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Submodularity of some classes of the combinatorial optimization games

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Abstract. Submodularity (or concavity) is considered as an important property in the field of cooperative game theory. In this article, we characterize submodular minimum coloring games and submodular minimum vertex cover games. These characterizations immediately show that it can be decided in polynomial time that the minimum coloring game or the minimum vertex cover game on a given graph is submodular or not. Related to these results, the Shapley values are also investigated.

Key words: Combinatorial optimization, Game theory, Submodularity

1 Introduction

In this article, we investigate minimum coloring games and minimum vertex cover games. Generally, a cooperative game arising from a combinatorial optimization problem is called a combinatorial optimization game, and so a minimum coloring game and a minimum vertex cover game are examples of combinatorial optimization games. Perhaps, the most classical combinatorial optimization game is an assignment game by Shapley–Shubik [17], and since then a lot of combinatorial optimization games have been introduced. See the books [1, 3] for some aspects of combinatorial optimization games.

Particularly, some combinatorial optimization games are used to model several situations concerning cost allocation. Here is an example. Consider a company which operates a cellular telephone network. It has some transmitters. Each transmitter covers the corresponding area (usually a disk centered at the transmitter) and frequency bands will be assigned to each

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transmitter. Here, distinct frequency bands should be assigned to two transmitters if they share an area. In the simplest assignment all the transmitters have distinct frequency bands, but it costs high. By cooperating with each other, they can reduce the cost. So now, the problem is how to allocate the total cost to each transmitter. We consider this problem as a game-theoretic situation, and a minimum coloring game is one of the possible models of the situation.

In this article, we discuss submodularity of the minimum coloring game and the minimum vertex cover game. In the next section, we characterize submodular minimum coloring games and submodular minimum vertex cover games in terms of forbidden subgraphs. In Section 3, we give formulae of the Shapley values for these games. In Section 4, we state the relationship of our results with matroids.

2 Minimum coloring games, minimum vertex cover games and submodularity

First we collect some terminology on graphs. If you have something missing, see a textbook of graph theory (as Diestel [6]). In this article, all graphs are finite and simple. For a graph $G = (V, E)$, the vertex set V and the edge set E are sometimes denoted as $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$.

A *vertex coloring* (or simply a *coloring*) of $G = (V, E)$ is a function $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any adjacent vertices $u, v \in V$. For a coloring $c : V \to \{1, \ldots, k\}$, each element in $\{1, \ldots, k\}$ is called a *color*, and a vertex $v \in V$ is said to be *colored by* $i \in \{1, \ldots, k\}$ if $c(v) = i$. A *minimum coloring* of G is a coloring $c : V \to \{1, \ldots, k\}$ such that k is as small as possible, and such k is called the *chromatic number* of G. The chromatic number of G is denoted by $\chi(G)$. If G is the empty graph, i.e., G has no vertex, we set $\chi(G) = 0$.

A *vertex cover* of $G = (V, E)$ is a subset $U \subseteq V$ such that every edge $e \in E$ is incident with some vertex $v \in U$, i.e., $v \in e$. A *minimum vertex cover* of G is a vertex cover $U \subseteq V$ such that $|U| \leq |U'|$ for any vertex cover $U' \subseteq V$. We denote the size of a minimum vertex cover of G by $\tau(G)$. Note that $\tau(G) = 0$ if G has no edge.

A set $S \subseteq V$ is called a *clique* of $G = (V, E)$ if $G[S]$ forms a complete graph (i.e., any two vertices in $G[S]$ form an edge). A set $S \subseteq V$ is called a *stable set* of G if $G[S]$ has no edge.

Now we introduce a cooperative game.

A *cooperative game* is defined as a function $\gamma : 2^X \to \mathbb{R}$ for a nonempty finite set X, which satisfies $\gamma(\emptyset) = 0$. A *player* of a cooperative game is an element of X. For a set $S \subseteq X$ of players, the value $\gamma(S)$ is regarded as a minimum cost owed by the players of S when they cooperate.

A cooperative game is *submodular* (or *concave*) if for any $S, T \subseteq X$ the inequality $\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$ holds. Submodularity can be interpreted as follows. If the players of $T \setminus S$ make a coalition with S and $S \cap T$, then the costs will increase by $\gamma(S \cup T) - \gamma(S)$ and $\gamma(T) - \gamma(S \cap T)$, respectively. Now, by submodularity, the set with more players (i.e., S than $S \cap T$) has the less cost increase (i.e., $\gamma(S \cup T) - \gamma(S) \leq \gamma(T) - \gamma(S \cap T)$). This means that the set with more players has potentially more incentive for other players to join, and we can see this is a natural condition in the real life. Also, in cooperative game theory submodularity is known to possess some important properties. For example, the core of a submodular game is nonempty, it is a unique von Neumann-Morgenstern solution, and the Shapley value is the barycenter of the core (when the degeneracy is taken into account) [16]. Moreover, in a submodular game, the core and the bargaining set coincide and the kernel and the nucleolus coincide [12]. In addition, the nucleolus and the τ-value can be computed in polynomial time for a submodular game ([11] and [18] respectively).

Refer to Bilbao [1] and Curiel [3] for cooperative game theory with emphasis on combinatorial optimization games. Also, see Fujishige [9] for an appearance of submodularity in the context of combinatorial optimization and network flows.

In this article, we investigate two kinds of combinatorial optimization games associated with a graph $G = (V, E)$. The *minimum coloring game* on a graph G is a function $\chi_G : 2^V \to \mathbb{N}$ defined as $\chi_G(S) = \chi(G[S])$ for $S \subseteq V$. Recall that $G[S]$ is the subgraph of G induced by S. The *minimum vertex cover game* on G is a function $\tau_G : 2^E \to \mathbb{N}$ defined as $\tau_G(S) = \tau((V, S))$ for $S \subseteq E$. Here, (V, S) is a subgraph of $G = (V, E)$ but is not necessarily an induced subgraph.

Our theorems are the following.

Theorem 1. *The following are equivalent for a graph* $G = (V, E)$.

(A) *The minimum coloring game* χ_G *is submodular.*
(B) G *contains no induced subgraph isomorphic to* $K^1 \cup K^2$. *Here,* $K^1 \cup K^2$ *is a graph with three vertices* a, b, c *and one edge* $\{a, b\}$.
(C) G *is a complete r-partite graph.*

Theorem 2. *The following are equivalent for a graph* $G = (V, E)$.

(A) *The minimum vertex cover game* τ_G *is submodular.*
(B) G *contains no subgraph isomorphic to* K^3 *or* P^3. *Here,* K^3 *is a complete graph with three vertices, and* P^3 *is a path of length three* (*i.e., with four vertices*).
(C) *Each connected component of* G *is a star. Here, a star is such a bipartite graph that one partition class has exactly one vertex. We regard a graph with one vertex as a star.*

These theorems immediately imply that the problem deciding the submodularity of minimum coloring games (and minimum vertex cover games, respectively) on a given graph can be solved in polynomial time.

Remark that the problems deciding the balancedness (and the total-balancedness, respectively) of minimum coloring games and minimum vertex cover games are investigated in Deng–Ibaraki–Nagamochi [4] (and Deng–Ibaraki–Nagamochi–Zang [5], respectively).

2.1 Proof of Theorem 1

First we show (A) $\Rightarrow$ (B). That is, if G contains an induced subgraph isomorphic to $K^1 \cup K^2$, then χ_G cannot be submodular. To show this, we use the fact that if H is an induced subgraph of G and χ_G is submodular, then χ_H is also submodular. From this fact, it suffices to show that $\chi_{K^1 \cup K^2}$ is not submodular. This is not so difficult to show.

Next we show (B) $\Rightarrow$ (C). Since that G is complete r-partite is equivalent to that all connected components of the complement $\bar{G}$ of G are complete graphs, we show that if $\bar{G}$ has a component C which is not complete, then G contains an induced subgraph isomorphic to $K^1 \cup K^2$.

Let $V(C)$ be the vertex set of C. This claim is obvious if $|V(C)| \leq 2$. So we assume that $|V(C)| \geq 3$. Since C is not a complete graph in $\bar{G}$, there exist two vertices $a, b \in V(C)$ which are not adjacent in $\bar{G}$. Then there must be a path from a to b in $\bar{G}$ since C is a connected component of $\bar{G}$. Let us denote this path by $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$, where $a = v_0$ and $b = v_k$. Here, put $i = \min\{j \in \{1, \ldots, k\} : a \text{ and } v_j \text{ are not adjacent}\}$. Note that such an i always exists because a and b are not adjacent and i cannot be 1. So $\bar{G}[\{a, v_{i-1}, v_i\}]$ is a path of length two in $\bar{G}$. This means that $G[\{a, v_{i-1}, v_i\}]$ is isomorphic to $K^1 \cup K^2$ in G. Thus the claim has been shown.

Finally, we show (C) $\Rightarrow$ (A). That is, if G is complete r-partite, then χ_G is submodular. Let $G = (V, E)$ be complete r-partite and $V_1, V_2, \ldots, V_r$ be the partition classes of V. Here, for any $S \subseteq V$ we have $\chi_G(S) = |\{i \in \{1, \ldots, r\} : V_i \cap S \neq \emptyset\}|$. Let us define a map $\pi : 2^V \to 2^{\{1,\ldots,r\}}$ as $\pi(S) = \{i \in \{1, \ldots, r\} : V_i \cap S \neq \emptyset\}$. With this map, we can write $\chi_G(S) = |\pi(S)|$. Observe that the map π has the following properties for a complete r-partite graph G: for any $S, T \subseteq V$

- $\pi(S \cup T) = \pi(S) \cup \pi(T)$,
- $\pi(S \cap T) \subseteq \pi(S) \cap \pi(T)$.

Using these properties, we can prove the submodularity of χ_G as follows.

$$\begin{aligned}
\chi_G(S) + \chi_G(T) &= |\pi(S)| + |\pi(T)| \\
&= |\pi(S) \cup \pi(T)| + |\pi(S) \cap \pi(T)| \\
&\geq |\pi(S \cup T)| + |\pi(S \cap T)| \\
&= \chi_G(S \cup T) + \chi_G(S \cap T).
\end{aligned}$$

Thus we have completed the proof of Theorem 1.

2.2 Proof of Theorem 2

First we show (A) $\Rightarrow$ (B). That is, if G contains a subgraph isomorphic to P^3 or K^3, then τ_G is not submodular. Here, we use the fact that if H is a subgraph of G and τ_G is submodular, then τ_H is also submodular. With this fact, it suffices to show that neither τ_{P^3} nor τ_{K^3} is submodular. This is not so difficult to show.

Next, we show (B) $\Rightarrow$ (C). That is, if there exists a connected component of G which is not a star, then G contains a subgraph isomorphic to P^3 or K^3.

Let C be a connected component of G which is not a star. Consider the case that C is not bipartite. Then C has a cycle of odd length. When this length is three, this cycle is isomorphic to K^3. When this length is five or more, along this cycle we can find P^3. Next consider the case that C is bipartite. In this case, since C is not a star, we have P^3 as a subgraph of C. Thus, we have shown (B) $\Rightarrow$ (C).

Finally we show (C) $\Rightarrow$ (A). That is, if every connected component of G is a star, then τ_G is submodular.

Let $G = (V, E)$ be a graph with every component being a star and $C_1, \ldots, C_r$ be the connected components of G. Then for any $S \subseteq E$ we have $\tau_G(S) = |\{i \in \{1, \ldots, r\} : E(C_i) \cap S \neq \emptyset\}|$. Here, define a map $\pi : 2^E \to 2^{\{1,\ldots,r\}}$ as $\pi(S) = \{i \in \{1, \ldots, r\} : E(C_i) \cap S \neq \emptyset\}$. With this map, we can write $\tau_G(S) = |\pi(S)|$.

Observe that the map π has the following properties when every connected component of G is a star: for $S, T \subseteq E$

- $\pi(S \cup T) = \pi(S) \cup \pi(T)$,
- $\pi(S \cap T) \subseteq \pi(S) \cap \pi(T)$.

By these properties, we can show the submodularity of τ_G as in the proof of Theorem 1.

This completes the whole proof of Theorem 2.

3 The Shapley value

The Shapley value is an important concept in cooperative game theory. First we will define marginal contributions which are needed in the definition of the Shapley value. Consider a linear order $\leq_\pi$ on a finite set X with n elements. Such an order has a one-to-one correspondence with a permutation $\pi \in S_n$ on X, and we have $n!$ possible linear orders. (S_n represents the set of all permutations on a set with n elements.) For a linear order $\leq_\pi$, put $P_\pi(i) = \{j \in X : j \leq_\pi i\}$. Note that $i \in P_\pi(i)$. For a cooperative game $\gamma : 2^X \to \mathbb{R}$, we define the *marginal contribution* $m^\gamma_\pi[i] \in \mathbb{R}$ of a player $i \in X$ with respect to a linear order $\leq_\pi$ as

$$m^\gamma_\pi[i] = \gamma(P_\pi(i)) - \gamma(P_\pi(i) \setminus \{i\}).$$

This means how much the cost increases due to the player i when the players gather one by one according to the order $\leq_\pi$ so as to form X. By the definition of the marginal contribution, we have $\sum\{m^\gamma_\pi[i] : i \in X\} = \gamma(X)$ for any $\pi \in S_n$.

The *Shapley value* $\varphi^\gamma[i] \in \mathbb{R}$ of a player $i \in X$ for a cooperative game $\gamma : 2^X \to \mathbb{R}$ is defined as

$$\varphi^\gamma[i] = \frac{1}{n!} \sum_{\pi \in S_n} m^\gamma_\pi[i].$$

This means, the Shapley value $\varphi^\gamma[i]$ of the player i is the expectation of the marginal contributions $m^\gamma_\pi[i]$ when a permutation π is chosen uniformly at random from S_n. We can see that $\sum\{\varphi^\gamma[i] : i \in X\} = \gamma(X)$.

Symmetry is an important property of the Shapley value. Here, two players $i, j \in X$ are said to be *symmetric* if for any $S \subseteq X \setminus \{i, j\}$ it holds that $\gamma(S \cup \{i\}) = \gamma(S \cup \{j\})$. Here is a well-known lemma on symmetric players.

Lemma 1. *For a cooperative game $\gamma : 2^X \to \mathbb{R}$, the Shapley values of symmetric players are equal: $\varphi^\gamma[i] = \varphi^\gamma[j]$ for symmetric $i, j \in X$.*

In this section, we explicitly write down the formulae of the Shapley values of submodular minimum coloring games and submodular minimum vertex cover games. From these formulae, we can compute the Shapley values for submodular minimum coloring games and submodular minimum vertex cover games. Note that generally it is NP-hard to compute the Shapley values for minimum coloring games and minimum vertex cover games. Otherwise, by computing $\sum\{\varphi^{\chi_G}[i] : i \in V\} = \chi(G)$ (or $\sum\{\varphi^{\tau_G}[i] : i \in E\} = \tau(G)$), we could tell what was the chromatic number (or the size of a minimum vertex cover, respectively) of G. However, computation of the chromatic number (or the size of a minimum vertex cover) is known to be NP-hard [9].

3.1 The Shapley value for submodular minimum coloring games

From Theorem 1, the minimum coloring game χ_G is submodular if and only if G is complete r-partite. In this subsection, we assume that G satisfies this property. Let $V_1, \dots, V_r$ be the partition classes of G. Then, we can see that two vertices from the same partition class V_i are symmetric players. That is because for $v \in V_i$ and $S \subseteq V \setminus \{v\}$, $\chi_G(S) = \chi_G(S \cup \{v\})$ holds if S contains a vertex of V_i, and otherwise $\chi_G(S) = \chi_G(S \cup \{v\}) - 1$ holds, which does not depend on the choice of v from V_i. Lemma 1 says that the Shapley values of symmetric players are equal, so we are allowed to denote the Shapley value of a player in V_i by φ_i.

Let us compute the Shapley value φ_i of a player $v \in V_i$. First we fix a linear order $\leq_\pi$ on the vertex set V and compute the marginal contribution $m_\pi^{\chi_G}[v]$. From the discussion above, we have

$$m_\pi^{\chi_G}[v] = \begin{cases} 1 & (P_\pi(v) \cap V_i = \{v\}) \\ 0 & (\text{otherwise}), \end{cases}$$

that is, $m_\pi^{\chi_G}[v]$ is 1 if v is the minimum of a linear order $\leq_\pi$ restricted to V_i and otherwise $m_\pi^{\chi_G}[v]$ is 0. Thus, the Shapley value φ_i is written as, with $n = |V|$,

$$\begin{aligned} \varphi_i &= \Pr(P_\pi(v) \cap V_i = \{v\} : \pi \text{ is chosen from } S_n \text{ u.a.r.}) \\ &= \Pr(v \leq_\pi u \quad \text{for all } u \in V_i : \pi \text{ is chosen from } S_n \text{ u.a.r.}) \\ &= \frac{1}{|V_i|}. \end{aligned}$$

Here, "u.a.r." means "uniformly at random".

3.2 The Shapley value for submodular minimum vertex cover games

By Theorem 2, the minimum vertex cover game τ_G is submodular if and only if every connected component of G is a star. In this subsection, we assume that G satisfies this property. Let $C_1, \dots, C_r$ be the connected components of G. Here, we can see that two edges from the same C_i are symmetric players. From Lemma 1 symmetric players have the same Shapley value, so we write φ_i for the Shapley value of a player from C_i.

Let us compute the Shapley value φ_i of $e \in E(C_i)$. First fix a linear order $\leq_\pi$ on the edge set E and compute the marginal contribution $m_\pi^{\tau_G}[e]$. Similarly to the case of the submodular minimum coloring games, we have

$$m_\pi^{\tau_G}[e] = \begin{cases} 1 & (P_\pi(e) \cap E(C_i) = \{e\}) \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore, the Shapley value φ_i is written as, with $n = |E|$,

$$\begin{aligned}\varphi_i &= \Pr(P_\pi(e) \cap E(C_i) = \{e\} : \pi \text{ is chosen from } S_n \text{ u.a.r.})\\ &= \Pr(e \leq_\pi f \text{ for all } f \in E(C_i) : \pi \text{ is chosen from } S_n \text{ u.a.r.})\\ &= \frac{1}{|E(C_i)|}.\end{aligned}$$

4 Relationship with matroids

In this section, we state the relationship between Theorems 1, 2 and matroids. For a detail of matroid theory, see Murota [13], Oxley [14] and so on.

A *matroid* is a pair $(X, \mathcal{I})$ of a nonempty finite set X and a family $\mathcal{I} \subseteq 2^X$ which satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$;
(I2) $T \in \mathcal{I}$ implies $S \in \mathcal{I}$ for any $S \subseteq T \subseteq X$;
(I3) for any $S, T \in \mathcal{I}$ with $|S| > |T|$ there exists $i \in S \setminus T$ such that $T \cup \{i\} \in \mathcal{I}$.

An *independent set* of a matroid $(X, \mathcal{I})$ is a member of $\mathcal{I}$. For a matroid $(X, \mathcal{I})$, we define a function $\rho : 2^X \to \mathbb{N}$ as $\rho(S) = \max\{|T| : T \subseteq S, T \in \mathcal{I}\}$. This function ρ is called the *rank function* of the matroid $(X, \mathcal{I})$. It is known that the rank function of a matroid satisfies the following properties:

(R1) $0 \leq \rho(S) \leq |S|$ for any $S \subseteq X$;
(R2) $\rho(S) \leq \rho(T)$ for any $S \subseteq T \subseteq X$;
(R3) $\rho(S) + \rho(T) \geq \rho(S \cup T) + \rho(S \cap T)$ for any $S, T \subseteq X$.

Conversely, if an integer-valued set function $\rho : 2^X \to \mathbb{N}$ satisfies these three conditions, then for a family $\mathcal{I} = \{I \subseteq X : |I| = \rho(I)\}$, $(X, \mathcal{I})$ is a matroid. In this sense, the properties (R1)–(R3) characterize a matroid.

Here, observe that χ_G satisfies the properties (R1)–(R3) if G contains no induced subgraph isomorphic to $K^1 \cup K^2$. The property (R1) is easy. The property (R3) is Theorem 1 itself. The property (R2) follows from the next lemma.

Lemma 2. *For any $S \subseteq V$ satisfying $|V \setminus S| \geq 1$ and any $v \in V \setminus S$, it holds that $\chi_G(S) \leq \chi_G(S \cup \{v\}) \leq \chi_G(S) + 1$.*

Proof. Let $c : S \to \{1, \ldots, k\}$ be a minimum coloring of $G[S]$, i.e., $\chi(G[S]) = k$. Then, construct a coloring $c' : S \cup \{v\} \to \{1, \ldots, k, k+1\}$ of $G[S \cup \{v\}]$ as $c'(v) = k + 1$ and $c'(i) = c(i)$ for $i \in S$. Hence, $\chi_G(S \cup \{v\}) \leq k + 1 = \chi_G(S) + 1$.

On the other hand, let $\bar{c}' : S \cup \{v\} \to \{1, \ldots, l\}$ be a minimum coloring of $G[S \cup \{v\}]$, i.e., $\chi(G[S \cup \{v\}]) = l$. Then, construct a coloring $\bar{c} : S \to \{1, \ldots, l\}$ of $G[S]$ as $\bar{c}(i) = \bar{c}'(i)$ for all $i \in S$. Hence, $\chi_G(S) \leq l = \chi_G(S \cup \{v\})$. $\square$

Here, remark that χ_G satisfies the properties (R1) and (R2) for any graph G. Moreover, χ_G satisfies the property (R3) if and only if G contains no induced subgraph isomorphic to $K^1 \cup K^2$ due to Theorem 1. Therefore, the conditions in Theorem 1 are equivalent to that χ_G is the rank function of a matroid.

Then, what is a matroid $(V, \mathcal{I})$ with the rank function χ_G when G contains no induced subgraph isomorphic to $K^1 \cup K^2$? An independent set is a set $I \in \mathcal{I}$ satisfying $|I| = \chi(G[I])$. This means that I is a clique of G. That is, if G contains no induced subgraph isomorphic to $K^1 \cup K^2$ and we denote the family of the cliques of G by $\mathcal{I}$, then $(V, \mathcal{I})$ is a matroid, and vice versa. In general, a family satisfying the properties (I1) and (I2) is called a *simplicial complex* or an *independence system*, and the family of the cliques of a graph is a simplicial complex. In the past literature, this is called a *clique complex*.

One of the most prominent feature of matroids is the validity of a greedy algorithm, which finds a maximum weighted base. A maximal independent set of a matroid is called a *base* of the matroid. It is known that all the bases have the same size, i.e., if B_1 and B_2 are bases of a matroid, then $|B_1| = |B_2|$. Later on, we use the terminology of matroids (as independent sets, bases, ...) even for a simplicial complex which may not satisfy (I3). For a simplicial complex $(X, \mathcal{I})$ and a non-negative weight vector $w \in \mathbb{R}^X_+$, we consider a problem computing a maximum weighted base. More formally, the maximum weighted base problem can be stated as

$$\begin{aligned} &\text{maximize } \sum_{i \in B} w[i] \\ &\text{subject to } B \text{ is a base of } (X, \mathcal{I}). \end{aligned}$$

It is known that this problem is related to matroids through the following greedy algorithm. More precisely speaking, the following greedy algorithm returns an optimal solution for any non-negative weight vector $w \in \mathbb{R}^X_+$ if and only if the simplicial complex $(X, \mathcal{I})$ is a matroid [7,15].

Algorithm: Greedy Algorithm
Input: a simplicial complex $(X, \mathcal{I})$ and a non-negative weight vector $w \in \mathbb{R}^X_+$;
Step 1: sort $X = \{1, \ldots, n\}$ so that $w[1] \geq w[2] \geq \cdots \geq w[n]$;
Step 2: $S \leftarrow \emptyset$;
Step 3: for $i = 1$ to n do
  Step 3-1: if $S \cup \{i\} \in \mathcal{I}$ then $S \leftarrow S \cup \{i\}$;
  end of for
Step 4: return S.

Remark that if a simplicial complex $(X, \mathcal{I})$ is a clique complex, then the maximum weighted base problem corresponds to the maximum weighted clique problem. What is known for arbitrary graph G is that the maximum weighted clique problem is NP-hard [9]. Some details can be found in a textbook of combinatorial optimization (like Cook–Cunningham–Pulleyblank–Schrijver [2] or Korte–Vygen [10]).

Synthesizing the discussion above, we have the following corollary.

Corollary 1. *Let $G = (V, E)$ be a graph and $\mathcal{I}$ be the family of all cliques of G. Then the conditions in Theorem 1 are equivalent to the following statements.*

1. *$(V, \mathcal{I})$ is a matroid.*
2. *For any non-negative weight vector $w \in \mathbb{R}^V_+$, the maximum weighted clique problem is solved by* Greedy Algorithm.

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