Z. angew. Math. Phys. 54 (2003) 815–821
0044-2275/03/050815-7
DOI 10.1007/s00033-003-3203-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

Estimates for solutions of quasilinear problems with dead cores

Catherine Bandle and Stella Vernier-Piro

To Larry Payne on the occasion of his 80th birthday

Abstract. A Rayleigh-Faber-Krahn type inequality is used to derive bounds for boundary value problems appearing in reaction-diffusion problems where the reactant is consumed. Interesting quantities are the minimum of the solution and the measure of the set where it vanishes. The proofs are rather elementary and apply to problems possessing solutions in a weak sense.

Mathematics Subject Classification (2000). 49K20, 35J28, 35J60.

Keywords. Boundary value problems, Rayleigh-Faber-Krahn inequalities, a priori estimates.

1. Introduction

Simple models describing a single steady state-reaction taking place in a bounded domain $D \subset \mathbb{R}^N$ are of the form

$$\triangle u = u^q \text{ in } D, \;\; u = 1 \text{ on } \partial D, \; q \geq 0.$$

Here the reactant is consumed in D and replaced through diffusion from the ambient region in such a way that a steady state is possible. It is well-known that this problem has a unique positive solution. In addition if $q \geq 1$ the strong minimum principle implies that this solution is strictly positive. However if $q < 1$ then a *dead core* $D_0 := \{x \in D : u(x) = 0\}$, i.e. a region where the solution vanishes identically may appear. Such a region is a waste from the engineering point of view. The mathematical discussion concerning its existence, geometry and estimates was carried out in a series of papers [7], [2], [6], [3].

It was shown in [2] for power nonlinearities and for more general functions that among all domains of given volume $|D|$ the Lebesgue measure of the dead core $|D_0|$ is maximal for the ball. The proof used heavily level set analysis and the isoperimetric inequality. Another physical quantity of interest is *the effectiveness* defined by $\mathcal{E}(D) = |D|^{-1} \int_D u^q dx$. It turns out [2] that among all domains of given volume $|D|$ the ball has the smallest effectiveness. These results were generalized in [3] and [4] to time dependent problems.

The goal of this paper is to extend those results to more general problems without making use of the isoperimetric inequality. We shall use as a substitute a Rayleigh-Faber-Krahn inequality. The role of these inequalities in connection with quasilinear boundary value problems has been studied in [1]. The same ideas carry over to the dead core problem.

Let $D \in \mathbb{R}^N$ be an arbitrary domain and $a, b : D \to \mathbb{R}^+$ be two continuous functions. In this paper we shall consider boundary value problems of the form

$$\sum_{i=1}^{N} \left(a(x)|\nabla u|^{p-2}u_{x_i}\right)_{x_i} = b(x)f(u) \text{ in } D, \quad u = 1 \text{ on } \partial D, \tag{1.1}$$

where $p \geq 1$ is a real number and $f(s)$ is a nondecreasing function such that $f(0) = 0$. It is well-known [6] that (1.1) has a unique solution such that $1 - u \in W_0^{1,p}(D)$. By the regularity result for the p-Laplacian we have $u \in C^{1,\alpha}(D)$.

As before D_0 denotes the dead core $\{x \in D : u(x) = 0\}$. It follows immediately from comparison principles that the dead core is monotone with respect to D in the following sense: if $D \subset D'$ then $D_0(D) \subset D_0(D')$. The existence of a dead core depends on the behaviour of the nonlinearity near zero and on the size of the domain. Diaz proved [6] that in the special case where a and b are constant and $\int_0 F^{-\frac{1}{p}}(s)ds < \infty$, F being the primitive of f, a dead core exists provided D is sufficiently large.

The estimates we will obtain are expressed in terms of a weighted L_1 - norm of the solution. The motivation came from a paper of L. Payne [8]. He considered the torsion problem

$$\triangle u + 1 = 0 \text{ in } D \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial D.$$

Using level line techniques he proved that $|u|_\infty \leq \left\{\pi^{-1} \int_D u dx\right\}^{1/2}$.

2. A priori bounds

In this section we shall derive lower bounds for the distribution function of the solution of (1.1)

$$m(t) := \int_{D(t)} b dx, \text{ where } D(t) = \{x \in D : u(x) < t\}$$

and for $u_m :=$ ess inf u. For this purpose we shall assume that the following *Rayleigh-Faber-Krahn type inequality* holds:

($\mathbf{A_1}$) There exists a positive constant c such that for all $K \subset D$

$$c\left(\int_K bv dx\right)^p m^{1-\frac{p}{N}-p}(t) \ \leq \int_K a|\nabla v|^p dx, \ \forall v \in C_0^\infty(K).$$

Observe that it is in general difficult to check such an inequality and to determine the best value for c. In this paper we will not focus on this question. We refer to [1] for some discussion.

We now are in position to formulate our key tool from which all estimates will follow.

Lemma 1. *Assume* (A_1) *and* $\alpha = p + \frac{p}{N} - 1$. *Then the distribution function of the solution* u *of (1.1) satisfies the inequality*

$$m(t) \geq \beta \left(\int_{u_m}^{t} f^{-\frac{1}{\alpha}} ds \right)^{\frac{(p-1)N}{p}} f^{-\frac{1}{\alpha}}(t).$$

for some constant β *which depends only on* c, p *and* N.

Proof. Multiplication of (1.1) by $(t-u)_+$ and integration over D yields

$$\int_{D(t)} a(x)|\nabla u|^p dx = \int_{D(t)} b(x) f(u)(t-u) dx. \tag{2.1}$$

By (A_1) we conclude that

$$c \left(\int_{D(t)} b(t-u) dx \right)^p m^{1-\frac{p}{N}-p}(t) \leq \int_{D(t)} b(x) f(u)(t-u) dx. \tag{2.2}$$

By Cavalieri's principle we have

$$\int_{D(t)} b(t-u) dx = \int_{u_m}^{t} (t-s) dm(s) = \int_{u_m}^{t} m(s) ds =: M(t).$$

Whence since f is non-decreasing, (2.2) leads to

$$cM^p(t) m^{1-\frac{p}{N}-p}(t) \leq f(t) M(t),$$

Observing that $m(t) = \frac{dM(t)}{dt} =: \dot{M}$ and since α is positive we obtain the differential inequality

$$c^{\frac{1}{\alpha}} f^{-\frac{1}{\alpha}}(t) \leq M^{\frac{1-p}{\alpha}}(t) \dot{M}(t) \text{ in } (u_m, 1). \tag{2.3}$$

Hence

$$\gamma \int_{u_m}^{t} f^{-\frac{1}{\alpha}}(s) ds \leq M^{\frac{1-p+\alpha}{\alpha}}(t), \ \gamma = \frac{c^{\frac{1}{\alpha}} p}{Np + p - N}. \tag{2.4}$$

This together with (2.3) yields

$$m(t) \geq c^{\frac{1}{\alpha}} \left(\gamma \int_{u_m}^{t} f^{-\frac{1}{\alpha}} ds \right)^{\frac{(p-1)N}{p}} f^{-\frac{1}{\alpha}}(t).$$

which completes the proof of the lemma. □

Notice that this inequality can never be optimal except possibly for $f(t) = t$.

From this lemma we obtain immediately a lower bound for u_m which is given implicitely by

$$\int_D b(x)dx \geq \left(\frac{c}{f(1)}\right)^{\frac{1}{\alpha}} \left(\gamma \int_{u_m}^1 f^{-\frac{1}{\alpha}} ds\right)^{\frac{(p-1)N}{p}}. \tag{2.5}$$

(2.5) yields also a necessary condition for the appearance of a dead core, namely

$$\int_D b(x)dx \left(\frac{f(1)}{c}\right)^{\frac{1}{\alpha}} \geq \left(\gamma \int_0^1 f^{-\frac{1}{\alpha}} ds\right)^{\frac{(p-1)N}{p}}.$$

This condition isn't sufficient in general. For instance if $f(t) = t^q$ and $p = 2$ it means that $q < 1 + 2/N$. Hence $q > 1$ is admissible. However the unique continuation principle implies that if u vanishes on a set of positive measure than it is zero everywhere.

Put for short

$$m^*(t) := \left(\frac{c}{f(t)}\right)^{\frac{1}{\alpha}} \left(\gamma \int_{u_m}^t f^{-\frac{1}{\alpha}} ds\right)^{\frac{(p-1)N}{p}}.$$

From Lemma 1 we obtain immediately the

Corollary 1. *Let ϕ be a positive, increasing function and u be the solution of (1.1). Then*

$$\int_D \phi(u)b(x)dx \leq \phi(1)\int_D bdx - \int_{u_m}^1 \phi'(s)m^*(s)ds.$$

For later purposes we shall need the lower bound for $M(1)$ given by (2.4)

$$\gamma \int_{u_m}^1 f^{-\frac{1}{\alpha}}(s)ds \leq M^{\frac{p}{N\alpha}}(1). \tag{2.6}$$

3. Estimates for the dead core and the effectiveness

We first assume that $u_m = 0$. Denote by m_0 the measure $\int_{D_0} bdx$ of the dead core with respect to the weight b. We shall also write $S(t) = D(t) \cap \text{supp}\{u\}$ and $\mu(t) = \int_{S(t)} bdx = m(t) - m_0$. We shall first derive a differential inequality similar to (2.3) taking into account m_0.

From (2.1) we have since $f(0) = 0$

$$\int_{D(t)} a(x)|\nabla u|^p dx = \int_{S(t)} b(x)f(u)(t-u)dx \leq$$
$$f(t)\int_{S(t)} b(x)(t-u)dx. \tag{3.1}$$

Hence (2.2) reads now as follows, using the same notation as in Section 2,

$$cM^p(t)m^{-\alpha}(t) \le f(t)\int_0^t \mu(s)ds = f(t)[M(t) - m_0 t]. \tag{3.2}$$

This yields the following differential inequality for $M(t)$

$$c^{\frac{1}{\alpha}} f^{-\frac{1}{\alpha}}(t) M^{\frac{p}{\alpha}}(t) \le \ [M(t) - m_0 t]^{\frac{1}{\alpha}} \dot{M}(t), \ \ M(0) = 0$$

which in general cannot be integrated by elementary means.

Setting $t = 1$ in (3.2) we obtain an upper bound for m_0

$$m_0 \le M(1) - \frac{cM^p(1)}{f(1)m^\alpha(1)} \le \left(\frac{m^\alpha(1)f(1)}{cp^p}\right)^{\frac{1}{p-1}} (p-1).$$

This inequality is in accordance with the fact the bigger the nonlinearity f the more likely there is a dead core cf. [2]. A lower bound for $M(1)$ is given in (2.6) and $M(1) \le m(1)$.

Next we shall derive a lower bound for the effectiveness

$$\mathcal{E}(D) = \frac{1}{\int_D b dx}\int_D f(u)b(x)dx.$$

It represents the ratio of the actual quantity of reactant consumed in D to the quantity that would be theoretically consumed if the internal concentration were everywhere equal to the ambient concentration. In most application, a high effectiveness is beneficial. An upper bound is obtained from Corollary 1.

From (2.2) we deduce

$$cM^p(1)m^{-\frac{p}{N}-p}(1)(1-u_m)^{-1} \le \mathcal{E}(D).$$

4. Special case

Let us first consider the case $f(u) = u^q$. We shall assume that $q/\alpha < 1$ in order to include solutions with dead core. Then the estimate for the distribution function obtained in Lemma 1 becomes

$$\begin{aligned} m(t) &\ge c^{\frac{1}{\alpha}} \left(\gamma \int_{u_m}^t s^{-\frac{q}{\alpha}} ds\right)^{\frac{(p-1)N}{p}} t^{-\frac{q}{\alpha}} = \\ &= \rho \left(t^{1-\frac{q}{\alpha}} - u_m^{1-\frac{q}{\alpha}}\right)^{\frac{(p-1)N}{p}} t^{-\frac{q}{\alpha}}, \end{aligned} \tag{4.1}$$

where

$$\rho = c^{\frac{N}{p}} \left(\frac{p}{N(p-q)+p-N}\right)^{\frac{(p-1)N}{p}}$$

Evaluating this inequality at $t = 1$ we get a lower bound cf. (2.5) for u_m

$$1 - \left(\frac{m(1)}{\rho}\right)^{\frac{p}{(p-1)N}} \le u_m^{1-q/\alpha}.$$

If $u_m = 0$ then (4.1) assumes the form

$$m(t) \geq \rho\, t^r, \qquad \text{where} \qquad r = \frac{N}{p}(p-1-q),$$

In order to have a dead core the measure of the domain has to be sufficiently large

$$m(1) \geq \rho.$$

If we assume $(q+1)/\alpha < 1$ and $p > N$ we can use inequality (3.2) as follows

$$cM^p m^{-\alpha} \leq t^{q+1}[m(1) - m_0]$$

which after integration implies

$$\left(\frac{c}{m(1) - m_0}\right)^{1/\alpha} \frac{\alpha - p}{\alpha - q - 1} \leq \; M(1)^{1-p/\alpha}.$$

Remark. This inequality shows that m_0 is small if $M(1)$ is small.

General Comment. We could have used in (2.2) a Rayleigh-Faber-Krahn inequality of the form

$$c_0 \left(\int_K b|v|^s dx\right)^{\frac{p}{s}} \; m^{1-\frac{p}{N}-\frac{p}{s}}(t) \; \leq \int_K a|\nabla v|^p dx \;\; \forall v \in C_0^\infty(K)$$

for some suitable $s > 0$ which according to a result in [1] is equivalent to $\mathbf{A_1}$. It turned out that the computations became more transparent and possibly more precise with $s = 1$.

Acknowledgement

The second author wishes to thank Professor Catherine Bandle and the Department of Mathematics of the University of Basel for the hospitality during this research.

References

[1] C.Bandle, *Rayleigh-Faber-Krahn inequalities and quasilinear elliptic boundary value problems*, Departement of Mathematics, University of Basel preprint 2002-18 (2002).

[2] C. Bandle, R. Sperb, I. Stakgold, Diffusion-reaction with monotone kinetics, *Nonlinear Analysis TMA* **8** (1984), 321-333.

[3] C. Bandle and I. Stakgold, The formation of dead core in parabolic reaction-diffusion problems, *Trans. Amer. Math. Soc.* **286** (1984).

[4] C. Bandle and I. Stakgold, *Isoperimetric inequalities for the effectiveness in semilinear parabolic equations*, General Inequalities 4 (ed. W.Walter) Birkhäuser ISNM 71 (1984).

[5] C. Bandle, *A note on optimal domains in a reaction-diffusion problem*, *ZAA* **4** (1985), 207-213.

[6] J.I. Díaz, *Nonlinear partial differential equations and free boundaries*, Pitman Research Notes in Mahematics 106 (1985).
[7] A. Friedman and D. Phillips, The free boundary of a semilinear elliptic equation, *Trans. Amer. Math. Soc.* **282** (1984), 153-182.
[8] L. E. Payne, *Torsion problem for multiply connected regions*, Studies in Mathematical Analysis and Related Topics, essays in honor of G.Pólya, Stanford University Press (1962).

Catherina Bandle
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel
Switzerland
e-mail: bandle@math.unibas.ch

Stella Vernier-Piro
Dipartimento di Matematica
Università di Cagliari
Viale Merello 92
I-09123 Cagliari
Italy
e-mail: svernier@unica.it

(Received: February 26, 2003)