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CONSTRUCTIVE APPROXIMATION

Sparse Approximation of Singularity Functions

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Abstract. We are concerned with the sparse approximation of functions on the d-dimensional unit cube $[0, 1]^d$, which contain powers of distance functions to lower-dimensional k-faces (corners, edges, etc.). These functions arise, e.g., from corners, edges, etc., of domains in solutions to elliptic PDEs. Usually, they deteriorate the rate of convergence of numerical algorithms to approximate these solutions.

We show that functions of this type can be approximated with respect to the H^1 norm by sparse grid wavelet spaces V_L, $\dim(V_L) = N_L$, of biorthogonal spline wavelets of degree p essentially at the rate p:

$$\|u - P_L u\|_{H^1([0,1]^d)} \leq C N_L^{-p} (\log_2 N_L)^s \|u\|, \qquad s = s(p, d),$$

where $\|\cdot\|$ is a weighted Sobolev norm and $P_L u \in V_L$.

1. Introduction

The efficient numerical approximation of functions of several variables is required in numerous applications: In the numerical solution of partial differential equations (PDEs) in dimension 3 or higher, in data mining, in numerical integration over high-dimensional domains, to name but a few.

Tensor products of piecewise polynomial functions of degree p on a uniform mesh of width $h > 0$ require $\mathcal{O}(h^{-d})$ degrees of freedom and achieve an asymptotic convergence rate in the H^1 norm of $\mathcal{O}(h^p)$ if the $(p+1)$st derivatives of the function to be approximated are square-integrable.

The growth as $h \to 0$ of the number of degrees of freedom in tensor product interpolants is excessive if d is large. So-called *sparse interpolants*—which require only $\mathcal{O}(h^{-1}|\log h|^{d-1})$ degrees of freedom rather than $O(h^{-d})$—have been proposed and successfully used in applications (see, e.g., [8], [16], [26]). There it was shown that despite their reduced number of degrees of freedom, sparse tensor product interpolants can achieve an H^1 convergence rate of $\mathcal{O}(h^p|\log h|^{d-1})$, i.e., up to logarithmic terms the rate of the full tensor product approximation is preserved. This is possible, however, only under very strong assumptions on the regularity of the function to be approximated:

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Certain derivatives up to order $d(p+1)$ must be square-integrable in order to realize the convergence rate $\mathcal{O}(h^p |\log h|^{d-1})$ in H^1.

In particular, in the numerical treatment of PDEs the solutions are *not* that regular, for example, due to corner singularities in elliptic problems. The potential gain in using sparse tensor product approximation is strongly limited in this case—the reduction in the convergence rate essentially offsets the reduction in degrees of freedom in the interpolant.

Singularities in solutions of elliptic problems contain as essential parts powers of distance functions to the singular support of the domain (corner, edges, etc.). For instance, for polygonal domains Ω in two dimensions, there holds a decomposition of the weak solution u to a uniformly elliptic boundary value problem into

$$u = u_{\text{reg}} + u_{\text{sing}},$$

with smooth u_{reg} and a finite sum

$$u_{\text{sing}} = \sum^{<\infty} \operatorname{Re}(r_i^{\lambda_i} (\log r_i)^{\mu_i} \Phi_i(\varphi_i) \eta_i(x)), \tag{1}$$

(in local polar coordinates (r_i, φ_i) associated to the corners (here not indexed)) with smooth cut-off functions η_i, smooth functions Φ_i of the angle coordinates φ_i, and parameters $\lambda_i \in \mathbf{C}$, $\mu_i \in \mathbf{N}_0$, see, e.g., [5], [6], [7]. For Dirichlet–Dirichlet corners, one knows Re $\lambda_i > \frac{1}{2}$, and for Dirichlet–Neumann corners, Re $\lambda_i > \frac{1}{4}$.

In higher dimensions, decomposition theorems are not that easy to write down, but estimates on the growth (of the function and its derivatives) toward the singular support of the domain are known, see, e.g., [19].

The rate of approximation of singular functions determines the convergence rate of Finite Element Methods for elliptic problems. The dimension N of FE spaces of continuous, piecewise polynomials of degree p on a quasi-uniform mesh of mesh width h is $N = \mathcal{O}(h^{-d})$. For solutions $u \in H^{s+1}(\Omega)$, the H^1 convergence rate of the best approximation is $\mathcal{O}(N^{-\min\{p,s\}/d})$. In dimension $d = 2$, singular terms of the above type belong to $H^{s+1}(\Omega)$ with $s < \operatorname{Re}(\lambda)$.

To overcome the reduced convergence rate, the nonuniform distribution of degrees of freedom by local mesh refinement has been proposed. This can be done a priori, i.e., by grading the mesh toward the singular support of (1), or a posteriori, by adaptive mesh refinement and nonlinear approximation.

In dimension 2, these techniques allow us to recover the maximal asymptotic rate $\mathcal{O}(N^{-p/2})$, see, e.g., [3], [20] for recent results and, in particular, [9], where it has been proven that the rate $\mathcal{O}(N^{-b})$ is achievable by adaptive (nonlinear) wavelet approximation whenever such a rate is possible by so-called best N-term approximation (for related numerical results, see also [2]). The result in [9] is, in particular, not restricted to functions of the special type (1), but applies to all functions which belong to a scale of suitable Besov spaces (see [13] for details).

The result [9] applies to certain FEM bases with "isotropic" support, or with "shape regular" elements in FE terminology. In polyhedra Ω in dimension $d = 3$, the appearance of edge singularities with *anisotropic* regularity does not allow us to achieve the rate $\mathcal{O}(N^{-p/d})$ with local mesh refinements, as long as the basis is isotropic. However, this rate can be achieved with suitable *anisotropic* mesh refinements (see, e.g., [1]).

In the present paper, we therefore analyze the approximation of singular functions built from powers of distance functions to corners, edges, etc., in anisotropically refined *sparse* tensor product spaces.

We prove that anisotropic sparse tensor product spaces (of piecewise polynomials of degree p) allow us to approximate corner and edge singularities in dimension 3 (defined on $[0, 1]^d$) at a rate

$$\mathcal{O}(N^{-p} (\log_2 N)^s), \qquad s = 2p + \tfrac{3}{2}, \tag{2}$$

in the H^1 norm, where N is the number of degrees of freedom in the sparse, anisotropic tensor product. In particular, (2) shows that the reduction in the convergence rate, due to the higher dimension and low Sobolev regularity, can be eliminated (up to logarithmic terms).

Our proof is not restricted to dimension 3, but applies to any dimension $d > 1$. In particular, and at first sight somewhat surprisingly, even in dimension 2, the use of sparse tensor products of wavelet bases with local refinement for the approximation of "isotropic" corner singularities (1) gives an improvement over the "optimal" rate $\mathcal{O}(N^{-p/2})$ in the H^1 norm: We obtain that, for each polynomial degree $p \geq 1$, there is a sequence of sparse spline wavelet approximants of degree p which converges at the rate $\mathcal{O}(N^{-p} (\log_2 N)^{p+1})$, as opposed to the rate $\mathcal{O}(N^{-p/2})$ obtained from nonlinear, adaptive approximation in the "isotropic" setting (i.e., support of wavelet bases and Besov spaces used to measure the regularity of (1) are isotropic).

Let us point out that our result immediately yields upper bounds on nonlinear and adaptive approximation schemes in anisotropic wavelet bases for the elliptic singularities, which are superior to those of [9] and [3] in the isotropic setting. These connections will be elaborated elsewhere.

The remainder of this paper is organized as follows: In Section 2 we introduce the wavelets we use and prove the one-dimensional results; the key result is Lemma 3. In Section 3 we consider the two-dimensional case and give a detailed proof of the two-dimensional consistency estimate using an abstract tensor product argument. In Section 4 we extend the two-dimensional results to d dimensions by iterating the arguments of Section 3. The main theorems are Theorems 4 and 5.

Concerning the application to singular functions arising from corners of polygonal domains or from changes in the type of boundary conditions, Theorem 3 (for the two-dimensional case) and Theorem 6 (for the three-dimensional case) specialize the results to these cases.

In the sequel, $\sim$ in expressions like $A \sim B$ means that both quantities can be uniformly bounded by constant multiples of each other. Likewise $\lesssim$ indicates inequalities up to constant factors.

2. Wavelet Approximation with Local Refinement in One Dimension

In this section we develop the one-dimensional means in order to approximate singular functions that are powers of distance functions to points. Instead of using explicit representations like (1), we derive an estimate in terms of weighted Sobolev norms. For this, we define a hierarchy of finite-dimensional spaces $V_L^\beta \subset V_{L+1}^\beta \subset H^1$ (where β is a parameter indicating the local refinement toward the singularity) and corresponding projectors P_L^β, and prove that

$$\dim(V_L^\beta) \le C(p)2^L \qquad \text{and} \qquad \|u - P_L^\beta u\|_{H^1([0,1])} \le C(p)L^{1/2}\, 2^{-pL}\|u\|_*,$$

for functions u with $\|u\|_* < \infty$ (p is the local polynomial degree of the ansatz functions). The norm $\|\cdot\|_*$ is a weighted Sobolev norm (and will be defined precisely in the sequel). The singularity functions of interest are in these weighted spaces.

The one-dimensional framework will later be used to infer d-dimensional results by sparse “tensor products” of the univariate results. In this regard, the use of hierarchic spaces is essential to our approach.

We begin by describing the wavelets we use. Then we define the locally (toward the singularity) refined projectors and ansatz spaces, as well as the weighted spaces. The last part of this section is devoted to the proof of the approximation property in terms of these weighted norms.

Let $\psi, \tilde{\psi}$ be compactly supported biorthogonal spline wavelets of (a fixed) degree $p \ge 1$ on the interval [0, 1], such that for

$$u = \sum_{j=0}^{\infty}\sum_{k=1}^{k_j} c_{jk}\psi_{jk} = \sum_{j=0}^{\infty}\sum_{k=1}^{k_j}\langle u, \tilde{\psi}_{jk}\rangle_{L^2}\, \psi_{jk}$$

the following norm equivalences (resp., estimates) hold:

$$(3) \qquad c_1\|u\|^2_{H^t([0,1])} \le \sum_{j,k} 2^{2tj}c_{jk}^2 \;\le\; c_2\|u\|^2_{H^t([0,1])} \qquad \text{for } 0 \le t < p + \tfrac{1}{2},$$

$$\sum_{\substack{j,k\\ j\le L}} 2^{2tj}c_{jk}^2 \;\le\; c_2\|u\|^2_{H^t([0,1])} \qquad \text{for } p + \tfrac{1}{2} \le t < p,$$

$$\sum_{\substack{j,k\\ j\le L}} 2^{2tj}c_{jk}^2 \;\le\; c_2 L\|u\|^2_{H^t([0,1])} \qquad \text{for } t = p,$$

where the constants c_1 and c_2 depend on t. Such wavelets are constructed, for instance, in [11]; see also [10]. For the one-sided norm estimates, see [24].

On each level j, the number of wavelets ψ_{jk} is bounded by $p(2^j + N)$ with some global constant $N \in \mathbf{N}$. To simplify notation, we take $N = 0$. We ignore the dependence of the indices k on p, thus regarding an index (j, k) standing for p local wavelets basis functions. The support of a wavelet ψ_{jk} is of order 2^{-j}. Note that the norm equivalences are a consequence of vanishing moment properties of the wavelets (resp., duals).

We define now one-dimensional wavelet projectors P_L^β, which are designed to approximate singular functions of type $x \mapsto x^\lambda$. For this, let $I = \{(j,k) \in \mathbf{N}\times\mathbf{N} : j \in \mathbf{N}, 1 \le k \le 2^j\}$, and define for $L \in \mathbf{N}$ and fixed $\beta \in [0, 1)$,

$$(4) \qquad \Lambda_L^\beta \equiv \{(j,k) \in I \;:\; (k2^{-j})^\beta \le 2^{L-j}\}.$$

The index set Λ_L^β contains all index pairs (j, k) with $j \le L$, but considerably more toward the left endpoint 0. But still it holds that

$$(5) \qquad \#\Lambda_L^\beta \lesssim p\, 2^L,$$

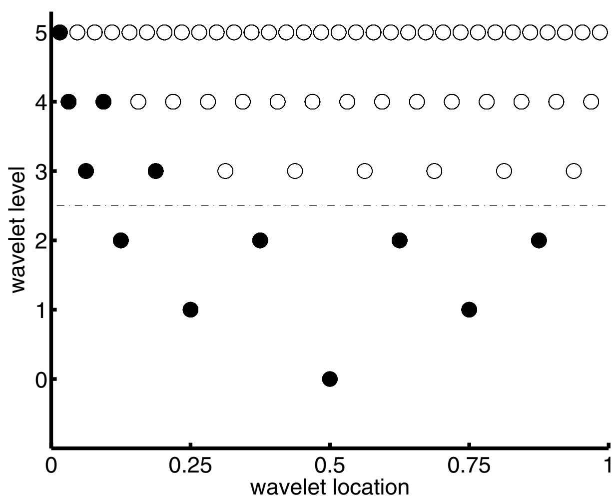
with a constant depending only on β. In the vicinity of 0, the minimal support of wavelets indexed by Λ_L^β is of order $2^{-L/(1-\beta)}$.

Definition 1 (Projector P_L^β). Let u be a function with wavelet decomposition $u = \sum_{j,k} c_{jk}\psi_{jk}$. Then the projector P_L^β is defined by

$$P_L^\beta u \equiv \sum_{(j,k)\in\Lambda_L^\beta} c_{jk}\psi_{jk}. \tag{6}$$

We denote by V_L^β the image of P_L^β, that is, $V_L^\beta = \text{span}\{\psi_{jk} : (j,k) \in \Lambda_L^\beta\} \subset H^p([0,1])$.

The following figure depicts a typical space V_L^β: The basis functions spanning V_L^β are marked by black dots. In this example, $L = 2$ and $\beta = 0.6$; the maximal occurring level in the vicinity of the origin is $L/(1-\beta) = 5$.



The projectors P_L^β are simultaneously stable with respect to the H^t norms, $0 \le t < p + \frac{1}{2}$.

Lemma 1 (Stability of P_L^β). *Let $t \in [0, p+\frac{1}{2})$. Then $P_L^\beta : H^t([0,1]) \to H^t([0,1])$ is bounded independently of L.*

Proof. Using (3), we have

$$\|P_L^\beta u\|^2_{H^t([0,1])} = \Bigg\| \sum_{(j,k)\in\Lambda_L^\beta} c_{jk}\psi_{jk} \Bigg\|^2_{H^t([0,1])} \sim \sum_{(j,k)\in\Lambda_L^\beta} 2^{2tj} c_{jk}^2$$

$$\le \textstyle\sum_{j,k} 2^{2tj} c_{jk}^2 \sim \|u\|^2_{H^t([0,1])}. \qquad \blacksquare$$

Next, we will prove an approximation result for P_L^β involving weighted Sobolev norms. For this, we first introduce these weighted spaces:

Definition 2 (Weighted spaces, $d = 1$). For $m \in \mathbf{N}$, let $H_\gamma^m([0,1])$ be the space of all measurable functions u such that

$$\|u\|_{H_\gamma^m([0,1])} := \left(\sum_{i=0}^{m} \|(D^i u) x^\gamma\|^2_{L^2([0,1])} \right)^{1/2}$$

is finite. The expression $\|\cdot\|_{H_\gamma^m([0,1])}$ is a norm on this space.

We have the following embedding result:

Lemma 2 (Embedding, $d = 1$). *If $\gamma > p + \frac{1}{2}$, then the following embedding is continuous*:

$$H_\gamma^{p+1}([0,1]) \hookrightarrow L^q([0,1]) \qquad \textit{for} \quad q < \frac{1}{\gamma - (p + \frac{1}{2})}.$$

Proof. We first note that there exists a continuous extension operator $E : H_\gamma^{p+1}([0,1]) \to H_\gamma^{p+1}([0,2])$ with the property supp $Eu \subset [0,2)$. This is due to the fact that close to the right boundary $\{1\}$ the space H_γ^{p+1} coincides (in a local sense) with the space H^{p+1}.

Let now $\gamma > p + \frac{1}{2}$. We write

$$\begin{aligned}
|u(x)| &= |Eu(x)| \le \int_x^2 |(Eu)'(s)|\,ds \le \int_x^2 \int_{s_1}^2 \cdots \int_{s_p}^2 |D^{p+1}(Eu)(t)|\,dt\,ds_p \cdots ds_1 \\
&= \int_x^2 \int_{s_1}^2 \cdots \int_{s_p}^2 |D^{p+1}(Eu)(t)\,t^\gamma|t^{-\gamma}\,dt\,ds_p \cdots ds_1 \\
&\le \int_x^2 \int_{s_1}^2 \cdots \int_{s_{p-1}}^2 \underbrace{\left(\int_{s_p}^2 |D^{p+1}(Eu)(t)\,t^\gamma|^2\,dt \right)^{1/2}}_{\substack{\le \|D^{p+1}(Eu)\,t^\gamma\|_{L^2([0,2))} \\ \le c\,\|D^{p+1}u\,t^\gamma\|_{L^2([0,1])}}} \\
&\quad \times \left(\int_{s_p}^2 t^{-2\gamma}\,dt \right)^{1/2} ds_p \cdots ds_1 \\
&\lesssim \|u\|_{H_\gamma^{p+1}([0,1])} \cdot \int_x^2 \int_{s_1}^2 \cdots \int_{s_{p-1}}^2 \left(\int_{s_p}^2 t^{-2\gamma}\,dt \right)^{1/2} ds_p \cdots ds_1 \\
&\lesssim \|u\|_{H_\gamma^{p+1}([0,1])} \cdot \int_x^2 \int_{s_1}^2 \cdots \int_{s_{p-1}}^2 s_p^{1/2-\gamma}\,ds_p \cdots ds_1 \\
&\lesssim \|u\|_{H_\gamma^{p+1}([0,1])} \cdot \int_x^2 \int_{s_1}^2 \cdots \int_{s_{p-2}}^2 s_{p-1}^{3/2-\gamma}\,ds_{p-1} \cdots ds_1 \\
&\lesssim \cdots \lesssim x^{p+1/2-\gamma} \|u\|_{H_\gamma^{p+1}([0,1])}.
\end{aligned}$$

Taking this to the power q, integrating over [0, 1], and taking to the power $1/q$ yields

$$\|u\|_{L^q([0,1])} \lesssim \|u\|_{H^{p+1}_{\gamma}([0,1])} \underbrace{\left(\int_0^1 x^{q(p+1/2-\gamma)}\, dx\right)^{1/q}}_{=:C}.$$

C is finite, if $q < 1/(\gamma - (p + \frac{1}{2}))$. ■

Now we are able to prove the one-dimensional approximation result with respect to a weighted norm of the above type.

Lemma 3 (Weighted Norm Consistency of P_L^β, $d = 1$). *For*

$$1 > \beta > \gamma > \frac{p + \frac{1}{2}}{p + 1}$$

and $u \in H^{p+1}_{\gamma(p+1)}([0, 1])$ *it holds that*

$$\|u - P_L^\beta u\|_{L^2([0,1])} \lesssim L^{1/2}\, 2^{-(p+1)L} \|u\|_{H^{p+1}_{\gamma(p+1)}([0,1])}. \tag{7}$$

For

$$\beta > \gamma > \frac{p - \frac{1}{2}}{p}$$

and $u \in H^{p+1}_{\gamma p}([0, 1])$ *it holds that*

$$\|u - P_L^\beta u\|_{H^1([0,1])} \lesssim L^{1/2}\, 2^{-pL} \|u\|_{H^{p+1}_{\gamma p}([0,1])}. \tag{8}$$

Proof. It suffices to prove the statements above for $u \in H^{p+1}([0, 1])$. The lemma then follows from a density argument.

We split the interval [0, 1] into subintervals of length $2^{-L/(1-\beta)}$, for simplicity assuming $L/(1 - \beta)$ to be an integer (otherwise rounded off). For $s = 1, \ldots, 2^{L/(1-\beta)}$, set

$$I(s) = [(s - 1)2^{-L/(1-\beta)}, s2^{-L/(1-\beta)}].$$

Let $j_0(s)$ be a bound for the minimal local level of wavelets ψ_{jk} with $(j, k) \notin \Lambda_L^\beta$ and $\operatorname{supp} \psi_{jk} \cap I(s) \neq \varnothing$. Examining the condition $(s2^{-L/(1-\beta)})^\beta \leq 2^{L-j}$, we see that $j_0(s)$ can be chosen to be

$$j_0(s) = \frac{L}{1 - \beta} - \beta \log_2 s.$$

Applying (3), we estimate

$$\|u - P_L^\beta u\|^2_{L^2(I(s))} \lesssim \sum_{\substack{(j,k) \notin \Lambda_L^\beta \\ \operatorname{supp} \psi_{jk} \cap I(s) \neq \varnothing}} c_{jk}^2$$

$$= \sum_{j \geq j_0(s)} 2^{-2(p+1)j} \sum_{\substack{k\,:\,(j,k) \notin \Lambda_L^\beta \\ \text{supp}\,\psi_{jk} \cap I(s) \neq \varnothing}} 2^{2(p+1)j} c_{jk}^2$$

$$\lesssim \sum_{j \geq j_0(s)} j 2^{-2(p+1)j} \|u\|_{H^{p+1}(I(s))}^2 \lesssim j_0(s) 2^{-2(p+1)j_0(s)} \|u\|_{H^{p+1}(I(s))}^2.$$

Using $2^{-2(p+1)j_0(s)} = 2^{-2(p+1)L}(s2^{-L/(1-\beta)})^{2(p+1)\beta}$ and $j_0(s) \lesssim L$, we infer

$$\|u - P_L^\beta u\|_{L^2(I(s))}^2 \lesssim L 2^{-2(p+1)L} \|u(s2^{-L/(1-\beta)})^{(p+1)\beta}\|_{H^{p+1}(I(s))}^2.$$

Away from 0, i.e., in the case of $s > 1$, we have $s2^{-L/(1-\beta)} \sim x$ for $x \in I(s)$. This yields

$$\|u - P_L^\beta u\|_{L^2([2^{-L/(1-\beta)},1])}^2 \lesssim L 2^{-2(p+1)L} \|u x^{(p+1)\beta}\|_{H^{p+1}([2^{-L/(1-\beta)},1])}^2. \tag{9}$$

In the case of $s = 1$, we employ L^2 stability of the projectors P_L^β (Lemma 1) and the embedding result, Lemma 2.

If $\gamma > (p + \frac{1}{2})/(p+1)$, i.e., $\gamma(p+1) > p + \frac{1}{2}$, we estimate

$$\|u - P_L^\beta u\|_{L^2(I(1))}^2 \lesssim \|u\|_{L^2(I(1))}^2 \leq |I(1)|^{1-2/q} \|u\|_{L^q(I(1))}^2 \lesssim |I(1)|^{1-2/q} \|u\|_{H_{\gamma(p+1)}^{p+1}([0,1])}^2.$$

Here, $q < 1/(\gamma(p+1) - (p+\frac{1}{2}))$ corresponding to Lemma 2. To guarantee $|I(1)|^{1-2/q} \leq 2^{-2(p+1)L}$, we get the constraint $q \geq 1/(\beta(p+1) - (p+\frac{1}{2}))$. Hence, if $\beta > \gamma$, we can choose such a q. Together with (9), the L^2 estimate follows.

The H^1 estimate is done in an analogous fashion. Using (3), we get

$$\begin{aligned}
\|u - P_L^\beta u\|_{H^1(I(s))}^2 &\lesssim \sum_{\substack{(j,k) \notin \Lambda_L^\beta \\ \text{supp}\,\psi_{jk} \cap I(s) \neq \varnothing}} 2^{2j} c_{jk}^2 \\
&= \sum_{j \geq j_0(s)} 2^{-2pj} \sum_{\substack{k\,:\,(j,k) \notin \Lambda_L^\beta \\ \text{supp}\,\psi_{jk} \cap I(s) \neq \varnothing}} 2^{2(p+1)j} c_{jk}^2 \\
&\lesssim \sum_{j \geq j_0(s)} j 2^{-2pj} \|u\|_{H^{p+1}(I(s))}^2 \lesssim j_0(s) 2^{-2pj_0(s)} \|u\|_{H^{p+1}(I(s))}^2.
\end{aligned}$$

Using $2^{-2pj_0(s)} = 2^{-2pL}(s2^{-L/(1-\beta)})^{2p\beta}$ and $j_0(s) \lesssim L$, we infer

$$\|u - P_L^\beta u\|_{H^1(I(s))}^2 \lesssim L 2^{-2pL} \|u(s2^{-L/(1-\beta)})^{p\beta}\|_{H^{p+1}(I(s))}^2.$$

Away from 0, i.e., in the case of $s > 1$, we have $s2^{-L/(1-\beta)} \sim x$ for $x \in I(s)$. This yields

$$\|u - P_L^\beta u\|_{H^1([2^{-L/(1-\beta)},1])}^2 \lesssim L 2^{-2pL} \|u x^{p\beta}\|_{H^{p+1}([2^{-L/(1-\beta)},1])}^2. \tag{10}$$

In the case $s = 1$, we estimate, using Lemma 2,

$$\begin{aligned}
\|D(u - P_L^\beta u)\|_{L^2(I(1))}^2 &\lesssim \|Du\|_{L^2(I(1))}^2 \leq |I(1)|^{1-2/q} \|Du\|_{L^q(I(1))}^2 \\
&\lesssim |I(1)|^{1-2/q} \|u\|_{H_{\gamma p}^{p+1}(I(1))}^2,
\end{aligned}$$

if $q < 1/(\gamma p - (p - \frac{1}{2}))$ and $\gamma p > p - \frac{1}{2}$. To guarantee $|I(1)|^{1-2/q} \leq 2^{-2pL}$, we get the constraint $q \geq 1/(\beta p - (p - \frac{1}{2}))$. Hence, if $\beta > \gamma$, we can choose such a q. Together with (10), the H^1 estimate (8) follows. ■

Remark 1. The function $u : x \mapsto x^\lambda$ belongs to $H^{p+1}_{\gamma p}([0,1])$, if

$$\gamma > 1 - \frac{2\lambda - 1}{2p}.$$

Thus, if $\lambda > \frac{1}{2}$, there is such a $\gamma < 1$.

Remark 2. In [4] a similar one-dimensional result for Chui prewavelet bases of degree 2 and 3 was proven, using a slightly different refinement toward the origin and a different method of proof.

A similiar approach was already undertaken in [23].

3. Wavelet Approximation with Local Refinement in Two Dimensions

We come now to the two-dimensional case. Consider the unit square $[0,1]^2 \subset \mathbf{R}^2$. We define a wavelet basis by taking tensor products of the univariate wavelets introduced above, i.e., the set $\{\psi_{jk} \otimes \psi_{j'k'} : (j,k) \in I, (j',k') \in I\}$. Note the anisotropy of the supports of these wavelets.

We will employ a sparse tensor product rather than a full tensor product of the projectors P^β_L. The sparse tensor product projector has comparable approximation properties to the full one, but uses a remarkably lower-dimensional subspace to project onto. We will give the details in the following.

Let $\beta \in [0,1)$ be fixed and let $L \in \mathbf{N}$. Then we define the index sets

$$\hat{\Lambda}^\beta_L \equiv \bigcup_{i+j \le L} \Lambda^\beta_i \times \Lambda^\beta_j \tag{11}$$

and the trial spaces

$$\hat{V}^\beta_L := \operatorname{span}\{\psi_{jk} \otimes \psi_{j'k'} : (j,k,j',k') \in \hat{\Lambda}^\beta_L\} = \bigoplus_{i+j \le L} V^\beta_i \otimes V^\beta_j. \tag{12}$$

From (5), we see

$$\#\hat{\Lambda}^\beta_L = \dim(\hat{V}^\beta_L) \lesssim p^2 L 2^L. \tag{13}$$

Definition 3 (Projector $\hat{P}^\beta_L$). Let u be a function with wavelet decomposition

$$u = \sum_{j,k,j',k'} c_{jkj'k'} \psi_{jk} \otimes \psi_{j'k'}.$$

Then the projector $\hat{P}^\beta_L$ is defined by

$$\hat{P}^\beta_L u \equiv \sum_{j,k,j',k' \in \hat{\Lambda}^\beta_L} c_{jkj'k'} \psi_{jk} \otimes \psi_{j'k'}. \tag{14}$$

Definition 4. Let $(s,t) \in \mathbf{N}^2$. Then we denote by

$$H^{s,t}([0,1]^2)$$

the space of all measurable functions $u : [0,1]^2 \to \mathbf{R}$, such that the norm

$$\|u\|_{H^{s,t}([0,1]^2)} := \left(\sum_{\substack{0 \le i \le s \\ 0 \le j \le t}} \|D_1^i D_2^j u\|^2_{L^2([0,1]^2)} \right)^{1/2}$$

is finite. That is, $H^{s,t}([0,1]^2) = H^s([0,1]) \otimes H^t([0,1])$.

Lemma 4. *The projectors $\hat{P}_L^\beta$ are stable in $H^1 \otimes L^2$ independently of L.*

Proof. From the one-dimensional norm equivalences, one readily derives norm equivalences for tensor product spaces in the following form:

$$\left\| \sum_{j,k,j',k'} c_{jkj'k'}\, \psi_{jk} \otimes \psi_{j'k'} \right\|^2_{H^{s,t}([0,1]^2)} \sim \sum_{j,k,j',k'} 2^{2sj+2tj'} c^2_{jkj'k'}, \qquad 0 \le s, t < p + \tfrac{1}{2}.$$

For $s = 1$ and $t = 0$, one accordingly has, for $u = \sum c_{jkj'k'}\, \psi_{jk} \otimes \psi_{j'k'}$,

$$\begin{aligned} \|\hat{P}_L^\beta u\|^2_{H^{1,0}([0,1]^2)} &= \left\| \sum_{(j,k,j',k') \in \hat{\Lambda}_L^\beta} c_{jkj'k'}\, \psi_{jk} \otimes \psi_{j'k'} \right\|^2_{H^{1,0}([0,1]^2)} \sim \sum_{(j,k,j',k') \in \hat{\Lambda}_L^\beta} 2^{2j} c^2_{jkj'k'} \\ &\le \sum_{j,k,j',k'} 2^{2j} c^2_{jkj'k'} \sim \|u\|^2_{H^{1,0}([0,1]^2)}. \end{aligned}$$ ■

Definition 5. For $\sigma, \tau \ge 0$ and $p, q \in \mathbf{N}$ let

$$H^{p,q}_{\sigma,\tau}([0,1]^2) := H^p_\sigma([0,1]) \otimes H^q_\tau([0,1]).$$

We will derive a consistency estimate for $\hat{P}_L^\beta$ by an abstract tensor product argument following the lines of [25]. Since $\|\varphi\|^2_{H^1([0,1]^2)} \le \|\varphi\|^2_{H^{1,0}([0,1]^2)} + \|\varphi\|^2_{H^{0,1}([0,1]^2)}$, it suffices (by symmetry) to estimate the $H^{1,0}([0,1]^2)$ norm of the error $u - \hat{P}_L^\beta u$. To simplify notation, we set

$$\|\cdot\| := \|\cdot\|_{H^1([0,1])} \otimes \|\cdot\|_{L^2([0,1])} = \|\cdot\|_{H^{1,0}([0,1]^2)}.$$

Let $Q^\beta_{1,i}$ be the orthogonal projection onto V_i^β with respect to the $H^1([0,1])$ norm, and let $Q^\beta_{2,i}$ be the orthogonal projection onto V_i^β with respect to the $L^2([0,1])$ norm.

We set

$$\hat{Q}_L^\beta := \sum_{i+j \le L} (Q^\beta_{1,i} - Q^\beta_{1,i-1}) \otimes (Q^\beta_{2,j} - Q^\beta_{2,j-1});$$

$\hat{Q}_L^\beta$ is the orthogonal projection onto $\hat{V}_L^\beta$ with respect to $\|\cdot\| = \|\cdot\|_{H^{1,0}([0,1]^2)}$. Note that, in general, $\hat{Q}_L^\beta \ne \hat{P}_L^\beta$ (but, of course, image $\hat{Q}_L^\beta =$ image $\hat{P}_L^\beta = \hat{V}_L^\beta$).

By stability of $\hat{P}_L^\beta$ in $H^1 \otimes L^2$, Lemma 4, we have

$$\begin{aligned}\|u - \hat{P}_L^\beta u\| &= \|u - \hat{Q}_L^\beta u + \hat{Q}_L^\beta u - \hat{P}_L^\beta u\| = \|u - \hat{Q}_L^\beta u + \hat{P}_L^\beta \hat{Q}_L^\beta u - \hat{P}_L^\beta u\| \\ &= \|(\mathrm{id} - \hat{P}_L^\beta)(u - \hat{Q}_L^\beta u)\| \le c\,\|u - \hat{Q}_L^\beta u\|.\end{aligned}$$

It therefore suffices to estimate the error $\|u - \hat{Q}_L^\beta u\|$ to get an approximation result for $\hat{P}_L^\beta$.

By orthogonality of the projectors $\hat{Q}_i^\beta$ we have, quite generally,

$$\begin{aligned}\|u - \hat{Q}_L^\beta u\|^2 &= \left\| u - \sum_{i+j\le L} (Q_{1,i}^\beta - Q_{1,i-1}^\beta) \otimes (Q_{2,j}^\beta - Q_{2,j-1}^\beta) u \right\|^2 \\ &= \left\| \sum_{i+j\ge L+1} (Q_{1,i}^\beta - Q_{1,i-1}^\beta) \otimes (Q_{2,j}^\beta - Q_{2,j-1}^\beta) u \right\|^2 \\ &= \left\| \sum_{i=0}^{L+1} (Q_{1,i}^\beta - Q_{1,i-1}^\beta) \otimes (\mathrm{id} - \hat{Q}_{2,L-i}^\beta) u \right\|^2 \\ &\quad + \left\| \sum_{i=L+2}^{\infty} (Q_{1,i}^\beta - Q_{1,i-1}^\beta) \otimes \mathrm{id}\ u \right\|^2 \\ &= \sum_{i=0}^{L+1} \|(Q_{1,i}^\beta - Q_{1,i-1}^\beta) \otimes (\mathrm{id} - \hat{Q}_{2,L-i}^\beta) u\|^2 + \|(\mathrm{id} - Q_{1,L+1}^\beta) \otimes \mathrm{id}\ u\|^2.\end{aligned}$$

Applying once more orthogonality, we estimate the last line by

$$\le \sum_{i=0}^{L+1} \|(\mathrm{id} - Q_{1,i-1}^\beta) \otimes (\mathrm{id} - \hat{Q}_{2,L-i}^\beta) u\|^2 + \|(\mathrm{id} - Q_{1,L+1}^\beta) \otimes \mathrm{id}\ u\|^2.$$

By stability of the orthogonal projections $Q_{r,i}^\beta$, we can translate this estimate back into an estimate involving the projectors P_L^β. To see this (using simplified notation), let Q, P, A be projectors with Q stable. Then it holds that

$$\begin{aligned}\|(\mathrm{id} - Q) \otimes Au\| &= \|(\mathrm{id} - P) \otimes Au + P \otimes Au - Q \otimes Au\| \\ &= \|(\mathrm{id} - P) \otimes Au + QP \otimes Au - Q \otimes Au\| \\ &= \|(\mathrm{id} - P) \otimes Au + (Q \otimes \mathrm{id})(P \otimes Au - \mathrm{id} \otimes A)u\| \\ &= \|(\mathrm{id} \otimes \mathrm{id} - Q \otimes \mathrm{id})((\mathrm{id} - P) \otimes A)u\| \le c\|(\mathrm{id} - P) \otimes Au\|.\end{aligned}$$

Hence, we finally get

$$\|u - \hat{P}_L^\beta u\|^2 \le c \left(\sum_{i=0}^{L+1} \|(\mathrm{id} - P_{i-1}^\beta) \otimes (\mathrm{id} - P_{L-i}^\beta) u\|^2 + \|(\mathrm{id} - P_{L+1}^\beta) \otimes \mathrm{id}\ u\|^2 \right).$$

Now using the one-dimensional consistency result, Lemma 3, we infer

$$\|u - \hat{P}_L^{\beta} u\|^2 \lesssim \sum_{i=0}^{L+1} L^2 2^{-2p(i-1)} 2^{-2(p+1)(L-i)} \|u\|^2_{H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)} + L 2^{-2p(L+1)} \|u\|^2_{H^{p+1}_{\gamma p}([0,1])\otimes L^2([0,1])},$$

hence,

$$\|u - \hat{P}_L^{\beta} u\|^2 \lesssim L^2 2^{-2pL} \|u\|^2_{H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)}.$$

We summarize:

Theorem 1. *For*

$$1 > \beta > \gamma > \frac{p + \frac{1}{2}}{p+1}$$

and $u \in H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2) \cap H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2)$, *it holds that*

$$\|u - \hat{P}_L^{\beta} u\|_{H^1([0,1]^2)} \lesssim L 2^{-pL} (\|u\|_{H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)} + \|u\|_{H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2)}).$$

Remark 3. Note, that it trivially holds that

$$H^{2(p+1)}([0,1]^2) \subset H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2) \cap H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2) \qquad \text{for} \quad \gamma \geq 0.$$

In an analogous fashion, one proves

Theorem 2. *For*

$$1 > \beta > \gamma > \frac{p + \frac{1}{2}}{p+1}$$

and $u \in H^{p+1,p+1}_{\gamma(p+1),\gamma(p+1)}([0,1]^2)$, *it holds that*

$$\|u - \hat{P}_L^{\beta} u\|_{L^2([0,1]^2)} \lesssim L^{3/2}\, 2^{-(p+1)L} \|u\|_{H^{p+1,p+1}_{\gamma(p+1),\gamma(p+1)}([0,1]^2)}.$$

Remark 4. In [21], P. Oswald has considered best N term approximation from sparse grid spaces built on the univariate Haar system ($p = 0$). He derives approximation rates in Sobolev spaces $H^s([0,1]^2)$, $-1 < s < \frac{1}{2}$. In the case of approximation in $L^2([0,1])$, i.e., $s = 0$, he achieves an approximation rate of $N^{-1}\,(\log N)^{3/2}$ for singularity functions of type $(1;\alpha,\alpha)$, $\alpha < \frac{1}{2}$. Here, a function $f(x_1,x_2)$ is said to be of type $(m;\alpha,\alpha)$, if $|\partial_1^k \partial_2^l f| \leq C x_1^{-(\alpha+k)}\, x_2^{-(\alpha+l)}$ for $0 \leq k, l \leq m$.

A singular function of type $(1;\alpha,\alpha)$ belongs to $H^{1,1}_{\gamma,\gamma}([0,1]^2)$ if $\gamma > \frac{1}{2} + \alpha$. Hence, our result is consistent with Oswald's.

3.1. *Application to Elliptic Singularities in Two Dimensions*

Consider an elliptic boundary value problem on a polygonal domain $\Omega \subset \mathbf{R}^2$:

$$Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \partial\Omega,$$

where L is a smooth, uniformly elliptic, linear second-order differential operator in divergence form, B is a smooth boundary operator of Dirichlet or Neumann type, and f and g are smooth functions.

The weak solution u is smooth (in the Sobolev scale) in the interior of Ω, but exhibits corner singularities, which reduce global regularity dramatically. In general, $u \notin H^2(\Omega)$. It is well-known, see [5], [6], [7], that there holds a decomposition

$$u = u_{\text{reg}} + u_{\text{sing}},$$

with (arbitrarily) smooth u_{reg}, which we specialize to $u_{\text{reg}} \in H^{2(p+1)}(\Omega)$, and (depending on the chosen smoothness of u_{reg}) a finite sum

$$u_{\text{sing}} = \sum^{<\infty} \operatorname{Re}(r_i^{\lambda_i}(\log r_i)^{\mu_i}\Phi_i(\varphi_i)\eta_i(x)),$$

(in local polar coordinates (r_i, φ_i) associated to the corners) with smooth cut-off functions η_i, smooth functions Φ_i of the angle coordinates φ_i, and parameters $\lambda_i \in \mathbf{C}$, $\mu_i \in \mathbf{N}_0$.

For Dirichlet–Dirichlet corners, one knows Re $\lambda_i > \frac{1}{2}$, and for Dirichlet–Neumann corners, Re $\lambda_i > \frac{1}{4}$.

We prove now that this type of singularity can be approximated at a rate p in the H^1 norm. Note that we can allow Re $\lambda > 0$.

Theorem 3. *Let*

$$u(x) = \operatorname{Re}(|x|^\lambda(\log|x|)^\mu\,\Phi(\varphi)\,\eta(x)) + g(x) \in H^1([0,1]^2)$$

with smooth η and Φ, parameters $\lambda \in \mathbf{C}$, $\operatorname{Re}(\lambda) > 0$, $\mu \in \mathbf{N}_0$, and a function $g \in H^{2(p+1)}([0,1]^2)$. Then there are β, γ (depending on λ) with

$$1 > \beta > \gamma > \frac{p+\frac{1}{2}}{p+1},$$

such that

$$u \in H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2) \cap H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2).$$

This means there is a constant C depending via β and γ only on λ, such that

$$\|u - \hat{P}^\beta_L u\|_{H^1([0,1]^2)} \le CL2^{-pL}(\|u\|_{H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)} + \|u\|_{H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2)}).$$

Concerning approximability in L^2, we have:

If $\operatorname{Re}(\lambda) > -1$ *and the remaining assumptions as above are met, then there are β, γ with*

$$1 > \beta > \gamma > \frac{p+\frac{1}{2}}{p+1},$$

such that

$$u \in H^{p+1,p+1}_{\gamma(p+1),\gamma(p+1)}([0,1]^2).$$

This means there is a constant C such that

$$\|u - \hat{P}^{\beta}_L u\|_{L^2([0,1]^2)} \le C L^{3/2} 2^{-(p+1)L} \|u\|_{H^{p+1,p+1}_{\gamma(p+1),\gamma(p+1)}([0,1]^2)}.$$

Remark 5. Note that we are only concerned with the approximation of certain singular terms, which are defined on the unit square. In this paper, we do not address the question of handling complex geometries with sparse grids within a numerical scheme.

Proof. The norm of $H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)$ is given by

$$\|u\|_{H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)} = \left(\sum_{0 \le i,j \le p+1} \int_{[0,1]^2} |(D_1^i D_2^j u) x_1^{\gamma p} x_2^{\gamma(p+1)}|^2 \, dx \right)^{1/2}.$$

Estimating $x_1^{\gamma p} x_2^{\gamma(p+1)}$ by $|x|^{\gamma(2p+1)}$, we see that $\mathrm{Re}(|x|^{\lambda}(\log|x|)^{\mu}) \in H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2)$ if

$$\gamma > 1 - \frac{\lambda}{2p+1}.$$

The same holds for the space $H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2)$.

The smooth factors η and Φ leave the regularity class unchanged, and g is trivially in the space $H^{p+1,p+1}_{\gamma p,\gamma(p+1)}([0,1]^2) \cap H^{p+1,p+1}_{\gamma(p+1),\gamma p}([0,1]^2)$, see Remark 3.

Hence, if $\lambda > 0$, we can find $1 > \beta > \gamma > 1 - \lambda/(2p+1)$ such that the assumptions of Theorem 1 are met.

The proof of the L^2 estimate is done analogously. ∎

4. Generalization to d Dimensions

The d-dimensional case with $d > 2$ follows from the two-dimensional one by iteration. We consider the domain $[0,1]^d$ and the tensor product basis

$$\{\psi_{j_1k_1\cdots j_dk_d} = \psi_{j_1k_1} \otimes \cdots \otimes \psi_{j_dk_d} : (j_i, k_i) \in I,\ i = 1, \ldots, d\}.$$

Let $\beta \in [0,1)$ be fixed and let $L \in \mathbf{N}$. Then we define the index sets

$$\hat{\Lambda}^{\beta}_{L,d} \equiv \bigcup_{j_1+\cdots+j_d \le L} \Lambda^{\beta}_{j_1} \times \cdots \times \Lambda^{\beta}_{j_d}$$

and the trial spaces

$$\hat{V}^{\beta}_{L,d} := \mathrm{span}\{\psi_{j_1k_1\cdots j_dk_d} : (j_1, k_1, \ldots, j_d, k_d) \in \hat{\Lambda}^{\beta}_{L,d}\} = \bigoplus_{j_1+\cdots+j_d \le L} V^{\beta}_{j_1} \otimes \cdots \otimes V^{\beta}_{j_d}.$$

From (5) we see that

$$\#\hat{\Lambda}^{\beta}_{L,d} = \dim(\hat{V}^{\beta}_{L,d}) \lesssim p^d L^{d-1} 2^L.$$

The constant depends here, and in the sequel, additionally on the dimension d.

Definition 6 (Projector $\hat{P}^{\beta}_{L,d}$). Let u be a function with wavelet decomposition

$$u = \sum_{j_1,k_1,\dots,j_d,k_d} c_{j_1k_1\cdots j_dk_d}\psi_{j_1k_1}\otimes\cdots\otimes\psi_{j_dk_d}.$$

Then the projector $\hat{P}^{\beta}_{L,d}$ is defined by

$$\hat{P}^{\beta}_{L,d}u \equiv \sum_{j_1,k_1,\dots,j_d,k_d\in\hat{\Lambda}^{\beta}_{L,d}} c_{j_1k_1\cdots j_dk_d}\psi_{j_1k_1}\otimes\cdots\otimes\psi_{j_dk_d}.$$

Definition 7. Let $(s_1,\dots,s_d)\in\mathbf{N}^d$. Then we denote by

$$H^{s_1,\dots,s_d}([0,1]^d)$$

the space of all measurable functions $u:[0,1]^d\to\mathbf{R}$, such that the norm

$$\|u\|_{H^{s_1,\dots,s_d}([0,1]^d)} := \left(\sum_{0\le j_i\le s_i}\|D_1^{j_1}\cdots D_d^{j_d}u\|^2_{L^2([0,1]^d)}\right)^{1/2}$$

is finite. That is, $H^{s_1,\dots,s_d}([0,1]^d)=\bigotimes_{i=1,\dots,d}H^{s_i}([0,1])$.

Definition 8. For $\sigma_1,\dots,\sigma_d\ge 0$ and $p_1,\dots,p_d\in\mathbf{N}$ let

$$H^{p_1,\dots,p_d}_{\sigma_1,\dots,\sigma_d}([0,1]^d) := \bigotimes_{i=1,\dots,d}H^{p_i}_{\sigma_i}([0,1]).$$

We further introduce the abbreviation

$$H(\gamma,p,d) := \bigcap_{k=1,\dots,d}\left(\bigotimes_{i=1,\dots,d}H^{p+1}_{\gamma(p+1-\delta_{ki})}([0,1])\right)$$

and take as a norm the summed up norms.

Remark 6. Note, that it trivially holds that

$$H^{d(p+1)}([0,1]^d)\subset H(\gamma,p,d) \qquad \text{for} \quad \gamma\ge 0.$$

With the latter notation, Theorem 1 may be rephrased as follows:

$$\beta>\gamma>\frac{p+\frac{1}{2}}{p+1},\quad u\in H(\gamma,p,d)\quad\Rightarrow\quad \|u-\hat{P}^{\beta}_{L,2}u\|_{H^1([0,1]^d)}\lesssim L2^{-pL}\|u\|_{H(\gamma,p,d)}.$$

Iterating the argument from Section 3, we infer the following two theorems:

Theorem 4 (H^1 consistency of $\hat{P}_L^\beta$). *For*

$$1 > \beta > \gamma > \frac{p+\frac{1}{2}}{p+1}$$

and $u \in H(\gamma, p, d)$, *it holds that*

$$\|u - \hat{P}_{L,d}^\beta u\|_{H^1([0,1]^d)} \lesssim L^{d/2}\, 2^{-pL} \|u\|_{H(\gamma,p,d)}.$$

Letting $N_L := \dim\, \hat{V}_L^\beta \lesssim p^d L^{d-1} 2^L$ *be the number of degrees of freedom, this means that*

$$\begin{aligned} \|u - \hat{P}_{L,d}^\beta u\|_{H^1([0,1]^d)} &\lesssim N_L^{-p}\,(\log_2 N_L)^s \|u\|_{H(\gamma,p,d)}, \\ s &= (d-1)p + d/2. \end{aligned}$$

Theorem 5 (L^2 consistency of $\hat{P}_L^\beta$). *For*

$$1 > \beta > \gamma > \frac{p+\frac{1}{2}}{p+1}$$

and $u \in H^{p+1,\ldots,p+1}_{\gamma(p+1),\ldots,\gamma(p+1)}([0,1]^d)$, *it holds that*

$$\|u - \hat{P}_{L,d}^\beta u\|_{L^2([0,1]^d)} \lesssim L^{d-1/2}\, 2^{-(p+1)L} \|u\|_{H^{p+1,\ldots,p+1}_{\gamma(p+1),\ldots,\gamma(p+1)}([0,1]^d)}.$$

In terms of numbers of degrees of freedom, this means that

$$\begin{aligned} \|u - \hat{P}_{L,d}^\beta u\|_{L^2([0,1]^d)} &\lesssim N_L^{-(p+1)}(\log_2 N_L)^s \|u\|_{H^{p+1,\ldots,p+1}_{\gamma(p+1),\ldots,\gamma(p+1)}([0,1]^d)}, \\ s &= (d-1)p + 2d - \tfrac{3}{2}. \end{aligned}$$

Remark 7. There holds the continuous inclusion

$$H(\gamma, p, d) \subset H^{p+1,\ldots,p+1}_{\gamma(p+1),\ldots,\gamma(p+1)}([0,1]^d).$$

Remark 8. By interpolating the estimates of Theorems 4 and 5, consistency estimates in $H^s([0,1]^d)$, $0 < s < 1$ are readily obtained.

Remark 9. The function $x \mapsto \operatorname{Re}\, |x|^\lambda$ on $[0,1]^d$ belongs to $H(\gamma, p, d)$, if

$$\gamma > 1 - \frac{(d-2) + 2\lambda}{2d(p+1) - 2}.$$

So, if

$$\lambda > 1 - d/2,$$

we can find such a $\gamma < 1$.

More generally, let ρ be the distance to a face of $[0,1]^d$ of dimension k containing the origin (for $k = 0$ the corner 0, for $k = 1$ an edge containing 0, etc.). Then the function $x \mapsto \operatorname{Re} \rho^\lambda$ on $[0,1]^d$ belongs to some $H(\gamma, p, d)$ with $\gamma < 1$, if

$$\lambda > 1 - (d-k)/2.$$

Note that this condition on the exponent λ coincides with the condition of membership in $H^1([0,1]^d)$, which indicates the sharpness of our result with respect to powers of distance functions to lower-dimensional faces.

4.1. *Application to Elliptic Singularities in Three Dimensions*

We apply our approximation result to elliptic corner-edge singularities in three dimensions. Consider on a polyhedral domain Ω (for instance, $[0,1]^3$) the boundary value problem

$$Lu = f \quad \text{in } \Omega, \qquad Bu = g \quad \text{on } \partial\Omega, \tag{15}$$

where L is a smooth, uniformly elliptic, linear second-order differential operator in divergence form, B is a smooth boundary operator of face-wise Dirichlet or Neumann type, and f and g are smooth functions.

As in the two-dimensional case, the weak solution u is smooth away from corners and edges, but exhibits singularities toward the singularities of the domain. The solution's asymptotics toward corners and edges can be described by weighted Sobolev spaces, and we quote only one result (in simplified form), which was proven by Maz'ya and Roßmann (Theorem 2 in [19]):

There is $\varepsilon > 0$ (depending on the domain), such that for $\delta \in \mathbf{R}$ and $\ell \in \mathbf{N}$ with $|\delta - \ell - 1| < \varepsilon$ there holds

$$f \in V_\delta^\ell \quad \Rightarrow \quad u \in V_\delta^{\ell+2}.$$

Here, the space V_δ^ℓ is defined as the set of functions v, for which

$$\left(\int_\Omega \left(\prod r_i^{2\delta} \right) \left(\prod (\rho_j / r)^{2\delta} \right) \sum_{|\alpha| \le \ell} \rho^{2(|\alpha| - \ell)} |D^\alpha u|^2 \, dx \right)^{1/2}$$

is finite. The functions r_i are the distance functions to the vertices of Ω, the functions ρ_j are the distance funtions to the edges of Ω, and $r = \min_i r_i$, $\rho = \min_j \rho_j$.

For the sake of simplicity of exposition, we specialize this to the case of one "active" cube-like corner (which is supposed to be the origin). Then the norm of V_δ^ℓ reads

$$\left(\int_\Omega r^{2\delta} \left(\prod_{j=1}^{3} (\rho_j / r)^{2\delta} \right) \sum_{|\alpha| \le \ell} \rho^{2(|\alpha| - \ell)} |D^\alpha u|^2 \, dx \right)^{1/2}.$$

Using the general relations $\rho \le \rho_j \le r$, a calculation shows that

$$f \in C^\infty(\overline{\Omega}) \quad \Rightarrow \quad f \in V_\delta^\ell \qquad \text{for all} \quad \delta \ge \ell.$$

Now choose $\ell = 3p+1$ and $\delta = \ell + 1 - \varepsilon_0 = 3p + 2 - \varepsilon_0$ for some $0 < \varepsilon_0 < \min\{\varepsilon, 1\}$. From the above shift theorem, we see that $u \in V_\delta^{3p+3}$. But this readily implies $u \in H(\gamma, p, 3)$ for

$$\frac{\delta}{3p+2} = \frac{3p+2-\varepsilon_0}{3p+2} < \gamma < 1.$$

We conclude:

Theorem 6. *The solution u of problem* (15) *lies locally in some space $H(\gamma, p, 3)$ with $\gamma < 1$ and may therefore* (*as a function transferred to the unit cube*) *be approximated at a rate p in the H^1 norm.*

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References

1. TH. APEL (1999): Anisotropic Finite Elements: Local Estimates and Applications. Advances in Numerical Mathematics. Stuttgart: Teubner.
2. A. BARINKA, T. BARSCH, P. CHARTON, A. COHEN, S. DAHLKE, W. DAHMEN, K. URBAN (2001): *Adaptive wavelet schemes for elliptic problems—Implementations and numerical experiments*. SIAM J. Sci. Comput., **23**:910–939.
3. P. BINEV, W. DAHMEN, R. DEVORE (2002): *Adaptive finite element methods with convergence rates*. Preprint.
4. C. BOURGEOIS, S. NICAISE (1999): *Approximation par préondelettes augmentée de l'équation de la plaque libre polygonale*. C. R. Acad. Sci. Paris Sér. I, **329**:727–730.
5. M. BOURLARD, M. DAUGE, M. LUBUMA, S. NICAISE (1990): *Coefficients des singularitétes pour des problèmes aux limites elliptiques sur un domaine à points coniques*. I: *Résultats généraux pour le probème de Dirichlet*. RAIRO Modél. Anal. Numér., **24**:27–52.
6. M. BOURLARD, M. DAUGE, M. LUBUMA, S. NICAISE (1990): *Coefficients des singularitétes pour des problèmes aux limites elliptiques sur un domaine à points coniques*. II: *Quelques opératuers particuliers*. RAIRO Modél. Anal. Numér., **24**:343–367.
7. M. BOURLARD, M. DAUGE, M. LUBUMA, S. NICAISE (1992): *Coefficients of the singularities for elliptic boundary value problems on domains with conical points*. III: *Finite element methods on polygonal domains*. SIAM J. Numer. Anal., **29**:136–155.
8. H. BUNGARTZ, M. GRIEBEL (1999): *A note on the complexity of solving Poisson's equation for spaces of bounded mixed derivatives*. J. Complity, **15**:167–199.
9. A. COHEN, W. DAHMEN, R. DEVORE (2000): *Adaptive wavelet methods for elliptic operator equations*: *Convergence rates*. Math. Comp., **70**(233):27–75.
10. W. DAHMEN, A. KUNOTH, K. URBAN (1999): *Biorthogonal spline-wavelets on the interval—Stability and moment conditions*. Appl. Comput. Harmonic Anal., **6**:132–196.
11. W. DAHMEN, R. SCHNEIDER (1998): *Wavelets with complementary boundary conditions—Function spaces on the cube*. Resultate Math., **34**:255–293.
12. M. DAUGE (1988): Elliptic Boundary Value Problems on Corner Domains: Smoothness and Asymptotics of Solutions. Berlin: Springer-Verlag.
13. R. DEVORE (1998): *Nonlinear approximation*. In: Acta Numerica 7. Cambridge: Cambridge University Press, pp. 51–150.

14. D. GILBARG, N. S. TRUDINGER (1998): Elliptic Partial Differential Equations of Second Order. New York: Springer-Verlag.
15. M. GRIEBEL, S. KNAPEK (2000): *Optimized tensor–product approximation spaces*. Constr. Approx., **16**:525–540.
16. M. GRIEBEL, P. OSWALD, T. SCHIEKOFER (1999): *Sparse grids for boundary integral equations*. Numer. Math., **83**:279–312.
17. P. GRISVARD (1985): Elliptic Problems in Nonsmooth Domains. Boston: Pitman.
18. V. A. KOZLOV, V. G. MAZYA, J. ROSSMANN (1997): Elliptic Boundary Value Problems in Domains with Point Singularities. Providence, RI: American Mathematical Society.
19. V. B. MAZ'YA, J. ROßMANN (1991): *On the Agmon–Miranda maximum principle*. Ann. Global Anal. Geom., **9**:253–303.
20. P. MORIN, R. NOCHETTO, K. SIEBERT (2000): *Data oscillation and convergence of adaptive FEM*. SIAM J. Numer. Anal., **38**(2).
21. P. OSWALD (1999): *On best n-term approximation by Haar functions in H^s-norms*. In: Metric Function Theory and Related Topics in Analysis (Nicolskij, Kashin, Izaak, eds.). Moscow: AFC, pp. 137–163.
22. P. OSWALD (1999): *Best n-term capacitance approximation on sparse grids*. Proc. 12th Intern. Conf. on Domain Decomp. Methods in Science and Engineering (T. Chan et al., eds.). Chiba, pp. 437–444.
23. T. VON PETERSDORFF, CH. SCHWAB (1995): *Boundary element methods with wavelets and mesh refinement*. Research Report No. 95-10, Seminar für Angewandte Mathematik, ETH Zürich.
24. T. VON PETERSDORFF, CH. SCHWAB, R. SCHNEIDER (1997): *Multiwavelets for second-kind integral equations*. SIAM J. Numer. Anal., **34**:2212–2227.
25. CH. SCHWAB, R. A. TODOR (2003): *Sparse finite elements for elliptic problems with stochastic data*. SAM ETHZ Report 2002-05 (available online under: http://www.sam.math.ethz.ch/reports/). To appear in Numer. Math.
26. CH. ZENGER (1991): *Sparse Grids*; *in Parallel Algorithms for PDE*. Proc. 6th GAMM Seminar, Kiel (W. Hackbusch, ed.). Braunschweig: Vieweg, pp. 241–251.

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