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Ptolemy circles and Ptolemy segments

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Abstract. In this paper we characterize Ptolemy circles and Ptolemy segments up to isometry. Moreover, we present an example of a metric sphere, which is Möbius equivalent but not homothetic to the standard metric sphere with its chordal metric.

1. Introduction. Recall the classical characterization of circles which goes back to Claudius Ptolemaeus (ca. 90-168).

Theorem 1.1. (Ptolemy's Theorem) *Consider four points in the Euclidean space, $x_1, x_2, x_3, x_4 \in \mathbb{E}^n = (\mathbb{R}^n, d)$. Then*

$$d(x_1, x_3)\, d(x_2, x_4) \leq d(x_1, x_2)\, d(x_3, x_4) + d(x_1, x_4)\, d(x_3, x_2). \tag{1}$$

Moreover, if the four points are distinct, equality holds if and only if they lie on a circle C such that x_2 and x_4 lie in different components of $C \backslash \{x_1, x_3\}$.

A metric space (X, d) is called a *Ptolemy metric space* if the inequality (1) holds for arbitrary quadruples in the space. We call a subset $\sigma \subset X$ a *Ptolemy circle* if σ is homeomorphic to S^1 and for any four points x_1, x_2, x_3, x_4 on σ (in this order) we have equality in (1).
Similarly, we call a subset $I \subset X$ a Ptolemy segment if I is homeomorphic to a closed interval and for any four points x_1, x_2, x_3, x_4 on σ (in this order) we have equality in (1).

The standard examples of Ptolemy metric spaces are the Euclidean space $\mathbb{E}^n$ or the sphere $S^n \subset \mathbb{E}^{n+1}$ with the induced (chordal) metric.

In this paper we study the Möbius and the isometry classes of Ptolemy segments and Ptolemy circles.
Recall that a Möbius map between metric spaces is a map that leaves cross-ratios of quadruples of points invariant. For a precise definition compare Section 2.
The Möbius classification is very simple. Namely, for circles we prove

Theorem 1.2. *Let C and C' be Ptolemy circles. Let x_1, x_2, x_3 and x'_1, x'_2, x'_3 be distinct points on C respectively on C'. Then there exists a unique Möbius homeomorphism $\varphi : C \to C'$ with $\varphi(x_i) = x'_i$.*

And for segments we prove

Theorem 1.3. *Let I, I' be Ptolemy segments. Let x_1, x_3 be the boundary points of I and x'_1, x'_3 be the boundary points of I'. Let in addition x_2 and x'_2 be inner points of I and I'. Then there exists a unique Möbius homeomorphism $\varphi : I \to I'$ with $\varphi(x_i) = x'_i$.*

The isometry type is, however, much more difficult to determine.
We show that the isometry type of a Ptolemy segment can be completely characterized by a convex domain in a Euclidean quadrant. The precise statement is Theorem 3.6 in Section 3.2.
Similarly we can describe the isometry type of two pointed Ptolemy circles in terms of convex domains in a Euclidean half plane. The precise statement will be given in Theorem 3.8 in Section 3.3.

A natural question is to study higher dimensional analoga of circles.
We call a subset Σ of a Ptolemy metric space X a *Ptolemy-n-sphere*, if Σ is homeomorphic to S^n and any triple of points $x, y, z \in \Sigma$ is contained in a circle $\sigma \subset \Sigma$.
As it follows from the main Theorem in [6], a Ptolemy-n-sphere is Möbius equivalent to the chordal sphere (S^n, d_0). Thus the Möbius classification is trivial. An isometry classification, however, remains an open problem.

To show the complexity we provide in Section 3.4 an example of a Ptolemy-n-sphere, which is not homothetic to a chordal sphere.

Our main motivation to study Ptolemy metric spaces is, that these spaces arise naturally in the context of negative curvature. Namely, it was proven in [4] that boundaries at infinity of CAT(-1)-spaces endowed with their Bourdon metrics are indeed Ptolemy.

As an example, we recall that the Bourdon boundary of a real hyperbolic space $\mathbb{H}^{n+1}$ is nothing but the sphere S^n endowed with its chordal metric d_0, i.e. the metric induced by its standard embedding $S^n \hookrightarrow \mathbb{E}^{n+1}$. Via the stereographic projection, (S^n, d_0) is Möbius equivalent to $\mathbb{E}^n \cup \{\infty\}$.
It was also shown in [4], that Ptolemy circles at boundaries at infinity of CAT(-1)-spaces, correspond to isometrically embedded real hyperbolic planes in the spaces themselves. For further information about Ptolemy metric spaces compare also [3].

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2. Elements of metric Möbius geometry.

2.1. Möbius maps. Let X be a set which contains at least two points. An *extended metric* on X is a map $d : X \times X \to [0, \infty]$, such that there exists a set $\Omega(d) \subset X$ with cardinality $\#\Omega(d) \in \{0, 1\}$, such that d restricted to the set $X \backslash \Omega(d)$ is a metric (taking only values in $[0, \infty)$) and such that $d(x, \omega) = \infty$ for all $x \in X \backslash \Omega(d)$, $\omega \in \Omega(d)$. Furthermore $d(\omega, \omega) = 0$.

If $\Omega(d)$ is not empty, we sometimes denote $\omega \in \Omega(d)$ simply as ∞ and call it the (infinitely) remote point of (X, d). We often write also $\{\omega\}$ for the set $\Omega(d)$ and X_ω for the set $X \backslash \{\omega\}$.

The topology considered on (X, d) is the topology with the basis consisting of all open distance balls $B_r(x)$ around points in $x \in X_\omega$ and the complements D^C of all closed distance balls $D = \overline{B}_r(x)$.

We say that a quadruple $(x, y, z, w) \in X^4$ is *admissible* if no entry occurs three or four times in the quadruple. We denote with $Q \subset X^4$ the set of admissible quadruples. We define the *cross ratio triple* as the map $\mathrm{crt}: \ Q \to \Sigma \subset \mathbb{R}P^2$ which maps admissible quadruples to points in the real projective plane defined by

$$\mathrm{crt}(x, y, z, w) = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)),$$

here Σ is the subset of points $(a : b : c) \in \mathbb{R}P^2$, where all entries a, b, c are nonnegative or all entries are non positive. Note that Σ can be identified with the standard 2-simplex, $\{(a, b, c) \,|\, a, b, c \geq 0,\ a + b + c = 1\}$.

We use the standard conventions for the calculation with ∞. If ∞ occurs once in Q, say $w = \infty$, then $\mathrm{crt}(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$. If ∞ occurs twice , say $z = w = \infty$ then $\mathrm{crt}(x, y, \infty, \infty) = (0 : 1 : 1)$.

A map $f : X \to Y$ between two extended metric spaces is called *Möbius*, if f is injective and for all admissible quadruples (x, y, z, w) of X,

$$\mathrm{crt}(f(x), f(y), f(z), f(w)) = \mathrm{crt}(x, y, z, w).$$

Möbius maps are continuous.

Two extended metric spaces (X, d) and (Y, d') are *Möbius equivalent* if there exists a bijective Möbius map $f : X \to Y$. In this case also f^{-1} is a Möbius map and f is in particular a homeomorphism.

2.2. Ptolemy spaces. An extended metric space (X, d) is called a *Ptolemy space* if for all quadruples of points $\{x, y, z, w\} \in X^4$ the *Ptolemy inequality* holds

$$d(x, y)\, d(z, w) \leq d(x, z)\, d(y, w) + d(x, w)\, d(y, z).$$

We can reformulate this condition in terms of the cross ratio triple. Let $\Delta \subset \Sigma$ be the set of points $(a : b : c) \in \Sigma$, such that the entries a, b, c satisfy the triangle inequality. This is obviously well defined. If we identify $\Sigma \subset \mathbb{R}P^2$ with the standard 2-simplex, i.e. the convex hull of the unit vectors e_1, e_2, e_3, then Δ is the convex subset spanned by $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. We denote by $\hat{e}_1 := (0 : 1 : 1)$, $\hat{e}_2 := (1 : 0 : 1)$ and $\hat{e}_3 := (1 : 1 : 0)$. Note that also Δ is homeomorphic to a 2-simplex and $\partial\Delta$ is homeomorphic to S^1. Then an extended space is Ptolemy if $\mathrm{crt}(x, y, z, w) \in \Delta$ for all allowed quadruples Q.

This description shows that the Ptolemy property is Möbius invariant.

2.3. Circles in Ptolemy spaces. A circle in a Ptolemy space (X, d) is a subset $\sigma \subset X$ homeomorphic to S^1 such that for distinct points $x, y, z, w \in \sigma$ (in this order)

$$d(x, z)d(y, w) = d(x, y)d(z, w) + d(x, w)d(y, z) \tag{2}$$

Here the phrase "in this order" means that y and w are in different components of $\sigma\setminus\{x,z\}$.

One can reformulate this via the crossratio triple. A subset σ homeomorphic to S^1 is a circle if and only if for all admissible quadruples (x,y,z,w) of points in σ we have $\operatorname{crt}(x,y,z,w)\in\partial\Delta$. This shows that the definition of a circle is Möbius invariant.

Let σ be a circle and let $\omega\in\sigma$. Consider $\sigma_\omega=\sigma\setminus\{\omega\}$ in a complete metric with infinitely remote point ω, then $\operatorname{crt}(x,y,z,\omega)\in\partial\Delta$ says that for $x,y,z\in\sigma_\omega$ (in this order) $d(x,y)+d(y,z)=d(x,z)$, i.e. it implies that σ_ω is a geodesic, actually a complete geodesic isometric to $\mathbb{R}$.

3. Classification of circles and Ptolemy segments. In this section we classify circles and segments in Ptolemy spaces. We start with a classification up to Möbius equivalence.

3.1. Möbius classification of circles and segments. We now prove Theorem 1.2.

Proof. Define $\varphi_C: C\to\partial\Delta$ by $\varphi_C(t)=\operatorname{crt}(t,x_1,x_2,x_3)$. Since C is a Ptolemy circle, the image of φ_C is actually in $\partial\Delta$. The map is continuous and maps x_1 to $\hat e_1=(0:1:1)$, x_2 to $\hat e_2=(1:0:1)$ and x_3 to $\hat e_3=(1:1:0)$.

One also easily checks that $\varphi^{-1}(\hat e_i)=x_i$. Now $C\setminus\{x_1,x_2,x_3\}$ consists of three open segments I_1,I_2,I_3, such that x_i and x_j are boundary points of I_k, and x_k is not in I_k (here $\{i,j,k\}=\{1,2,3\}$). Correspondingly $\partial\Delta\setminus\{\hat e_1,\hat e_2,\hat e_3\}$ consists of three open segments J_1,J_2,J_3. Note that J_i consists of all $(a_1:a_2:a_3)\in\partial\Delta$, such that $|a_i|>\max\{|a_j|,|a_k|\}$. Since φ_C gives a bijection $x_i\leftrightarrow\hat e_i$, φ_C maps I_i to J_i. If $t\in I_i$, this implies that the equality (2) is written in the following way:

$$d(t,x_j)d(x_i,x_k)+d(t,x_k)d(x_i,x_j)=d(t,x_i)d(x_j,x_k).$$

We now show that φ_C is injective. Assume that $\varphi_C(s)=\varphi_C(t)$. This implies that there exists $\lambda>0$ such that $d(s,x_i)=\lambda d(t,x_i)$ for $i=1,2,3$. In particular we have

$$d(x_1,t)d(s,x_2)=d(x_1,s)d(t,x_2) \tag{3}$$

Since φ_C maps I_i to J_i this implies that s and t are in the same component of $C\setminus\{x_1,x_2,x_3\}$ (namely the component I_3). Thus we have (eventually after permuting t and s)

$$d(x_1,t)d(s,x_2)+d(t,s)d(x_1,x_2)=d(x_1,s)d(t,x_2),$$

which implies $d(t,s)=0$ because of (3).

Since C is homeomorphic to S^1 and $\varphi_C: C\to\partial\Delta$ is injective and continuous, it is also surjective and a homeomorphism.

Now the map $\varphi: C\to C'$, $\varphi:=\varphi_{C'}^{-1}\circ\varphi_C$ is a Möbius homeomorphism and maps x_i to x_i'. Assume on the other side that $\psi: C\to C'$ is a Möbius homeomorphism with $\psi(x_i)=(x_i')$. Then $\varphi_C(t)=\operatorname{crt}(t,x_1,x_2,x_3)=\operatorname{crt}(\psi(t),x_1',x_2',x_3')=\varphi_{C'}(\psi(t))$, which implies $\psi=\varphi_{C'}^{-1}\circ\varphi_C$. $\square$

The proof of Theorem 1.3 is completely analogous. The map $\varphi_C(t) := \mathrm{crt}(t, x_1, x_2, x_3)$ now maps I homeomorphically on the path in $\partial\Delta$, which goes from $\hat{e}_1$ via $\hat{e}_2$ to $\hat{e}_3$. Again $\varphi = \varphi_{C'}^{-1} \circ \varphi_C$ is the required homeomorphism.

3.2. Classification of Ptolemy segments up to isometry. We now study Ptolemy segments and classify them up to isometry.

We first consider the special case that one boundary point of the segment is the point ∞. Let $x \in X\backslash\{\infty\}$ be the other boundary point. Let $x < s < t < \infty$ be points on the segment (where the order is induced by the homeomorphism of the segment to $[0,1]$). Then the Ptolemy equality implies $d(x,t) = d(x,s) + d(s,t)$. This implies that the segment is isometric to an interval and actually isometric to $[0,\infty] \subset \mathbb{R} \cup \{\infty\}$.

We now consider a Ptolemy segment $([0,1], d)$, with $R := d(0,1)$ a positive and finite number. Let $Q := [0,\infty) \times [0,\infty) \subset \mathbb{R}^2$. We define a map $\psi : [0,1] \to Q$ by $t \mapsto p_t = \binom{a_t}{b_t}$, where $a_t = d(t,1)$ and $b_t = d(t,0)$. Thus p_t is a curve in Q from $e_R^1 = \binom{R}{0}$ to $e_R^2 = \binom{0}{R}$.

Note that the Ptolemy condition applied to the quadruple $0 \le t_1 \le t_2 \le 1$ implies that $Rd(t_1,t_2) = a_{t_1} b_{t_2} - b_{t_1} a_{t_2} = \langle Jp_{t_1}, p_{t_2}\rangle$, where $\langle\, ,\, \rangle$ is the standard scalar product on Q and J the standard rotation $J\binom{a}{b} = \binom{-b}{a}$.

Now for two points $p, q \in Q$ we have $\langle Jp, q\rangle \ge 0$ iff $\arg(p) \le \arg(q)$. As a consequence this implies that $t \mapsto \arg(p_t)$ is a strictly increasing function.

This motivates that we consider for two points $p, q \in Q$ the expression $\langle Jp, q\rangle$ as a kind of "signed distance". This "signed distance" is related to the Ptolemy equality, as a trivial computation shows:

Lemma 3.1. *For $p_1, p_2, p_3, p_4 \in Q$ (actually for arbitrary $p_1, p_2, p_3, p_4 \in \mathbb{R}^2$) we have*

$$\langle Jp_1, p_2\rangle\langle Jp_3, p_4\rangle + \langle Jp_2, p_3\rangle\langle Jp_1, p_4\rangle = \langle Jp_1, p_3\rangle\langle Jp_2, p_4\rangle$$

Remark 3.2. Thus the expression $R(p_1, p_2, p_3, p_4) = \langle Jp_1, p_2\rangle\langle Jp_3, p_4\rangle$ has the symmetries of a curvature tensor. The above Lemma corresponds to the Bianchi identity. The other symmetries are obvious.

The expression $\langle Jp, q\rangle$ does not satisfy the triangle inequality and we have to study more precisely what conditions correspond to the triangle inequality. Let us therefore consider three points $u, v, w \in Q$ such that $\arg(u) \le \arg(v) \le \arg(w)$. This implies that $v = \lambda\, u + \mu\, w$, with $\lambda, \mu \ge 0$. We say that u, v, w *satisfies the triangle inequality*, if the following three inequalities hold:

1. $\langle Ju, v\rangle + \langle Jv, w\rangle \ge \langle Ju, w\rangle$
2. $\langle Jv, w\rangle + \langle Ju, w\rangle \ge \langle Ju, v\rangle$
3. $\langle Ju, w\rangle + \langle Ju, v\rangle \ge \langle Jv, w\rangle$

Clearly for $0 \le t_1 \le t_2 \le t_3 \le 1$ the points $p_{t_1}, p_{t_2}, p_{t_3} \in Q$ have to satisfy the triangle inequality.

An easy computation shows the following:

Lemma 3.3. *Let $u, v, w \in Q$ be as above. Then the three triangle inequalities are respectively equivalent to: $\lambda + \mu \ge 1$, $\lambda + 1 \ge \mu$ and $1 + \mu \ge \lambda$, i.e. that the three non-negative numbers $\lambda, \mu, 1$ satisfy the triangle inequality.*

Let $u, w \in Q$ with $\arg(u) < \arg(w)$. We define the region

$$T(u,w) = \{(\lambda u + \mu w) \in Q |\, 0 \leq \lambda,\, 0 \leq \mu,\, \lambda + \mu \geq 1,\, \lambda + 1 \geq \mu,\, \mu + 1 \geq \lambda\}.$$

Note that $T(u,w)$ is the convex region in Q which is bounded by the line segment $s\,u + (1-s)\,w$, where $0 \leq s \leq 1$ from u to w, and the two parallel rays $u + s\,(u+w)$ resp. $w + s\,(u+w)$ for $0 \leq s < \infty$. These three affine segments correspond to the three triangle inequalities. This can easily be verified, since the equation $\lambda + \mu = 1$ defines the line $\ell(u,w)$, the equation $\lambda + 1 = \mu$ the line $\ell_w = \ell(w, w + (u+w))$ and $\mu + 1 = \lambda$ the line $\ell_u = \ell(u, u + (u+w))$. The last two lines are parallel, The inequalities define corresponding half spaces. Here we denote for different $p, q \in Q$ with $\ell(p,q)$ the affine line determined by p and q.

If we consider a Ptolemy interval, then the corresponding curve p_t satisfies: if $0 \leq t_1 < t_2 < t_3 \leq 1$ then $p_{t_2} \in T(p_{t_1}, p_{t_3})$. In particular the whole curve is contained in the set $T(e_R^1, e_R^2)$.

The first of the three inequalities (namely $\lambda + \mu \geq 1$) is easy to understand. It just means that the curve p_t is convex in the following sense.

Definition 3.4. We call a curve p_t in Q from e_R^1 to e_R^2 *convex*, if p_t is continuous, $\arg(p_t)$ is strictly increasing and the bounded component of $Q \backslash \{p_t | t \in [0,1]\}$ is convex.

Surprisingly this condition together with the condition that $p_t \in T(e_R^1, e_R^2)$ imply all other triangle conditions.

Lemma 3.5. *Let p_t be a convex curve in $T(e_R^1, e_R^2)$ from e_R^1 to e_R^2, then for $0 \leq t_1 < t_2 < t_3 \leq 1$ we have $p_{t_2} \in T(p_{t_1}, p_{t_3})$.*

Proof. Let p_t be a convex curve in $T(e_R^1, e_R^2)$ from e_R^1 to e_R^2. We denote $u = p_{t_1}, v = p_{t_2}, w = p_{t_3}$. We have to show that $v \in T(u,w)$. Consider the lines $\ell(e_R^1, u)$ and $\ell(e_R^2, w)$.

There are two cases. First assume that the lines are parallel (and hence do not intersect). In this case u and w lie on the two boundary rays of $T(e_R^1, e_R^2)$ and one easily checks that $T(u,w)$ is the closure of the unbounded component of $T(e_R^1, e_R^2) \backslash \ell(u,w)$. Since p_t is convex, this implies that $v \in T(u,w)$.

Thus we consider the second case that the two lines intersect in a point z. Since p_t is a convex curve containing e_R^1, u, w, e_R^2, we see that z is in the closure of the unbounded component of $T(e_R^1, e_R^2) \backslash \{p_t | t \in [0,1]\}$ and $v \in \Delta(u,w,z)$, where $\Delta(u,w,z)$ is the corresponding triangle.

It remains to prove that $z \in T(u,w)$. Let $\alpha = \arg(u+w) = \angle_o(e_R^1, u+w)$, where o is the origin, and $\angle$ the usual Euclidean angle. Let $\beta = \angle_o(u+w, e_R^2)$. Thus $\alpha + \beta = \pi/2$. We assume (without loss of generality) that $\alpha \geq \pi/4$. Consider the two lines $\ell_u = \ell(u, u + (u+w))$ and $\ell_w = \ell(w, w + (u+w))$, which are the lines containing the boundary rays of $T(u,w)$. Then ℓ_u intersects the boundary of Q in a point $q_1 = \binom{a}{0}$ and ℓ_w intersects the boundary of Q in $q_2 = \binom{0}{b}$. Since $\alpha \geq \pi/4$, we have $a \leq b$. Since $\angle_{q_2}(w, 2q_2) = \beta \leq \pi/4$ and $\angle_{e_R^2}(w, 2e_R^2) \geq \pi/4$ (since $w \in T(e_R^1, e_R^2)$), we see that $b \leq R$ and hence also $a \leq R$. This implies that $z = \ell(e_R^1, u) \cap \ell(e_R^2, w)$ is in the strip bounded by ℓ_u and ℓ_w and hence in $T(u,w)$. $\square$

Collecting all results we can now state:

Theorem 3.6. *The isometry classes of Ptolemy intervals* $([0,1],d)$ *with* $d(0,1) = R$ *stay in* $1-1$ *relation to the convex curves in* $T(e_R^1, e_R^2)$ *from* e_R^1 *to* e_R^2 *modulo reflection at the bisecting line in* Q*.*

Indeed given such a Ptolemy interval, we obtain such a convex curve. If we have otherwise given such a convex curve $\psi : [0,1] \to Q$, $\psi(t) = p_t = \binom{a_t}{b_t}$, then for given $s \leq t \in [0,1]$ we have $d(0,s) = b_s$, $d(1,s) = a_s$, $d(0,t) = b_t$, $d(1,t) = a_t$. Now $d(t,s)$ is determined by the Ptolemy equality:

$$d(s,t)d(0,1) + d(0,s)d(t,1) = d(0,t)d(s,1).$$

Thus the curve determines the isometry class completely.
Note that two Ptolemy segments are isometric to each other if and only if their Ptolemy parameterizations as above either coincide or if they are obtained from one another by reflection at the bisecting line in Q.

Remark 3.7. (i) On Ptolemy intervals which can be realized in $\mathbb{E}^2$.
Let $p, q \in \mathbb{E}^2$ be two points of distance $|pq| = R$. Then for $r = \frac{R}{2}$ there is exactly one Ptolemy segment (namely a half circle of radius r) in $\mathbb{E}^2$ connecting two points of distance R up to isometry, whereas for $r > \frac{R}{2}$ there are exactly two such segments. These segments are precisely the Ptolemy segments which can be isometrically embedded in $\mathbb{E}^2$ (this follows essentially from the classical characterization of circles by the Ptolemy equality). We claim now, that their images in Q are the intersections of Q with the ellipses in $\mathbb{R}^2$ through e_R^1 and e_R^2 with axis along $\operatorname{span}\{e_R^1 + e_R^2\}$ and $\operatorname{span}\{e_R^1 - e_R^2\}$. Indeed, for $R > 0$, $r \geq \frac{R}{2}$ consider a circle in $\mathbb{E}^2$ with radius r connecting two points p and q in distance $|pq| = R$ of each other. Seen from the circle‘s origin, the points p and q enclose an angle α with $\sin\frac{\alpha}{2} = \frac{R}{2r}$. In each point m on the circle segment considered, let β denote the angle of p and q at m. This angle $\beta = [\pi - \frac{\alpha}{2}]$ is constant along the segment and using the law of cosine in $\mathbb{E}^2$ the image of this circle segment in Q is given in the coordinates a and b of Q through $R^2 = a^2 + b^2 - 2ab\cos\beta$. Since the last equation describes the ball of radius R around the origin w.r.t. the scalar product on $\mathbb{R}^2$ with $< e_R^1, e_R^1 >= 1 =< e_R^2, e_R^2 >$ and $< e_R^1, e_R^2 >= -\cos\beta$, it determines an ellipse as claimed.
This remark shows that the possible isometry classes of Ptolemy segments are much richer than those of circles which can be realized in the Euclidean plane (Figure 1).

(ii) *The Ptolemy parameterization.*
Given a Ptolemy segment (I,d), there exists a unique Ptolemy parameterization $\mathfrak{pt} : [0, \frac{\pi}{2}] \longrightarrow Q$ as above, parameterizing the segment by the angle $\alpha \in [0, \frac{\pi}{2}]$ its image point in Q encloses with e_1^R.
Consider two Ptolemy segments I_i, $i = 1,2$, and denote their Ptolemy parameterizations by $\mathfrak{pt}_i$, $i = 1,2$. Then the map $\varphi : I_1 \longrightarrow I_2$ satisfying $\alpha(\varphi(x)) = \alpha(x)$ for all $x \in I_1$ is a Möbius map.

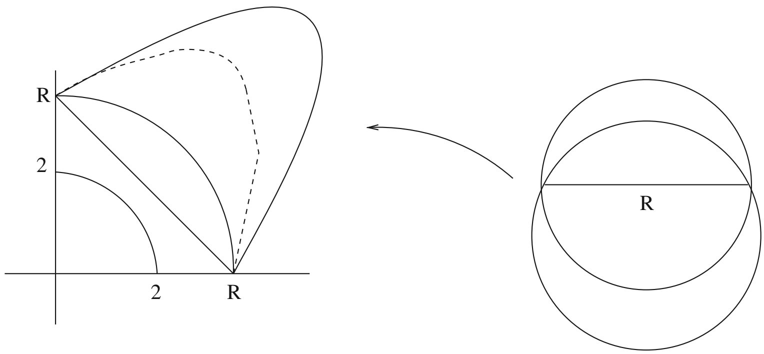

Figure 1. The figure shows a variety of Ptolemy parameterizations of different Ptolemy segments. The *dashed curve* on the *left hand side* corresponds to a non Euclidean configuration. The others are pieces of ellipses as described in Remark 3.7. Their corresponding Euclidean configurations are shown on the *right hand side*

3.3. Classification of Ptolemy circles up to isometry. We now study Ptolemy circles. Let $S^1 = \{e^{\pi it} \in \mathbb{C} \,|\, 0 \le t \le 2\}$. We will assume that $d(-1,1) = R$. Now S^1 consists of two "segments" from 1 to -1. Let $H = [0,\infty) \times \mathbb{R}$ be the upper half space. We now define a map $\varphi : [0,2] \to H = [0,\infty) \times \mathbb{R}$, by $t \mapsto p_t = \binom{a_t}{b_t}$, where $b_t = d(e^{\pi it}, 1)$ and $a_t = d(e^{\pi it}, -1)$ for $0 \le t \le 1$ and $a_t = -d(e^{\pi it}, -1)$ for $1 \le t \le 2$.

Let (S^1, d) be a Ptolemy circle and let $0 \le t_1 < t_2 \le 2$. Then the Ptolemy condition for the circle applied to the three possible cases $0 \le t_1 < t_2 \le 1$, $0 \le t_1 \le 1 \le t_2$ and $0 < 1 \le t_1 \le t_2$ always gives the distance $d(t_1, t_2) = \langle Jp_{t_1}, p_{t_2} \rangle$.

This implies that $\arg(p_t)$ is strictly increasing with t. The discussion with the triangle inequality is similar as in the case of a Ptolemy segment. However, note that in the proof of Lemma 3.3 we used the fact that $\langle Ju, v \rangle \neq 0$. Thus the argument does not work for $u = e^1_R$ and $w = -e^1_R$. We can, however, say the following: if $0 < t_1 < t_2 < t_3 < 2$ are three points in the open interval $(0,2)$, then the triangle condition is equivalent to $p_{t_2} \in T(p_{t_1}, p_{t_3})$ as above. The same is true if $t_1 = 0$ and $t_3 < 2$ (resp. $0 < t_1$ and $t_3 = 2$).

We want to understand the limit case $t_1 = 0, t_3 = 2$.

Therefore we define for a unit vector $x \in S^1$ with $0 \le \arg(x) \le \pi$ the sector $T_x(e^1_R, -e^1_R) := \{(se^1_R + tx) \,|\, -1 \le s \le 1, 0 \le t < \infty\}$.

The analogon of Proposition 3.6 for Ptolemy circles now reads as follows.

Theorem 3.8. *The Ptolemy circles (S^1, d) with $d(1,-1) = R$ are in $1-1$ relation to the convex curves p_t in H from e^1_R via e^2_R to $-e^1_R$ which are contained in $T_x(e^1_R, -e^1_R)$ for some $x \in S^1$ with $\pi/2 \le x \le 3\pi/2$.*

Proof. The condition that (for $0 \le t_1 < t_2 < t_3 < 2$) $p_{t_2} \in T(p_{t_1}, p_{t_3})$ implies that the curve p_t is convex, i.e. the bounded component of $H \setminus \{p_t | \, 0 \le t \le 2\}$ is

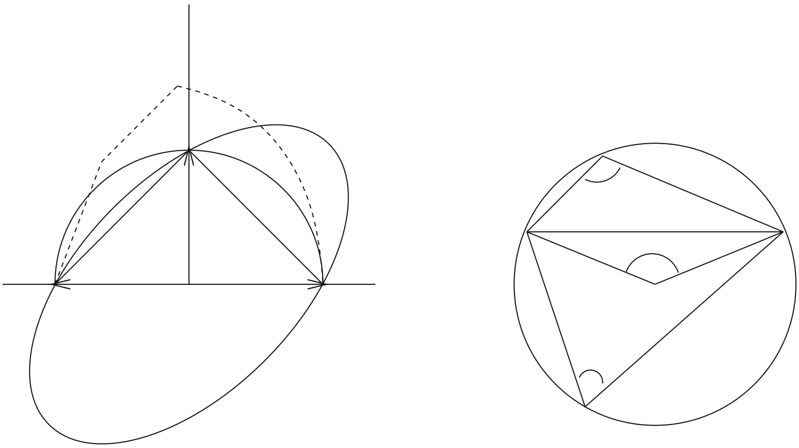

Figure 2. The figure shows a variety of different Ptolemy circles

convex. The condition also implies for $t_1 = 0$ and $t_3 \to 2$ a limit condition that $p_{t_2} \in T_x(e_R^1, -e_R^1)$ for some $x \in S^1$. Since $e_R^2 = p_1$ we have $\pi/2 \leq x \leq 3\pi/2$.

Conversely let us assume that $p_t = \binom{a_t}{b_t}$, $0 \leq t \leq 2$ is a convex curve in H from $p_0 = e_R^1$ via $p_1 = e_R^2$ to $p_2 = -e_R^1$ which is contained in $T_x(e_R^1, -e_R^1)$ for some $x \in S^1$ with $\pi/2 \leq x \leq 3\pi/2$. For $0 \leq t_1 \leq t_2 \leq 2$ define $d(e^{\pi i t_1}, e^{\pi i t_2}) = \langle Jp_{t_1}, p_{t_2}\rangle$. By Lemma 3.1 (S^1, d) satisfies the Ptolemy condition. We have to show that d is indeed a metric, hence that for $0 \leq t_1 < t_2 < t_3 < 2$ we have $p_{t_2} \in T(p_{t_1}, p_{t_3})$.

The proof is similar to the proof of Proposition 3.5

We write $u = p_{t_1}, v = p_{t_2}, w = p_{t_3}$ and have to show that $v \in T(u, w)$. Consider the lines $\ell(e_R^1, u)$ and $\ell(-e_R^1, w)$.

If these lines are parallel, then they are the boundary rays of $T_x(e_R^1, -e_R^1)$ and $T(u, w)$ is the closure of the unbounded component of $T_x(e_R^1, -e_R^1)\backslash\ell(u, w)$.

If the lines are not parallel, let $z = \ell(e_R^1, u) \cap \ell(-e_R^1, v)$ be the intersection point and we have to show $z \in T(u, w)$.

Consider the parallel lines $\ell_u = \ell(u, u+(u+w))$ and $\ell_w = \ell(w, w+(u+w))$, which are the lines containing the boundary rays of $T(u, w)$. These lines intersect the boundary of H in points $-\lambda e_1$ and λe_1 for some $0 \leq \lambda$. The condition that $u, w \in T_x(e_R^1, -e_R^1)$ implies $\lambda \leq 1$. This implies that z is in the strip bounded by ℓ_u and ℓ_w. $\square$

Remark 3.9. (i) On the Ptolemy circles which can be realized in $\mathbb{E}^2$.

Now we can have a similar discussion of the variety of Ptolemy circles as the one in Remark 3.7 (i) for Ptolemy segments. Once again, the Ptolemy circles which admit isometric embeddings in $\mathbb{E}^2$ are precisely given by the intersections of the upper half plane H with the ellipses in $\mathbb{R}^2$ considered above (Figure 2).

(ii) The Ptolemy parameterization.

The analogon of Remark 3.7 (ii) also applies to Ptolemy circles once one fixes additional two points on the circle, fixing its Ptolemy parameterization.

Note that in contrast to the situation for Ptolemy segments, Ptolemy circles do not admit the pair of an initial- and an endpoint. Our characterization of isometry classes therefore requires the additional choice of two distinct points on the circle. It therefore is a characterization of two-pointed Ptolemy circles.

3.4. An example of a Möbius sphere. In this section we provide an example of a metric sphere which is Möbius equivalent, but not homothetic to the standard chordal sphere.

Let X be a CAT(κ)-space, $\kappa < 0$, and let $o \in X$. We recall that the Bourdon metric $\rho_o : \partial_\infty X \times \partial_\infty X \longrightarrow \mathbb{R}_0^+$ can also be expressed in terms of the Gromov product on $\partial_\infty X$.

Given $x, y \in X$, the *Gromov product* $(x \cdot y)_o$ of x and y w.r.t. the basepoint o is defined as

$$(x \cdot y)_o \;:=\; \frac{1}{2}\left[d(x,o) \,+\, d(y,o) \,-\, d(x,y)\right].$$

This Gromov product naturally extends to points at infinity, by

$$(\xi \cdot \xi')_o \;:=\; \lim_{i \longrightarrow \infty} (x_i \cdot x_i')_o \qquad \forall \xi, \xi' \in \partial_\infty X,$$

where $\{x_i\}$ and $\{x_i'\}$ are sequences in X converging to ξ and ξ', respectively, i.e. sequences which satisfy $\lim_{i\to\infty}(\gamma_{o,\xi}(i) \cdot x_i)_o = \infty$ and $\lim_{i\to\infty}(\gamma_{o,\xi'}(i) \cdot x_i')_o = \infty$, respectively, where $\gamma_{o,\eta}$ denotes the unique geodesic ray connecting o to $\eta \in \{\xi, \xi'\}$.

This limit exists and does not depend on the choice of sequence. This phenomenon is referred to as the so called boundary continuity of CAT(κ)-spaces (cf. Section 3.4.2 in [2] and note that one can generalize the proof given there for proper CAT(κ)-spaces to the non-proper case).

With this notation one can write the Bourdon metric ρ_o as

$$\rho_o(\xi, \xi') \;=\; e^{-\sqrt{-\kappa}(\xi \cdot \xi)_o} \qquad\qquad \forall \xi, \xi' \in \partial_\infty X \qquad\qquad \text{(cf.[1,4])}.$$

Example 3.10. Consider the 3-dimensional real hyperbolic space $\mathbb{H}^3_{-1}$ of constant curvature -1 in the Poincaré ball model. Let o denote the center of the ball and consider a complete geodesic γ through o.

Now we glue a real hyperbolic half plane H of curvature -1 along γ. The resulting space X is a CAT(-1)-space.

Let $o' \in H$ be some point, the projection of which in H on γ coincides with o. The Bourdon metric ρ_o of ∂X w.r.t. o when restricted to the boundary of $\mathbb{H}^3_{-1}$, $S^2 = \partial_\infty \mathbb{H}^3_{-1} \subset \partial_\infty X$, is isometric to S^2 when endowed with half of its chordal metric.

The Bourdon metric $\rho_{o'}$ of ∂X w.r.t. o' when restricted to $\partial_\infty \mathbb{H}^3_{-1}$ is Möbius equivalent to ρ_o. We now verify that $(S^2, \rho_{o'})$ is not homothetic to the standard chordal sphere (S^2, d_0).

Let $N = \{\gamma(i)\}_i$ and $S = \{\gamma(-i)\}_i$ denote the endpoints of γ in S^2. They are diametrical points w.r.t. ρ_o and define the equator A as such sets of points

with coinciding distances to N and S, respectively. Note that by the symmetry of the construction, A has the very same property w.r.t. the metric $\rho_{o'}$
Since every geodesic ray in X from o' to some $a \in A$ contains o, A endowed with the Bourdon metric $\rho_{o'}$ with respect to o' is isometrically a Euclidean circle of radius e^{-l}, where $l := |oo'|$.
In contrast to the points $a \in A$, the points N and S also lie in the boundary of the halfplane H. Hence the geodesic rays connecting o' to N and S, respectively, do not contain o. In fact, denote by b_γ the Busemann function associated to γ normalized such that $b_\gamma(o) = 0$, then

$$\rho_{o'}(N, S) \;=\; e^{-(N \cdot S)_{o'}} \;=\; e^{-b_\gamma(o')} > e^{-l}.$$

It follows that $(S^2, \rho_{o'})$ is not homothetic to (S^2, d_0).

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