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Small Mass Implies Uniqueness of Gibbs States of a Quantum Crystal

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Abstract: A model of interacting quantum particles performing one-dimensional anharmonic oscillations around their equilibrium positions which form a lattice $\mathbb{Z}^d$ is considered. For this model, it is proved that the set of tempered Euclidean Gibbs measures is a singleton provided the particle mass is less than a certain bound m_*, which is independent of the temperature β^{-1}. This settles a problem that was open for a long time and is an essential improvement of a similar result proved before by the same authors [5], where the bound m_* depended on β in such a way that $m_*(\beta) \to 0$ as $\beta \to +\infty$.

Contents

1. The Model and the Main Result

We consider the following model of a quantum crystal. To each point of the lattice $l \in \mathbb{Z}^d$ there is attached a quantum particle of mass m performing polarized (one-dimensional) oscillations in the crystalline field around the equilibrium position at l and described by its momentum operator $\mathfrak{p}_l$ and displacement operator $\mathfrak{q}_l$. The particles interact via a nearest neighbor ferromagnetic potential. The heuristic Hamiltonian of this model is of the following form:

$$H = \sum_{l} \left[\frac{1}{2m} \mathfrak{p}_l^2 + U(\mathfrak{q}_l) \right] + \frac{J}{4} \sum_{\text{nn: } l,l'} (\mathfrak{q}_l - \mathfrak{q}_{l'})^2 , \quad J > 0. \tag{1.1}$$

Here the sums run through the lattice $\mathbb{Z}^d$ and "nn" means that the sum is taken over all pairs l, l' satisfying the condition $|l - l'| = 1$. The potential energy U in the crystalline field is supposed to be a smooth even function $U : \mathbb{R} \to \mathbb{R}$, which satisfies a stability condition of the following type:

$$U(t) \geq A + Bt^2, \quad t \in \mathbb{R},$$

with certain $A \in \mathbb{R}$ and $B > 0$. Similar models have been studied for many years as providing quite realistic description of a crystalline substance undergoing structural phase transitions [19, 34, 38]. They (and their simplified versions) are also used as a base of models describing strong electron-electron correlations caused by the interaction of electrons with vibrating ions [21, 35, 36].

If the potential U has a double well shape, the system may undergo a phase transition [15, 24] connected with the appearance of macroscopic displacements of particles (see also [20, 30] where a particular case of U was studied). This phase transition occurs for $d \geq 3$ and large enough values of the inverse temperature β and of the particle mass m. The large mass limit of this model gives a model of interacting classical particles moving in the field U (see [4] and Sect. 3 in [6]), which certainly undergoes a phase transition. Hence one may say that a phase transition occurs if the system is close to its classical limit.

Starting from the pioneering paper [33] many efforts were made to show that "the more quantum is the model, the less possible is a phase transition". The first fully rigorous proof of the suppression of the long range order in models of this type was done in [40]. This effect was also demonstrated in certain exactly solvable models [31, 39]. In [3] (see also [25] for the case of multi-dimensional oscillations) it was shown that not only the long range order but any critical anomaly is suppressed if the model is strongly quantum. The latter occurs in particular if the particle mass is small enough. In the present paper we get the strongest result of this type, which settles the above problem[1]. Namely, for a class of potentials U, we show that the Euclidean Gibbs state of the model is unique if the particle mass belongs to the interval $(0, m_*)$, where the bound m_* depends solely on the parameters of the system Hamiltonian and is independent of the temperature. This effect may be called *a quantum stabilization* of the crystal since the corresponding condition may be written in a form similar to the stability condition for harmonic oscillators (see below).

Now let us make more precise the model, the methods and the result mentioned above. The potential U is assumed to be of the form

$$U(t) = b_1 t^2 + b_2 t^4 + \cdots + b_r t^{2r}, \quad b_1 \in \mathbb{R}, \quad b_s \geq 0, \quad b_r > 0, \quad r \geq 2. \tag{1.2}$$

The momentum $\mathfrak{p}_l$ and displacement $\mathfrak{q}_l$ are defined as unbounded operators on a dense subset (e.g., on $C_0^\infty(\mathbb{R})$) of the complex Hilbert space $\mathcal{H}_l = L^2(\mathbb{R})$. For quantum models with finite dimensional phase spaces $\mathcal{H}_l$, Gibbs states are constructed as positive normalized functionals on von Neumann algebras of observables (see e.g., [18]). For the model considered, the usual way of constructing Gibbs states may lead (and leads in fact, see e.g., the discussion in [6] and [23] Chapter IV, pp. 169, 170) to a number

[1] This result was announced in [7].

of difficulties. In this paper we use the approach, initiated in [2], in which the Gibbs states are constructed as probability measures with infinite dimensional spin spaces. This enables us to apply the technique of conditional probabilities (see e.g., [22]) and to define Euclidean Gibbs states as solutions of the Dobrushin-Lanford-Ruelle (DLR) equation. A full exposition of the Euclidean approach as applied to the model (1.1) and an extended related bibliography may be found in the review article [6].

Like for the other models with unbounded spins, the set of all Euclidean Gibbs states $\mathcal{G}_\beta$ of our model existing at a given β may contain elements of no physical relevance. In order to exclude them certain conditions restricting the support of these measures are imposed. The measures which satisfy such conditions and solve the DLR equation are called *tempered Euclidean Gibbs measures*. The set of all such measures will be denoted by $\mathcal{G}^{\mathrm{t}}_\beta$. Since the mentioned restrictions may be different, there are different kinds of tempered Gibbs measures. We are not going to discuss this aspect of the Euclidean approach and just mention that the restrictions used in this paper to define $\mathcal{G}^{\mathrm{t}}_\beta$ are the weakest ones. Hence our set $\mathcal{G}^{\mathrm{t}}_\beta$ includes all tempered Gibbs measures considered so far (for more details see e.g., [12, 13, 28] for the quantum case, and [16, 26, 32] for the classical case).

One of the possible ways to study Euclidean Gibbs states is the method of cluster expansions applied in [27] where, for small values of the mass, these expansions were shown to converge uniformly with respect to β. As a consequence, the existence of a Gibbs state was proved and its certain properties were described. However, such a convergence does not imply uniqueness because it is impossible to obtain it uniformly with respect to boundary conditions.

For the model considered in this article, uniqueness of tempered Euclidean Gibbs measures (with a more restrictive condition on the supporting sets) first was proven under conditions, which did not involve m [12, 13]. Later, in [5] the uniqueness was proven for $m \in (0, m_*(\beta))$ with $m_*(\beta)$ tending to zero as $\beta \to +\infty$. In this paper we remove the β-dependence of the bound m_* and prove the following result.

Main Theorem. *There exists $m_* > 0$ such that, for all $m \in (0, m_*)$ and all $\beta > 0$,*

$$|\mathcal{G}^{\mathrm{t}}_\beta| = 1.$$

The paper is organized as follows. In Sect. 2 we describe the main aspects of the Euclidean approach and give all necessary definitions. In Sect. 3 we give the proof of the above theorem, which is performed in four steps: (i) it is shown that the uniqueness holds provided all tempered Euclidean Gibbs measures have coinciding first local moments, which occurs if for every such a measure, its Duhamel function has an exponential decay; (ii) a uniform bound for all these Duhamel functions is proved; (iii) it is proved that this bound has an exponential decay if a certain condition is satisfied; (iv) it is shown that this condition is satisfied if the mass m is less than some m_*, which is independent of β. The corresponding statements are proved in Sect. 4. Consequences of our result particularly relevant for physics have been published in [8].

2. Euclidean Gibbs States

The heuristic Hamiltonian (1.1) cannot be defined directly as a mathematical object and is "represented" by local Hamiltonians H_Λ, which are essentially self-adjoint and lower bounded operators in the complex Hilbert space $\mathcal{H}_\Lambda = L^2(\mathbb{R}^{|\Lambda|})$ ($|\cdot|$ stands for cardinality). They are indexed by finite subsets $\Lambda \subset \mathbb{Z}^d$. In standard situations it is enough to take these subsets as boxes. The local Hamiltonian of the subsystem in such a box Λ is of the following form:

$$H_\Lambda = -\frac{J}{2} \sum_{\text{nn: } l,l' \in \Lambda} \mathfrak{q}_l \mathfrak{q}_{l'} + \sum_{l \in \Lambda} H_l^{(0)}, \quad J > 0. \tag{2.1}$$

Here the one-particle Hamiltonian is

$$H_l^{(0)} = \frac{1}{2m} \mathfrak{p}_l^2 + \frac{1}{2} \mathfrak{q}_l^2 + V(\mathfrak{q}_l^2), \tag{2.2}$$

$$V(t) = (b_1 + dJ - 1/2)t + b_2 t^2 + \cdots + b_r t^r, \tag{2.3}$$

where b_j, $j = 1, 2, \ldots, r$ are the same as in (1.2). It defines a local Gibbs state

$$\gamma_{\beta,\Lambda}(A) = \frac{\text{trace}\,(A \exp(-\beta H_\Lambda))}{\text{trace}\,(\exp(-\beta H_\Lambda))}, \tag{2.4}$$

where β is the inverse temperature and the observable A is an element of the C^*-algebra of bounded linear operators on $\mathcal{H}_\Lambda$. These states may be completely determined by means of the corresponding Matsubara functions, which for observables $A_1, \ldots, A_k$ and $0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_k \le \beta$ are

$$\begin{aligned} \Gamma^{\beta,\Lambda}_{A_1,\ldots,A_k}(\tau_1, \ldots, \tau_k) = \gamma_{\beta,\Lambda}\, \{ & A_1 \exp[-(\tau_2 - \tau_1) H_\Lambda] \\ & \times A_2 \exp[-(\tau_3 - \tau_2) H_\Lambda] \\ & \times \ldots A_k \exp[(\tau_k - \tau_1) H_\Lambda]\}. \end{aligned} \tag{2.5}$$

For the remaining values of $(\tau_1, \tau_2, \ldots, \tau_k) \in [0, \beta]^k$, the Matsubara functions are defined as follows. Given a tuple $(\tau_1, \tau_2, \ldots, \tau_k)$, one takes the permutation $\sigma \in \Sigma_k$ such that $\tau_{\sigma(1)} \le \tau_{\sigma(2)} \le \cdots \le \tau_{\sigma(k)}$. Then one sets

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_k}(\tau_1, \ldots, \tau_k) = \Gamma^{\beta,\Lambda}_{A_{\sigma(1)},\ldots,A_{\sigma(k)}}(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(k)}),$$

where the latter function is defined by (2.5).

In constructing the states (2.4) a special role is played by multiplication operators. For a bounded continuous function $A : \mathbb{R}^{|\Lambda|} \to \mathbb{C}$, the corresponding multiplication operator is defined as follows:

$$(A\psi)(x) = A(x)\psi(x), \quad \psi \in \mathcal{H}_\Lambda.$$

It is known (see e.g., [6], Prop. 2.1) that, for a given Λ, the Matsubara functions constructed only with these multiplication operators already determine completely the state $\gamma_{\beta,\Lambda}$. The essence of the Euclidean approach lies in the fact that such Matsubara functions may be written as moments of probability measures (see [2] and [6] for a detailed description). To construct them we start by introducing the basic measure χ_β – a symmetric Gaussian measure defined on the Banach space of continuous periodic paths

$$\mathcal{C}_\beta = \{\omega \in C([0, \beta]) \mid \omega(0) = \omega(\beta)\}. \tag{2.6}$$

It is uniquely determined by its covariance

$$\int_{\mathcal{C}_\beta} \omega(\tau)\omega(\tau')\chi_\beta(\mathrm{d}x) = \frac{1}{2\sqrt{m}} \cdot \frac{\exp\left((\beta - |\tau - \tau'|)/\sqrt{m}\right) + \exp\left(|\tau - \tau'|/\sqrt{m}\right)}{\exp\left(\beta/\sqrt{m}\right) - 1}, \tag{2.7}$$

where $\tau, \tau' \in [0, \beta]$. A full account of the properties of χ_β is given in Sect. 2 of [6].

Given a box Λ, we set

$$\Omega_{\beta,\Lambda}=\{\omega_\Lambda = (\omega_l)_{l\in\Lambda} \mid \omega_l \in \mathcal{C}_\beta, \quad l \in \Lambda\}. \tag{2.8}$$

This set equipped with the supremum norm becomes a Banach space. By $\mathcal{B}_{\beta,\Lambda}$ we denote the σ-algebra of its Borel subsets. Furthermore, we introduce the following measure on $\Omega_{\beta,\Lambda}$:

$$\chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda) = \bigotimes_{l\in\Lambda} \chi_\beta(\mathrm{d}\omega_l). \tag{2.9}$$

By means of it, one defines *the local Euclidean Gibbs measure*

$$\begin{aligned} \nu_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda) &= \frac{1}{Z_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(\omega_\Lambda)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \\ Z_{\beta,\Lambda} &= \int_{\Omega_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(\omega_\Lambda)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} E_{\beta,\Lambda}(\omega_\Lambda) = &-\frac{J}{2} \sum_{\text{nn: } l,l'\in\Lambda} \int_0^\beta \omega_l(\tau)\omega_{l'}(\tau)\mathrm{d}\tau \\ &+ \sum_{l\in\Lambda} \int_0^\beta V([\omega_l(\tau)]^2)\mathrm{d}\tau. \end{aligned} \tag{2.11}$$

The measure $\nu_{\beta,\Lambda}$ determines the state $\gamma_{\beta,\Lambda}$ due to the following representation of the functions (2.5):

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1, \ldots, \tau_n) = \int_{\Omega_{\beta,\Lambda}} A_1(\omega(\tau_1)) \ldots A_n(\omega(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \tag{2.12}$$

which holds for all $n \in \mathbb{N}$ and all bounded multiplication operators $A_1, \ldots, A_n$.

Furthermore, we define

$$\Omega_\beta = \mathcal{C}_\beta^{\mathbb{Z}^d} = \{\omega = (\omega_l)_{l\in\mathbb{Z}^d} \mid \omega_l \in \mathcal{C}_\beta\}. \tag{2.13}$$

This set is endowed with the product topology and the corresponding Borel σ-algebra $\mathcal{B}_\beta$. The set of *tempered configurations* is

$$\Omega^{\mathrm{t}}_\beta = \{\omega \in \Omega_\beta \mid \forall\delta > 0 : \sum_{l\in\mathbb{Z}^d} e^{-\delta|l|}\|\omega_l\|_{L^2[0,\beta]} < \infty\}, \tag{2.14}$$

where $|\cdot|$ stands for the Euclidean distance.

Given two configurations $\xi, \eta \in \Omega$ and a box Λ, by $\xi_\Lambda \times \eta_{\Lambda^c}$ we denote the configuration whose components labelled by $l \in \Lambda$ (resp. $l \in \Lambda^c = \mathbb{Z}^d \setminus \Lambda$) coincide with the corresponding components of ξ (resp. η). Then, for any finite box Λ, each $\omega_\Lambda \in \Omega_{\beta,\Lambda}$ can be associated with the element $\omega = \omega_\Lambda \times 0_{\Lambda^c}$ of Ω_β, where 0_{Λ^c} is the zero configuration. This determines an embedding $\Omega_{\beta,\Lambda} \to \Omega_\beta$, thus we have $\mathcal{B}_{\beta,\Lambda} \subset \mathcal{B}_\beta$.

Along with (2.10) we introduce *conditional local Gibbs measures*. Given $\xi \in \Omega_\beta$, we set

$$E_{\beta,\Lambda}(\omega_\Lambda|\xi) = E_{\beta,\Lambda}(\omega_\Lambda) - J \sum_{\text{nn: } l\in\Lambda,\ l'\in\Lambda^c} \int_0^\beta \omega_l(\tau)\xi_{l'}(\tau)\mathrm{d}\tau, \tag{2.15}$$

and

$$\nu_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda|\xi) = \frac{1}{Z_{\beta,\Lambda}(\xi)} \exp\left(-E_{\beta,\Lambda}(\omega_\Lambda|\xi)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda),$$
$$Z_{\beta,\Lambda}(\xi) = \int_{\Omega_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(\omega_\Lambda|\xi)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda). \tag{2.16}$$

For every fixed $\xi \in \Omega_\beta$, $\nu_{\beta,\Lambda}$ is a probability measure on $\mathcal{B}_{\beta,\Lambda}$.

Together with the boundary conditions defined by configurations outside of the box Λ we will use also *periodic* boundary conditions. Let $T(\Lambda)$ stand for the torus which one obtains by identifying the opposite walls of the box Λ. Then we set

$$\begin{aligned} E_{\beta,\Lambda}(\omega_\Lambda|p) = -\frac{J}{2} \sum_{\text{nn: } l,l'\in T(\Lambda)} \int_0^\beta \omega_l(\tau)\omega_{l'}(\tau)\mathrm{d}\tau \\ + \sum_{l\in\Lambda} \int_0^\beta V([\omega(\tau)]^2)\mathrm{d}\tau, \end{aligned} \tag{2.17}$$

and

$$\nu_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda|p) = \frac{1}{Z_{\beta,\Lambda}(p)} \exp\left(-E_{\beta,\Lambda}(\omega_\Lambda|p)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda). \tag{2.18}$$

In the sequel, by $\nu_{\beta,\Lambda}(\cdot|b)$ we will denote the local Euclidean Gibbs measure with the boundary condition b which is either the one defined by a configuration $\xi \in \Omega_\beta$ (including the zero configuration) or the periodic boundary condition. In these cases we write $b = \xi$ or $b = p$ respectively.

Given $B \subset \Omega_\beta$ and $\omega \in \Omega_\beta$, let

$$\mathbf{1}_B(\omega) = \begin{cases} 1 & \text{if } \omega \in B; \\ 0 & \text{otherwise} \end{cases}.$$

Then for a box Λ and $B \in \mathcal{B}_\beta$, we set

$$\pi_{\beta,\Lambda}(B|\xi) = \int_{\Omega_{\beta,\Lambda}} \mathbf{1}_B(\omega_\Lambda \times \xi_{\Lambda^c})\nu_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda|\xi). \tag{2.19}$$

Definition 2.1. *A probability measure μ on Ω_β is said to be a Euclidean Gibbs measure at temperature β^{-1} if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equation*

$$\int_{\Omega_\beta} \pi_{\beta,\Lambda}(B|\omega)\mu(\mathrm{d}\omega) = \mu(B), \tag{2.20}$$

for all boxes Λ and all Borel subsets $B \subset \Omega_\beta$.

The set of all Euclidean Gibbs measures existing for given β will be denoted by $\mathcal{G}_\beta$. For the model considered, this set is nonempty [13, 14]. The set of *tempered Euclidean Gibbs measures* is

$$\mathcal{G}_\beta^{\mathrm{t}} \stackrel{\mathrm{def}}{=} \{\mu \in \mathcal{G}_\beta \mid \mu(\Omega_\beta^{\mathrm{t}}) = 1\}. \tag{2.21}$$

By [9 – 11] (see Theorem 1 of those papers), $|\mathcal{G}_\beta^{\mathrm{t}}| \neq \emptyset$. Our main theorem states that this set is a singleton provided $m \in (0, m_*)$.

Usually, tempered configurations are defined by more restrictive conditions than (2.14) (see e.g., [12, 13] for the quantum case, and [16, 26, 32] for the classical case). Therefore, the uniqueness stated above is the strongest result of this type. In fact, as it will be clear from the proof of our main theorem, for any temperature, one cannot expect uniqueness for Gibbs measures supported by larger sets than $\Omega_\beta^{\mathrm{t}}$ as in (2.14).

3. The Proof of the Main Theorem

The proof of our main theorem will be carried out in four steps. First, inspired by the pioneering paper [26], we use monotonicity arguments and a priori estimates for Euclidean Gibbs measures [9, 11] to show that the uniqueness may be a consequence of the fact that the infinite volume limits of the conditional local Gibbs measures coincide for all possible boundary conditions in the set of tempered configurations. Then we employ a zero boundary domination estimate to get rid of the boundary conditions, which finally yields a condition for the uniqueness. As the last step, we show that this condition is satisfied if the particle mass m belongs to the interval $(0, m_*)$, where the bound m_* depends on the parameters of the model only and does not depend on β. These steps are mainly implemented by Theorems 3.1–3.4 and Corollary 3.3 stated below in this section, the proofs of which will be given in the subsequent section.

Given a measure μ, for a μ-integrable function f, we write

$$\langle f \rangle_\mu = \int f \mathrm{d}\mu.$$

We denote by $\mathcal{L}$ an increasing sequence of boxes Λ, which exhausts the lattice $\mathbb{Z}^d$, i.e.,

$$\bigcup_{\mathcal{L}} \Lambda = \mathbb{Z}^d.$$

The infinite volume limit taken along such $\mathcal{L}$ will be denoted by $\Lambda \stackrel{\mathcal{L}}{\to} \mathbb{Z}^d$.

Step 1.

Theorem 3.1. *Suppose that, for every $l_0 \in \mathbb{Z}^d$ and $\tau_0 \in [0, \beta]$, for every sequence $\mathcal{L}$, such that l_0 belongs to each of its elements, and for any two configurations $\xi, \eta \in \Omega_\beta^{\mathrm{t}}$, one has the following convergence:*

$$\langle \omega_{l_0}(\tau_0) \rangle_{\nu_\Lambda(\cdot|\xi)} - \langle \omega_{l_0}(\tau_0) \rangle_{\nu_\Lambda(\cdot|\eta)} \longrightarrow 0, \tag{3.1}$$

as $\Lambda \xrightarrow{\mathcal{L}} \mathbb{Z}^d$. *Then* $|\mathcal{G}^{\rm t}_\beta| = 1$.

Given a box Λ, $l, l' \in \Lambda$, $\tau, \tau' \in [0, \beta]$ and a boundary condition b, we set

$$K^\Lambda_{ll'}(\tau, \tau'|b) = \langle \omega_l(\tau)\omega_{l'}(\tau') \rangle_{\nu_\Lambda(\cdot|b)} - \langle \omega_l(\tau) \rangle_{\nu_\Lambda(\cdot|b)} \langle \omega_{l'}(\tau') \rangle_{\nu_\Lambda(\cdot|b)}. \tag{3.2}$$

Let also

$$w_\Lambda(t) \stackrel{\rm def}{=} \langle \omega_{l_0}(\tau_0) \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta + t\zeta)}, \quad \zeta = \xi - \eta, \quad t \in [0, 1]. \tag{3.3}$$

Obviously, this function is differentiable for all $t \in \mathbb{R}$, hence

$$|\langle \omega_{l_0}(\tau_0) \rangle_{\nu_\Lambda(\cdot|\xi)} - \langle \omega_{l_0}(\tau_0) \rangle_{\nu_\Lambda(\cdot|\eta)}| \le \sup_{t\in[0,1]} |w'_\Lambda(t)|. \tag{3.4}$$

By (3.3) and (2.16), (2.15), the derivative $w'_\Lambda(t)$ is

$$w'_\Lambda(t) = J \sum_{\text{nn: } l\in\Lambda, l'\in\Lambda^c} \int_0^\beta K^\Lambda_{ll_0}(\tau, \tau_0|\eta + t\zeta)\zeta_{l'}(\tau){\rm d}\tau. \tag{3.5}$$

To estimate it we use the Schwarz inequality, which yields

$$|w'_\Lambda(t)| \le J \sum_{\text{nn: } l\in\Lambda, l'\in\Lambda^c} \left\{ \int_0^\beta \left[K^\Lambda_{ll_0}(\tau, \tau_0|\eta + t\zeta) \right]^2 {\rm d}\tau \right\}^{1/2} \|\zeta_{l'}\|_{L^2[0,\beta]}. \tag{3.6}$$

Suppose now that the function

$$T^\Lambda_{ll_0}(\tau_0|\xi) \stackrel{\rm def}{=} \left\{ \int_0^\beta \left[K^\Lambda_{ll_0}(\tau, \tau_0|\xi) \right]^2 {\rm d}\tau \right\}^{1/2}, \quad \xi \in \Omega^{\rm t}_\beta, \tag{3.7}$$

obeys the following estimate:

$$T^\Lambda_{ll_0}(\tau_0|\xi) \le C \exp\left(-\alpha|l - l_0|\right), \tag{3.8}$$

where the parameters $C > 0$ and $\alpha > 0$ do not depend on Λ, τ_0 and the boundary condition ξ. Then having in mind that the sum in (3.6) is taken under the condition $|l - l'| = 1$ and that $\zeta \in \Omega^{\rm t}_\beta$, one concludes that the right-hand side of (3.6) tends to zero as $\Lambda \xrightarrow{\mathcal{L}} \mathbb{Z}^d$ for any sequence of boxes $\mathcal{L}$.

Step 2. To prove (3.8) we first get rid of the boundary condition in (3.7)[2]. To this end we consider the model described by the local Hamiltonians $\widetilde{H}_\Lambda$ defined by (2.1) but with the following one-particle Hamiltonian:

$$\widetilde{H}^{(0)}_l = \frac{1}{2m}\mathfrak{p}_l^2 + \frac{1}{2}\mathfrak{q}_l^2 + \widetilde{V}(\mathfrak{q}_l^2), \tag{3.9}$$

[2] Here we apply a technique already used in [5]. Its detailed description may be found in Subsect. 7.2 of [6].

where

$$\widetilde{V}(t) = (b_1 + dJ - 1/2)t + 2^{-1}b_2t^2 + \cdots + 2^{1-r}b_rt^r, \tag{3.10}$$

and the parameters $m, b_1, \ldots, b_r$ are the same as in (1.2). Each such a Hamiltonian $\widetilde{H}_\Lambda$ defines local Gibbs measures $\tilde{\nu}_{\beta,\Lambda}(\cdot|b)$ corresponding to the boundary condition $b = p$ or $b = \xi \in \Omega_\beta$. Therefore, similarly to (3.2) we may set

$$\widetilde{K}^{\Lambda}_{ll'}(\tau,\tau'|b) = \langle\omega_l(\tau)\omega_{l'}(\tau')\rangle_{\tilde{\nu}_\Lambda(\cdot|b)} - \langle\omega_l(\tau)\rangle_{\tilde{\nu}_\Lambda(\cdot|b)}\langle\omega_{l'}(\tau')\rangle_{\tilde{\nu}_\Lambda(\cdot|b)}. \tag{3.11}$$

Theorem 3.2. *For any boxes Λ, Λ' such that $\Lambda' \supset \Lambda$, for all $l, l' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$, and for any $\xi \in \Omega_\beta$,*

$$0 \le K^{\Lambda}_{ll'}(\tau,\tau'|\xi) \le \widetilde{K}^{\Lambda}_{ll'}(\tau,\tau'|0) \le \widetilde{K}^{\Lambda}_{ll'}(\tau,\tau'|p); \tag{3.12}$$

$$K^{\Lambda}_{ll'}(\tau,\tau'|0) \le K^{\Lambda'}_{ll'}(\tau,\tau'|0). \tag{3.13}$$

Corollary 3.3. *There exists a constant $C_\beta > 0$, which depends on β only, such that for all boxes Λ, for all $l, l' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$, and for any $\xi \in \Omega_\beta$,*

$$0 \le K^{\Lambda}_{ll'}(\tau,\tau'|\xi) \le C^2_\beta. \tag{3.14}$$

Now by (3.12) and (3.14), one gets for (3.7),

$$T^{\Lambda}_{ll_0}(\tau_0|\xi) \le C_\beta\sqrt{D^{\Lambda}_{ll_0}(0)}, \tag{3.15}$$

which holds for all $\tau_0 \in [0, \beta]$ and $\xi \in \Omega_\beta$. Here we employed one of the two Duhamel functions

$$D^{\Lambda}_{ll_0}(b) \stackrel{\text{def}}{=} \int_0^\beta \widetilde{K}^{\Lambda}_{ll_0}(\tau,\tau_0|b)\mathrm{d}\tau, \quad b = p, 0. \tag{3.16}$$

Clearly these functions do not depend on τ_0.

Step 3. Set

$$\begin{aligned}
\varkappa = D^{\{l\}}_{ll}(0) &= \int_0^\beta\left\{\int_{\mathcal{C}_\beta}\omega_l(0)\omega_l(\tau)\tilde{\nu}_{\beta,\{l\}}(\mathrm{d}\omega_l)\right\}\mathrm{d}\tau \\
&= \frac{1}{\widetilde{Z}_l}\int_0^\beta\left\{\int_{\mathcal{C}_\beta}\omega_l(0)\omega_l(\tau)\exp\left(-\int_0^\beta\widetilde{V}([\omega_l(t)]^2)\mathrm{d}t\right)\chi_\beta(\mathrm{d}\omega_l)\right\}\mathrm{d}\tau \quad (3.17)\\
&= \frac{1}{\widetilde{Z}_l}\int_0^\beta \mathrm{trace}\left\{\mathrm{q}_l\exp\left(-\tau\widetilde{H}^{(0)}_l\right)\mathrm{q}_l\exp\left(-(\beta-\tau)\widetilde{H}^{(0)}_l\right)\right\}\mathrm{d}\tau, \quad (3.18)
\end{aligned}$$

where $\widetilde{H}^{(0)}_l$ is defined by (3.9) and

$$\widetilde{Z}_l = \int_{\mathcal{C}}\exp\left(-\int_0^\beta\widetilde{V}([\omega_l(t)]^2)\mathrm{d}t\right)\chi_\beta(\mathrm{d}\omega_l) = \mathrm{trace}\exp\left(-\beta\widetilde{H}^{(0)}_l\right).$$

Set also

$$I(q) = 2J\sum_{j=1}^d(1-\cos(q_j)), \quad q \in (-\pi,\pi]^d. \tag{3.19}$$

Theorem 3.4. *Let $\varkappa$, d and J obey the condition*

$$2dJ\varkappa < 1. \tag{3.20}$$

Then for any box Λ, for any $\beta > 0$, $l, l_0 \in \Lambda$, the following is true:

$$D_{ll_0}^{\Lambda}(0) \leq \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\exp(i(q, l - l_0))\mathrm{d}q}{\varkappa^{-1} - 2dJ + I(q)}. \tag{3.21}$$

Clearly, the right-hand side of (3.21) may be estimated by the right-hand side of (3.8) with proper C and α, hence the estimate (3.8) will hold provided (3.20) holds.

Step 4. By standard arguments, the spectrum of $\widetilde{H}_l^{(0)}$ consists of non-degenerated eigenvalues ϵ_s, $s \in \mathbb{N}$, such that $\epsilon_s \to +\infty$ as $s \to \infty$. We denote the corresponding eigenfunctions by ψ_s and set

$$\delta(m) \stackrel{\text{def}}{=} \inf\{\epsilon_{s+1} - \epsilon_s \ : \ s \in \mathbb{N}\}. \tag{3.22}$$

By means of ϵ_s, ψ_s, $s \in \mathbb{N}$, we rewrite (3.18) as follows:

$$\varkappa = \frac{1}{\widetilde{Z}_l} \sum_{s,s'=1,\ s\neq s'}^{+\infty} |Q_{ss'}|^2 \frac{(\epsilon_s - \epsilon_{s'})[e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s}]}{(\epsilon_s - \epsilon_{s'})^2},$$

where $Q_{ss'}$ stands for the matrix element $(\mathfrak{q}_l\psi_s, \psi_{s'})_{L^2(\mathbb{R})}$. Now we may estimate the denominator by means of (3.22), which yields

$$\begin{aligned} \varkappa &\leq \frac{1}{[\delta(m)]^2\widetilde{Z}_l} \sum_{s,s'=1}^{+\infty} |Q_{ss'}|^2(\epsilon_s - \epsilon_{s'})[e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s}] \\ &= \frac{1}{[\delta(m)]^2\widetilde{Z}_l} \text{trace}\left(\left[\mathfrak{q}_l, \left[\widetilde{H}_l^{(0)}, \mathfrak{q}_l\right]\right] e^{-\beta\widetilde{H}_l^{(0)}}\right) = \frac{1}{m[\delta(m)]^2}, \end{aligned} \tag{3.23}$$

where $[\cdot, \cdot]$ stands for commutator. In what follows, the uniqueness condition (3.20) now may be written as

$$m[\delta(m)]^2 > 2dJ. \tag{3.24}$$

For the harmonic oscillator, the parameter $m[\delta(m)]^2$ is nothing else but its rigidity and (3.24) is the stability condition for the system of such oscillators, interacting via the nearest neighbor potential. Therefore, the uniqueness condition (3.24) may be considered as a *stability-due-to-quantum-effects* condition and its left-hand side may be called *quantum rigidity*. As was proved in [3], for the potential (3.10), the parameter $m[\delta(m)]^2$ is a continuous function of m and $m[\delta(m)]^2 \sim Cm^{-(r-1)/(r+1)}$ as $m \to 0$ for a certain $C > 0$. Thus, one may find a positive m_*, which depends on d, J and b_j, $j = 1, \ldots, r$, such that the condition (3.24) will be satisfied for all $m \in (0, m_*)$. This completes the proof of our main theorem. $\square$

Discussion. The quantum rigidity $m[\delta(m)]^2$ introduced above may be made large either by substituting "heavy" particles by "light" ones or by increasing $\delta(m)$ at fixed m (recall that $\delta(m)$ also depends on several other parameters). Both effects were observed experimentally (see [1 and 38]) and are known as *the isotopic effect* (i.e., substitution of deuterons by protons on hydrogen bounds in the KDP-type ferroelectrics) and as *the stabilization-by-pressure effect* p. 188 in [17 and 37] (increasing $\delta(m)$ by applying hydrostatic pressure, which makes minima of the wells closer to each other and increases tunneling).

The main significant feature of the above proof is that it is based on the control of the first local moments only and does not use Dobrushin's contractivity technique, which constitutes the base of the technique used in [5, 12, 13]. This made it possible for us to reduce the uniqueness condition to (3.20) and then to apply the "quantum" arguments (*Step 4*), similar to those used in [3, 5]. In the latter work, being in the framework of Dobrushin's technique, we could use such arguments only partially, which resulted in the β-dependence of the bound m_* for the uniqueness interval $(0, m_*)$. In [12, 13] Dobrushin's contractivity technique and the logarithmic Sobolev inequality, applied directly to the Euclidean Gibbs measures, led to a uniqueness condition, which does not contain the mass m, hence it is valid also in the quasi-classical limit $m \to +\infty$, i.e., for the classical analog of the model considered (see [4] and Sect. 3 in [6]). On the other hand, the proofs of Theorems 3.1–3.4 and Corollary 3.3 are based on correlation inequalities and are independent of the value of the particle mass m. Hence these statements hold true also in the quasi-classical limit $m \to +\infty$ and it would make sense to obtain a bound for $\varkappa$, independent of the mass m, and to compare this with the results of [12, 13].

Let $\widetilde{U}$ be the polynomial (1.2) with the coefficients b_j, $j = 1, \ldots, r$ replaced by $2^{1-j} b_j$ (cf., (3.10)). According to [12, 13] we write it in the form

$$\widetilde{U}(t) = W(t) + \widetilde{U}_0(t), \quad t \in \mathbb{R}, \tag{3.25}$$

where W is a bounded and twice differentiable function on $\mathbb{R}$ such that $\widetilde{U}_0 = \widetilde{U} - W$ is strictly convex. Hence there exists $b^2 > 0$ such that, for all $t \in \mathbb{R}$,

$$\widetilde{U}_0''(t) \geq b^2. \tag{3.26}$$

Set also

$$f(\omega_l) = \frac{1}{\sqrt{\beta}} \int_0^\beta \omega_l(\tau) \mathrm{d}\tau, \tag{3.27}$$

which is a Lipschitz-continuous function $f : L^2[0, \beta] \to \mathbb{R}$ with the Lipschitz constant equal to one. Here l is the same as in (3.17). Then by (3.17), the parameter $\varkappa$ is nothing else but the variance of f taken with respect to the measure $\tilde{\nu}_{\beta,\{l\}}$. By means of the logarithmic Sobolev inequality, this variance may be estimated as follows (see [12], Eq. (4.17))

$$\varkappa = \mathrm{Var} f \leq \frac{e^{\beta\delta(W)}}{2dJ + 1 + b^2},$$

where

$$\delta(W) \stackrel{\mathrm{def}}{=} \sup_{\mathbb{R}} W - \inf_{\mathbb{R}} W.$$

In what follows, the condition (3.20) may be written

$$\frac{J e^{\beta\delta(W)}}{2dJ + 1 + b^2} < \frac{1}{2d}, \tag{3.28}$$

which is a version of the uniqueness condition obtained in [6], Theorem 4.1.

4. The Proof of Theorems 3.1–3.4 and Corollary 3.3

4.1. Theorem 3.1. In the sequel, *a local function* will mean the function $f : \Omega_\beta \to \mathbb{R}$ for which there exist $n \in \mathbb{N}, l_1, \ldots, l_n \in \mathbb{Z}^d, \tau_1, \ldots, \tau_n \in [0, \beta]$ and a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, such that

$$f(\omega) = \varphi(\omega_{l_1}(\tau_1), \ldots, \omega_{l_n}(\tau_n)), \quad \omega \in \Omega_\beta. \tag{4.1}$$

Local polynomials are those local functions for which φ are real polynomials. The set of local polynomials will be denoted by $\mathcal{P}$. In our analysis an important role is played by the following fact proved in [11].

Proposition 4.1. *For the model considered, the set $\mathcal{G}^{\mathrm{t}}_\beta$ is nonempty for all values of β and $m, d, J, a, b_2, \ldots, b_r$. For any $p \in \mathcal{P}$, there exists a constant $C(p)$ such that, for every $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$,*

$$\langle |p| \rangle_\mu \le C(p). \tag{4.2}$$

The set of *polynomially bounded continuous local functions* $\mathcal{F}^{\mathrm{pb}}$ consists of the local functions, for each of which: (a) the function φ in (4.1) is continuous; (b) there exists $p \in \mathcal{P}$ such that

$$|f(\omega)| \le |p(\omega)|, \quad \omega \in \Omega_\beta. \tag{4.3}$$

Given a box Λ, $\mathcal{F}^{\mathrm{pb}}_\Lambda$ will denote the set of all polynomially bounded continuous local functions such that the corresponding $l_1, \ldots, l_n$ belong to Λ. Clearly,

$$\mathcal{F}^{\mathrm{pb}} = \bigcup_{\Lambda \in \mathcal{L}} \mathcal{F}^{\mathrm{pb}}_\Lambda,$$

for any increasing sequence of boxes $\mathcal{L}$, which exhausts the lattice $\mathbb{Z}^d$.

Given $\alpha > 0$ and $t \in \mathbb{R}$, we set

$$\vartheta_\alpha(t) \stackrel{\mathrm{def}}{=} \begin{cases} t, & \text{if } |t| \le \alpha, \\ \alpha \operatorname{sgn}(t), & \text{otherwise,} \end{cases} \tag{4.4}$$

and Q_β to be the set of all rational elements of the interval $[0, \beta]$.

Let us introduce one more set of local functions on Ω_β. It consists of all functions, such that there exist $n \in \mathbb{N}, l_1, \ldots, l_n \in \mathbb{Z}^d, \tau_1, \ldots, \tau_n \in Q_\beta$, positive rational numbers $\alpha_1, \ldots, \alpha_n$, possibly coinciding, such that

$$f(\omega) = \vartheta_{\alpha_1}(\omega_{l_1}(\tau_1)) \ldots \vartheta_{\alpha_n}(\omega_{l_n}(\tau_n)). \tag{4.5}$$

The set of all such functions will be denoted by $\mathcal{F}$.

Proposition 4.2. *For any two probability measures μ_1, μ_2 on Ω_β, let*

$$\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2},$$

for all $f \in \mathcal{F}$. Then $\mu_1 = \mu_2$.

Proof. The set $\mathcal{F}$ is countable, closed with respect to multiplication and separates the points of Ω_β. By standard monotone class arguments it is a measure determining set. □

Now we introduce an ordering on Ω_β. We write $\xi \geq \xi'$ if, for every $l \in \mathbb{Z}^d$ and $\tau \in [0, \beta]$, $\xi_l(\tau) \geq \xi'_l(\tau)$. In the same sense we define the ordering on every $\Omega_{\beta,\Lambda}$. A function $f \in \mathcal{F}^{\rm pb}_\Lambda$ is called increasing if $f(\omega_\Lambda) \geq f(\omega'_\Lambda)$ for $\omega_\Lambda \geq \omega'_\Lambda$. A significant role in the proof of Lemma 3.1 is played by the FKG inequality, which, for the measures (2.16), was proved in Sect. 6 of [6].

Proposition 4.3 (FKG). *For any box Λ, for every two increasing functions $f, g \in \mathcal{F}^{\rm pb}_\Lambda$ and any $\xi \in \Omega_\beta$, the following is true*

$$\langle fg \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} \geq \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} \langle g \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)}. \tag{4.6}$$

Corollary 4.4. *For every increasing $f \in \mathcal{F}^{\rm pb}_\Lambda$ and any $\xi, \eta \in \Omega_\beta$, $\xi \geq \eta$, implies*

$$\langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} \geq \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)}. \tag{4.7}$$

Proof of Theorem 3.1. By Proposition 4.2, it is enough to show that the condition (3.1) implies that, for any two extreme elements $\mu_1, \mu_2 \in \mathcal{G}^{\rm t}_\beta$, the following equality:

$$\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}, \tag{4.8}$$

holds for all $f \in \mathcal{F}$. For a box Λ, let $\mathcal{F}_\Lambda$ denote the subset of $\mathcal{F}$, consisting of functions depending on ω_l with $l \in \Lambda$ only. By Theorem 7.12, p. 122, [22], the equality (4.8) is a consequence of the following convergence:

$$\langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \longrightarrow 0, \qquad \text{as} \quad \Lambda \overset{\mathcal{L}}{\to} \mathbb{Z}^d, \tag{4.9}$$

which has to hold for every $f \in \mathcal{F}$, for any sequence of boxes $\mathcal{L}$ such that $f \in \mathcal{F}_\Lambda$ for all $\Lambda \in \mathcal{L}$, and for any two $\xi, \eta \in \Omega^{\rm t}_\beta$.

Obviously, for every $f \in \mathcal{F}$, there exists $\lambda > 0$ such that the function

$$\phi(\omega) = \lambda \sum_{j=1}^{n} \omega_{l_j}(\tau_j) + \theta f(\omega), \tag{4.10}$$

is increasing for both values $\theta = \pm 1$.

First let us show that (3.1) implies (4.9) for an ordered pair $\xi \geq \eta$ of elements of $\Omega^{\rm t}_\beta$. By Corollary 4.4, for such a pair, one has

$$\langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} \geq \langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)}, \quad j = 1, 2, \ldots, n,$$

and

$$\langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} \geq \langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)},$$

which holds for both values $\theta = \pm 1$. This yields

$$\begin{aligned} 0 &\le \langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \\ &= \lambda \sum_{j=1}^{n} \left[\langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \right] \\ &\quad + \theta \left[\langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \right], \end{aligned} \tag{4.11}$$

also for both values of θ. Hence

$$\left| \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \right| \le \lambda \sum_{j=1}^{n} \left[\langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \right],$$

which yields (4.9) by (3.1). Now let us consider the case of arbitrary $\xi, \eta \in \Omega^{\mathrm{t}}_{\beta}$. Define

$$\begin{aligned} A_\Lambda(\xi, \eta) &= \lambda \sum_{j=1}^{n} \left[\langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle \omega_{l_j}(\tau_j) \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \right], \\ B_\Lambda(\xi, \eta) &= \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)}, \\ C_\Lambda(\xi, \eta|\theta) &= \langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle \phi \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} = A_\Lambda(\xi, \eta) + \theta B_\Lambda(\xi, \eta), \end{aligned}$$

and set $\bar{\xi} = \max\{\xi, \eta\}$. Then for both values $\theta = \pm 1$, $C_\Lambda(\bar{\xi}, \eta|\theta) \ge C_\Lambda(\xi, \eta|\theta)$, since $C_\Lambda(\xi, \eta|\theta)$ is increasing in ξ. This yields

$$A_\Lambda(\bar{\xi}, \eta) - A_\Lambda(\xi, \eta) \ge \theta \left[B_\Lambda(\xi, \eta) - B_\Lambda(\bar{\xi}, \eta) \right], \quad \theta = \pm 1.$$

By (3.1), the left-hand side of the latter inequality tends to zero as $\Lambda \xrightarrow{\mathcal{L}} \mathbb{Z}^d$. The same is true for $B_\Lambda(\bar{\xi}, \eta)$, because $\bar{\xi} \ge \eta$. Since this holds for both $\theta = \pm 1$, one has

$$B_\Lambda(\xi, \eta) = \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\eta)} \longrightarrow 0. \quad \square$$

4.2. Theorem 3.2 and Corollary 3.3 . The proof of Theorem 3.2 is based on the GKS inequalities, which hold for the measures $\nu_{\beta,\Lambda}(\cdot|b)$, $\tilde{\nu}_{\beta,\Lambda}(\cdot|b)$, $b = 0, p$ by Theorem 6.2 in [6]. Here we will use them in the following form.

Proposition 4.5. *Let ν denote one of the following measures $\nu_{\beta,\Lambda}(\cdot|b)$, $\tilde{\nu}_{\beta,\Lambda}(\cdot|b)$, $b = 0$, p. Then, for any $l_1, l_2, \ldots, l_{2n} \in \Lambda$, $\tau_1, \tau_2, \ldots, \tau_{2n} \in [0, \beta\,]$ (possibly coinciding) and for any positive integer $p \le n$, one has*

$$\langle \omega_{l_1}(\tau_1) \ldots \omega_{l_{2n}}(\tau_{2n}) \rangle_\nu \ge 0, \tag{4.12}$$

$$\begin{aligned} &\langle \omega_{l_1}(\tau_1) \ldots \omega_{l_{2n}}(\tau_{2n}) \rangle_\nu \\ &\quad \ge \langle \omega_{l_1}(\tau_1) \ldots \omega_{l_{2p}}(\tau_{2p}) \rangle_\nu \langle \omega_{l_{2p+1}}(\tau_{2p+1}) \ldots \omega_{l_{2n}}(\tau_{2n}) \rangle_\nu. \end{aligned} \tag{4.13}$$

Let $\Phi = (\Phi_{ll'})_{l,l' \in \Lambda}$, with $\Phi_{ll'} \geq 0$ for all $l, l' \in \Lambda$. Set

$$\mu_{\beta,\Lambda}^{\Phi}(\mathrm{d}\omega_\Lambda) = \frac{1}{Z_{\beta,\Lambda}^{\Phi}} \exp\left\{ \frac{1}{2} \sum_{l,l' \in \Lambda} \Phi_{ll'} \int_0^\beta \omega_l(\tau)\omega_{l'}(\tau)\mathrm{d}\tau \right. \\ \left. - \sum_{l \in \Lambda} \int_0^\beta \hat{V}([\omega_l(\tau)]^2)\mathrm{d}\tau \right\} \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \tag{4.14}$$

where $\hat{V}$ denotes either V or $\widetilde{V}$ and $1/Z_{\beta,\Lambda}^{\Phi}$ is a normalization constant. A corollary of (4.13) is the following statement, whose proof is standard hence omitted.

Proposition 4.6. *If $\Phi_{ll'} \leq \Phi'_{ll'}$ for all $l, l' \in \Lambda$, then the following inequality*

$$\langle \omega_l(\tau)\omega_{l'}(\tau') \rangle_{\mu_{\beta,\Lambda}^{\Phi}} \leq \langle \omega_l(\tau)\omega_{l'}(\tau') \rangle_{\mu_{\beta,\Lambda}^{\Phi'}}, \tag{4.15}$$

holds for all $l, l' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$.

Proof of Theorem 3.2. Positiveness in (3.12) follows from the FKG inequality (4.6). The estimate (3.13) and the periodic boundary domination in (3.12) follow from the inequality (4.15). To prove the zero boundary estimate in (3.12) we rewrite (3.2) as follows:

$$K_{ll'}^{\Lambda}(\tau, \tau'|\xi) = \frac{1}{[Z_{\beta,\Lambda}(\xi)]^2} \int\int_{\Omega_{\beta,\Lambda}\times\Omega_{\beta,\Lambda}} \frac{\omega_l(\tau) - \omega'_l(\tau)}{\sqrt{2}} \cdot \frac{\omega_{l'}(\tau') - \omega'_{l'}(\tau')}{\sqrt{2}} \\ \times \exp\left\{ \frac{J}{2} \sum_{\mathrm{nn:}\ l,l' \in \Lambda} \int_0^\beta [\omega_l(\tau)\omega_{l'}(\tau) + \omega'_l(\tau)\omega'_{l'}(\tau)]\mathrm{d}\tau \right. \\ + J \sum_{\mathrm{nn:}\ l \in \Lambda,\ l' \in \Lambda^c} \int_0^\beta [\omega_l(\tau) + \omega'_l(\tau)]\xi_{l'}(\tau)\mathrm{d}\tau \\ \left. - \sum_{l \in \Lambda} \int_0^\beta [V([\omega_l(\tau)]^2) + V([\omega'_l(\tau)]^2)]\mathrm{d}\tau \right\} \bigotimes_{l \in \Lambda} (\chi_\beta \otimes \chi_\beta)(\mathrm{d}\omega_l, \mathrm{d}\omega'_l).$$

By means of the substitutions

$$x_l(\tau) = 2^{-1/2}[\omega_l(\tau) - \omega'_l(\tau)], \quad y_l(\tau) = 2^{-1/2}[\omega_l(\tau) + \omega'_l(\tau)],$$

we transform this into

$$K_{ll'}^{\Lambda}(\tau, \tau'|\xi) = \frac{1}{[Z_{\beta,\Lambda}(\xi)]^2} \int\int_{\Omega_{\beta,\Lambda}\times\Omega_{\beta,\Lambda}} x_l(\tau)x_{l'}(\tau') \\ \times \exp\left\{ \frac{J}{2} \sum_{\mathrm{nn:}\ l,l' \in \Lambda} \int_0^\beta [x_l(\tau)x_{l'}(\tau) + y_l(\tau)y_{l'}(\tau)]\mathrm{d}\tau \right.$$

$$+\sqrt{2}J \sum_{\text{nn: } l\in\Lambda,\ l'\in\Lambda^c} \int_0^\beta y_l(\tau)\xi_{l'}(\tau)\mathrm{d}\tau - \sum_{l\in\Lambda} B(x_l, y_l)$$
$$- \sum_{l\in\Lambda} \int_0^\beta [\widetilde{V}([x_l(\tau)]^2) + \widetilde{V}([y_l(\tau)]^2)]\mathrm{d}\tau \Bigg\}$$
$$\times \bigotimes_{l\in\Lambda} (\chi_\beta \otimes \chi_\beta)(\mathrm{d}x_l, \mathrm{d}y_l). \tag{4.16}$$

Here $\widetilde{V}$ is defined by (3.10) and

$$B(x_l, y_l) \stackrel{\text{def}}{=} \sum_{j=1}^{r-1} \int_0^\beta b_j(y_l(\tau))[x_l(\tau)]^{2j}\mathrm{d}\tau,$$
$$b_j(y_l(\tau)) \stackrel{\text{def}}{=} \sum_{k=j+1}^{r} 2^{1-k} b_k \binom{2k}{2j} [y_l(\tau)]^{2(k-j)} \geq 0, \quad j = 2, \ldots, r-1. \tag{4.17}$$

Recall that we suppose $b_k \geq 0$ for all $k = 2, \ldots, r-1$. Further, for $t \in [0, 1]$, set

$$\mu^{(t)}(\mathrm{d}x_\Lambda|y_\Lambda) = \frac{1}{Y^{(t)}(y_\Lambda)} \exp\Bigg\{ -t\sum_{l\in\Lambda} B(x_l, y_l) + \frac{J}{2} \sum_{\text{nn: } l,l'\in\Lambda} \int_0^\beta x_l(\tau)x_{l'}(\tau)\mathrm{d}\tau$$
$$- \sum_{l\in\Lambda} \int_0^\beta \widetilde{V}([x_l(\tau)]^2)\mathrm{d}\tau \Bigg\} \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{4.18}$$

$$Y^{(t)}(y_\Lambda) = \int_{\Omega_{\beta,\Lambda}} \exp\Bigg\{ -t\sum_{l\in\Lambda} B(x_l, y_l) + \frac{J}{2} \sum_{\text{nn: } l,l'\in\Lambda} \int_0^\beta x_l(\tau)x_{l'}(\tau)\mathrm{d}\tau$$
$$- \sum_{l\in\Lambda} \int_0^\beta \widetilde{V}([x_l(\tau)]^2)\mathrm{d}\tau \Bigg\} \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{4.19}$$

Now we set

$$\Xi^{(t)}_{ll'}(\tau, \tau'|y_\Lambda) = \langle x_l(\tau)x_{l'}(\tau')\rangle_{\mu^{(t)}(\cdot|y_\Lambda)}. \tag{4.20}$$

By standard arguments, this is a differentiable function of $t \in (0, 1)$, which is continuous on $[0, 1]$. Then by the definitions (4.18)–(4.20), one has

$$\frac{\partial}{\partial t}\Xi^{(t)}_{ll'}(\tau, \tau'|y_\Lambda) = -\frac{1}{Y^{(t)}(y_\Lambda)} \sum_{j=2}^{r-1}\sum_{\lambda\in\Lambda} \int_0^\beta b_j(y_\lambda(\vartheta)) \Big\{ \langle [x_\lambda(\vartheta)]^{2j} x_l(\tau)x_{l'}(\tau')\rangle_{\mu^{(t)}(\cdot|y_\Lambda)}$$
$$- \langle [x_\lambda(\vartheta)]^{2j}\rangle_{\mu^{(t)}(\cdot|y_\Lambda)} \langle x_l(\tau)x_{l'}(\tau')\rangle_{\mu^{(t)}(\cdot|y_\Lambda)} \Big\} \mathrm{d}\vartheta. \tag{4.21}$$

For every $t \in [0, 1]$ and $y_\Lambda \in \Omega_{\beta,\Lambda}$, the moments of the measure (4.18) satisfy the GKS inequalities, thus, the expression in $\{\cdot\}$ in the latter formula is non-negative. Taking into account (4.17), one concludes that, for all $t \in [0, 1]$ and $y_\Lambda \in \Omega_{\beta,\Lambda}$, $l, l' \in \Lambda$, $\tau, \tau' \in [0, \beta]$,

$$\frac{\partial}{\partial t} \Xi^{(t)}_{ll'}(\tau, \tau'|y_\Lambda) \leq 0,$$

which immediately yields

$$\Xi^{(1)}_{ll'}(\tau, \tau'|y_\Lambda) \leq \Xi^{(0)}_{ll'}(\tau, \tau'|y_\Lambda) = \widetilde{K}^{\Lambda}_{ll'}(\tau, \tau'|0). \tag{4.22}$$

The latter fact holds since, by (4.18), one has $\mu^{(0)}(\cdot|y_\Lambda) = \tilde{\nu}_{\beta,\Lambda}(\cdot|0)$. On the other hand, by (4.16), (4.18), (4.20), one has

$$\begin{aligned} K^{\Lambda}_{ll'}(\tau, \tau'|\xi) = \frac{1}{[Z_{\beta,\Lambda}(\xi)]^2} \int_{\Omega_{\beta,\Lambda}} & Y^{(1)}(y_\Lambda) \Xi^{(1)}_{ll'}(\tau, \tau'|y_\Lambda) \\ & \times \exp\left\{ \sqrt{2} J \sum_{\text{nn: } l \in \Lambda,\ l' \in \Lambda^c} \int_0^\beta y_l(\tau) \xi_{l'}(\tau) \mathrm{d}\tau \frac{J}{2} \sum_{\text{nn: } l,l' \in \Lambda} \int_0^\beta y_l(\tau) y_{l'}(\tau) \mathrm{d}\tau \right. \\ & \left. - \sum_{l \in \Lambda} \int_0^\beta \widetilde{V}([y_l(\tau)]^2) \mathrm{d}\tau \right\} \chi_{\beta,\Lambda}(\mathrm{d}y_\Lambda). \end{aligned}$$

Taking here into account (4.22) and (4.19) one obtains

$$K^{\Lambda}_{ll'}(\tau, \tau'|\xi) \leq \widetilde{K}^{\Lambda}_{ll'}(\tau, \tau'|0). \quad \square$$

Proof of Corollary 3.3. By the Schwarz inequality, one has

$$\widetilde{K}^{\Lambda}_{ll'}(\tau, \tau'|0) \leq \sqrt{\widetilde{K}^{\Lambda}_{ll}(\tau, \tau|0) \widetilde{K}^{\Lambda}_{l'l'}(\tau', \tau'|0)}. \tag{4.23}$$

Let $\tilde{\pi}_{\beta,\Lambda}(\cdot|\xi)$ be the probability kernel defined by (2.19) for the measure $\tilde{\nu}_{\beta,\Lambda}(\cdot|\xi)$. Then

$$\widetilde{K}^{\Lambda}_{ll}(\tau, \tau|0) = \langle [\omega_l(\tau)]^2 \rangle_{\tilde{\nu}_{\beta,\Lambda}(\cdot|0)} = \langle [\omega_l(\tau)]^2 \rangle_{\tilde{\pi}_{\beta,\Lambda}(\cdot|0)} \leq C^2_\beta, \tag{4.24}$$

with a certain C_β independent of Λ, $l \in \Lambda$ and $\tau \in [0, \beta]$. The latter estimate was proven in [5] (see Eq. (4.57)). This yields (3.14). $\square$

4.3. Theorem 3.4. By periodic boundary domination in (3.12) and by (3.13), one concludes that for any pair of boxes Λ, Λ', such that $\Lambda \subset \Lambda'$, the following is true:

$$D^{\Lambda}_{ll'}(0) \leq D^{\Lambda}_{ll'}(p), \qquad D^{\Lambda}_{ll'}(0) \leq D^{\Lambda'}_{ll'}(0). \tag{4.25}$$

Let us choose the box Λ as follows:

$$\Lambda = (-L, L]^d \bigcap \mathbb{Z}^d, \qquad L \in \mathbb{N}. \tag{4.26}$$

Set

$$\begin{aligned}\Lambda_* = \{q = (q_1, \dots, q_d) \mid q_j = -\pi + (\pi/L)\kappa_j, \\ \kappa_j = 1, 2, \dots, 2L, \quad j = 1, 2, \dots, d\}.\end{aligned} \tag{4.27}$$

Lemma 4.7. *Let $\varkappa$ (cf., (3.17)), d and J satisfy the condition (3.20). Then, for every box Λ of the form (4.26), the following holds:*

$$D^{\Lambda}_{ll'}(p) \leq \frac{1}{|\Lambda|} \sum_{q \in \Lambda_*} \frac{\exp\left(i(q, l - l')\right)}{\varkappa^{-1} - 2dJ + I(q)}, \tag{4.28}$$

where $I(q)$ is given by (3.19).

Proof. For $t \in [0, 1]$, we set

$$\begin{aligned}\widetilde{E}^{(t)}_{\beta,\Lambda}(\omega_\Lambda | p) = -\frac{tJ}{2} \sum_{\text{nn: } l,l' \in T(\Lambda)} \int_0^\beta \omega_l(\tau)\omega_{l'}(\tau)\mathrm{d}\tau \\ + \sum_{l \in \Lambda} \int_0^\beta \widetilde{V}([\omega_l(\tau)]^2)\mathrm{d}\tau,\end{aligned} \tag{4.29}$$

and

$$\tilde{\nu}^{(t)}_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda | p) = \frac{1}{\widetilde{Z}^{(t)}_{\beta,\Lambda}(p)} \exp\left(-\widetilde{E}^{(t)}_{\beta,\Lambda}(\omega_\Lambda | p)\right) \chi_{\beta,\Lambda}(\mathrm{d}\omega_\Lambda), \tag{4.30}$$

where $1/\widetilde{Z}^{(t)}_{\beta,\Lambda}(p)$ is the normalization constant and $\widetilde{V}$ is given by (3.10). To shorten notations by the end of this proof we will write $\langle\cdot\rangle_t$ instead of $\langle\cdot\rangle_{\tilde{\nu}^{(t)}_{\beta,\Lambda}(\cdot|p)}$. Furthermore, for $l, l', l_1, \dots, l_4 \in \Lambda$ and $\tau, \tau', \tau_1, \dots, \tau_4 \in [0, \beta]$, we set

$$X_{ll'}(\tau, \tau' | t) = \langle \omega_l(\tau)\omega_{l'}(\tau')\rangle_t, \tag{4.31}$$

$$\begin{aligned}R_{l_1 l_2 l_3 l_4}(\tau_1, \tau_2, \tau_3, \tau_4 | t) = \langle \omega_{l_1}(\tau_1)\omega_{l_2}(\tau_2)\omega_{l_3}(\tau_3)\omega_{l_4}(\tau_4)\rangle_t \\ - X_{l_1 l_2}(\tau_1, \tau_2 | t) X_{l_3 l_4}(\tau_3, \tau_4 | t) \\ - X_{l_1 l_3}(\tau_1, \tau_3 | t) X_{l_2 l_4}(\tau_2, \tau_4 | t) \\ - X_{l_1 l_4}(\tau_1, \tau_4 | t) X_{l_2 l_3}(\tau_2, \tau_3 | t).\end{aligned} \tag{4.32}$$

These functions are differentiable with respect to $t \in (0, 1)$ and continuous at the endpoints for all possible values of the rest of their arguments. Taking into account (4.29), (4.30) one concludes that the functions $X_{ll'}(\tau, \tau' | t)$ solve the following Cauchy problem:

$$\begin{aligned}\frac{\partial}{\partial t} X_{ll'}(\tau, \tau' | t) = \frac{J}{2} \sum_{l_1, l_2 \in \Lambda} \varepsilon_{l_1 l_2} \int_0^\beta R_{l_1 l_2 l l'}(\tau_1, \tau_1, \tau, \tau' | t)\mathrm{d}\tau_1 \\ + J \sum_{l_1, l_2 \in \Lambda} \varepsilon_{l_1 l_2} \int_0^\beta X_{l l_1}(\tau, \tau_1 | t) X_{l' l_2}(\tau', \tau_1 | t)\mathrm{d}\tau_1,\end{aligned} \tag{4.33}$$

with the initial condition

$$X_{ll'}(\tau,\tau'|0) = \frac{\delta_{ll'}}{\widetilde{Z}_\beta}\int_{\mathcal{C}_\beta}\omega_l(\tau)\omega_l(\tau')\exp\left(-\int_0^\beta \widetilde{V}([\omega_l(\tau)]^2)\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}\omega_l)$$
$$= \frac{\delta_{ll'}}{\widetilde{Z}_l}\mathrm{trace}\left\{\mathrm{q}_l\exp\left(-(\tau'-\tau)\widetilde{H}_l^{(0)}\right)\mathrm{q}_l\exp\left(-(\beta-\tau'+\tau)\widetilde{H}_l^{(0)}\right)\right\}. \tag{4.34}$$

Here $\delta_{ll'}$ is the Kronecker delta, $\varepsilon_{ll'} = 1$ if l, l' are nearest neighbors on the torus $T(\Lambda)$ and $\varepsilon_{ll'} = 0$ otherwise, $1/\widetilde{Z}_\beta$, $1/\widetilde{Z}_l$ are normalizing constants. Moreover, comparing (4.31), (4.30), (4.29) with (3.2), (2.18), one gets

$$X_{ll'}(\tau,\tau'|1) = \widetilde{K}^\Lambda_{ll'}(\tau,\tau'|p). \tag{4.35}$$

Since we are interesting in the Duhamel functions (4.25) only, we will study the following function:

$$Y_{ll'}(t) = \int_0^\beta X_{ll'}(\tau,\tau'|t)\mathrm{d}\tau' = \int_0^\beta X_{ll'}(0,\tau'|t)\mathrm{d}\tau', \tag{4.36}$$

for which we have from (4.33), (4.34),

$$\frac{\partial}{\partial t}Y_{ll'}(t) = \frac{J}{2}\sum_{l_1,l_2\in\Lambda}\varepsilon_{l_1l_2}\int_0^\beta\int_0^\beta R_{l_1l_2ll'}(\tau_1,\tau_1,0,\tau|t)\mathrm{d}\tau_1\mathrm{d}\tau$$
$$+J\sum_{l_1,l_2\in\Lambda}\varepsilon_{l_1l_2}Y_{ll_1}(t)Y_{l'l_2}(t), \tag{4.37}$$

subject to the following conditions (see (4.25) and (3.17)):

$$Y_{ll'}(0) = \delta_{ll'}\varkappa, \qquad Y_{ll'}(1) = D^\Lambda_{ll'}(p). \tag{4.38}$$

By [6], Theorem 6.3, the Lebowitz inequality

$$R_{l_1l_2l_3l_4}(\tau_1,\tau_2,\tau_3,\tau_4|t) \le 0$$

holds for all $t \in [0, 1]$, thus Eq. (4.36) may be rewritten as

$$\frac{\partial}{\partial t}Y_{ll'}(t) = S_{ll'}(t) + J\sum_{l_1,l_2\in\Lambda}\varepsilon_{l_1l_2}Y_{ll_1}(t)Y_{l'l_2}(t), \quad S_{ll'}(t) \le 0. \tag{4.39}$$

Along with the latter let us consider the following Cauchy problem:

$$\frac{\partial}{\partial t}Q_{ll'}(t) = J\sum_{l_1,l_2\in\Lambda}\varepsilon_{l_1l_2}Q_{ll_1}(t)Q_{l'l_2}(t), \quad Q_{ll'}(0) = \delta_{ll'}\varkappa. \tag{4.40}$$

Due to the translation symmetry on the torus $T(\Lambda)$ it may be diagonalized by means of the Fourier transformation

$$Q_{ll'}(t) = \frac{1}{|\Lambda|}\sum_{q\in\Lambda_*}\hat{Q}_q(t)\exp\left(i(q,l-l')\right), \quad \hat{Q}_q(t) = \sum_{l'\in\Lambda}Q_{ll'}(t)\exp\left(-i(q,l-l')\right), \tag{4.41}$$

where Λ_* is given by (4.27). This yields

$$\frac{\partial}{\partial t}\hat{Q}_q(t) = (2dJ - I(q))\left[\hat{Q}_q(t)\right]^2, \quad \hat{Q}_q(0) = \varkappa. \tag{4.42}$$

Under the condition (3.20) the latter may be solved for $t \in [0, 1]$,

$$\begin{aligned}\hat{Q}_q(t) &= \frac{1}{\varkappa^{-1} - t(2dJ - I(q))}, \\ Q_{ll'}(t) &= \frac{1}{|\Lambda|}\sum_{q\in\Lambda_*}\frac{\exp\left(i(q, l - l')\right)}{\varkappa^{-1} - t(2dJ - I(q))}.\end{aligned} \tag{4.43}$$

Given $\alpha \in (0, 1)$, we set

$$Q^{(\alpha)}_{ll'}(t) = Q_{ll'}(t + \alpha), \quad t \in [0, 1 - \alpha]. \tag{4.44}$$

These functions solve the Cauchy problem

$$\frac{\partial}{\partial t}Q^{(\alpha)}_{ll'}(t) = J\sum_{l_1, l_2\in\Lambda}\varepsilon_{ll_1}\varepsilon_{l'l_1}Q^{(\alpha)}_{ll_1}(t)Q^{(\alpha)}_{l'l_2}(t), \tag{4.45}$$

$$Q^{(\alpha)}_{ll'}(0) = Q_{ll'}(\alpha) > Q_{ll'}(0) = \delta_{ll'}\varkappa. \tag{4.46}$$

Now let us compare the latter problem with the problem (4.38), (4.39), which has a unique solution defined by (4.31), (4.36). By Theorem V, p. 65, [41], one has

$$Y_{ll'}(t) < Q^{(\alpha)}_{ll'}(t) = Q_{ll'}(t + \alpha), \quad t \in [0, 1 - \alpha],$$

which holds for all $\alpha \in (0, 1)$. Since both above functions are continuous, this yields (see (4.38), (4.43))

$$D^{\Lambda}_{ll'}(p) = Y_{ll'}(1) \leq Q_{ll'}(1) = \frac{1}{|\Lambda|}\sum_{q\in\Lambda_*}\frac{\exp\left(i(q, l - l')\right)}{\varkappa^{-1} - 2dJ + I(q)}. \tag{4.47}$$

□

Proof of Theorem 3.4. Given a box Λ, let $\mathcal{L}$ be the sequence of boxes, each element of which contains Λ and is of the form (4.26). Then, for every $\Lambda' \in \mathcal{L}$, by (4.25) and (4.28), one has

$$D^{\Lambda}_{ll_0}(0) \leq D^{\Lambda'}_{ll_0}(p) \leq \frac{1}{|\Lambda'|}\sum_{q\in\Lambda'_*}\frac{\exp\left(i(q, l - l_0)\right)}{\varkappa^{-1} - 2dJ + I(q)}.$$

Passing here to the limit $\Lambda' \xrightarrow{\mathcal{L}} \mathbb{Z}^d$ one gets (3.21). □

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