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Quantization for a fourth order equation with critical exponential growth

Michael Struwe

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Abstract For concentrating solutions $0 < u_k \rightharpoonup 0$ weakly in $H^2(\Omega)$ to the equation $\Delta^2 u_k = \lambda_k u_k e^{2u_k^2}$ on a domain $\Omega \subset \mathbb{R}^4$ with Navier boundary conditions the concentration energy $\Lambda = \lim_{k\to\infty} \int_\Omega |\Delta u_k|^2 \, dx$ is shown to be strictly quantized in multiples of the number $\Lambda_1 = 16\pi^2$.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^4$ and let $u_k > 0$ be solutions to the equation

$$\Delta^2 u_k = \lambda_k u_k e^{2u_k^2} \quad \text{in } \Omega \tag{1}$$

with constants $\lambda_k > 0$, where we prescribe Navier boundary conditions

$$u_k = \Delta u_k = 0 \quad \text{on } \partial\Omega. \tag{2}$$

We assume that $\lambda_k \to 0$ and

$$\int_\Omega |\Delta u_k|^2 \, dx = \int_\Omega u_k \Delta^2 u_k \, dx = \lambda_k \int_\Omega u_k^2 e^{2u_k^2} \, dx \to \Lambda > 0 \tag{3}$$

as $k \to \infty$. In view of the boundary condition $u_k = 0$ on $\partial\Omega$, then by standard elliptic estimates we also have the uniform estimate

$$\int_\Omega |\nabla^2 u_k|^2 \, dx \le C \int_\Omega |\Delta u_k|^2 \, dx \le C \tag{4}$$

for all k. Since $\lambda_k \to 0$, from (3), (4) we conclude that $\Delta^2 u_k \to 0$ in $L^1(\Omega)$ and $u_k \rightharpoonup 0$ weakly in $H^2(\Omega)$ as $k \to \infty$, but not strongly. In fact, as shown in [10], the sequence

M. Struwe (✉)
Departement für Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland
e-mail: struwe@math.ethz.ch

(u_k) blows up in a finite number of points where after rescaling spherical bubbles form in the following sense.

Theorem 1.1 *Let Ω be a bounded domain of $\mathbb{R}^4$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of positive solutions to problem (1), (2), satisfying (3) for some $\Lambda > 0$ as above.*

Then there exist a subsequence (u_k) and finitely many points $x^{(i)} \in \Omega$, $1 \le i \le I \le C\Lambda$, such that for each i with suitable points $x_k = x_k^{(i)} \to x^{(i)}$ and scale factors $0 < r_k = r_k^{(i)} \to 0$ satisfying

$$\lambda_k r_k^4 u_k^2(x_k) e^{2u_k^2(x_k)} = 96 \tag{5}$$

we have

$$\eta_k(x) = \eta_k^{(i)}(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) \to \eta_0 = \log\left(\frac{1}{1+|x|^2}\right) \tag{6}$$

locally C^3-uniformly on $\mathbb{R}^4$ as $k \to \infty$, where $\eta = \eta_0 + \log 2 = \log\left(\frac{2}{1+|x|^2}\right)$ solves the fourth order analogue of Liouville's equation

$$\Delta^2 \eta = \Delta^2 \eta_0 = 96 e^{4\eta_0} = 6e^{4\eta} \quad \text{on } \mathbb{R}^4. \tag{7}$$

In addition we have

$$\frac{\left|x_k^{(i)} - x_k^{(j)}\right|}{r_k^{(i)}} \to \infty \quad \text{for all } 1 \le i \ne j \le I, \tag{8}$$

and there holds the pointwise estimate

$$\lambda_k \inf_i \left|x - x_k^{(i)}\right|^4 u_k^2(x) e^{2u_k^2(x)} \le C, \tag{9}$$

uniformly for all $x \in \Omega$ and all k.

Geometrically, the solutions η to the limit equation (7) correspond to conformal metrics $g = e^{2\eta} g_{\mathbb{R}^4}$ on $\mathbb{R}^4$ of constant Q-curvature $Q = \frac{1}{2} e^{-4\eta} \Delta^2 \eta = 3 = Q_{S^4}$, which are obtained by pull-back of the spherical metric on S^4 under stereographic projection and with total Q-curvature

$$2\int_{\mathbb{R}^4} Q \, \mathrm{d}\mu_g = \int_{\mathbb{R}^4} 6e^{4\eta} \, \mathrm{d}x = 2\int_{S^4} Q_{S^4} \, \mathrm{d}\mu_{g_{S^4}} = 16\pi^2 =: \Lambda_1. \tag{10}$$

This geometric interpretation of η is the reason why we prefer to state the preceding result in the present form rather than choosing scaling constants as in [10].

Continuing our previous work, here we show that the concentration energy Λ is quantized in multiples of Λ_1.

Theorem 1.2 *In the context of Theorem 1.1 we have $\Lambda = L\Lambda_1$ for some $L \in \mathbb{N}$.*

Theorem 1.2 is the four-dimensional analogue of a recent result by Druet [4] for the corresponding 2-dimensional equation

$$-\Delta u_k = \lambda_k u_k e^{2u_k^2} \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{11}$$

which refines our previous result with Adimurthi in [2], characterizing only the first blow-up energy level.

A similar quantization has been observed by Wei [11] for the fourth order analogue of Liouville's equation

$$\Delta^2 u_k = \lambda_k e^{4u_k} \quad \text{in } \Omega \subset \mathbb{R}^4, \tag{12}$$

with Navier boundary conditions (2), assuming the uniform L^1-bound

$$\int_\Omega \lambda_k e^{4u_k}\, \mathrm{d}x \le \Lambda \tag{13}$$

and with $\lambda_k \to 0$ as $k \to \infty$. Quite remarkably, Wei is able to show that for (12) each blow-up point is simple in the sense that $L = I$. In the geometric context of the problem of prescribed Q-curvature on S^4, an analogous result was obtained by Malchiodi and this author [8]. It is an interesting open question whether the same strong quantization property holds true for Eq. (1) as well.

Related results on compactness issues for fourth order equations can be found in [1,5,9], or [7].

In the following two sections we first present the proof of Theorem 1.2 in the rotationally symmetric case. In Sect. 4 then we show how these results can be extended to the general case by means of the gradient estimate Proposition 4.1, whose proof is given in Sect. 5. Clearly, we may pass to further subsequences (u_k), when needed. Throughout we let $\Delta = \sum_{i=1}^4 (\partial/\partial x_i)^2$ be the Laplacian (with the analysts' sign). The letter C denotes a generic constant that may change from line to line.

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2 The radial case

The proof of Theorem 1.2 is most transparent in the radial case where $\Omega = B_R = B_R(0)$ and when $u_k(x) = u_k(|x|)$. Indeed, in this case in the notations of Theorem 1.1 we have $I = 1$, $x_k^{(1)} = x^{(1)} = 0$, and

$$\eta_k(x) = \eta_k^{(1)}(x) = u_k(0)(u_k(r_k x) - u_k(0)) \le 0,$$

where $r_k = r_k^{(1)}$ is given by (5). Moreover, the functions u_k and $-\Delta u_k$ are positive and radially decreasing, and $u_k \to 0$ locally uniformly away from $x = 0$ as $k \to \infty$.

For brevity we let

$$u_k \Delta^2 u_k = \lambda_k u_k^2 e^{2u_k^2} =: e_k \quad \text{in } \Omega$$

and we denote as

$$w_k(x) = u_k(0)(u_k(x) - u_k(0))$$

the unscaled function η_k, satisfying the equation

$$\begin{aligned} \Delta^2 w_k &= \lambda_k u_k(0) u_k e^{2u_k^2} \\ &= \lambda_k \left(u_k^2(0) + w_k\right) e^{2u_k^2(0)} e^{4\left(1+\frac{w_k}{2u_k^2(0)}\right) w_k} =: f_k \ge e_k \quad \text{in } \Omega = B_R. \end{aligned} \tag{14}$$

The function η_k likewise satisfies the equation

$$\Delta^2\eta_k = V_k e^{4a_k\eta_k} \quad \text{in } \Omega_k = B_{R/r_k}(0) \tag{15}$$

with

$$0 \le V_k = \lambda_k r_k^4 \left(u_k^2(0) + \eta_k\right) e^{2u_k^2(0)} \le 96,\ 1/2 \le a_k = 1 + \frac{\eta_k}{2u_k^2(0)} \le 1. \tag{16}$$

In view of Theorem 1.1 we have $V_k \to 96, a_k \to 1$ locally uniformly on $\mathbb{R}^4$ as $k \to \infty$.
For $0 < r < R$ let

$$\Lambda_k(r) = \int_{B_r} e_k \, \mathrm{d}x, \quad \sigma_k(r) = \int_{B_r} f_k \, \mathrm{d}x,$$

Observe that $\Lambda_k(r) \le \sigma_k(r)$, and both functions are non-decreasing in r; moreover, (10) implies

$$\lim_{L\to\infty}\lim_{k\to\infty} \Lambda_k(Lr_k) = \lim_{L\to\infty}\lim_{k\to\infty} \sigma_k(Lr_k) = \lim_{L\to\infty}\lim_{k\to\infty} 96 \int_{B_L} e^{4\eta_k} \, \mathrm{d}x = \Lambda_1. \tag{17}$$

We can now show our first decay estimate.

Lemma 2.1 *For any $\varepsilon > 0$, letting $T_k > 0$ be such that $u_k(T_k) = \varepsilon u_k(0)/2$, for any constant $b < 2$ and sufficient large k there holds*

$$w_k(r) \le b \log\left(\frac{r_k}{r}\right) + C \quad \text{on } B_{T_k} \tag{18}$$

and we have

$$\lim_{k\to\infty} \Lambda_k(T_k) = \lim_{k\to\infty} \sigma_k(T_k) = \Lambda_1. \tag{19}$$

Proof Note that $T_k \to 0$ as $k \to \infty$ in view of the locally uniform convergence $u_k \to 0$ away from 0.

For $0 \le t \le R$ decompose $w_k = g_k + h_k$ on B_t, where g_k satisfies $g_k = \partial_\nu g_k = 0$ on ∂B_t and where $\Delta^2 h_k = 0$. Then we have $\Delta h_k \equiv 8d_k$ for some $d_k \in \mathbb{R}$ and it follows that $h_k(x) = w_k(t) + d_k(|x|^2 - t^2)$. The equations $\partial_\nu w_k = \partial_\nu g_k + \partial_\nu h_k = 2td_k$ and $\Delta w_k = \Delta g_k + \Delta h_k = \Delta g_k + 8d_k$ on ∂B_t then imply the identity

$$\Delta w_k - 4t^{-1}\partial_\nu w_k = \Delta g_k \quad \text{on } \partial B_t.$$

If we now choose $t = t_k \ge Lr_k$ from (17) we obtain

$$\begin{aligned}\int_{\partial B_t} t\Delta g_k \, \mathrm{d}o &= \int_{\partial B_t} \partial_\nu (x \cdot \nabla g_k) \, \mathrm{d}o = \int_{B_t} (2\Delta g_k + x \cdot \nabla \Delta g_k) \, \mathrm{d}x \\ &= \int_{B_t} x \cdot \nabla \Delta g_k \, \mathrm{d}x = \int_0^t r \int_{\partial B_r} \partial_\nu \Delta g_k \, \mathrm{d}o \, \mathrm{d}r = \int_0^t r\sigma_k(r) \, \mathrm{d}r \\ &\ge \frac{t^2}{2}(\Lambda_1 - o(1))\end{aligned} \tag{20}$$

with error $o(1) \to 0$ if first $k \to \infty$ and then $L \to \infty$. For any $b < 2$ and sufficiently large $L \geq L(b)$ for $k \geq k_0(L)$ then we conclude the estimate

$$\Delta g_k \geq \frac{1}{2\omega_3 t^2}(\Lambda_1 - o(1)) \geq \frac{2b}{t^2} \quad \text{on } \partial B_t$$

for such $t = t_k$, where $\omega_3 = 2\pi^2$ is the volume of S^3. Writing $w_k(x) = w_k(r)$ for convenience and denoting $w_k' = \frac{\partial w_k}{\partial r}$ and so on, we thus obtain that

$$r\left(\frac{w_k'}{r} + \frac{b}{r^2}\right)' = w_k'' - \frac{w_k'}{r} - \frac{2b}{r^2} = \Delta w_k - 4r^{-1}\partial_\nu w_k - \frac{2b}{r^2} \geq 0$$

on ∂B_t, provided $t \geq Lr_k$ with $L \geq L(b)$ and k sufficiently large; that is, the expression $\frac{w_k'}{r} + \frac{b}{r^2}$ is non-decreasing for $r \geq Lr_k$. Since $w_k' \leq 0$ we conclude that

$$w_k'(r) \leq -\frac{b}{r} + \frac{rw_k'(t)}{t} + \frac{rb}{t^2} \leq -\frac{b}{r} + \frac{rb}{t^2} \tag{21}$$

for all $t = t_k \geq r \geq Lr_k$. For any $r \in [Lr_k, T_k]$, upon choosing $t = t_k$ where $T_k/t_k \to 0$ as $k \to \infty$ and integrating from Lr_k to r, with error $o(1) \to 0$ as $k \to \infty$ from (6) we find

$$\begin{aligned} w_k(r) &\leq w_k(Lr_k) - b\log\left(\frac{r}{Lr_k}\right) + o(1) \\ &= \eta_k(L) + b\log L + b\log\left(\frac{r_k}{r}\right) + o(1) \leq b\log\left(\frac{r_k}{r}\right) + o(1), \end{aligned}$$

as claimed in (18). For $r \leq Lr_k$ the asserted bound already follows from Theorem 1.1.

Inserting (18) in the definition (14) of f_k and recalling (5), for $Lr_k \leq r \leq T_k$ with sufficiently large $L > 0$ and k then we obtain

$$\begin{aligned} f_k &\leq \lambda_k u_k^2(0) e^{2u_k^2(0)} e^{4\left(1+\frac{w_k}{2u_k^2(0)}\right)w_k} \\ &\leq \lambda_k r_k^4 u_k^2(0) e^{2u_k^2(0)} r_k^{-4} e^{(2+\varepsilon)w_k} \leq Cr_k^{-4}\left(\frac{r_k}{r}\right)^{(2+\varepsilon)b}. \end{aligned}$$

Choosing $b < 2$ such that $(2+\varepsilon)b = 4+\varepsilon$, upon integrating over B_{T_k} we obtain

$$\begin{aligned} \sigma_k(T_k) &= \int_{B_{T_k}} f_k \,\mathrm{d}x \leq \Lambda_1 + \int_{B_{T_k}\setminus B_{Lr_k}} f_k \,\mathrm{d}x \\ &\leq \Lambda_1 + Cr_k^{-4} \int_{B_{T_k}\setminus B_{Lr_k}} \left(\frac{r_k}{r}\right)^{4+\varepsilon} \mathrm{d}x \leq \Lambda_1 + C\varepsilon^{-1}\left(\frac{r_k}{Lr_k}\right)^{\varepsilon} \leq \Lambda_1 + \varepsilon, \end{aligned}$$

if first $L > L_0(\varepsilon)$ and then $k \geq k_0(L)$ is chosen sufficiently large. Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

For a suitable sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$ and corresponding numbers $s_k = T_k(\varepsilon_k)$ then we have

$$\lim_{k\to\infty} \Lambda_k(s_k) = \lim_{k\to\infty} \sigma_k(s_k) = \Lambda_1, \qquad \lim_{k\to\infty} \frac{u_k(s_k)}{u_k(r_k)} = \lim_{k\to\infty} \frac{r_k}{s_k} = 0, \tag{22}$$

where $r_k/s_k \to 0$ as $k \to \infty$ by Theorem 1.1. In addition from (17) we obtain that

$$\lim_{L\to\infty}\lim_{k\to\infty}(\Lambda_k(s_k)-\Lambda_k(Lr_k))=0. \tag{23}$$

Let $r_k = r_k^{(1)}, s_k = s_k^{(1)}$. We now proceed by iteration. Suppose that for some integer $l \geq 1$ we already have determined numbers $r_k^{(1)} < s_k^{(1)} < \cdots < r_k^{(l)} < s_k^{(l)}$ such that

$$\lim_{k\to\infty}\Lambda_k(s_k^{(l)}) = l\Lambda_1 \tag{24}$$

and

$$\lim_{L\to\infty}\lim_{k\to\infty}\left(\Lambda_k\left(s_k^{(l)}\right)-\Lambda_k\left(Lr_k^{(l)}\right)\right)=\lim_{k\to\infty}\frac{u_k\left(s_k^{(l)}\right)}{u_k\left(r_k^{(l)}\right)}=\lim_{k\to\infty}\frac{r_k^{(l)}}{s_k^{(l)}}=0. \tag{25}$$

For $0 < s < t < R$ let

$$N_k(s,t)=\int_{B_t\setminus B_s} e_k\,\mathrm{d}x=\int_{B_t\setminus B_s}\lambda_k u_k^2 e^{2u_k^2}\,\mathrm{d}x=\omega_3\int_s^t \lambda_k r^3 u_k^2 e^{2u_k^2}\,\mathrm{d}r$$

and define

$$P_k(t)=t\frac{\partial}{\partial t}N_k(s,t)=t\int_{\partial B_t}e_k\,\mathrm{d}o=\omega_3\lambda_k t^4 u_k^2(t)e^{2u_k^2(t)}.$$

Note that monotonicity of u_k implies

$$\begin{aligned}P_k(t)&=\omega_3\lambda_k t^4 u_k^2(t)e^{2u_k^2(t)}=C_0\omega_3\lambda_k u_k^2(t)e^{2u_k^2(t)}\int_{t/2}^t r^3\mathrm{d}r\\&\leq C_0 N_k(t/2,t)\leq C_1 P_k(t/2)\end{aligned} \tag{26}$$

with uniform constants $C_0 = 4(1-1/16)^{-1}$, $C_1 = 16$, respectively.

A preliminary quantification now can be achieved, as follows.

Lemma 2.2
(i) *Suppose that for some $t_k > s_k^{(l)}$ there holds*

$$\sup_{s_k^{(l)}<t<t_k} P_k(t)\to 0 \quad \text{as } k\to\infty.$$

Then we have

$$\lim_{k\to\infty} N_k\left(s_k^{(l)},t_k\right)=0.$$

(ii) *Conversely, if for some $t_k > s_k^{(l)}$ and a subsequence (u_k) there holds*

$$\lim_{k\to\infty} N_k\left(s_k^{(l)},t_k\right)=c_0>0,$$

then either $c_0 \geq 2\omega_3 = 4\pi^2$, or we have

$$\liminf_{k\to\infty} P_k(t_k)\geq 2c_0$$

and

$$\lim_{L\to\infty}\liminf_{k\to\infty} N_k(s_k^{(l)}, Lt_k) \geq 2\omega_3, \quad \lim_{L\to\infty}\limsup_{k\to\infty} N_k\left(s_k^{(l)}, t_k/L\right) = 0.$$

Proof (i) For $s = s_k^{(l)} < t$ we integrate by parts to obtain

$$\begin{aligned}
4N_k(s,t) &= 4\int_{B_t\setminus B_s} \lambda_k u_k^2 e^{2u_k^2} \mathrm{d}x = 4\omega_3 \int_s^t \lambda_k r^3 u_k^2 e^{2u_k^2}\,\mathrm{d}r \\
&= \omega_3\lambda_k\left(r^4 u_k^2 e^{2u_k^2}\right)\Big|_s^t - 4\omega_3\int_s^t \lambda_k r^4\left(u_k/2 + u_k^3\right) u_k' e^{2u_k^2}\,\mathrm{d}r \\
&\leq P_k(t) - 4\omega_3\int_s^t \lambda_k r^4\left(1/2 + u_k^2\right)\frac{u_k}{u_k(0)} w_k' e^{2u_k^2}\,\mathrm{d}r. \qquad (27)
\end{aligned}$$

In order to further estimate the right hand side we need to derive a lower bound for w_k'. For $0 < r < t$ decompose $w_k = g_k + h_k$ on B_r as in the proof of Lemma 2.1, where g_k satisfies $g_k = \partial_\nu g_k = 0$ on ∂B_r and where $\Delta^2 h_k = 0$. Then

$$\Delta w_k - 4r^{-1}\partial_\nu w_k = \Delta g_k \quad \text{on } \partial B_r$$

as before while (20) yields the equation

$$\Delta g_k(r) = \frac{\int_{\partial B_r}\Delta g_k\,\mathrm{d}o}{\omega_3 r^3} = \frac{\int_0^r r'\sigma_k(r')\,\mathrm{d}r'}{\omega_3 r^4} \leq \frac{\sigma_k(r)}{2\omega_3 r^2}.$$

We conclude that

$$\begin{aligned}
&r\left(\frac{w_k'}{r} + \frac{\int_0^r r'\sigma_k(r')\,dr'}{2\omega_3 r^4}\right)' \\
&= w_k'' - \frac{w_k'}{r} + \frac{\sigma_k(r)}{2\omega_3 r^2} - \frac{2\int_0^r r'\sigma_k(r')\,dr'}{\omega_3 r^4} \geq \Delta w_k - 4r^{-1}\partial_\nu w_k - \Delta g_k = 0
\end{aligned}$$

on ∂B_r. Thus, for $0 \leq r_0 \leq r$ we have

$$\frac{w_k'(r)}{r} + \frac{\int_0^r r'\sigma_k(r')\,\mathrm{d}r'}{2\omega_3 r^4} \geq \frac{w_k'(r_0)}{r_0} + \frac{\int_0^{r_0} r'\sigma_k(r')\,\mathrm{d}r'}{2\omega_3 r_0^4}.$$

Upon choosing $r_0 = Lr_k$ for some fixed L, from Theorem 1.1 we can explicitly compute the limit as $k \to \infty$ of the term on the right to see that

$$\frac{w_k'(r)}{r} + \frac{\int_0^r r'\sigma_k(r')\,\mathrm{d}r'}{2\omega_3 r^4} \geq 0,$$

for $r \geq Lr_k$ when L and $k \geq k_0(L)$ are sufficiently large; that is, there holds

$$rw_k'(r) \geq -\frac{\int_0^r r'\sigma_k(r')\,\mathrm{d}r'}{2\omega_3 r^2} \geq -\frac{\sigma_k(r)}{4\omega_3} \quad \text{for all } r \geq Lr_k. \qquad (28)$$

Inserting the bound (28) into (27) and observing that by (25) for $s = s_k^{(l)} < r < t$ with error $o(1) \to 0$ as $k \to \infty$ we have

$$\begin{aligned}\frac{u_k(r)}{u_k(0)}\sigma_k(r) &= \omega_3 \int_0^r \lambda_k u_k(r) u_k(r') e^{2u_k^2(r')} r'^3 \, \mathrm{d}r' \\ &\le \omega_3 \int_s^r \lambda_k u_k^2(r') e^{2u_k^2(r')} r'^3 \, \mathrm{d}r' + o(1) = N_k(s,r) + o(1),\end{aligned}$$

we arrive at the estimate

$$\begin{aligned}4N_k(s,t) &\le P_k(t) + \int_s^t \lambda_k r^3 \left(1/2 + u_k^2\right) e^{2u_k^2} N_k(s,r) \, \mathrm{d}r + o(1) \\ &\le P_k(t) + \omega_3^{-1} N_k(s,t)^2 + o(1).\end{aligned} \tag{29}$$

If we now assume that

$$\sup_{s<t<t_k} P_k(t) \to 0 \quad \text{as } k \to \infty,$$

upon letting t increase from $t = s = s_k^{(l)}$ to t_k we find

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) = 0,$$

as claimed.

(ii) On the other hand, if we suppose that for some $t_k > s_k^{(l)}$ we have

$$0 < \lim_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) = c_0 < 2\omega_3, \tag{30}$$

from (29) with error $o(1) \to 0$ as $k \to \infty$ we conclude that

$$2c_0 + o(1) \le \left(4 - c_0\omega_3^{-1}\right) N_k\left(s_k^{(l)}, t_k\right) \le P_k(t_k) + o(1). \tag{31}$$

It also follows that

$$\lim_{L\to\infty} \liminf_{k\to\infty} N_k\left(s_k^{(l)}, Lt_k\right) \ge 2\omega_3.$$

Otherwise, (26) and (31) yield the uniform bound

$$C_0 \liminf_{k\to\infty} N_k(Lt_k/2, Lt_k) \ge \liminf_{k\to\infty} P_k(Lt_k) \ge 2c_0$$

for all $L \in \mathbb{N}$. Choosing $L = 2^m$, where $m \in \mathbb{N}$, and summing over $1 \le m \le M$, we obtain

$$C_0 \liminf_{k\to\infty} \Lambda_k(2^M t_k) \ge C_0 \liminf_{k\to\infty} N_k\left(t_k, 2^M t_k\right) \ge 2c_0 M \to \infty \quad \text{as } M \to \infty,$$

contrary to assumption (3). Upon replacing t_k by t_k/L in the previous argument and recalling our assumption (30), by the same reasoning we also arrive at the estimate

$$\lim_{L\to\infty} \liminf_{k\to\infty} N_k\left(s_k^{(l)}, t_k/L\right) = 0.$$

This completes the proof. □

Suppose that for some $t_k > s_k^{(l)}$ there holds

$$\liminf_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) > 0.$$

Then we can find a subsequence (u_k) and numbers $r_k^{(l+1)} \in]s_k^{(l)}, t_k[$ such that

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, r_k^{(l+1)}\right) = c_0 > 0. \tag{32}$$

Replacing our original choice of $r_k^{(l+1)}$ by a smaller number, if necessary, we may assume that $c_0 < 2\omega_3$. Lemma 2.2 then implies that

$$\lim_{L\to\infty}\liminf_{k\to\infty} N_k\left(s_k^{(l)}, Lr_k^{(l+1)}\right) \geq 2\omega_3, \quad \lim_{L\to\infty}\limsup_{k\to\infty} N_k\left(s_k^{(l)}, r_k^{(l+1)}/L\right) = 0, \tag{33}$$

and that

$$\liminf_{k\to\infty} P_k\left(r_k^{(l+1)}\right) > 0. \tag{34}$$

Note that (25) and (26) also yield that $r_k^{(l+1)}/s_k^{(l)} \to \infty$ as $k \to \infty$. Indeed, from (25) and (26) for any $m \in \mathbb{N}$ with error $o(1) \to 0$ as $k \to \infty$ we obtain that

$$\begin{aligned} o(1) &\geq N_k\left(s_k^{(l)}/2, s_k^{(l)}\right) \geq C_0^{-1} P_k\left(s_k^{(l)}\right) \\ &\geq C_1^{-1} N_k\left(s_k^{(l)}, 2s_k^{(l)}\right) \geq \cdots \geq C_1^{-m} N_k\left(2^{m-1}s_k^{(l)}, 2^m s_k^{(l)}\right). \end{aligned}$$

Thus, if we assume that $r_k^{(l+1)} \leq 2^M s_k^{(l)}$ for some fixed number M, we find that

$$N_k\left(s_k^{(l)}, r_k^{(l+1)}\right) \leq \sum_{m=1}^{M} N_k\left(2^{m-1}s_k^{(l)}, 2^m s_k^{(l)}\right) \leq o(1),$$

contradicting (32).

Similar to the blow-up analysis in [10], we then obtain the following result.

Proposition 2.3 *There exist a subsequence (u_k) and a constant $c^{(l+1)}$ such that*

$$\eta_k^{(l+1)}(x) := u_k\left(r_k^{(l+1)}\right)\left(u_k(r_k^{(l+1)}x) - u_k(r_k^{(l+1)})\right) \to \eta_0^{(l+1)}(x)$$

locally C^3-uniformly on $\mathbb{R}^4 \setminus \{0\}$ as $k \to \infty$, where $\eta^{(l+1)} = \eta_0^{(l+1)} + c^{(l+1)} \in C^\infty(\mathbb{R}^4)$ solves equation (7) on $\mathbb{R}^4$ with

$$\int_{\mathbb{R}^4} 6e^{4\eta^{(l+1)}}\mathrm{d}x = \Lambda_1.$$

Postponing the details of the proof of Proposition 2.3 to the next section, we now complete the proof of Theorem 1.2.

For a suitable subsequence (u_k) Proposition 2.3 implies

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k\left(r_k^{(l+1)}/L, Lr_k^{(l+1)}\right) = \Lambda_1,$$

and from (33) we obtain that

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k\left(s_k^{(l)}, Lr_k^{(l+1)}\right) = \lim_{L\to\infty}\lim_{k\to\infty}\left(N_k\left(s_k^{(l)}, r_k^{(l+1)}/L\right) + N_k\left(r_k^{(l+1)}/L, Lr_k^{(l+1)}\right)\right) = \Lambda_1. \tag{35}$$

Our induction hypothesis (24) then yields

$$\lim_{L\to\infty}\lim_{k\to\infty}\Lambda_k\left(Lr_k^{(l+1)}\right)=\lim_{L\to\infty}\lim_{k\to\infty}\left(\Lambda_k\left(s_k^{(l)}\right)+N_k\left(s_k^{(l)}, Lr_k^{(l+1)}\right)\right)=(l+1)\Lambda_1. \tag{36}$$

Denote as

$$w_k^{(l+1)}(x) = u_k\left(r_k^{(l+1)}\right)\left(u_k(x) - u_k\left(r_k^{(l+1)}\right)\right)$$

the unscaled function $\eta_k^{(l+1)}$, satisfying the equation

$$\Delta^2 w_k^{(l+1)} = \lambda_k u_k\left(r_k^{(l+1)}\right) u_k e^{2u_k^2} =: f_k^{(l+1)} \quad \text{in } \Omega = B_R. \tag{37}$$

Observe that $f_k^{(l+1)} \geq e_k$ for $r \geq r_k^{(l+1)}$. We have the analogue of Lemma 2.1, which may be proved in the same fashion.

Lemma 2.4 *For any $\varepsilon > 0$, letting $T_k = T_k^{(l+1)}(\varepsilon) > 0$ be such that $u_k(T_k) = \varepsilon u_k(r_k^{(l+1)})/2$, for any constant $b < 2$ and sufficiently large k and L there holds*

$$w_k^{(l+1)}(r) \leq b\log\left(\frac{r_k^{(l+1)}}{r}\right) + C \quad \text{on } B_{T_k}\setminus B_{Lr_k^{(l+1)}} \tag{38}$$

and we have

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, T_k\right) = \Lambda_1. \tag{39}$$

For suitable numbers $s_k^{(l+1)} = T_k^{(l+1)}(\varepsilon_k)$ where $\varepsilon_k \downarrow 0$ as $k \to \infty$ then we have

$$\lim_{k\to\infty}\Lambda_k\left(s_k^{(l+1)}\right) = (l+1)\Lambda_1 \tag{40}$$

and

$$\lim_{L\to\infty}\lim_{k\to\infty}\left(\Lambda_k\left(s_k^{(l+1)}\right) - \Lambda_k\left(Lr_k^{(l+1)}\right)\right) = \lim_{k\to\infty}\frac{r_k^{(l+1)}}{s_k^{(l+1)}} = \lim_{k\to\infty}\frac{u_k\left(s_k^{(l+1)}\right)}{u_k\left(r_k^{(l+1)}\right)} = 0, \tag{41}$$

completing the induction step. In view of (3) and Lemma 2.2 the iteration must terminate after finitely many steps $1 \leq l \leq L$, after which

$$N_k(s_k^{(L)}, R) \to 0 \quad \text{as } k \to \infty.$$

This concludes the proof.

3 Proof of Proposition 2.3

Throughout this section we let $r_k = r_k^{(l+1)}$, $s_k = s_k^{(l)}$, etc., and we set $r_k^- = r_k^{(l)}$. Note that monotonicity of u_k and (26) give rise to the following estimate.

Lemma 3.1 *There exists a constant C independent of k such that*

$$r^2 u_k(r) \le C r_k^2 u_k(r_k) \quad \textit{for all } r < r_k. \tag{42}$$

Proof For $r \le r_k$ we use (26) to estimate

$$\frac{u_k^2(r) r^4}{u_k^2(r_k) r_k^4} P_k(r_k) = \omega_3 \lambda_k r^4 u_k^2(r) e^{2u_k^2(r_k)} \le P_k(r) \le C_0 N_k(r/2, r) \le C\Lambda.$$

The claim follows from (34). □

Denote as $v_k(x) = u_k(r_k x)$ the scaled function u_k. Also write $v_k(x) = v_k(r)$ for $r = |x|$. The following result is similar to Lemma 2.2 in [10].

Lemma 3.2 *As $k \to \infty$ we have $v_k(1) \to \infty$ and there holds $v_k(x) - v_k(1) \to 0$ locally uniformly on $\mathbb{R}^4 \setminus \{0\}$).*

Proof Convergence $v_k(1) = u_k(r_k) \to \infty$ immediately follows from (34).

Let $\tilde{v}_k(x) = v_k(x) - v_k(1)$, satisfying the equation

$$\Delta^2 \tilde{v}_k = \lambda_k r_k^4 v_k e^{2v_k^2} =: g_k \ge 0.$$

We claim that $g_k \to 0$ locally uniformly away from 0. Indeed, for $|x| = r \ge 1$ monotonicity implies that

$$g_k(x) \le \lambda_k r_k^4 v_k(1) e^{2v_k^2(1)} = \lambda_k r_k^4 u_k(r_k) e^{2u_k^2(r_k)} = C P_k(r_k)/u_k(r_k) \to 0$$

as $k \to \infty$. On the other hand, for $1/L \le |x| \le 1$, letting $t = |r_k x|$ from (26) we have the estimate

$$g_k(x) = \lambda_k r_k^4 u_k(r_k x) e^{2u_k^2(r_k x)} \le L^4 \lambda_k t^4 u_k(t) e^{2u_k^2(t)} = C L^4 P_k(t)/u_k(t) \to 0.$$

Since from (4) we also have the uniform L^2-bound

$$||\nabla^2 \tilde{v}_k||_{L^2} = ||\nabla^2 u_k||_{L^2} \le C, \tag{43}$$

we may extract a subsequence (u_k) such that $\Delta \tilde{v}_k =: \tilde{w}_k \to \tilde{w}$ weakly in $L^2_{loc}(\mathbb{R}^4)$ and in $C^{1,\alpha}_{loc}(\mathbb{R}^4 \setminus \{0\})$, where $\tilde{w}$ is harmonic away from the origin. In addition, $\tilde{w} \in L^2(\mathbb{R}^4)$; since the point $x = 0$ has vanishing H^2-capacity, we then have $\Delta \tilde{w} = 0$ in the distribution sense on all of $\mathbb{R}^4$ and $\tilde{w}$ is a smooth, everywhere harmonic function. Again invoking the fact that $\tilde{w} \in L^2(\mathbb{R}^4)$, then we see that $\tilde{w}$ vanishes identically; that is, we have $\Delta \tilde{v}_k \to 0$ weakly in $L^2_{loc}(\mathbb{R}^4)$ and in $C^{1,\alpha}_{loc}(\mathbb{R}^4 \setminus \{0\})$.

The L^2-bound (43) together with the fact that $\tilde{v}_k(1) = 0$ by the H^2-estimates for the Laplace operator also yields weak H^2-convergence $\tilde{v}_k \to \tilde{v}$ in B_1 and then, by a variant of the Poincaré inequality, also local weak H^2-convergence on all of $\mathbb{R}^4$, where $\tilde{v}$ is smooth and harmonic. Since $\tilde{v}(1) = 0$, moreover, by the maximum principle $\tilde{v}$ vanishes throughout B_1. By the unique continuation property of harmonic functions then $\tilde{v}$ vanishes everywhere. Recalling that for radially symmetric functions weak H^2-convergence implies locally uniform convergence away from the origin, we obtain the claim. □

From the definition

$$\eta_k(x) = u_k(r_k)(u_k(r_k x) - u_k(r_k)) = v_k(1)(v_k(x) - v_k(1)),$$

with $r_k = r_k^{(l+1)}$, etc., and with $\hat{v}_k = v_k/v_k(1) \to 1$ locally uniformly on $\mathbb{R}^4 \setminus \{0\}$) by Lemma 3.2, we derive the equation

$$\begin{aligned}\Delta^2\eta_k &= r_k^4 u_k(r_k)(\Delta^2 u_k)(r_k\cdot) = \lambda_k r_k^4 u_k^2(r_k) e^{2u_k^2(r_k)}\hat{v}_k e^{2(v_k^2 - v_k^2(1))} \\ &= \mu_k \hat{v}_k e^{2\eta_k(\hat{v}_k+1)},\end{aligned} \tag{44}$$

where $\mu_k = \omega_3^{-1} P_k(r_k) \geq c_0 > 0$ by (34). Note that by monotonicity of u_k we have

$$\eta_k \leq 0,\ \hat{v}_k \leq 1,\ 0 \leq \Delta^2\eta_k \leq \mu_k \leq C\Lambda \text{ outside } B_1(0), \tag{45}$$

whereas $\eta_k \geq 0$, $\hat{v}_k \geq 1$ in $B_1(0)$. Thus, monotonicity of u_k together with (26) yield

$$0 \leq \mu_k e^{4\eta_k(x)} \leq \Delta^2\eta_k(x) \leq r_k^4 e_k(r_k x) = |x|^{-4}\omega_3^{-1} P_k(r_k|x|) \leq C|x|^{-4} \tag{46}$$

for any $x \in B_1(0)$; moreover, for any $L \geq 1$ and sufficiently large k by (3) we have

$$\int_{B_1(0)} \Delta^2\eta_k \,\mathrm{d}x \leq \int_{B_L(0)} \hat{v}_k \Delta^2\eta_k \,\mathrm{d}x = \int_{B_{Lr_k}(0)} u_k \Delta^2 u_k \,\mathrm{d}x \leq C\Lambda. \tag{47}$$

Finally, in view of (25) and (33) there holds

$$\begin{aligned}\int_{B_{1/L}(0)} \Delta^2\eta_k \,\mathrm{d}x &= \int_{B_{r_k/L}(0)} u_k(r_k)\Delta^2 u_k \,\mathrm{d}x \\ &\leq \int_{B_{r_k/L}\setminus B_{Lr_k^-}(0)} u_k\Delta^2 u_k \,\mathrm{d}x + \frac{u_k(r_k)}{u_k(Lr_k^-)} \int_{B_{Lr_k^-}(0)} u_k \Delta^2 u_k \,\mathrm{d}x \\ &\leq N_k(Lr_k^-, r_k/L) + C\Lambda \frac{u_k(s_k)}{u_k(Lr_k^-)} \to 0,\end{aligned} \tag{48}$$

if we first let $k \to \infty$ and then pass to the limit $L \to \infty$.

Recall that

$$v_k := -\Delta\eta_k = -v_k(1)\Delta v_k \geq 0$$

in view of (2) and the maximum principle. The following result is similar to [10], estimate (21).

Lemma 3.3 *For any $K > 3$ there exists a constant $C(K)$ such that*

$$\int_{B_L(0)} v_k \,\mathrm{d}x \leq C(K)L^2 + CL^4 \frac{1 + \log K}{1 + K^2}$$

for any $L > 1$ and any sufficiently large k.

Proof Extend v_k to $v_k \in H_0^2(\mathbb{R}^4)$ by letting

$$v_k(x) = -\tau(r_k x) v_k(R_k^2 x/|x|^2) \quad \text{for } |x| > R_k = R/r_k,$$

with a fixed cut-off function $\tau \in C_0^\infty(B_{2R}(0))$ such that $\tau \equiv 1$ in a neighborhood of $B_R(0)$. Observe that we have the uniform bound

$$||\nabla v_k||_{L^4} \leq C||\nabla^2 v_k||_{L^2} \leq C||\nabla^2 u_k||_{L^2} \leq C\Lambda.$$

For any $y \in \mathbb{R}^4$ and any $r > 0$ define

$$c_k^{(y,r)} = \fint_{B_r(y)} v_k \, dx,$$

where we denote as $\fint_A = \frac{1}{Vol(A)} \int_A$ the mean value on a domain $A \subset \mathbb{R}^4$. Similar to [2], Lemma 4.2, the Poincaré inequality (or the embedding $W^{1,4}(\mathbb{R}^4) \to BMO(\mathbb{R}^4)$) together with Lemma 3.2 yields the uniform estimate

$$\left|c_k^{(y,1)} - v_k(1)\right| \leq \left|c_k^{(y,1)} - c_k^{(1,1/2)})\right| + \left|c_k^{(1,1/2)} - v_k(1)\right| \leq C \log(1 + |y|) + C$$

for all $y \in \mathbb{R}^4$ and sufficiently large k. The proof of [10], estimate (21), now may be carried over essentially unchanged to the present situation to yield our claim. The only modification required is replacing [10], estimate (19), by the estimate

$$\begin{aligned} \int_{|y| \leq K} \lambda_k r_k^4 u_k(r_k) v_k(y) e^{2v_k^2(y)} \, dy &= \int_{|y| \leq 1} \cdots + \int_{1 \leq |y| \leq K} \cdots \\ &\leq \int_{|y| \leq 1} \lambda_k r_k^4 v_k^2(y) e^{2v_k^2(y)} \, dy \\ &\quad + CK^4 \lambda_k r_k^4 u_k^2(r_k) e^{2u_k^2(r_k)} \leq C(1 + K^4), \end{aligned} \tag{49}$$

which follows from (3), (26), and monotonicity of u_k. □

Lemma 3.4 *There exist a subsequence (u_k) such that $v_k \to v$ locally uniformly in $C^{1,\alpha}$ on $\mathbb{R}^4 \setminus \{0\}$ as $k \to \infty$.*

Proof For any $L > 1$ decompose $v_k = H_k + N_k$ on $B_L \setminus B_{1/L}(0)$, where $\Delta H_k = 0$ in $B_L \setminus B_{1/L}(0)$, and where $N_k = 0$ on $\partial(B_L \setminus B_{1/L}(0))$. After passing to a further subsequence, if necessary, in view of (45), (46) we may assume that $N_k \to N$ in $C^{1,\alpha}$ on $B_L \setminus B_{1/L}(0)$ as $k \to \infty$. On the other hand, by the mean value property of harmonic functions and Lemma 3.3 together with the fact that $v_k \geq 0$ a subsequence $H_k \to H$ locally C^2-uniformly on $B_L \setminus B_{1/L}(0)$ as $k \to \infty$, and the proof is complete. □

Lemma 3.5 *There exist a subsequence (u_k) such that $\eta_k \to \eta_0$ locally uniformly in $C^{3,\alpha}$ on $\mathbb{R}^4 \setminus \{0\}$ as $k \to \infty$.*

Proof Similar to the proof of Lemma 3.4 for any $L > 1$ we decompose $\eta_k = h_k + l_k$ on $B_L \setminus B_{1/L}(0)$, where $\Delta h_k = 0$ in $B_L \setminus B_{1/L}(0)$, and where $l_k = 0$ on $\partial(B_L \setminus B_{1/L}(0))$. By Lemma 3.4 we may assume that $l_k \to l$ in $C^{3,\alpha}$ on $B_L \setminus B_{1/L}(0)$ as $k \to \infty$. Taking account of the estimates $\eta_k(r) \geq \eta_k(1) = 0$ for $1/L < r < 1$ and $\eta_k(r) \leq \eta_k(1) = 0$ for $1 < r < L$, respectively, from Harnack's inequality we conclude that either a subsequence $h_k \to h$ locally C^4-uniformly on $B_L \setminus B_{1/L}(0)$, or $h_k \to \infty$ and hence $\eta_k \to \infty$ locally uniformly on $B_1 \setminus B_{1/L}(0)$ as $k \to \infty$. But the latter possibility is excluded by (46), and the assertion follows. □

Proof of Proposition 2.3 Since $\Delta^2\eta_k \geq 0$ by (47) is uniformly bounded in $L^1(B_L(0))$, from Lemma 3.5 and elliptic regularity we also obtain that the sequence (η_k) is bounded in $W^{2,q}(B_L(0))$ for any $q < 2$ and any $L \geq 1$. In addition to the assertion of Lemma 3.5 we thus may assume that $\eta_k \to \eta_0$ weakly locally in $W^{2,q}$ as $k \to \infty$.

By Lemmas 3.2 and 3.5 we may pass to the limit $k \to \infty$ in Eq. (44) to see that η_0 solves the equation

$$\Delta^2\eta_0 = \mu_0 e^{4\eta_0} \quad \text{on } \mathbb{R}^4 \setminus \{0\}, \tag{50}$$

for some constant $\mu_0 = \lim_{k\to\infty} \mu_k > 0$. Moreover, from Lemma 3.3 we obtain that

$$\lim_{L\to\infty} \left(L^{-4} \int_{B_L(0)} \Delta\eta_0 \, dx \right) = 0. \tag{51}$$

In addition, by Fatou's lemma, Lemma 3.2, and (47), for any $L > 1$ with a uniform constant C we have

$$\begin{aligned} \int_{B_L\setminus B_{1/L}(0)} e^{4\eta_0} \, dx &\leq \liminf_{k\to\infty} \int_{B_L\setminus B_{1/L}(0)} \hat{v}_k^2 e^{2\eta_k(\hat{v}_k(x)+1))} \, dx \\ &= \liminf_{k\to\infty} \int_{B_L\setminus B_{1/L}(0)} \mu_k^{-1} \hat{v}_k \Delta^2\eta_k \, dx \leq C\Lambda. \end{aligned} \tag{52}$$

Passing to the limit $L \to \infty$, we see that $e^{4\eta_0} \in L^1(\mathbb{R}^4)$. Since in view of (48) we also have

$$\int_{B_{1/L}(0)} \Delta^2\eta_0 \, dx \leq \limsup_{k\to\infty} \int_{B_{1/L}(0)} \Delta^2\eta_k \, dx \to 0$$

as $L \to \infty$, it follows that η_0 extends to a distribution solution of (50) on all of $\mathbb{R}^4$. Writing $\mu_0 = 6e^{4c}$ for some constant $c = c^{(l+1)} \in \mathbb{R}$, and letting $\eta = \eta_0 + c$ we find that η solves Eq. (7). The desired characterization of η then can be obtained exactly as in [10], Proposition 2.4, from the bound (51) and Lin's [6] classification of all solutions to equation (7) on $\mathbb{R}^4$ with $e^{4\eta} \in L^1(\mathbb{R}^4)$. □

4 The general case

For the proof of Theorem 1.2 in the general case fix an index $1 \leq i \leq I$ and let $x_k = x_k^{(i)} \to x^{(i)}$, $0 < r_k = r_k^{(i)} \to 0$ be determined as in Theorem 1.1 so that $u_k(x_k) = \max_{|x-x_k|\leq Lr_k} u_k(x)$ for any $L > 0$ and sufficiently large k and such that

$$\eta_k(x) = \eta_k^{(i)}(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) \to \log\left(\frac{1}{1+|x|^2}\right) \tag{53}$$

as $k \to \infty$. For each k we may shift the origin so that henceforth we may assume that $x_k = 0$ for all k. Denote as $\Omega_k = \Omega_k^{(i)}$ the shifted domain Ω. We also extend u_k to $u_k \in W_0^{2,2}(\mathbb{R}^4)$, still satisfying the uniform bound (4) up to a constant. Again we let $e_k = \lambda_k u_k^2 e^{2u_k^2}$, $f_k = \lambda_k u_k(0) u_k e^{2u_k^2}$, and for $0 < r < R$ we set

$$\Lambda_k(r) = \int_{B_r} e_k \, dx, \quad \sigma_k(r) = \int_{B_r} f_k \, dx,$$

satisfying Eq. (17).

Also introduce the spherical mean $\bar{u}_k(r) = \fint_{\partial B_r} u_k \, do$ of u_k on ∂B_r, and so on, and set $\tilde{e}_k = \lambda_k \bar{u}_k^2 e^{2\bar{u}_k^2}$. Recall that in view of (2) and the maximum principle we have $\Delta u_k \le 0$ in Ω_k and then also

$$\bar{u}_k'(r) = \fint_{\partial B_r} \frac{x}{|x|} \cdot \nabla u_k \, do = \frac{r}{4} \fint_{B_r} \Delta u_k \, dx \le 0;$$

that is, $\bar{u}_k(r)$ is non-increasing in r as long as $B_r \subset \Omega_k$.

The spherical mean $\bar{w}_k$ of the function

$$w_k(x) = u_k(0)(u_k(x) - u_k(0)),$$

satisfies the equation

$$\Delta^2 \bar{w}_k = \lambda_k u_k(0) \overline{u_k e^{2u_k^2}} = \bar{f}_k. \tag{54}$$

Note that by Jensen's inequality we have

$$\tilde{e}_k \le \bar{e}_k; \tag{55}$$

hence

$$\tilde{\Lambda}_k(r) := \int_{B_r} \tilde{e}_k \, dx \le \Lambda_k(r), \quad \int_{B_r} \bar{f}_k \, dx = \sigma_k(r).$$

Observe that in analogy with (17) Theorem 1.1 implies

$$\lim_{L\to\infty} \lim_{k\to\infty} \tilde{\Lambda}_k(Lr_k) = \lim_{L\to\infty} \lim_{k\to\infty} \Lambda_k(Lr_k) = \lim_{L\to\infty} \lim_{k\to\infty} \sigma_k(Lr_k) = \Lambda_1. \tag{56}$$

To proceed, we also need the following gradient estimate similar to Druet [4], Proposition 2. For any $k \in \mathbb{N}$, $x \in \Omega$ we let

$$R_k(x) = \inf_{1\le j\le I} |x - x_k^{(j)}|.$$

Proposition 4.1 *There exists a uniform constant C such that for all $x \in \Omega$ there holds*

$$R_k(x) u_k(x) |\nabla u_k(x)| \le C,$$

uniformly in $k \in \mathbb{N}$.

The proof of Proposition 4.1 is given in the next section.

Recalling that $x_k^{(i)} = 0$, we let

$$\rho_k = \rho_k^{(i)} = \frac{1}{2} \inf_{j\ne i} |x_k^{(j)}|,$$

and we set $\rho_k = \infty$ if $\{j;\ j \ne i\} = \emptyset$, that is, if there is no other concentration point but $x_k^{(i)}$. We now use Proposition 4.1 to deal with concentrations around the point $x_k^{(i)}$

at scales which are small with respect to ρ_k. Indeed, for $|x| \le \rho_k$ we have $|x| = R_k(x)$; therefore, by Proposition 4.1 for any $0 < r \le \rho_k$ with a uniform constant C there holds

$$0 \le \sup_{x\in\partial B_r} u_k^2(x) - \inf_{x\in\partial B_r} u_k^2(x) \le Cr \sup_{x\in\partial B_r} |\nabla u_k^2(x)| \le C. \tag{57}$$

Hence

$$\sup_{x\in\partial B_r} e^{2u_k^2(x)} \le Ce^{2\bar{u}_k^2(r)}, \tag{58}$$

and we conclude the estimate

$$\sup_{x\in\partial B_r} u_k^2(x)e^{2u_k^2(x)} \le C_3\left(1+\bar{u}_k^2(r)\right)e^{2\bar{u}_k^2(r)} \tag{59}$$

with a uniform constant C_3.

Also note that in view of (9) and our choice of $x_k^{(i)}$ there holds $u_k(x) \le u_k(0)$ for all $|x| \le \rho_k$; hence at this scale there also holds the inequality $e_k \le f_k$.

Similarly to Lemma 2.1 then we obtain

Lemma 4.2 *For any $\varepsilon > 0$, if there is $0 < T_k \le \rho_k$ such that $\bar{u}_k(T_k) = \varepsilon u_k(0)/2$, then for any constant $b < 2$ and sufficiently large k there holds*

$$\bar{w}_k(r) \le b\log\left(\frac{r_k}{r}\right) + C \quad \text{for } 0 \le r \le T_k \tag{60}$$

and we have

$$\lim_{k\to\infty} \tilde{\Lambda}_k(T_k) = \lim_{k\to\infty} \sigma_k(T_k) = \Lambda_1. \tag{61}$$

Proof First observe that (2) together with (57) and the fact that $u_k(0) \to \infty$ as $k \to \infty$ implies that $dist(0, \partial\Omega_k) \ge T_k$ for sufficiently large k. Hence for such k the spherical averages $\bar{u}_k(r)$ are well-defined for any $r \le T_k$.

As in the proof of Lemma 2.1 for $t = t_k \le T_k$ decompose $\bar{w}_k = g_k + h_k$ on B_t, where g_k satisfies $g_k = \partial_\nu g_k = 0$ on ∂B_t and where $h_k(r) = \bar{w}_k(t) + d_k(r^2 - t^2)$ satisfies $\Delta^2 h_k = 0$. Then as before we derive the identity

$$\Delta\bar{w}_k - 4t^{-1}\partial_\nu\bar{w}_k = \Delta g_k \quad \text{on } \partial B_t$$

and for any $b < 2$, any $t = t_k \ge Lr_k$, for sufficiently large L and k from (20) and (56) we obtain

$$\Delta g_k \ge \frac{2b}{t^2} \quad \text{on } \partial B_t.$$

Again it follows that the expression $\frac{\bar{w}_k'}{r} + \frac{b}{r^2}$ is non-decreasing for $r \ge Lr_k$ when $L > 0$ and k are chosen sufficiently large. Since $\bar{w}_k' = u_k(0)\bar{u}_k' \le 0$ on B_{T_k} we may then deduce (60) in the same way as (18) from (21).

Using (14) and (58), for $Lr_k \le r \le T_k \le \rho_k$ with sufficiently large $L > 0$ and k from (5) and (60) then we obtain

$$\begin{aligned}\bar{f}_k &\le C\lambda_k u_k^2(0)e^{2u_k^2(0)}e^{4\left(1+\frac{\bar{w}_k}{2u_k^2(0)}\right)\bar{w}_k} \\ &\le C\lambda_k r_k^4 u_k^2(0)e^{2u_k^2(0)}r_k^{-4}e^{(2+\varepsilon)\bar{w}_k} \le Cr_k^{-4}\left(\frac{r_k}{r}\right)^{(2+\varepsilon)b},\end{aligned}$$

and we complete the proof as before. □

With the help of Proposition 4.1 we can improve the estimate (61) to obtain the analogue of (19). For $0 \le s < t \le \rho_k$ we let

$$N_k(s,t) = \int_{B_t \setminus B_s} e_k \,\mathrm{d}x = \int_{B_t \setminus B_s} \lambda_k u_k^2 e^{2u_k^2} \mathrm{d}x,$$

as before and define

$$\tilde{N}_k(s,t) = \int_{B_t \setminus B_s} \tilde{e}_k \,\mathrm{d}x = \omega_3 \int_s^t \lambda_k r^3 \bar{u}_k^2 e^{2\bar{u}_k^2} \,\mathrm{d}r \le N_k(s,t). \tag{62}$$

Also let

$$P_k(t) = t\frac{\partial}{\partial t} N_k(s,t) = t \int_{\partial B_t} e_k \,\mathrm{d}o$$

and set

$$\tilde{P}_k(t) = t\frac{\partial}{\partial t} \tilde{N}_k(s,t) = t \int_{\partial B_t} \tilde{e}_k \,\mathrm{d}o = \omega_3 \lambda_k t^4 \bar{u}_k^2(t) e^{2\bar{u}_k^2(t)} \le P_k(t).$$

The analogue of (26) then holds for $\tilde{N}_k$ and $\tilde{P}_k$. Also note that (59) implies the estimate

$$N_k(s,t) \le C_3 \tilde{N}_k(s,t) + o(1), \quad P_k(t) \le C_3 \tilde{P}_k(t) + o(1), \tag{63}$$

with error $o(1) \to 0$ as $k \to \infty$, uniformly in $s \le t \le \rho_k$.

Lemma 4.3 *For any $\varepsilon > 0$, if $T_k = T_k(\varepsilon) \le \rho_k$ is as in Lemma 4.2, then we have*

$$\lim_{k\to\infty} \Lambda_k(T_k) = \Lambda_1.$$

Proof In view of (56) and Lemma 4.2 we have

$$\lim_{L\to\infty} \lim_{k\to\infty} \tilde{N}_k(Lr_k, T_k) \le \lim_{k\to\infty} \tilde{\Lambda}_k(T_k) - \lim_{L\to\infty} \lim_{k\to\infty} \tilde{\Lambda}_k(Lr_k) = 0.$$

Since by (63) with error $o(1) \to 0$ as $k \to \infty$ we can estimate

$$\Lambda_k(Lr_k) \le \Lambda_k(T_k) = \Lambda_k(Lr_k) + N_k(Lr_k, T_k) \le \Lambda_k(Lr_k) + C_3 \tilde{N}_k(Lr_k, T_k) + o(1),$$

upon letting $k \to \infty$ and then passing to the limit $L \to \infty$, from (56) we obtain the claim. □

Under the assumptions of Lemma 4.2, as in the radially symmetric case for a suitable sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$ and corresponding numbers $s_k = T_k(\varepsilon_k) \le \rho_k$ now we have

$$\lim_{k\to\infty} \Lambda_k(s_k) = \Lambda_1,$$

and

$$\lim_{L\to\infty} \lim_{k\to\infty} (\Lambda_k(s_k) - \Lambda_k(Lr_k)) = \lim_{k\to\infty} \frac{\bar{u}_k(s_k)}{\bar{u}_k(r_k)} = \lim_{k\to\infty} \frac{r_k}{s_k} = 0.$$

By slight abuse of notation we let $r_k = r_k^{(1)}$, $s_k = s_k^{(1)}$ and proceed by iteration as before. Suppose that for some integer $l \geq 1$ we already have determined numbers $r_k^{(1)} < s_k^{(1)} < \cdots < r_k^{(l)} < s_k^{(l)} \leq \rho_k$ such that

$$\lim_{k\to\infty} \Lambda_k\left(s_k^{(l)}\right) = l\Lambda_1 \tag{64}$$

and

$$\lim_{L\to\infty}\lim_{k\to\infty}\left(\Lambda_k\left(s_k^{(l)}\right) - \Lambda_k\left(Lr_k^{(l)}\right)\right) = \lim_{k\to\infty}\frac{u_k\left(s_k^{(l)}\right)}{u_k\left(r_k^{(l)}\right)} = \lim_{k\to\infty}\frac{r_k^{(l)}}{s_k^{(l)}} = 0. \tag{65}$$

Similar to Lemma 2.2 then we have the following result.

Lemma 4.4
(i) *Suppose that for some* $s_k^{(l)} < t_k \leq \rho_k$ *there holds*

$$\sup_{s_k^{(l)}<t<t_k} P_k(t) \to 0 \quad \text{as } k \to \infty.$$

Then we have

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) = 0.$$

(ii) *Conversely, if for some* $t_k > s_k^{(l)}$ *and a subsequence* (u_k) *there holds*

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) = c_0 > 0, \ \lim_{k\to\infty} t_k/\rho_k = 0,$$

then either $c_0 \geq 2\omega_3$*, or we have*

$$\liminf_{k\to\infty} P_k(t_k) \geq 2c_0$$

and

$$\lim_{L\to\infty}\liminf_{k\to\infty} N_k\left(s_k^{(l)}, Lt_k\right) \geq 2\omega_3, \quad \lim_{L\to\infty}\limsup_{k\to\infty} N_k\left(s_k^{(l)}, t_k/L\right) = 0.$$

Proof (i) In view of (63) we may consider $\tilde{N}_k(s,t)$ and $\tilde{P}_k(t)$ instead of $N_k(s,t)$ and $P_k(t)$. For $s = s_k^{(l)} < t$ similar to (27) we integrate by parts to obtain

$$\begin{aligned} 4\tilde{N}_k(s,t) &= \tilde{P}_k(r)\big|_s^t - \int_{B_t\setminus B_s} x\cdot\nabla\tilde{e}_k \,\mathrm{d}x \\ &\leq \tilde{P}_k(t) - 4\int_{B_t\setminus B_s} \frac{\bar{u}_k}{u_k(0)} r\bar{w}_k'\tilde{e}_k \,\mathrm{d}x + o(1), \end{aligned}$$

where $r\bar{w}_k' = x\cdot\nabla\bar{w}_k$, as usual, and with error $o(1) \to 0$ as $k \to \infty$, uniformly in $s \leq t$. As in the proof of Lemma 2.2 then for sufficiently large L and k we obtain the estimate

$$r\bar{w}_k'(r) \geq -\frac{\int_0^r r'\sigma_k(r')\,\mathrm{d}r'}{2\omega_3 r^2} \geq -\frac{\sigma_k(r)}{4\omega_3} \quad \text{for all } r \geq Lr_k$$

analogous to (28) and we conclude that

$$-\frac{4\bar{u}_k}{u_k(0)} r\bar{w}_k' \leq \omega_3^{-1} N_k(s,r) + o(1).$$

It follows that

$$4\tilde{N}_k(s,t) \leq \tilde{P}_k(t) + \omega_3^{-1} N_k(s,r)\tilde{N}_k(s,t) + o(1). \tag{66}$$

Thus, if we assume that

$$\sup_{s<t<t_k} \tilde{P}_k(t) \leq \sup_{s<t<t_k} P_k(t) \to 0 \quad \text{as } k \to \infty,$$

upon letting t increase from $t = s_k^{(l)}$ to t_k as before we find

$$\lim_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) \leq C_3 \lim_{k\to\infty} \tilde{N}_k\left(s_k^{(l)}, t_k\right) = 0,$$

as claimed.

(ii) On account of (62), (63), the second assertion follows from (66) exactly as in the proof of Lemma 2.2. □

Continuing to argue as in the radial case, if for some $s_k^{(l)} < t_k \leq \rho_k$ there holds

$$\limsup_{k\to\infty} N_k\left(s_k^{(l)}, t_k\right) > 0, \quad \lim_{k\to\infty} t_k/\rho_k = 0,$$

from Lemma 4.4 we infer that for a subsequence (u_k) and suitable numbers $r_k^{(l+1)} \in]s_k^{(l)}, t_k[$ we have

$$0 < \lim_{k\to\infty} N_k\left(s_k^{(l)}, r_k^{(l+1)}\right) = c_0 < 2\omega_3, \quad \liminf_{k\to\infty} P_k\left(r_k^{(l+1)}\right) > 0,$$

while

$$\lim_{L\to\infty} \limsup_{k\to\infty} N_k\left(s_k^{(l)}, r_k^{(l+1)}/L\right) = 0. \tag{67}$$

Note that as in the radial case we have $r_k^{(l+1)}/s_k^{(l)} \to \infty$ as $k \to \infty$ by (26), (62), and (65); in addition, there holds

$$\lim_{k\to\infty} r_k^{(l+1)}/\rho_k \leq \lim_{k\to\infty} t_k/\rho_k = 0. \tag{68}$$

Moreover, we find the analogue of Proposition 2.3.

Proposition 4.5 *There exist a subsequence (u_k) and a constant $c^{(l+1)}$ such that*

$$\eta_k^{(l+1)}(x) := \bar{u}_k\left(r_k^{(l+1)}\right)\left(u_k\left(r_k^{(l+1)}x\right) - \bar{u}_k\left(r_k^{(l+1)}\right)\right) \to \eta_0^{(l+1)}(x)$$

locally C^3-uniformly on $\mathbb{R}^4 \setminus \{0\}$ as $k \to \infty$, where $\eta^{(l+1)} = \eta_0^{(l+1)} + c^{(l+1)} \in C^\infty(\mathbb{R}^4)$ solves equation (7) on $\mathbb{R}^4$ with

$$\int_{\mathbb{R}^4} 6e^{4\eta^{(l+1)}} \mathrm{d}x = \Lambda_1.$$

Proposition 4.5 is a special case of Proposition 4.7, whose proof will be presented in Sect. 6.

From Proposition 4.5 the energy estimate at the scale $r_k^{(l+1)}$ follows as in the radial case. Likewise, under assumptions similar to Lemma 4.2 we obtain the analogue of Lemma 2.4. By iteration, as in the radial case, we determine the concentration profiles and concentration energies for secondary concentrations around the point $x_k^{(i)} = 0$ at scales up to some $s_k^{(l_0)} < r_k^{(l_0+1)}$ satisfying (68) and such that (64), (65) are valid for all l up to some maximal index $l_0 \geq 0$. (We set $s_k^{(0)} = 0$.)

If $\rho_k = \infty$, moreover, for any $\varepsilon > 0$ we can find numbers $0 < T_k = T_k(\varepsilon) < \rho_k$ such that $\bar{u}_k(T_k) = \varepsilon \bar{u}_k(r_k^{(l_0+1)})/2$ and hence can determine radii $s_k^{(l_0+1)} < \rho_k$ such that (64), (65) hold also for $l = l_0 + 1$, which on account of maximality of l_0 in view of Lemma 4.4 completes the concentration analysis.

If $\rho_k < \infty$, we may distinguish two cases. First assume that for some $L \geq 1$ there is a sequence (x_k) such that $\rho_k/L \leq R_k(x_k) \leq |x_k| \leq L\rho_k$ and

$$\lambda_k |x_k|^4 u_k^2(x_k) e^{2u_k^2(x_k)} \geq c_0 > 0. \tag{69}$$

By Proposition 4.1 we may assume that $|x_k| = \rho_k$.

Lemma 4.6 *Assuming* (69)*, we have* $\bar{u}_k(\rho_k)/\bar{u}_k(r_k^{(l_0+1)}) \to 0$ *as* $k \to \infty$*. Thus, for any* $\varepsilon > 0$ *there is* $T_k = T_k(\varepsilon) \in [r_k^{(l_0+1)}, \rho_k]$ *such that* $\bar{u}_k(T_k) = \varepsilon \bar{u}_k(r_k^{(l_0+1)})/2$.

Proof If we suppose that $\bar{u}_k(\rho_k) \geq \varepsilon \bar{u}_k(r_k^{(l_0+1)})/2$ for some $\varepsilon > 0$, then, letting

$$w_k(x) = w_k^{(l_0+1)}(x) := \bar{u}_k\left(r_k^{(l_0+1)}\right)\left(u_k(x) - \bar{u}_k\left(r_k^{(l_0+1)}\right)\right)$$

and also writing $r_k = r_k^{(l_0+1)}$ for brevity, from (59) and the analogue of Lemma 2.4 for any constant $b < 2$ we obtain

$$\begin{aligned}
\lambda_k |x_k|^4 u_k^2(x_k) e^{2u_k^2(x_k)} &\leq C\lambda_k \rho_k^4 \bar{u}_k^2(\rho_k) e^{2\bar{u}_k^2(\rho_k)} \\
&\leq C\lambda_k \left(\frac{\rho_k}{r_k}\right)^4 r_k^4 \bar{u}_k^2(r_k) e^{2\bar{u}_k^2(r_k)} e^{2(\bar{u}_k^2(\rho_k) - \bar{u}_k^2(r_k))} \\
&= C\tilde{e}_k(r_k)) \left(\frac{\rho_k}{r_k}\right)^4 e^{2\frac{\bar{u}_k(\rho_k)+\bar{u}_k(r_k)}{\bar{u}_k(r_k)} \bar{w}_k(\rho_k)} \\
&\leq C\left(\frac{\rho_k}{r_k}\right)^4 e^{(2+\varepsilon)\bar{w}_k(\rho_k)} \leq C\left(\frac{\rho_k}{r_k}\right)^{4-(2+\varepsilon)b} \to 0
\end{aligned}$$

as $k \to \infty$, if we choose $b < 2$ such that $(2+\varepsilon)b > 4$. This, however, contradicts our assumption (69). □

Thus we can define numbers $s_k^{(l_0+1)} < \rho_k$ such that (64), (65) also hold for $l = l_0+1$. The concentration analysis at the scale $\rho_k =: r_k^{(l_0+2)}$ again uses a blow-up result.

Proposition 4.7 *Assuming* (69)*, there exist a finite set* $S^{(0)} \subset \mathbb{R}^4$*, a subsequence* (u_k)*, and a constant* $c^{(l_0+2)}$ *such that*

$$\eta_k^{(l_0+2)}(x) := \bar{u}_k\left(r_k^{(l_0+2)}\right)\left(u_k\left(r_k^{(l_0+2)}x\right) - \bar{u}_k\left(r_k^{(l_0+2)}\right)\right) \to \eta_0^{(l_0+2)}(x)$$

locally C^3-uniformly on $\mathbb{R}^4 \setminus S^{(0)}$ as $k \to \infty$, where $\eta^{(l_0+2)} = \eta_0^{(l_0+2)} + c^{(l_0+2)}$ extends to a solution of Eq. (7) *of class $C^\infty(\mathbb{R}^4)$ with*

$$\int_{\mathbb{R}^4} 6e^{4\eta^{(l_0+2)}} \mathrm{d}x = \Lambda_1.$$

From Proposition 4.7 the energy estimate at the scale $r_k^{(l_0+2)}$ follows as in the radial case and the concentration analysis at scales up to ρ_k is complete. We then deal with secondary concentrations around $x_k^{(i)} = 0$ at scales exceeding ρ_k.

Define

$$X_{k,1} = X_{k,1}^{(i)} = \left\{ x_k^{(j)};\ \exists C > 0\colon |x_k^{(j)}| \leq C\rho_k \text{ for all } k \right\}$$

and denote as

$$\rho_{k,1} = \rho_{k,1}^{(i)} = \frac{1}{2} \inf_{\{j;\, x_k^{(j)} \notin X_{k,1}\}} |x_k^{(j)}|.$$

Again we set $\rho_{k,1} = \infty$ if $\{j;\, x_k^{(j)} \notin X_{k,1}\} = \emptyset$. Note that $\rho_{k,1}/\rho_k \to \infty$ as $k \to \infty$. Then (57)–(59) and the analogues of Lemmas 4.2–4.4, Lemma 4.6, and Lemma 2.4 hold for $r \in [L\rho_k, \rho_{k,1}]$ for sufficiently large L, and we may continue as above to resolve concentrations in this range of scales.

In the remaining case when (69) fails to hold we have

$$\lim_{L\to\infty} \limsup_{k\to\infty} \int_{\{x\in\Omega;\, \rho_k/L \leq R_k(x) \leq |x| \leq L\rho_k\}} e_k \,\mathrm{d}x = 0. \tag{70}$$

Then either

$$\lim_{L\to\infty} \limsup_{k\to\infty} N_k(L\rho_k, \rho_{k,1}/L) = 0,$$

and we iterate to the next scale, or there exist radii $t_k \leq \rho_{k,1}$ such that $t_k/\rho_k \to \infty$, $t_k/\rho_{k,1} \to 0$ as $k \to \infty$ and a subsequence (u_k) such that

$$P_k(t_k) \geq c_0 > 0 \quad \text{for all } k. \tag{71}$$

Lemma 4.8 *Assume* (71) *holds true. Then $\bar{u}_k(t_k)/\bar{u}_k(r_k^{(l_0+1)}) \to 0$ as $k \to \infty$.*

Proof Set $r_k = r_k^{(l_0+1)}$ for brevity and define $w_k = w_k^{(l_0+1)}$ as in the proof of Lemma 4.6. Going through the proof of Lemma 4.2, we note that (60) still holds for $r_k \leq r \leq t_k$. If we then suppose by contradiction that $\bar{u}_k(t_k) \geq \varepsilon \bar{u}_k(r_k)/2$ for some $\varepsilon > 0$ and sufficiently large k a contradiction follows exactly as in the proof of Lemma 4.6. □

By Lemma 4.8 and on account of (70), if we assume that (71) holds we can find numbers $s_k^{(l_0+1)}$ such that $s_k^{(l_0+1)}/\rho_k \to \infty$ as $k \to \infty$ and such that the analogues of (64), (65) hold for $l = l_0 + 1$ in the following sense. Similar to Lemma 4.2 we have

$$\lim_{k\to\infty} \Lambda_k \left(s_k^{(l_0+1)} \right) = I_1 \Lambda_1,$$

where I_1 is the total number of bubbles concentrating at the points $x_k^{(j)} \in X_{k,1}^{(i)}$. Since we may repeat the previous argument with any $x_k^{(j)} \in X_{k,1}^{(i)}$ in place of $x_k^{(i)}$ we also obtain the estimates

$$\lim_{L\to\infty}\lim_{k\to\infty}\left(\Lambda_k\left(s_k^{(l_0+1)}\right)-\sum_{x_k^{(j)}\in X_{k,1}^{(i)}}\Lambda_k^{(j)}\left(Lr_k^{(l_0^{(j)}+1)}\right)\right)=0,$$

$$\lim_{k\to\infty}\frac{u_k\left(s_k^{(l_0+1)}\right)}{u_k\left(r_k^{(l_0^{(j)}+1)}\right)}=\lim_{k\to\infty}\frac{r_k^{(l_0^{(j)}+1)}}{s_k^{(l_0+1)}}=0\quad \text{for all } x_k^{(j)}\in X_{k,1}^{(i)},$$

where $\Lambda_k^{(j)}(r)$ and $r_k^{(l_0^{(j)}+1)}$ are computed as above with reference to the concentration point $x_k^{(j)}$.

We then proceed by iteration. For $l\geq 2$ we inductively define the sets

$$X_{k,l}=X_{k,l}^{(i)}=\left\{x_k^{(j)};\,\exists C>0\colon |x_k^{(j)}|\leq C\rho_{k,l-1}\text{ for all } k\right\}$$

and let

$$\rho_{k,l}=\rho_{k,l}^{(i)}=\frac{1}{2}\inf_{\{j;\,x_k^{(j)}\notin X_{k,l}^{(i)}\}}|x_k^{(j)}|,$$

where $\rho_{k,l}=\infty$ if $\{j;\,x_k^{(j)}\notin X_{k,l}^{(i)}\}=\emptyset$. Successively performing the above analysis at all scales $\rho_{k,l}$ and at all concentration points $x_k^{(j)}$, upon passing to further subsequences, if needed, we obtain Theorem 1.2.

5 Proof of Proposition 4.1

We argue indirectly, closely following the proof of Druet [4], Proposition 2. Let $y_k\in\Omega$ satisfy

$$R_k(y_k)u_k(y_k)|\nabla u_k(y_k)|=\sup_{x\in\Omega}R_k(x)u_k(x)|\nabla u_k(x)|=:L_k,$$

and suppose by contradiction that $L_k\to\infty$ as $k\to\infty$. From (9) and elliptic regularity it follows that $s_k=R_k(y_k)\to 0$ as $k\to\infty$. Set

$$\Omega_k=\{y;\,y_k+s_ky\in\Omega\}$$

and scale

$$v_k(y)=u_k(y_k+s_ky),\quad y\in\Omega_k.$$

Observe that (3), (4) again yield the uniform bounds

$$||\nabla v_k||_{L^4}\leq C||\nabla^2 v_k||_{L^2}\leq C,\quad \int_{\Omega_k}v_k\Delta^2 v_k\,dx\leq C. \tag{72}$$

Letting $x_k^{(i)}$ be as in the statement of Theorem 1.1, we set

$$y_k^{(i)}=\frac{x_k^{(i)}-y_k}{s_k},\quad 1\leq i\leq I,$$

and let

$$S_k = \left\{ y_k^{(i)}; 1 \le i \le I \right\}.$$

Note that in the scaled coordinates we have

$$dist(0, S_k) = \inf \left\{ |y_k^{(i)}|; 1 \le i \le I \right\} = 1$$

and there holds

$$\sup_{y \in \Omega} (dist(y, S_k) v_k(y) |\nabla v_k(y)|) \le v_k(0) |\nabla v_k(0)| = L_k \to \infty \tag{73}$$

as $k \to \infty$; moreover, (9) implies the bound

$$0 \le v_k \Delta^2 v_k = \lambda_k s_k^4 v_k^2 e^{2v_k^2} \le \frac{C}{dist^4(y, S_k)}. \tag{74}$$

Since $s_k = R_k(y_k) \to 0$ we may assume that as $k \to \infty$ the domains Ω_k exhaust a half-space

$$\Omega_0 = \mathbb{R}^3 \times] - \infty, R_0[,$$

where $0 < R_0 \le \infty$. We also may assume that as $k \to \infty$ either $|y_k^{(i)}| \to \infty$ or $y_k^{(i)} \to y^{(i)}$, $1 \le i \le I$, and we let S_0 be the set of accumulation points of S_k, satisfying $dist(0, S_0) = 1$. For $R > 0$ denote as

$$K_R = K_{k,R} = \Omega_k \cap B_R(0) \setminus \bigcup_{y \in S_0} B_{1/R}(y).$$

Recalling that $\lambda_k s_k^4 \to 0$, from (74), finally, for any $R > 0$ we obtain

$$||\Delta^2 v_k||_{L^\infty(K_R)} \to 0 \quad \text{as } k \to \infty. \tag{75}$$

We are now ready to show

Lemma 5.1 *We have $R_0 = \infty$; that is, $\Omega_0 = \mathbb{R}^4$.*

Proof Suppose by contradiction that $R_0 < \infty$. Choosing $R = 2R_0$, from (73) and (2) we conclude the uniform bound

$$\sup_{y \in K_R} \frac{v_k^2(y)}{v_k(0) |\nabla v_k(0)|} \le C = C(R).$$

Letting $w_k = \frac{v_k}{\sqrt{v_k(0)|\nabla v_k(0)|}}$, we then have $0 \le w_k \le C$ while (72) and (75) give

$$||\nabla w_k||_{L^4} + ||\nabla^2 w_k||_{L^2} + ||\Delta^2 w_k||_{L^\infty(K_R)} \to 0 \quad \text{as } k \to \infty.$$

Since $w_k = 0 = \Delta w_k$ on $\partial \Omega_k$, it follows that $w_k \to 0$ in $C^{3,\alpha}(K_R)$, contradicting the fact that $w_k(0) |\nabla w_k(0)| = 1$. □

Lemma 5.2 *As $k \to \infty$ we have $v_k(0) \to \infty$ and*

$$\frac{v_k}{v_k(0)} \to 1 \quad \text{in } C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0).$$

Proof First observe that

$$c_k := \sup_{y \in B_{1/2}} v_k(y) \to \infty \quad \text{as } k \to \infty.$$

Indeed, if by contradiction we assume that $c_k \le C$, the uniform bounds (72) and (75) by elliptic regularity imply a uniform C^1-bound for v_k near $y = 0$, contradicting (73). Letting $w_k = c_k^{-1} v_k$, from (72) and (75) for any $R > 0$ then we have

$$||\nabla w_k||_{L^4} + ||\nabla^2 w_k||_{L^2} + ||\Delta^2 w_k||_{L^\infty(K_R)} \to 0 \quad \text{as } k \to \infty$$

and we conclude that w_k converges to a constant limit function w in $C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0)$. Recalling that $dist(0, S_0) = 1$, we obtain that

$$w \equiv \sup_{y \in B_{1/2}} w(y) = \lim_{k \to \infty} \sup_{y \in B_{1/2}} w_k(y) = 1.$$

In particular, we conclude that $c_k^{-1} v_k(0) = w_k(0) \to 1$ as $k \to \infty$ and therefore $v_k(0) = c_k w_k(0) \to \infty$, $\frac{v_k}{v_k(0)} = \frac{w_k}{w_k(0)} \to 1$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0)$, as claimed. □

Defining

$$\tilde{v}_k(y) = \frac{v_k(y) - v_k(0)}{|\nabla v_k(0)|},$$

from (73) and Lemma 5.2 with error $o(1) \to 0$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0)$ as $k \to \infty$ we obtain the bound

$$|\nabla \tilde{v}_k(y)| = \left(\frac{v_k(y)}{v_k(0)} + o(1) \right) \frac{|\nabla v_k(y)|}{|\nabla v_k(0)|} \le \frac{1 + o(1)}{dist(y, S_0)}. \tag{76}$$

Since $\tilde{v}_k(0) = 0$, this shows that $\tilde{v}_k$ is uniformly bounded in $C^1(K_R)$ for any $R > 0$. Moreover, from (74) and Lemma 5.2 with a constant $C = C(R)$ we obtain

$$\Delta^2 \tilde{v}_k = \frac{v_k(0)}{v_k} \frac{v_k \Delta^2 v_k}{v_k(0)|\nabla v_k(0)|} \le C \frac{v_k(0)}{L_k v_k} \to 0 \tag{77}$$

uniformly on K_R as $k \to \infty$, for any $R > 0$; the sequence $(\tilde{v}_k)$ thus is in fact bounded in $C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0)$ and we may assume that $\tilde{v}_k \to \tilde{v}$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4 \setminus S_0)$, where $\tilde{v}$ satisfies

$$\Delta^2 \tilde{v} = 0, \quad \tilde{v}(0) = 0, \quad |\nabla \tilde{v}(0)| = 1, \quad |\nabla \tilde{v}(y)| \le \frac{1}{dist(y, S_0)}. \tag{78}$$

Fix any point $x_0 \in S_0$. For any $r > 0$ by the divergence theorem we have

$$\begin{aligned} \int_{\partial B_r(x_0)} v_k \frac{x}{|x|} \cdot \nabla \Delta v_k \, do &= \int_{B_r(x_0)} \left(v_k \Delta^2 v_k + \nabla v_k \cdot \nabla \Delta v_k \right) dx \\ &= \int_{B_r(x_0)} \left(v_k \Delta^2 v_k - |\Delta v_k|^2 \right) dx + \int_{\partial B_r(x_0)} \frac{x}{|x|} \cdot \nabla v_k \Delta v_k \, do. \end{aligned} \tag{79}$$

By (72) we can bound

$$\left| \int_{B_r(x_0)} \left(v_k \Delta^2 v_k - |\Delta v_k|^2 \right) dx \right| \le C, \tag{80}$$

uniformly in k. The remaining term equals

$$\int_{\partial B_r(x_0)} \frac{x}{|x|}\cdot\nabla v_k\Delta v_k\,\mathrm{d}o = |\nabla v_k(0)| \int_{\partial B_r(x_0)} \frac{x}{|x|}\cdot\nabla\tilde{v}_k\Delta v_k\,\mathrm{d}o.$$

Since $\Delta v_k \le 0$ by (2) and the maximum principle, the estimate (76) for sufficiently small $r > 0$ yields the bound

$$\left|\int_{\partial B_r(x_0)} \frac{x}{|x|}\cdot\nabla\tilde{v}_k\Delta v_k\,\mathrm{d}o\right| \le -(1+o(1))r^{-1}\int_{\partial B_r(x_0)} \Delta v_k\,\mathrm{d}o \tag{81}$$

with error $o(1)\to 0$ as $k\to\infty$. By Fubini's theorem for any $r > 0$ there is a subsequence (u_k) and a radius $r/2 \le r_0 \le r$ such that there holds

$$\left|r_0\int_{\partial B_{r_0}(x_0)} \Delta v_k\,\mathrm{d}o\right| \le 2\int_{B_r\setminus B_{r/2}(x_0)} |\Delta v_k|\,\mathrm{d}x \quad \text{for all } k.$$

From Hölder's inequality and (72) for such r_0 we conclude that

$$\begin{aligned}\left|r_0^{-1}\int_{\partial B_{r_0}(x_0)} \Delta v_k\,\mathrm{d}o\right| &\le 8r^{-2}\int_{B_r\setminus B_{r/2}(x_0)} |\Delta v_k|\,\mathrm{d}x\\ &\le C\left(\int_{B_r\setminus B_{r/2}(x_0)} |\Delta v_k|^2\,\mathrm{d}x\right)^{1/2} \le C.\end{aligned}$$

Replacing r by r_0, from (81) for sufficiently large k we then obtain

$$\left|\int_{\partial B_{r_0}(x_0)} \frac{x}{|x|}\cdot\nabla v_k\Delta v_k\,\mathrm{d}o\right| \le C|\nabla v_k(0)|. \tag{82}$$

Together with (79), (80) this estimate implies

$$\left|\int_{\partial B_{r_0}(x_0)} v_k\frac{x}{|x|}\cdot\nabla\Delta v_k\,\mathrm{d}o\right| \le C(1+|\nabla v_k(0)|).$$

On the other hand, by Lemma 5.2 the integral on the left-hand side equals

$$\int_{\partial B_{r_0}(x_0)} v_k\frac{x}{|x|}\cdot\nabla\Delta v_k\,\mathrm{d}o = (1+o(1))v_k(0)|\nabla v_k(0)| \int_{\partial B_{r_0}(x_0)} \frac{x}{|x|}\cdot\nabla\Delta\tilde{v}_k\,\mathrm{d}o$$

and we obtain that

$$v_k(0)|\nabla v_k(0)|\Big|\int_{\partial B_{r_0}(x_0)} \frac{x}{|x|}\cdot\nabla\Delta\tilde{v}_k\,\mathrm{d}o\Big| \le C(1+|\nabla v_k(0)|).$$

But then by (73), and since by Lemma 5.2 we also have $v_k(0) \to \infty$, it follows that

$$\int_{B_{r_0}(x_0)} \Delta^2 \tilde{v}_k \, dx = \int_{\partial B_{r_0}(x_0)} \frac{x}{|x|} \cdot \nabla \Delta \tilde{v}_k \, do \to 0$$

as $k \to \infty$. Since $\Delta^2 \tilde{v}_k \geq 0$, together with (77) this estimate implies that $\Delta^2 \tilde{v}_k \to 0$ in $L^1_{loc}(\mathbb{R}^4)$ as $k \to \infty$. The sequence $(\tilde{v}_k)$ therefore is uniformly locally bounded in $W^{2,q}$ for any $q < 2$ and the limit $\tilde{v} \in W^{2,q}_{loc}(\mathbb{R}^4)$ extends as a weakly biharmonic function to all of $\mathbb{R}^4$.

The function $\Delta \tilde{v} \in L^q_{loc}(\mathbb{R}^4)$ then is weakly harmonic on $\mathbb{R}^4$. Since $\Delta \tilde{v} \leq 0$, from Harnack's inequality we conclude that $\Delta \tilde{v}$ everywhere equals a constant $c \leq 0$. If $c \neq 0$, as shown for instance in [1], Theorem 2.4, the function $\tilde{v}$ is a non-trivial quadratic polynomial, which violates the decay condition (78) for $\nabla \tilde{v}$. Thus we must have $c = 0$ and $\tilde{v}$ is harmonic. The decay condition (78) then implies that $\tilde{v}$ is a constant; that is, $\tilde{v} \equiv \tilde{v}(0) = 0$. But by (78) we have $|\nabla \tilde{v}(0)| = 1$, which is the desired contradiction completing the proof of Proposition 4.1.

6 Proof of Proposition 4.7

As in Sect. 4, we fix an index $i \in \{1, \ldots, I\}$. Writing $r_k = r_k^{(l_0+2)}$ for simplicity, we define

$$v_k(y) = u_k\left(x_k^{(i)} + r_k y\right), \quad y \in \Omega_k,$$

where by slight abuse of notation we denote as

$$\Omega_k = \Omega_k^{(i)} = \left\{y; x_k^{(i)} + r_k y \in \Omega\right\}$$

the shifted and scaled set Ω. For $j \in \{1, \ldots, I\}$ also let

$$y_k^{(j)} = \frac{x_k^{(j)} - x_k^{(i)}}{r_k}$$

and set

$$S_k = S_k^{(i)} = \left\{y_k^{(j)}; 1 \leq j \leq I\right\}.$$

We may assume that as $k \to \infty$ either $|y_k^{(j)}| \to \infty$ or $y_k^{(j)} \to y^{(j)}$, $1 \leq j \leq I$, and we let $S_0 = S_0^{(i)}$ be the set of accumulation points of S_k. Note that S_0 contains the origin. Finally, let

$$y_k^{(0)} = \frac{x_k - x_k^{(i)}}{r_k}$$

be the scaled points x_k for which (69) is valid, satisfying $|y_k^{(0)}| = 1$. Again we may assume that $y_k^{(0)} \to y^{(0)}$ as $k \to \infty$. Observe that Proposition 4.1 implies the uniform bound

$$\begin{aligned} &\inf_{1 \leq j \leq I} |y - y_k^{(j)}| v_k(y) |\nabla v_k(y)| \\ &= R_k\left(x_k^{(i)} + r_k y\right) u_k\left(x_k^{(i)} + r_k y\right) |\nabla u_k\left(x_k^{(i)} + r_k y\right)| \leq C. \end{aligned} \quad (83)$$

Since $v_k(y_k^{(0)}) \to \infty$ as $k \to \infty$ by (69), we conclude that $|\nabla v_k| \to 0$ as $k \to \infty$ locally uniformly on $\mathbb{R}^4 \setminus S_0$ and therefore, since $\mathbb{R}^4 \setminus S_0$ is connected, that

$$v_k - v_k\left(y_k^{(0)}\right) \to 0 \quad \text{as } k \to \infty \tag{84}$$

locally uniformly on $\mathbb{R}^4 \setminus S_0$; moreover, as $k \to \infty$ the sets Ω_k exhaust all of $\mathbb{R}^4$.

For $\eta_k = \eta_k^{(l_0+2)}$ we then have the equation

$$\begin{aligned}\Delta^2\eta_k(y) &= r_k^4 v_k\left(y_k^{(0)}\right)\Delta^2 u_k,\left(x_k^{(i)}\right) + r_k y)\\ &= \lambda_k r_k^4 v_k^2\left(y_k^{(0)}\right)e^{2v_k^2\left(y_k^{(0)}\right)}\frac{v_k(y)}{v_k\left(y_k^{(0)}\right)}e^{2\left(v_k^2(y)-v_k^2\left(y_k^{(0)}\right)\right)}\\ &= \mu_k \hat{v}_k(y)e^{2\eta_k(\hat{v}_k(y)+1))},\end{aligned} \tag{85}$$

where

$$\mu_k = \lambda_k r_k^4 v_k^2\left(y_k^{(0)}\right)e^{2v_k^2(y_k^{(0)})} = \lambda_k|x_k|^4 u_k^2(x_k)e^{2u_k^2(x_k)} \to \mu_0 > 0$$

as $k \to \infty$ by (69), and where

$$\hat{v}_k = \frac{v_k}{v_k\left(y_k^{(0)}\right)} \to 1 \quad \text{locally uniformly on } \mathbb{R}^4 \setminus S_0$$

on account of (84). The estimate (83) together with the condition $\bar{\eta}_k(1) = 0$ also implies locally uniform C^1-bounds for (η_k) away from S_0. By equation (85) then as $k \to \infty$ a subsequence (η_k) converges locally C^3-uniformly on $\mathbb{R}^4 \setminus S_0$ and weakly locally in $W^{2,q}$ to some limit $\eta_0 \in W^{2,q}_{loc}(\mathbb{R}^4)$ which is smooth away from S_0 and solves the equation $\Delta^2\eta_0 = \mu_0 e^{4\eta_0}$ on $\mathbb{R}^4 \setminus S_0$. Similar to (52), moreover, we can estimate

$$\int_{\mathbb{R}^4} e^{4\eta_0}\,\mathrm{d}y \le \lim_{L\to\infty}\liminf_{k\to\infty}\int_{B_L\setminus\bigcup_{y_0\in S_0}B_{1/L}(y_0)}\hat{v}_k^2 e^{2\eta_k(\hat{v}_k+1))}\,\mathrm{d}y \le C\Lambda,$$

and $e^{4\eta_0} \in L^1(\mathbb{R}^4)$. Taking account of (57), (65), and (67), we obtain the analogue of (48); that is,

$$\int_{B_{1/L}(y_0)}\Delta^2\eta_k\,\mathrm{d}x \to 0$$

for any $y_0 \in S_0$, if we first let $k \to \infty$ and then pass to the limit $L \to \infty$. Thus, as in the proof of Proposition 2.3 we see that, in fact, η_0 solves the equation $\Delta^2\eta_0 = \mu_0 e^{4\eta_0}$ on all of $\mathbb{R}^4$. For a suitably chosen constant c_0 the function $\eta = \eta_0 + c_0$ then again solves Eq. (7) on $\mathbb{R}^4$. Finally, by (59) also (49) and hence the conclusion of Lemma 3.3 remain valid and we obtain the estimate (51). Again using Lin's [6] classification result we complete the proof of Proposition 4.7.

In the case of Proposition 4.5 we argue similarly, scaling with $r_k = r_k^{(l_0+1)}$. Note that $S_0 = \{0\}$ in this case.

7 Concluding remarks

Since positivity of $-\Delta u_k$ is only used near the blow-up points, it should not be too difficult to carry over our analysis to the case of Eq. (1) with Dirichlet boundary condition $u_k = \partial_\nu u_k = 0$ on $\partial\Omega$ or to general nonlinearities of critical exponential growth, as studied in [2] or [4].

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