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ORIGINAL ARTICLE

Game and decision theory in mathematics education: epistemological, cognitive and didactical perspectives

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Abstract In the 1950s, game and decision theoretic modeling emerged—based on applications in the social sciences—both as a domain of mathematics and interdisciplinary fields. Mathematics educators, such as Hans Georg Steiner, utilized game theoretical modeling to demonstrate processes of mathematization of real world situations that required only elementary intuitive understanding of sets and operations. When dealing with *n*-person games or voting bodies, even students of the 11th and 12th grade became involved in what Steiner called the evolution of mathematics from situations, building of mathematical models of given realities, mathematization, local organization and axiomatization. Thus, the students could participate in processes of epistemological evolutions in the small scale. This paper introduces and discusses the epistemological, cognitive and didactical aspects of the process and the roles these activities can play in the learning and understanding of mathematics and mathematical modeling. It is suggested that a project oriented study of game and decision theory can develop situational literacy, which can be of interest for both mathematics education and general education.

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1 The rise of mathematical modeling in the social sciences

After the 1940s, mathematizing in the social sciences went beyond applications of differential equations (see Walras 1874/2003; Hicks 1937 for modeling equilibrium and market models in economy). Starting from the groundbreaking book *Game theory and economic behavior* by Neumann and Morgenstern (1944), new fields such as operations research or game, decision, bargaining and learning theory developed (see, e.g., Nash 1950; Lazarsfeld 1954; Kemeny and Snell 1962; Bush and Estes 1959). For instance, as Raiffa (2002) pointed out, operations research "...was not so much a collection of mathematical techniques but an approach to complex, strategic decision making. Typically the decision entity was some branch of government ...and they were confronted with an ill-formed problem. Part of the task ...was to crystallize the problem and structure it in such a way that systematic thinking could help the decision entity to make a wise choice" (Raiffa 2002, p. 179). The analytically motivated abstractions started from problem understanding through elementary analysis of complex situations to advanced analysis of idealized analysis of well-structured problems. These approaches have been clearly linked to models of uncertainty and probability theory. The work of the statistician Wald demonstrates that operations research and statistics were strongly influenced by questions of social sciences and decisions in economy, production and military actions (Morgenstern 1951).

From a mathematics education perspective, this new field of mathematical activity offered new possibilities. This was also enhanced in view of the fact that many applications in social science have been based on elementary mathematics from logic, set or probability theory. The new field of mathematical modeling allowed one to discover social realities when applying mathematics; it was possible to refer to the ''personal, social and political dimension of mathematics'' (Törner and Sriraman 2007, in this issue). Since this approach initially centered on revealing general structures in social settings, it is clear that the discussions on axiomatizing could be linked to it. Yet, as seen in the following chapters, this new branch of linking mathematics and social sciences also played an important role in the history of mathematics education. As Vollrath (2007) conveys, Steiner elaborated that this new branch did not only rely on sets, structures, and representations by functions, but also allowed for a non-deductive, constructivistic approach for working with social reality. Thus, mathematics has not been seen solely as a ''science of formal systems'' (Steiner 1965b) or as a tool or language for physics and natural sciences.

2 Societal values and education under discussion: challenges from the 1960s

Trends in science correspond with societal change. In the Western World, the post-World War II era was highlighted by, among others, economic growth, rapid technological development, the Cold War and the belief in the conception of *(hu)man as a rational being*. However, as Shulman and Carey (1984) pointed out, with the end of the Vietnam War around 1968, a political crisis and riots arose initiated by university teachers and students. This crisis was accompanied by a fundamental change in the notion of ''(hu)man''. The belief in (hu)man as a rational being, which implicitly or explicitly underlied most theories of economics, was substituted in psychology and other social sciences as well as in education and humanities by the conception of *(hu)man as a bounded rational being* (Simon 1982). The latter has been characterized as a being full of intent and deliberation but with limited memory and operational capabilities. Clearly, the constructivist perspective has been promoted by this new trend in the academic world.

Nevertheless, the 1968 student riots also penetrated many German universities and, in some cases, the mathematics departments as well. This was true, for instance, at the University of Marburg where this author studied. Quite remarkably, the complaints of the mathematics students did not focus purely on political problems, but also challenged didactical issues. The undergraduate instruction (i.e., the first 2 years of the diploma curriculum) in Marburg, as was the case throughout Germany, was held almost exclusively in a traditional style of one-way communication. For students, mathematics appeared as closed, abstract (in a Bourbacian sense), inert, philosophy-like nature and body of knowledge. The teaching was widely free from any real world applications of mathematics, which endangered to spoil this nature. Almost all professors enjoyed strict selection rules and applied the well known ''take it or leave it strategy''. Examinations were often focused on solving tasks under time constraints and led to break-off rates far above 50%.

The societal and educational crisis that reached university mathematics departments challenged educational discussions—which were almost taboo at that time—and demanded reforms in mathematics education. This author (pro)actively participated in a series of student strikes at the University of Marburg. As reported in the Frankfurter Allgemeine Zeitung (Reumann 1972), the focus of these strikes was a push for didactical changes, in particular, innovative forms of teaching and examinations. Thus, the students challenged the traditional mathematics professors to offer new types of teaching and fairer, alternative, and innovative forms of examinations. The idea was to assess and to evaluate a broader scope of aptitudes than that which could be assessed by solving a set of traditional tasks under time pressure.

In the context of these riots on teaching and exam methods, discussions about the genesis and history of mathematics, constructivist approaches to mathematics and the upcoming challenges of computer sciences for mathematics were also induced. The students also scrutinized the relationship between mathematics and society. This included exploring the contributions which pure mathematics could provide to practice, computer sciences and vocational aptitudes. Thus, the role and epistemics of modeling and of societal applications of mathematics were also challenged.

Presumably, the Marburg mathematics department staff was overburdened by actively stepping into such discussions with unfriendly, insistent students at that time. In order to calm the waters, the staff invited Hans-Georg Steiner to present a lecture at the colloquium of the mathematics department. On 31 January 1972, Steiner faced a group of overly critical mathematics students and presented an intriguing, fascinating lecture on how to build mathematical models on *voting systems*. The undergraduate students were fully engaged and able to grasp the lecture due to the elementary character of the concepts. Many of them, including the author, were highly attracted by the contents and by the charismatic, inductive, Socratic teaching style used when dealing with mathematics and real world examples.

For most of the critical Marburg mathematics students of the year 1972, Steiner opened a new chapter in the book of mathematics. The mathematics he presented strongly differed from what was taught to students in traditional mathematics classes at both senior high school and undergraduate university level. Steiner's aspirations and visions can be seen from his subtitle in one of the first papers on game theoretical topics. His treatise dealt with subject matter for which Arrow later received the Nobel Prize in economics, i.e., voting power, social choice, and individual preferences (Arrow 1951). Steiner entitled his paper: ''Contributions to the topic: mathematical models of reality'' (German: Material zum Unterrichtsthema: Mathematische Modelle der Wirklichkeit; Steiner 1968). In the following, we show which didactics Steiner could demonstrate using this approach.

3 From inductive to deductive approaches on *n*-person coalition games

The mathematization of voting systems on the secondary school level has been Steiner's favorite topic to depict ''evolutions of mathematics from situations, building of mathematical models of given realities, mathematization, local organization, axiomatization'' (Steiner 1968, p. 181; see also 1966, 1967, 1969, 1976, 1982, 1984, 1988). The latter, axiomatization, is unmistakably a core identity of structure-oriented mathematics of the 1960s and 1970s. The process of activity-based instruction on voting bodies ''consists in forming a complete deductive representation of the theory which before has been developed inductively with only local deductive analysis'' (Steiner 1968, p. 182). In the following, we sketch the discursive, dynamic process that arose in Steiner's aforementioned presentation at Marburg in 1972 and his typical approach in presentations and in papers; we sketch the simple, but also the formal arrangements.

[A] *Introducing situations*: Steiner used voting *situations* found in society as springboards for mathematization. Take, for example, a board of examiners, a jury court, a city council deciding on a new stadium, or decisions of the United Nations Security Council (UNSC). The UNSC, for instance, is an *n*-person game in which five permanent members (i.e., the big players) and six non-permanent members (i.e., the small players), which changed every 2 years, each had to vote. The minimal winning coalition consisted of all five ''big players'' and two of the six ''small players''.

In general, a voting body includes of member of parties, players or persons, i.e., a *voting set* $N = \{a_1, \ldots, a_n\}$. Each *member* or *player* a_i is assumed to have a number of votes v_i, which he or she can set out ''for'' or ''against'' an alternative. Subsets $C \subseteq N$ are called *coalitions*. A coalition is called a *winning coalition*, if the sum of votes is sufficiently large enough to determine the decision. The sum of the votes is sufficiently large if it is *above* a critical *majority quotient* or *quorum* $q \leqslant m/n \leqslant 1$. The *majority quotient* is a positive real number, which depends on the decision rule. It can be, e.g., a *simple majority* ($p = 1/2$), *two-thirds* ($q = 2/3$) or *unanimous* ($q = 1$) rule.

Steiner also introduced simple, abstract examples such as a voting set with four players $N = \{a_1, a_2, a_3, a_4\}$, in which:

Example 1: Each member of N has one vote and the simple majority rule holds.
Example 2: Each member of N has one vote, and the simple majority rule holds; however, in case of a tie, the chairperson a_1 decides the majority.
Example 3: The players of N have the votes $v_1 = 5, v_1 = 2, v_1 = 1, v_1 = 1$ and the simple majority rule holds.

[B] *Utilizing observations for definitions*: In the second step, the situations are utilized to make and to compile certain *observations* and *definitions*. By ''observations'' Steiner means statements, which allow for defining concepts that are useful in understanding certain situations (examples). For instance, many observations can be taken directly from Example 1 and 2 of above:

(i) If a subset $C \subseteq N$ is a *winning coalition*, then so are all supersets.
(ii) There are *minimal winning coalitions*.
(iii) A complementary coalition of a winning coalition is a *losing coalition*.
(iv) There are *blocking coalitions* which are neither winning nor losing coalitions.
(v) In any voting set, there exists at least one *winning coalition*.
(vi) There are *powerless players* (dummies) and *dictators*.

The observations that can be taken from the pupils' intuitive understanding allow for elementary mathematical definitions such as:

Definition 1: A subset $C \subseteq N$ is called a *minimal winning coalition* if and only if no proper subset $C' \subset C$ is a winning coalition.

As Steiner (1968) remarked, to ''clarify the meaning of *mathematization* it is decisive that the students recognize that by using a mathematical approach, generally only certain sectors and only a part of the aspects of a real situation can be covered'' (p. 187).

[C] *Logical penetration*: The third step of mathematization has been called ‘‘logical penetration’’, which provides the ‘‘analysis of positions of power’’ (Steiner 1968, p. 188). For instance one can infer from the above observation (v) and (vi) that if C is a winning coalition, its complement $\bar{C}$ is a losing coalition.

[D] *Theorems*: Based on the logical penetration, the formulation of theorems becomes possible. Steiner introduced theorems such as the following in senior high school classrooms, which can be derived by simple logical and set-theoretical methods:

Theorem 1: *A member $a_i \in N$ is called a dictator in N if and only if $\{a_i\}$ is a minimal winning coalition.*

Theorem 2: *There exists no blocking coalition, for which all members are powerless (‘‘dummies’’).*

[E] *Axiomatization*: Most notably, and keeping in line with the mathematics of the 1960s and 1970s, axiomatization was the goal and most innovative part of the process (Steiner 1988, p. 199). The aim was to provide a set of consistent, independent axioms that provide a deductive approach and to dissipate potential inconsistencies that may rise from the introduction of observations based on voting bodies and from definitions that emerge from majority quotients (see Steiner 1968, pp. 192, 1988, pp. 206).

The axiomatization can result in a more abstracted mathematical notation and provide a mathematical *structure*. Thus, for instance, a vote distribution can be defined as mapping of a set ‘‘A’’ into natural numbers including zero such as $v : N \rightarrow \mathbb{N}_0$. Here the mapping v represents the voting power the player has at start-up. Yet, one can argue whether a pure set-theoretic approach is sufficient to deduce all knowledge obtained in the above steps [A] to [D]. Steiner (1988, p. 207) questioned whether a voting body of the UNSC type can be aptly represented as a pair $(N, \mathfrak{W})$, where N is a finite, non-empty set and $\mathfrak{W}$ is a non-empty set of subsets, called winning situations, such that the following holds:

I. $C \in \mathfrak{W}$ and $C \subseteq D \subseteq N \Rightarrow D \in \mathfrak{W}$
II. $C \in \mathfrak{W} \Rightarrow \bar{C} \notin \mathfrak{W}$

The students’ approach to these simple axioms consisted of checking whether all examples could be incorporated in these axioms. Steiner (1988, p. 206) noted this as ‘‘a breakthrough to *structural axiomatics*’’. We do not deal in detail with the experiences gained from axiom-based reasoning as opposed to the students’ expectations. We want to show, rather, how useful the elementary game theoretical approach can be to both research and analysis of political voting bodies. If we refer to Example 3, we can define a voting situation by a set of players N, an individual voting power function v and a utility function u, which is an indicator function demonstrating whether a coalition is a winning coalition [if $u(C) = 1$] or whether it is a losing coalition [if $u(C) = 0$].

Definition 2: A pair (N, v) with $N = \{1, \ldots, n\}$ and $u : \mathfrak{P}(N) \rightarrow \{0, 1\}$, for which $u(\emptyset) = 0, u(N) = 1$ and $u(C_1) \leqslant u(C_2)$ if $C_1 \subset C_2 \subset N$, is called a *simple, monotone n-person game*.

When applying the definition to power analysis of voting bodies in parliaments, we can link the axiomatic and set theoretic approach with the quorum and voting power and the *voting distribution*-based approach for n political parties. Let us analyze the voting situation of the Weimar Republic on March 3, 1933 (see Table 1). The *majority level* is $V = (v_1,\ldots,v_n)$. Thus, we can represent the situation by

$$(q, V) = (q, (v_1, v_2, \ldots, v_n)) \\ = (\underbrace{324}_{\text{Majority level}}, (\underbrace{288}_{\text{Grand player}}, \underbrace{120, 81, 74, 52}_{\text{Small players,}}, \underbrace{18, 5, 4, 5}_{\text{Dummies}}))$$

If one assumes that all votes of each party are unanimous, one can easily see that this distribution of votes is equivalent to a simplified game of $(q',V') = (4;3,1,1,1,1,0,0,0,0)$ because the same sets of players can build the majority in both games. Based on this process, one can represent the power of factions in one-chamber parliaments with s factions by structured sets.

This representation shows that the Weimar Parliament had a very straightforward structure. There were four powerless parties, i.e., the dummies. These dummies could neither win with any subset of dummies together with the grand player nor contribute to gaining majority level for any coalition, which had not been a winning coalition before. The *small players* could only win, i.e., contribute to a majority, if they would join the Nazi-Party NSDAP or if they would build a counter-coalition of all small players.

As Steiner noted (1984, p. 35), Shapley’s model of coalition games assigns a power of 19.74 to each of the five grand players whereas that of the small ones is only 0.22. Table 1 depicts the game theoretic power indices of the Shapley and another value, the Banzhaf value. Both values are developed through different assumptions on how winning- or grand-coalitions are formed (Banzhaf 1965; Laan and Brink 1998). We will not discuss these values too deeply here (see Osborne 2003; Holler and Illing 2003), but only want to mention that concepts such as the Shapley value include all possibilities of stepwise coalition formation and calculate the average increase of gains for all sequences by which one player transforms a coalition to

Table 1 The parliament of the Weimar Republic as an example of a one-chamber parliament (Statistical Yearbook of the German Empire, 1933)

Parties	Seats	Votes	Power indices	
			Shapley-Shubik value	Banzhaf-value
NSDAP	288	3	60	63.6
USPD	120	1	10	9.1
KPD	81	1	10	9.1
Zentrum	74	1	10	9.1
DNVP	52	1	10	9.1
BVP	18	0	0	0.0
Dt. Staatspartei	5	0	0	0.0
CSVD	4	0	0	0.0
Others	5	0	0	0.0
Sums	647	7	100	100.0

a winning coalition. Contrary to the Shapley value, which allocates different values (i.e., outcomes or utilities) to different coalitions, the idea of the Banzhaf value is based on the number of cases a party can ‘‘swing’’ a winning coalition to a non-winning coalition. The Shapley value can also be applied simply to measure the power of shareholders in hierarchical capital networks (Ostmann 1988), to allocate pollution costs or reduction (Petrosjan and Zaccour 2003) and to many other real life situations. However, we want to note that hierarchical parliaments usually correspond to games that do not always generate measures. In other words, the strengths of the players cannot be represented by real numbers. Today, intriguing questions on the measurability of strengths in simple n-person games or voting structures are still unanswered. In addition, questions as to the necessary and sufficient prerequisites for measurability and of the ordinal structures of strengths in simple n-person games or voting structures also remain unsettled.

4 Steiner's didactical principles of mathematization

Steiner utilized game theory, the theory of voting bodies and the above questions to demonstrate his concept of learning and teaching mathematics in courses with 11th and 12th grade students in Germany and in the US. When discussing Steiner's approach, we will distinguish between: (i) *epistemics and didactical principles of mathematization* in instruction (German: Mathematisierender Unterricht), (ii) the *process characteristics* of introducing and teaching mathematics, and (iii) *student activities* in these two stages.

(i) *Epistemological and didactical principles*: A basic component of Steiner's mathematization in game theory was that it ‘‘does not require specific mathematical tools except the naïve understanding of sets and set operations’’ (Steiner 1968, p. 181). The dynamics and the otherness of mathematics instruction came through the ‘‘evolution of mathematics from situations, building of mathematical models of given realities, mathematization, local organization, axiomatization’’ (Steiner 1968, p. 181). Clearly, a major intention of this activity was the *‘‘breakthrough of the axiomatic standpoint’’*; the ‘‘forming of a complete deductive representation of the theory which before has been developed inductively with only local deductive analysis.’’ (Steiner 1968, p. 182) Thus, mathematization in Steiner's sense was bridging the gap between constructivism and structuralism. This runs parallel to other positions, which claim that ‘‘that the axiomatic construction of the concept to teach is the neatest form of its presentation to the student’’ (Negrete 2000, p. 490).

(ii) *Process characteristics*: The process of teaching is characterized as an open, dialogue-driven, ‘‘*project type* ...’’ of a ‘‘quasi-empirical approach’’ (Steiner 1988, p. 199). In effect, it is a discourse in which mathematical concepts build ‘‘on an everyday-life background of *pre-knowledge and personal interests* ...In this process axiomatics and axiomatization’’ could be introduced as creative parts, in which ‘‘students can take the role of experts’’ and ‘‘*concerned people*’’ (Steiner 1988, p. 200). The goal of the teaching process was to allow the students to utilize the ‘‘*methodological* and epistemological experiences and insights’’ for ‘‘comparisons with and transfers to other fields in the mathematical and social dimensions. This contributes to a broader and more flexible understanding of mathematics as a human, i.e., both as cognitive and social activity.’’ (Steiner 1988, p. 200)

(iii) The cognitive and social dimension of the *student activities* were not placed in a ‘‘functional social constructivism perspective’’ (see Stauffacher et al. 2006) in the earlier writings of Steiner. By this approach, we mean that a student is constructing a representation of reality, which meets the intentions of that student. The student activities were conceived instead as an epistemological process, which included the ‘‘fundamental activities that belong to mathematization: *observation, description, idealization, logical analysis, axiomatization, application*’’ (Steiner 1968, p. 181.)

Taking (i) to (iii) together, the goal of mathematical education when mathematizing judgments, decisions, voting bodies, or other fields has been an *epistemic revolution on the small scale* (Kahane 1988). The student should participate or become an eyewitness in this process, which embodies a genetic mode of doing mathematics (Wittenberg 1963).

The first critical question is whether game theory presents a typical process of mathematization. Steiner acknowledged the specificities of game and decision analysis (Steiner 1976, pp. 221). However, he shared Neumann and Morgenstern's position who postulated: ''For economic and social problems the games fulfill—or should fulfill—the same function which various geometric—mathematical models have performed. Such models are theoretical constructs with a precise, exhaustive and not too complicated definition; and they must be similar to reality in those respects which are essential in the investigation at hand'' (Neumann and Morgenstern 1944, p. 32). Lucas and Billera, who compiled applications of mathematics to introduce mathematics teachers to new developments of applied mathematics, have doubted this position. In their introduction to a chapter on modeling coalition values they argued: ''The objectives ...are to present some non-typical illustrations of mathematical modeling that assume only elementary concepts ...intended for use by the instructor in more open-ended modeling courses, as well as in traditional courses'' (Lucas and Billera 1982, p. 97).

A second critical question is, whether or in what way the mathematization emerges from a *real world problem*. In a later paper on voting bodies, Steiner (1988) reflected on the concept of a dictator in voting bodies: ''We are building a mathematical model for a concrete situation. ...Epistemologically speaking we have replaced the original intuitive concept by a technical one, i.e., we explicated the intuitive idea by a construct within a theoretical framework which we are developing'' (Steiner 1988, p. 205). We agree with Steiner's approach to focus on a discourse and a sequence that explicates concepts which rely on the intuitive, i.e., ''direct accessible knowledge'' of a student (Scholz 1987; Hogarth 2001). However, in the aforementioned example of the UNSC and the Weimar Republic, the students usually began with an already-highly abstracted, conceptual and even numerical representation of the number of votes.

In this context, it makes sense to distinguish between *concrete, real systems* (i.e., real-world cases), *conceptual systems*, and *abstract, theoretical concepts*. *Conceptual* systems are built from natural language terms. *Abstract systems* are formal, mathematical systems presented in a formal, symbolic language or refer explicitly to natural or social science theories (Sneed 1971; Jahnke 1978). This differentiation could help to separate different levels of epistemics (i.e., voting experience in the family of a child, the intuitive concept of a voting procedure, previous lessons on the parliamentary process, dominant themes in TV and other media, etc.) and the mathematical model of voting bodies. As the history of game and decision theory shows (see Raiffa 2002; Aumann 2005), this separation between real world observations and game theoretical models has been inherent in game theory from its beginnings. Leading mathematical game theorists, such as Maschler (1962), performed experiments with subjects to relate abstracted mathematical models (Aumann and Maschler 1964) with real world behavior, albeit as an in vitro, laboratory type.

However, laboratory observations are, of course, part of a constrained real world; the reality of political voting bodies would require a broader methodology. Starting from real world problems would require a rather more extended case study approach (see Scholz and Tietje 2002), which must include the concrete, historical contexts of the parliaments, e.g., in Germany around 1923/1933. We only want to mention that the case study approach enables mathematical modeling of the principles that underlie, for instance, the voting dynamics of parliaments and allows us to then make inferences and generalities.

5 Perspectives: from epistemology via cognitive, motivational and situational constraints of learning mathematics to real world decision making

In the following, we discuss different perspectives from which game and decision theoretic issues as part of mathematics education can be considered.

5.1 Distinguishing different types of knowledge

Steiner (1968, 1988) demonstrated that game and decision theoretic modeling can be dealt with on different epistemological levels. In the context of secondary high school education, the didactic discourse set off from a prepared, but not completely well-defined, conceptual or even abstract situation such as the proportion of votes that the factions in the parliament of the Weimar Republic had. We pointed out earlier that such situations are presented in a conceptual if not semi-abstract way. Note that this semi-abstract starting point is also characteristic for other game theoretical subjects. Take, for example, the study of variants of 2×2 conflict games (e.g., the Prisoner's Dilemma Game, Luce and Raiffa 1989). Here, the essence of the conflict structure is presented by a story and data on the utilities resulting from the choices of the players. In general, a specific situation of (real) players is not given.

Appropriately prepared game theoretic situations, such as simple *n*-person games or the example of the Weimar Republic, allow for competing definitions of power, normative solutions, etc. depending on what process dynamics or concepts of rationality are postulated in coalition making or in the finding of a solution (see the remarks on the Shapley and the Banzhaf value above). Here, different concepts of justice, models of rationality, ideas about the process of coalition formation, etc. can play a role. Thus, the definitions and not merely the theorems also become an interesting part of mathematics. Moreover, in the presented didactical demonstrations of mathematization, the student encounters simple axioms.

We do not deal with the question of whether an average senior high school student is able to understand the nature of a proof (Schoenfield 2000). However, based on a simple, intuitive understanding of sets and functions, the student can certainly participate and contribute to the processes of mathematization. Without a doubt, this can also be achieved in the domains of the calculus, geometry etc. However, one can suspect that the relation between the knowledge represented in the mathematical definitions or proof and the knowledge represented in the individual, intuitive, everyday-life experience-based observations differ from other (physical, technological etc.) fields of mathematical application. This has been the focal point of Steiner's interest; he linked the students' intuitive knowledge (i.e., the *observations* he could make) with mathematical game theory (e.g., *definitions*). Steiner was interested in the nature of knowledge regarding voting situations, its presuppositions, foundations, and validity. Case in point, Steiner (1988) declared, ''Epistemologically speaking we have replaced the original intuitive concept ...by a construct within a theoretical framework...'', (p. 205).

5.2 An understanding of learning also requires a psychological view

Steiner did not make a real reference to cognitive psychology or psychological theories on the subject of teaching based on voting bodies. Psychological theories can stress different types of cognitive operations, representations and modes of thought (see e.g. Neisser 1967; Simon 1982; Resnick and Ford 1981; Scholz 1987; Skemp 1987). A psychological approach, for instance, would include discussion on the students' previous knowledge of voting procedures in parliaments, concepts of fairness, perspectives on the roles and power of minorities etc. Perhaps one can state that Steiner's approach showed some similarities to the approach of Piaget's conception of cognitive development (1973), which also focused on logical and relational aspects of tasks and operations. One can state that the way knowledge is processed by the individual student, i.e., a psychological model of the students learning, was not the focus of Steiner's interest. Thus, epistemics, which contrary to epistemology also incorporates the interdisciplinary study of the way knowledge is processed, including linguistics, psychology, logic and philosophy (cf. Collins and Ferguson 1993; Shaffer 2006) was not the center of Steiner's concern.

5.3 A well-chosen learning situation matters

From a *didactical view*, eliciting students' *motivation through appropriate situations is important*; the learning situation must meet the interests of the student. One can certainly argue that the social and cultural aspects of learning consists of treating mathematics as both an individual, constructive activity and as a communal, social practice (Olivier 1999). One can also discern that this can be met brilliantly with game and decision theoretical topics in a *project-like instructional setting* (Frey 1982). Well-chosen, semi-abstract game and decision theoretic situations allow for dealing with the ''dualism created between mathematics in students' heads and mathematics in their environment'' (Cobb et al. 1992). In this manner, learning becomes a social process in which students gain knowledge from each other (and from the teacher) through discussion, communication and the sharing of ideas. When actively comparing different arguments on solutions and reflecting on their own thinking and when being coached by a teacher, a group of students can negotiate a shared meaning. Clearly, this approach offers a way to go beyond the traditional tripartite scheme of the teacher, the student, and mathematics that underlies many theories on mathematics education. We should mention that we do not deal with the prerequisites on the teacher's side here. Neither game theory nor the open teaching method nor further key qualifications required are part of the standard curriculum. We think that a special in-service training should be considered as a prerequisite. This training should stress the role of enhancing key qualifications such as communication and problem structuring, but should also stress training for formulating research questions (Scholz et al. 2004). The pronounced interrelation to research has been shown above.

5.4 The game theoretic situation must be of interest for the student

Notwithstanding, not all well-prepared teaching situations are accepted by the students. Teaching requires a learning contract (Brousseau 1997). This learning contract must be primarily based on *intrinsic* motivation, which comes from the *internal* satisfaction pupils receive from solving problems and not from *extrinsic* motivation, i.e., the *external* satisfaction, for instance, from obtaining praise from the teacher. Further, both ''teacher and students must believe that mathematics is not a finished formal body of knowledge'' (Olivier 1999, p. 30). As Steiner has strongly suggested, a discourse-oriented process in an appropriately introduced learning arrangement motivates the students. Yet, the question remains: under which conditions are the students actually developing this intrinsic motivation in a game theoretical situation? From a socio-cultural constructivist perspective (Bruner 1966, 1990), one can argue that students are interested in jointly constructing a political, cultural and social context and reality (Berger and Luckmann 1966). Nevertheless, mental constructions are tied to certain contexts and purposes (Brunswik 1952). In the context of this paper, this means that from the students' perspective, motivation arises from what they consider to be purposeful; to state it more simply, questions are posed which can motivate the student. When looking back at the featured example of the Weimar Republic, it seems apparent that the success of Steiner's instruction in the late 1960s also relied on the fact that the students were intrinsically motivated to reflect on and understand the origins of the Nazi Republic and World War II. Today, the problem of voting power in parliamentarian systems is certainly still of interest for some students. Yet, for others, the idea might be too general. However, this has yet to be investigated in practice or through empirical studies.

5.5 From mathematical game theory to real world decision-making

Mathematizable game and decision theoretic situations in the classroom confront the question of which abilities are or could be promoted to use the practiced mathematics in real-world situations. Traditionally, not social situations but rather technology ''and technology-intensive mathematics curricula are catalysts for the mathematics education reform movement'' (Heid 1997). Game and decision theory, however, allow one to bridge mathematics and society, seeing as it deals with social and behavioral situations. A critical question is whether and in what way the instruction on these topics could help one to deal more effectively *with social situations outside of the original context of learning*. Clearly, from an instructional and motivational perspective, a vital issue is that the students should bring in their own experience or anticipate situations of social reality, which might be of interest for them. This can be accomplished by arranging settings in which students learn ''together in small groups on problems that might reasonably occur in the normal lives of the students and their families'' (Lesh 1985, 439).

Thus, the above presented view and the examples of voting bodies suggest that game and decision theory allow for going beyond the ''halo effect'', in which a real world situation from the student's world is only used for having a positive impact on the mathematics that follows (Pierce and Stacey 2006). A critical question is whether one should strive to implement game and decision theory in a similar role in other fields such as *statistical literacy* (Ben-Zvi and Garfield 2004). Here, Straf stresses to ''permeate the mathematics curricula at all elementary and secondary levels, and all children should understand variability and uncertainty, how to make sense from data, and the elements involved in making decisions'' (Straf 2003, 461). This author is convinced that game and decision theory (and related fields such as risk literacy, Zint 2001) have a similar potential. The issue is that game theory is the basic tool of *situational literacy*. By the term situational literacy, we delineate the competence of analyzing and understanding the *type or character of conflict* of a situation (i.e., whether one is facing a malignant or benign situation, Scholz and Tietje 2002, p. 203), and whether the player's *aspiration toward the outcomes* can be fulfilled if we reflect on their power and investments. Finally, questions in which *solutions* are considered as right, fair, just, stable, adequate, justifiable, acceptable, standard, normal etc. are of interest. In addition, an analysis of which concepts of rationality (see above) are coupled to different solutions can be of interest.

Presumably, *situational literacy* is best developed in a dynamic, discourse-oriented manner. ''It follows that one should undertake very early to teach children the practice of 'situations of rational validation'. These are the situations where two players cooperate dialectically with the goal of establishing or rejecting the truth of an assertion. They cooperate, but without concessions, the one proposing, the other opposing him whenever he sees the need, until he arrives at the point of sincerely accepting the evidence. But what is the type of situation that can require and permit the development of different axioms and theorems of logic and make the student conscious of them?'' (Brousseau 2004) It is most remarkable that this statement on searching for proper teaching processes of mathematical reasoning and verification utilizes game theoretical language such as ''players'', ''cooperate'', ''opposing'' and ''concession'' (see also Brousseau 1997). When looking back to the examples utilized by Steiner (1968, 1988), the answer to his question could be that the imparted ''non-deductive'', ''constructivistic'', ''quasi-empirical approach'' game and decision theoretic situations, which provide access to students' personal interests, previous knowledge and everyday-life background, would offer an excellent choice of situation and of a forum for learning.

6 Conclusions

Game and decision theory allows for a discursive, dynamic process of mathematical modeling and mathematization. Project-type instruction can originate in *situations* from the political world, from environmental disputes on resources, or from everyday situations in which different groups in a class vote for certain leisure activities. Hans-Georg Steiner (1966, 1968, 1988) has been a pioneer in utilizing game theory as a subject in secondary high school. In a series of papers, he utilized students' *observations* for generating *definitions*. Based on elementary, intuitive understanding of sets and operations, a *logical penetration, proofs of theorems* and even *axiomatization* was achieved. The students could participate and deal with different types of epistemology and could even go so far as to touch on questions from game and decision theoretic research.

Mathematics education with game and decision can link the social and the mathematical world when referring to simple, everyday-life intuitions in project-like instruction. Critical question in this context are, however, whether students' interest are invoked and under which conditions successful didactical situations and contracts are established. As Steiner demonstrated, game theory has the potential to go ''beyond the halo effect'' in which real world situations are simply used as an isolated icebreaker. Game and decision theory allows for presenting real world situations, which are structured and visible in the mathematics that follows. Game theory further incorporates many problems in which questions of fairness, rationality or adequacy of solutions arise. In addition, questions may also arise in the troublesome decision-making processes of dealing with malignant and benign situations. For many students, dealing with these issues and gaining *situational literacy* is intriguing, motivating and encouraging, even if they do not have a particular interest in mathematics. We would like to suggest that the potentials of game and decision theory should be discussed, investigated and explored.

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