - \delta)^2} \int_{T^d} \langle \omega(0) \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \omega(0) \chi_{\Omega} \rangle \\
&= \lambda^2 \frac{1}{(\epsilon - \delta)^2} \int_{\Omega} \omega(0)^2 \langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle d\mathbb{P}(\omega),
\end{aligned}$$

where in the last equality we have inverted the Fourier transform, using Equation (17). We may estimate the right hand side with Hölder's inequality and the Wegner estimate:

$$\begin{aligned}
&\int_{\Omega} \omega(0)^2 \langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle d\mathbb{P}(\omega) \\
&\leq \|\omega(0)\|_q^2 \left(\int_{\Omega} \langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle d\mathbb{P}(\omega) \right)^{1-\frac{2}{q}} \\
&\leq \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_{\infty}}{\lambda} 2\delta \right)^{1-\frac{2}{q}},
\end{aligned}$$

since $\langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle^p \leq \langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle$ for $p > 1$ (because $\langle \delta_0, f_{\delta}(H_{\omega}) \delta_0 \rangle \leq 1$). Here $\|\omega(0)\|_q^q = \int \omega(0)^q d\mathbb{P}(\omega)$ for $q < \infty$ and $\|\omega(0)\|_{\infty} = \operatorname{ess-sup}_{\omega} |\omega(0)|$.

Therefore

$$\int_{T^d} \langle \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \chi_{\Omega} \rangle \leq \Gamma(E) \epsilon + \lambda^2 \frac{1}{(\epsilon - \delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_{\infty}}{\lambda} 2\delta \right)^{1-\frac{2}{q}}, \tag{22}$$

where the first term on the right hand side is an upper bound for

$$\int_{\{|\varepsilon(\mathbf{k}) - E| \leq \epsilon\}} \langle \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}}) \chi_{\Omega} \rangle \frac{d\mathbf{k}}{(2\pi)^d} \leq \int_{\{|\varepsilon(\mathbf{k}) - E| \leq \epsilon\}} \frac{d\mathbf{k}}{(2\pi)^d}.$$

Upon optimizing over $\epsilon \in (\delta, \infty)$, this implies

$$\int_{\Omega} \langle \delta_0, f_{\delta}(H_{\omega})\delta_0 \rangle \leq \Gamma(E)\delta + C_{\rho,q,\Gamma} \lambda^{\frac{1}{3}\left(1+\frac{2}{q}\right)} \delta^{\frac{1}{3}\left(1-\frac{2}{q}\right)},$$

which completes the proof of Theorem 1. \square

3.2. PROOF OF THEOREM 3

If instead of being a regular point, E is a point of order α then the proof goes through up to Equation (22), in place of which we have

$$\int_{T^d} \langle \chi_{\Omega}, f_{\delta}(\widehat{H}_{\mathbf{k}})\chi_{\Omega} \rangle \leq \Gamma(E; \alpha)\epsilon^{\alpha} + \lambda^2 \frac{1}{(\epsilon - \delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_{\infty}}{\lambda} \delta \right)^{1-\frac{2}{q}}.$$

Setting $\epsilon = \delta + \lambda^{\gamma} \delta^{\beta}$ and choosing γ, β such that the two terms are of the same order yields

$$\gamma = \frac{1}{2+\alpha} \left(1 + \frac{2}{q} \right), \quad \beta = \frac{1}{2+\alpha} \left(1 - \frac{2}{q} \right),$$

which implies

$$\int_{\Omega} \langle \delta_0, f_{\delta}(H_{\omega})\delta_0 \rangle \leq \Gamma(E; \alpha)\delta^{\alpha} + C_q \lambda^{\frac{\alpha}{2+\alpha}\left(1+\frac{2}{q}\right)} \delta^{\frac{\alpha}{2+\alpha}\left(1-\frac{2}{q}\right)},$$

completing the proof. \square

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