J Nonlinear Sci (2012) 22:463–497
DOI 10.1007/s00332-012-9143-4

Journal of Nonlinear Science

Reduced Variational Formulations in Free Boundary Continuum Mechanics

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Received: 18 April 2011 / Accepted: 2 July 2012 / Published online: 18 July 2012

Abstract We present the material, spatial, and convective representations for elasticity and fluids with a free boundary from the Lagrangian reduction point of view, using the material and spatial symmetries of these systems. The associated constrained variational principles are formulated and the resulting equations of motion are deduced. In addition, we introduce general free boundary continua that contain both elasticity and free boundary hydrodynamics, extend for them various classical notions, and present the constrained variational principles and the equations of motion in the three representations.

Keywords Symmetry · Reduction · Variational principle · Constraints · Material representation · Spatial representation · Convective representation · Lagrangian · Rigid body · Ideal fluid · Elastic material

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We fondly dedicate this paper to the memory of our dear friend and coauthor Jerry Marsden who sadly passed away as this project was being finished

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Mathematics Subject Classification 37K05 · 37K65 · 75B20 · 35R35

1 Introduction

In continuum mechanics, the material, spatial, and convective representations are fundamental for the mathematical description of the motion. Depending on the problem and on the questions posed, one of these representations may be more convenient than the others. Nevertheless, insight from the other representations may be crucial in the analysis of the equations of motion.

For example, in free rigid body dynamics, the material description is preferred when one wants to emphasize that the attitude matrix moves along a geodesic of the metric characterized by the mass distribution of the body. In this case, the associated variational formulation (i.e., the critical action principle) is the standard Hamilton's principle on the tangent bundle of the proper rotation group. However, the equations of motion are more easily handled when written in terms of the convective or spatial angular velocities since these variables belong to the Lie algebra rather than to the Lie group. As opposed to what happens for the material representation, the critical action principles underlying the convective and spatial formulations are not standard since they require the use of constrained variations. It is, however, important to recall that these principles can be naturally deduced and justified by applying the techniques of Lagrangian reduction by symmetries. Among the two symmetry reduced formulations (convective and spatial), the convective representation is preferred since it is associated to a natural symmetry of the material Lagrangian and does not require the introduction of additional variables. However, the spatial angular momentum, a quantity that naturally appears in the spatial representation, is a conserved quantity which plays a fundamental role in the description of the dynamics, and makes the spatial formulation also attractive, at the price of introducing a new dynamic variable representing the spatial inertia tensor, as we will recall below.

For ideal homogeneous incompressible flow, one usually works in spatial representation, since it corresponds to the natural invariance of the material Lagrangian, namely, relabeling symmetry. But if one is interested in Lagrangian coherent states and transport in ocean motion, for example, it is the material representation that is best suited. In elasticity, as opposed to hydrodynamics, the convective representation dominates because it is associated to a symmetry that is always assumed, namely, material frame indifference. However, if the material is isotropic, the material Lagrangian also possesses a relabeling symmetry and the spatial representation gives a lot of insight, see e.g. Chap. 12, Sect. 15 in Antman (2004). Similarly, in both fluid–solid interactions as well as in the dynamics of complex fluids, it is desirable to have a consistent description in the spatial, the material, or the convective pictures.

The goal of this paper is to develop such methods systematically and in an unified way for free boundary nonlinear continuum mechanics. More precisely, the validity of the relevant variational principles in the various representation is not postulated in an ad hoc way but rather is derived from the standard Hamilton principle in material representation using the geometric methods of reduction by symmetry. As we already mentioned for the rigid body case, these principles are not trivial when written uniquely in terms of the convective or spatial variables, since they involve constrained variations. In addition, we propose a general class of continua that contains both free boundary hydrodynamics and nonlinear elasticity, and develop the variational principles as well as the motion equations in material, spatial, and convective representation for them. The condition under which the spatial reduction is allowable for this general class is also determined.

In this paper we address only the Lagrangian reduction approach (as opposed to the Hamiltonian approach related to Poisson structures) for two reasons. First, from a practical point of view, if one needs to derive the equations of motions of a continuum system whose energy is known, it is more efficient to work directly via the critical action principle and its reduced version, rather than take the route of the Poisson reduction approach. The Lagrangian reduction approach also yields Kelvin circulation theorems in both the spatial and the convective pictures. Second, the critical action principle is crucial to derive discretization algorithms. For example, it is well known that, in some particular cases, the standard (i.e., in material representation) Hamilton's principle in continuum mechanics is intimately related to the weak formulation of the equations, which is of fundamental importance for finite element methods, as indicated by the huge literature on the subject. We refer, e.g., to Chap. 5 in Marsden and Hughes (1983) for further information about how these weak formulations in material representation can be recast into the modern context of geometric Lagrangian mechanics. However, this is not the kind of application we have in mind when developing the constrained variational principle for continua in this paper. We are thinking of discretization methods that derive the discrete equations of motion in a geometric way directly via a discretization of the action principle and therefore respect the geometry of the continuous system. For these discretization methods, a geometric reduced description, as provided in this paper, is of crucial importance. These so called variational integrators (see Marsden and West 2001 and references therein) are examples of symplectic integrators (Hairer et al. 2000) and therefore exhibit an excellent energy behavior; some of them even preserve exactly various conservation laws of the original continuum system. For example, in Pavlov et al. (2011) a variational integrator was developed for the Euler equations of an ideal fluid, by using the constrained variational principle in pure Eulerian variables. Thanks to its geometrical character, it was then possible to extend this approach to include the case of magnetohydrodynamics and planar liquid crystals (Gawlik et al. 2011) and the case of rotating Boussinesq equations (Desbrun et al. 2012) by using in a crucial way the constrained variational principles associated to the spatial formulation of these systems (Holm et al. 1998; Gay-Balmaz and Ratiu 2009). This program has just started and its extension to free boundary continuum mechanics needs the developments made in this present paper and is the subject of current work.

The results of this paper are achieved by implementing the philosophy of Euler–Poincaré reduction for a semidirect product involving the diffeomorphism group of the fluid's domain. We hasten to add, however, that we cannot just apply this existing theory since the continuum models we consider (except for the case of fixed boundary fluid) do not satisfy the hypotheses of the Euler–Poincaré reduction theorem which requires the configuration space to be a group. Rather, these theories are based on a manifold of embeddings.

Before giving the plan of the paper and recalling the Euler–Poincaré theory, we mention now some historical considerations. It is difficult to know when variational principles were first used in continuum mechanics but Walter (1868) and Kirchhoff (1883) already had such formulations. The introduction of Clebsch potentials (Clebsch 1857, 1859) allowed a classical variational formulation at the expense of the introduction of new variables. The same situation occurred in the search for a Hamiltonian formulation of the Euler equations of the free rigid body in three dimensional space and its associated variational principle. This was accomplished by the introduction of the four Cayley–Klein parameters (Klein 1897, equivalently, Kustaanheimo–Stiefel coordinates, or the Hopf fibration) that played the same role in rigid body dynamics as the Clebsch variables in hydrodynamics. We refer to the extensive surveys (Truesdell and Toupin 1960) and (Serrin 1959) for a historical account and many references up to 1960. The classical Lin constraints in hydrodynamics, introduced in Lin (1963) and applied to many examples in Seliger and Whitham (1968), were intimately related to the attempt to formulate a variational principle using exclusively spatial quantities.

On the Hamiltonian side, the noncanonical Poisson structures for various systems in continuum mechanics and its relationship with the Clebsch representation was developed in Sudarshan and Mukunda (1974), Iwinski and Turski (1976), Dzyaloshinskiĭ and Volovick (1980), Morrison and Greene (1980), Holm and Kupershmidt (1983) and others. Poisson brackets in plasma physics and continuum mechanics were further studied and obtained by reduction of the canonical symplectic form on the material phase space in Marsden and Weinstein (1982, 1983) and Marsden et al. (1984) for plasmas and fluids in spatial representation, in Simo et al. (1988) for nonlinear elasticity in convective representation, in Holm et al. (1986) for fluids in the convective representation, in Lewis et al. (1986) and Mazer and Ratiu (1989) for free boundary fluids in spatial representation.

On the Lagrangian side, a first approach to constrained variational principles on Lie groups and its relation with Lin constraints and Clebsch potentials for fixed boundary fluid dynamics was established in Cendra and Marsden (1987). Lagrangian reduction for fixed boundary fluids in the spatial representation and the associated variational principles were further developed in Holm et al. (1998), via Euler–Poincaré reduction for semidirect products. So far, the constrained variational principle for fixed boundary fluid in convective variables, for free boundary fluids and nonlinear elasticity in convective and spatial variables, as well as the associated Lagrangian reduction processes have not been carried out, and the goal of this paper is to develop these approaches.

Recently, the techniques of Hamiltonian and Lagrangian reductions have been extended to more general hydrodynamical systems such as complex fluids (e.g. liquid crystals), Yang–Mills fluids, or other models of nonabelian fluids (see e.g., Gay-Balmaz and Ratiu 2008a, 2008b, 2009, 2011; Gay-Balmaz and Tronci 2010).

The plan of the paper is the following. Section 2 applies the Euler–Poincaré reduction theorem for semidirect products to develop the Lagrangian convective formulation of compressible hydrodynamics; both the constrained variational principle and the equations of motion are presented. Nonlinear elasticity is studied in Sect. 3. In this case, Euler–Poincaré reduction theory does not apply but we use it as a guide. The constrained variational principles and the equations of motion are deduced from the material representation in both convective and spatial formulation. The same program is carried out in Sect. 4 for free boundary compressible fluids with surface tension. Section 5 proposes a model of continua that contains both elasticity and free boundary hydrodynamics. For this model we extend some of the classical notions such as material and spatial symmetries, deduce the constrained variational principles in pure spatial or convective variables, and derive the equations of motion in both convective and spatial representations.

In this paper we do not go into the analytic details of the relevant function spaces for the problems, although for some of them, this is fairly straightforward. Our main focus is the geometric setting. In particular, the solutions we study are assumed to exist (prior to shocks) and to be smooth.

We close this introduction with a brief review of Euler–Poincaré reduction for semidirect products since it will be used for the spatial and convective formulations of fixed boundary fluids and because it serves as a guide for the variational formulation of problems with free boundaries of general continua.

Generalities on Semidirect Products We briefly recall some general facts and formulas for semidirect products. Consider a *left* (respectively, *right*) representation of a Lie group G on a vector space V. We denote by $v \mapsto gv$ (respectively, $v \mapsto vg$) this representation. As a set, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$ whose group multiplication is given by

$$\begin{aligned} &(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \quad \text{respectively,} \\ &(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2). \end{aligned}$$

The Lie algebra of S is the semidirect product Lie algebra, $\mathfrak{s} = \mathfrak{g} \circledS V$, whose bracket has the expression

$$\begin{aligned} &\big[(\xi_1, v_1), (\xi_2, v_2)\big] = \big([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1\big), \quad \text{respectively,} \\ &\big[(\xi_1, v_1), (\xi_2, v_2)\big] = \big([\xi_1, \xi_2], v_1 \xi_2 - v_2 \xi_1\big), \end{aligned}$$

where ξv (respectively, $v\xi$) denotes the induced action of the Lie algebra $\mathfrak{g}$ on V. From the expression for the Lie bracket, it follows that for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ the infinitesimal coadjoint action reads

$$\begin{aligned} &\mathrm{ad}^*_{(\xi,v)}(\mu, a) = \big(\mathrm{ad}^*_\xi \mu - v \diamond a, -\xi a\big), \quad \text{respectively,} \\ &\mathrm{ad}^*_{(\xi,v)}(\mu, a) = \big(\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi\big), \end{aligned}$$

where $\xi a \in V^*$ (respectively, $a\xi \in V^*$) denotes the induced action of $\mathfrak{g}$ on V^* and $v \diamond a \in \mathfrak{g}^*$ is defined by

$$\langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle \xi a, v \rangle_V \quad \text{and} \quad \langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V,$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Lagrangian Semidirect Product Theory We recall some needed facts about Euler–Poincaré reduction for semidirect products (see Holm et al. 1998).

- Assume that we have a function $L : TG \times V^* \to \mathbb{R}$ which is *left* (respectively, *right*) G-invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right (respectively, left) invariant under the lift to TG of the left (respectively, right) action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.
- G-invariance of L permits us to define the reduced Lagrangian $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l\big(g^{-1}v_g, g^{-1}a_0\big) = L(v_g, a_0), \quad \text{respectively,} \quad l\big(v_g g^{-1}, a_0 g^{-1}\big) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := g(t)^{-1}\dot{g}(t)$ (respectively, $\xi(t) := \dot{g}(t)g(t)^{-1}$). Define the curve $a(t)$ as the unique solution of the following linear differential equation with time-dependent coefficients:

$$\dot{a}(t) = -\xi(t)a(t), \quad \text{respectively,} \quad \dot{a}(t) = -a(t)\xi(t),$$

  with initial condition $a(0) = a_0$. The solution can be written as $a(t) = g(t)^{-1}a_0$, respectively, $a(t) = a_0 g(t)^{-1}$.

Theorem 1.1 *With the preceding notations, the following are equivalent:*

(i) *With a_0 held fixed, Hamilton's variational principle,*

$$\delta \int_{t_1}^{t_2} L_{a_0}\big(g(t), \dot{g}(t)\big)\,\mathrm{d}t = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

(iii) *The constrained variational principle,*

$$\delta \int_{t_1}^{t_2} l\big(\xi(t), a(t)\big)\,\mathrm{d}t = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} + [\xi, \eta], \qquad \delta a = -\eta a, \quad \textit{respectively,}$$
$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \qquad \delta a = -a\eta,$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

(iv) *The Euler–Poincaré equations hold on $\mathfrak{g} \times V^*$:*

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} &= \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a, \quad \textit{respectively,} \\ \frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} &= -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a. \end{aligned} \tag{1.1}$$

As a basic example, for the illustration of the convective and spatial descriptions in mechanics and its associated variational principles, we consider the case of a free rigid body moving around a fixed point (see Holm et al. 1986 for the general case of a heavy top). The attitude of the body is completely described by a matrix A in the Lie group $SO(3)$. The Lagrangian in material representation is obtain by integrating the kinetic energy of each material point X of the body over the reference configuration space $\mathcal{B} \subset \mathbb{R}^3$ of the body, a compact subset of $\mathbb{R}^3$ with non-empty interior. Hence

$$L(A, \dot{A}) = \frac{1}{2} \int_{\mathcal{B}} \rho(X) \| \dot{A} X \| \, \mathrm{d}^3 X$$

and the equations of motion in material representation are given by the associated Euler–Lagrange equations, producing geodesic equations for the associated Riemannian metric on $SO(3)$. As is well known, these equations are obtained by applying Hamilton's principle,

$$\delta \int_{t_0}^{t_1} L(A, \dot{A}) \, \mathrm{d}t = 0,$$

for arbitrary variations δA vanishing at the endpoints. It is readily checked that the material Lagrangian is invariant under left translation by $SO(3)$, so that it can be written uniquely in terms of the convective angular velocity $\Omega = A^{-1}\dot{A} \in \mathfrak{so}(3)$. This allows us to write the reduced Lagrangian as

$$\ell_{\mathrm{conv}}(\boldsymbol{\Omega}) = \frac{1}{2} \mathbf{I} \boldsymbol{\Omega} \cdot \boldsymbol{\Omega},$$

where $\boldsymbol{\Omega} \in \mathbb{R}^3$ is the vector associated to the Lie algebra element $\Omega \in \mathfrak{so}(3)$ (i.e., $\Omega \mathbf{u} = \boldsymbol{\Omega} \times \mathbf{u}$ for any $\mathbf{u} \in \mathbb{R}^3$) and $\mathbf{I}$ is the inertia tensor of the body. To obtain the classical Euler equations in convective representation

$$\mathbf{I}\dot{\boldsymbol{\Omega}} = \mathbf{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}$$

from a variational principle in pure convective variables, one has to use constrained variations, namely

$$\delta \int_{t_0}^{t_1} \ell_{\mathrm{conv}}(\boldsymbol{\Omega}) \, \mathrm{d}t = 0, \qquad \text{for variations } \delta \boldsymbol{\Omega} = \partial_t \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Psi},$$

where $\boldsymbol{\Psi}$ is a arbitrary curve in $\mathbb{R}^3$ vanishing at the endpoints, consistently with the general Euler–Poincaré theory recalled above.

If one wants to write the equations of motion in spatial representation, namely by using the spatial angular momentum $\omega = \dot{A}A^{-1}$, one has to implement reduction relative to the right translation by $SO(3)$. Such a symmetry can be obtained only by introducing a new variable, namely, the spatial inertial tensor $\mathbf{I}_{\mathrm{spat}} = A\mathbf{I}A^{-1}$. In this case, the associated reduction process yields the Lagrangian

$$\ell_{\mathrm{spat}}(\boldsymbol{\omega}, \mathbf{I}_{\mathrm{spat}}) = \frac{1}{2} \mathbf{I}_{\mathrm{spat}} \boldsymbol{\omega} \cdot \boldsymbol{\omega}.$$

By applying the constrained variational principle arising from the Euler–Poincaré theory, namely

$$\delta \int_{t_0}^{t_1} \ell(\boldsymbol{\omega}, \mathbf{I}_{\text{spat}})\, \mathrm{d}t = 0 \quad \text{for variations } \delta\boldsymbol{\omega} = \partial_t \boldsymbol{\varphi} - \boldsymbol{\omega} \times \boldsymbol{\varphi},\ \delta\mathbf{I}_{\text{spat}} = [\boldsymbol{\varphi}, \mathbf{I}_{\text{spat}}],$$

where $\boldsymbol{\varphi}$ is a arbitrary curve in $\mathbb{R}^3$ vanishing at the endpoints, we get the equations of the free rigid body in pure spatial variables as

$$\partial_t (\mathbf{I}_{\text{spat}} \boldsymbol{\omega}) = 0, \qquad \partial_t \mathbf{I}_{\text{spat}} = [\omega, \mathbf{I}_{\text{spat}}],$$

where $\omega \in \mathfrak{so}(3)$ is the matrix corresponding to $\boldsymbol{\omega} \in \mathbb{R}^3$. From the preceding theorem, we know that all these formulations are equivalent.

2 Lagrangian Convective Formulation of Hydrodynamics

Consider the motion of an ideal compressible barotropic fluid on a smooth orientable n-dimensional Riemannian manifold $\mathcal{D}$ with smooth boundary $\partial\mathcal{D}$.

The geometric description presented below is valid only for smooth solutions up to the first shock time. The motion of the fluid is completely described by a curve η_t in the diffeomorphism group $\mathrm{Diff}(\mathcal{D})$. We shall denote the material points by capital letters X and spatial points by lower case letters x so that we have the relation $x = \eta_t(X)$, where $t \mapsto \eta_t(X)$ is the trajectory of a fluid particle that at time $t = 0$ is at $X \in \mathcal{D}$. The *material velocity field* is defined by

$$V_t(X) := \frac{\partial \eta_t(X)}{\partial t}$$

and the *spatial velocity field* by $\mathbf{v}_t(x) := V_t(X) = (\dot{\eta}_t \circ \eta_t^{-1})(x)$. Note that

$$V_t \in T_{\eta_t} \mathrm{Diff}(\mathcal{D}) \quad \text{and} \quad \mathbf{v}_t \in T_e \mathrm{Diff}(\mathcal{D}) = \mathfrak{X}_{\|}(\mathcal{D}),$$

where $T_{\eta_t} \mathrm{Diff}(\mathcal{D})$ denotes the tangent space to the diffeomorphism group at η_t and $T_e \mathrm{Diff}(\mathcal{D})$ the tangent space at the identity. The latter consists of vector fields on $\mathcal{D}$ parallel to the boundary, denoted by $\mathfrak{X}_{\|}(\mathcal{D})$. The *convected velocity* $\mathcal{V}_t$ is the negative of the spatial velocity of the inverse motion (or back to labels map) $l_t := \eta_t^{-1}$, i.e.,

$$\mathcal{V}_t(X) := -\big(\dot{l}_t \circ l_t^{-1}\big)(X).$$

We thus we have the following relations between the three velocities:

$$\mathcal{V}_t = T\eta_t^{-1} \circ \dot{\eta}_t = \eta_t^* \mathbf{v}_t.$$

Choosing a Riemannian metric g on $\mathcal{D}$ and denoting by $\rho : \mathcal{D} \to \mathbb{R}$ the *mass density* of the fluid, the equations of motions of a barotropic fluid in spatial representation are

$$\begin{cases} \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\dfrac{1}{\rho} \operatorname{grad}_g p, \quad p = \rho^2 \dfrac{\partial e}{\partial \rho}, \\ \partial_t \rho + \operatorname{div}_g(\rho \mathbf{v}) = 0, \quad \mathbf{v} \| \partial\mathcal{D}, \end{cases} \tag{2.1}$$

where the function $e = e(\rho)$ is the internal energy of the fluid, and the operators ∇, div_g, and grad_g denote, respectively, the Levi-Civita covariant derivative, the divergence, and the gradient associated to g. It is simpler, and geometrically more natural, to consider as variable the density $\bar{\rho} := \rho\mu(g)$ rather than the function ρ. Here, $\mu(g)$ denotes the volume form associated to the Riemannian metric g. In terms of the density $\bar{\rho}$, the mass conservation equation (the second equation in (2.1)) reads $\partial_t \bar{\rho} + \boldsymbol{\pounds}_{\mathbf{u}} \bar{\rho} = 0$.

Spatial Formulation As shown in Holm et al. (1998), the equations of motion (2.1) are the (*right*) Euler–Poincaré equations (1.1) for the semidirect product $\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})$, where $\mathcal{F}(\mathcal{D})$ denotes the vector space of smooth functions on $\mathcal{D}$, associated to the Lagrangian

$$\ell_{\text{spat}}(\mathbf{v}, \bar{\rho}, g) = \frac{1}{2} \int_{\mathcal{D}} g(x)\big(\mathbf{v}(x), \mathbf{v}(x)\big)\bar{\rho}(x) - \int_{\mathcal{D}} e\big(\rho(x)\big)\bar{\rho}(x). \tag{2.2}$$

Thus, in material representation, the equations of motion are given by the ordinary Euler–Lagrange equations associated to the material Lagrangian $L_{(\bar{\varrho},g)}$ on the tangent bundle $T\operatorname{Diff}(\mathcal{D})$. In order to obtain $L_{(\bar{\varrho},g)}$, we use a change of variables in the spatial Lagrangian given by a diffeomorphism $\eta : \mathcal{D} \to \mathcal{D}$, $x = \eta(X)$. We obtain the expression

$$\begin{aligned} L_{(\bar{\varrho},g)}(V_\eta) = {} & \frac{1}{2} \int_{\mathcal{D}} g\big(\eta(X)\big)\big(V_\eta(X), V_\eta(X)\big)\bar{\varrho}(X) \\ & - \int_{\mathcal{D}} E\big(\bar{\varrho}(X), g\big(\eta(X)\big), T_X\eta\big)\bar{\varrho}(X), \end{aligned}$$

where ϱ is the *mass density of the fluid in the reference configuration*, related to $\bar{\rho}$ by the formula $\varrho(X)\mu(g)(X) := \bar{\varrho}(X) := (\eta^*\bar{\rho})(X)$, and the *Lagrangian internal energy* E is given by

$$E\big(\bar{\varrho}(X), g\big(\eta(X)\big), T_X\eta\big) := e\left(\frac{\bar{\varrho}(X)}{\mu(\eta^* g)(X)}\right). \tag{2.3}$$

The function E is introduced in order to facilitate a comparison with the Lagrangian formulation of elasticity in Sect. 3.

By construction, the Lagrangian L is *right*-invariant under the action of $\varphi \in \operatorname{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto \big(V_\eta \circ \varphi, \varphi^*\bar{\varrho}, g\big)$$

and the reduction map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathbf{v}, \bar{\rho}, g) := \big(V_\eta \circ \eta^{-1}, \eta_*\bar{\varrho}, g\big)$$

induces the spatial Lagrangian $\ell_{\text{spat}} = \ell_{\text{spat}}(\mathbf{v}, \rho, g)$. Note, in particular, that when $(\eta, \bar{\varrho}) \mapsto (\eta \circ \varphi, \varphi^*\bar{\varrho})$ the Lagrangian internal energy transforms as

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E\big(\varphi^*\bar{\varrho}, g \circ \eta \circ \varphi, T\eta \circ T\varphi\big) = E(\bar{\varrho}, g \circ \eta, T\eta) \circ \varphi. \tag{2.4}$$

The Riemannian metric g is not acted on by the diffeomorphism group. It can be therefore considered as a constant in the spatial formulation. This will not be the case for the convective description.

In order to write the Euler–Poincaré equations, we identify the dual of $\mathcal{F}(\mathcal{D})$ with the vector space $|\Omega^n(\mathcal{D})|$ of densities on $\mathcal{D}$ and the dual of $\mathfrak{X}_\parallel(\mathcal{D})$ with the vector space $\Omega^1_\parallel(\mathcal{D}) \otimes |\Omega^n(\mathcal{D})|$ of one-form densities tangent to $\partial\mathcal{D}$, that is,

$$\Omega^1_\parallel(\mathcal{D}) := \{\alpha \in \Omega^1(\mathcal{D}) \mid \iota^*(\star\alpha) = 0\},$$

where $\iota : \partial\mathcal{D} \hookrightarrow \mathcal{D}$ is the inclusion and $\star : \Omega^1(\mathcal{D}) \to \Omega^{n-1}(\mathcal{D})$ is the Hodge star operator associated to the Riemannian metric g and the given orientation $\mu(g)$ on $\mathcal{D}$. The corresponding L^2 duality pairings are given by

$$\langle h, \bar{\rho}\rangle = \int_{\mathcal{D}} h(x)\bar{\rho}(x), \quad h \in \mathcal{F}(\mathcal{D}),\ \bar{\rho} \in \left|\Omega^n(\mathcal{D})\right|,$$

$$\langle \overline{\mathbf{m}}, \mathbf{v}\rangle = \int_{\mathcal{D}} \overline{\mathbf{m}}(x) \cdot \mathbf{v}(x), \quad \overline{\mathbf{m}} \in \Omega^1_\parallel(\mathcal{D}) \otimes \left|\Omega^n(\mathcal{D})\right|,\ \mathbf{v} \in \mathfrak{X}_\parallel(\mathcal{D}),$$

where $\overline{\mathbf{m}} \cdot \mathbf{v} \in |\Omega^n(\mathcal{D})|$ is the density obtained by contracting the one-form density $\overline{\mathbf{m}}$ with the vector field $\mathbf{v}$.

By Euler–Poincaré theory for semidirect products (with respect to a right action and a right representation), the motion equations are obtained by the variational principle

$$\delta \int_{t_0}^{t_1} \ell_{\text{spat}}(\mathbf{v}, \bar{\rho}, g)\,\mathrm{d}t = 0,$$

relative to the constrained variations

$$\delta\mathbf{v} = \dot{\xi} + [\mathbf{v}, \xi], \qquad \delta\bar{\rho} = -\boldsymbol{\pounds}_\xi \bar{\rho},$$

where ξ is an arbitrary curve in $\mathfrak{X}_\parallel(\mathcal{D})$ vanishing at the endpoints. Using the spatial Lagrangian (2.2), we get the Euler equations (2.1); see (Holm et al. 1998).

Remark 2.1 (Ideal homogeneous incompressible fluids) In this case the configuration space is the group $\mathrm{Diff}_\mu(\mathcal{D}) := \{\eta \in \mathrm{Diff}(\mathcal{D}) \mid \eta^*\mu(g) = \mu(g)\}$, where $\mu(g)$ is the volume form associated to the Riemannian metric g on $\mathcal{D}$. The Lagrangian is

$$\ell_{\text{spat}}(\mathbf{v}, g) = \frac{1}{2}\int_{\mathcal{D}} g(x)\big(\mathbf{v}(x), \mathbf{v}(x)\big)\mu(g)(x). \tag{2.5}$$

The Lie algebra of $\mathrm{Diff}_\mu(\mathcal{D})$ is the space $\mathfrak{X}_{\text{div},\parallel}(\mathcal{D})$ of divergence free vector fields relative to the volume form $\mu(g)$ tangent to $\partial\mathcal{D}$. The formal dual $\mathfrak{X}_{\text{div},\parallel}(\mathcal{D})^*$ has two possible convenient representations (see Marsden and Weinstein 1983). The first formal dual is obtained using the natural pairing between one-forms and vector fields relative to integration with respect to $\mu(g)$; then $\mathfrak{X}_{\text{div},\parallel}(\mathcal{D})^*$ is identified with $\Omega^1_{\delta,\parallel}(\mathcal{D}) := \{\mathbf{v}^{\flat_g} \mid \mathbf{v} \in \mathfrak{X}_{\text{div},\parallel}(\mathcal{D})\}$, where $\flat_g$ is the index lowering operator given by the metric g. The second formal dual has an easy description if $H^1(\mathcal{D}, \mathbb{R}) = 0$ and equals $\Omega^2_{\rm ex}(\mathcal{D})$, the vector space of exact two forms on $\mathcal{D}$; the pairing is in this case given by

$$\langle \omega, \mathbf{v}\rangle = \int_{\mathcal{D}} \alpha(\mathbf{v})\mu(g), \quad \text{where } \omega = \mathbf{d}\alpha.$$

Writing the Euler–Poincaré equations for the first choice of the dual yields the classical Euler equations $\partial_t \mathbf{v} + \nabla_{\mathbf{v}}\mathbf{v} = -\operatorname{grad} p$. Using the second description of the dual one gets $\partial_t \omega + \boldsymbol{\pounds}_{\mathbf{v}}\omega = 0$, where $\omega := \mathbf{d}\mathbf{v}^{\flat_g}$ is the vorticity.

Convective Formulation Recall that the convective velocity $\mathcal{V}_t$ is obtained from the material velocity V_t by the formula $\mathcal{V}_t = T\eta_t^{-1} \circ V_t$. From the Lie group point of view, this corresponds to the relation $\mathcal{V}_t = TL_{\eta_t^{-1}}(V_{\eta_t})$, that is, the convective representation of the equations of motion is obtained by a reduction relative to the tangent lifted action of left translation. Thus, in order to recover the equations of motion in convective form by Euler–Poincaré reduction, we need to show that the material Lagrangian L is left-invariant under the tangent lift of left translation $V_\eta \mapsto T\psi \circ V_\eta$. This is accomplished by letting the diffeomorphism group act on the left also on the Riemannian metric g that used to be constant in the spatial representation. Indeed, the material Lagrangian L is *left*-invariant under the action of $\psi \in \operatorname{Diff}(\mathcal{D})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto (T\psi \circ V_\eta, \bar{\varrho}, \psi_* g).$$

To see this, it suffices to observe that the Lagrangian internal energy E is invariant under the transformation $(\eta, g) \mapsto (\psi \circ \eta, \psi_* g)$:

$$E(\bar{\varrho}, g \circ \eta, T\eta) \mapsto E\big(\bar{\varrho}, \psi_* g \circ (\psi \circ \eta), T\psi \circ T\eta\big) = E(\bar{\varrho}, g \circ \eta, T\eta).$$

In contrast to the spatial representation, the mass density $\bar{\varrho}$ is not acted on by the diffeomorphism group. It is therefore a constant in the convective description.

By left invariance, L induces the convective Lagrangian $\ell_{\rm conv} : \mathfrak{X}_{\|}(\mathcal{D}) \times |\Omega^n(\mathcal{D})| \times S_2(\mathcal{D}) \to \mathbb{R}$ via the quotient map

$$(V_\eta, \bar{\varrho}, g) \mapsto (\mathcal{V}, \bar{\varrho}, C) := \big(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g\big),$$

where $S_2(\mathcal{D})$ denotes the space of two-covariant symmetric tensor fields, and $C := \eta^* g$ is the *Cauchy–Green tensor*. We obtain the expression

$$\ell_{\rm conv}(\mathcal{V}, \bar{\varrho}, C) = \frac{1}{2}\int_{\mathcal{D}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{D}} \mathcal{E}(\bar{\varrho}, C)\bar{\varrho}, \tag{2.6}$$

where $\mathcal{E}$ denotes the *internal energy in convective representation* given by

$$\mathcal{E}(\bar{\varrho}, C) := e\left(\frac{\bar{\varrho}}{\mu(C)}\right).$$

In order to write the Euler–Poincaré equation, we choose the L^2 duality pairings

$$\langle \overline{\mathcal{M}}, \mathcal{V}\rangle = \int_{\mathcal{D}} \overline{\mathcal{M}}(X) \cdot \mathcal{V}(X), \quad \overline{\mathcal{M}} \in \Omega^1_{\|}(\mathcal{D}) \otimes \big|\Omega^n(\mathcal{D})\big|,$$

$$\langle\overline{T}, C\rangle = \int_{\mathcal{D}} \overline{T} : C, \quad \overline{T} \in S^2(\mathcal{D}) \otimes \left|\Omega^n(\mathcal{D})\right|,$$

where $S^2(\mathcal{D})$ denotes the vector space of two-contravariant symmetric tensors and $\overline{T} : C \in |\Omega^n(\mathcal{D})|$ is the contraction on both indices. By the Euler–Poincaré theory for semidirect products (with respect to a left action and a left representation), the motion equations are obtained by the variational principle

$$\delta \int_{t_0}^{t_1} \ell_{\text{conv}}(\mathcal{V}, \bar{\varrho}, C)\,\mathrm{d}t = 0,$$

relative to the constrained variations

$$\delta\mathcal{V} = \dot{\zeta} - [\mathcal{V}, \zeta], \qquad \delta C = \boldsymbol{\pounds}_\zeta C,$$

where ζ is an arbitrary curve in $\mathfrak{X}_{\parallel}(\mathcal{D})$ vanishing at the endpoints. We compute

$$\delta \int_{t_0}^{t_1} \ell_{\text{conv}}(\mathcal{V}, \bar{\varrho}, C)\,\mathrm{d}t = \int_{t_0}^{t_1} \int_{\mathcal{D}} \left(-\partial_t \frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}} + \boldsymbol{\pounds}_{\mathcal{V}} \frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}}\right) \cdot \zeta \\ - \int_{\mathcal{D}} \boldsymbol{\pounds}_{\mathcal{V}} \left(\frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}} \cdot \zeta\right) + \int_{\mathcal{D}} \frac{\delta \ell_{\text{conv}}}{\delta C} : \boldsymbol{\pounds}_\zeta C. \tag{2.7}$$

In order to obtain the equations, we make use of the following particular instances of Stokes' theorem (see e.g. Marsden and Hughes 1983, Problem 7.6, Chap. 1, for the second identity):

$$\int_{\mathcal{D}} \boldsymbol{\pounds}_{\mathcal{V}} \left(\frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}} \cdot \zeta\right) = \int_{\partial\mathcal{D}} \widetilde{\frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}}} \cdot \zeta C(\mathcal{V}, \mathbf{N}_C)\gamma(C), \tag{2.8}$$

$$\int_{\mathcal{D}} \frac{\delta \ell_{\text{conv}}}{\delta C} : \boldsymbol{\pounds}_\zeta C = 2 \int_{\mathcal{D}} \frac{\delta \ell_{\text{conv}}}{\delta C} : \nabla \zeta^{\flat_C} \\ = -2 \int_{\mathcal{D}} \zeta \cdot \operatorname{Div}_C \left(\frac{\delta \ell_{\text{conv}}}{\delta C}\right)^{\flat_C} + 2 \int_{\partial\mathcal{D}} \left(\widetilde{\frac{\delta \ell_{\text{conv}}}{\delta C}}\right)^{\flat_C} (\mathbf{N}_C, \zeta)\gamma(C), \tag{2.9}$$

where $\gamma(C)$ denotes the volume form induced by $\mu(C)$ on the boundary, $\widetilde{T}$ denotes the tensor field defined by $T = \widetilde{T}\mu(C)$ for a given tensor density T, and $\mathbf{N}_C$ is the unit outward-pointing unit normal vector field, relative to the Cauchy–Green tensor. The operator $\operatorname{Div}_C : S^2(\mathcal{D}) \otimes |\Omega^n(\mathcal{D})| \to \mathfrak{X}(\mathcal{D}) \otimes |\Omega^n(\mathcal{D})|$ is the divergence operator associated to the Riemannian metric C operating on $S^2(\mathcal{D})$.

Using these formulas and the fact that ζ is arbitrary in $\mathfrak{X}_{\parallel}(\mathcal{D})$ we get the equation

$$\partial_t \frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}} - \boldsymbol{\pounds}_{\mathcal{V}} \left(\frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}}\right) = -2 \operatorname{Div}_C \left(\frac{\delta \ell_{\text{conv}}}{\delta C}\right)^{\flat_C} \quad \text{on } \mathcal{D}. \tag{2.10}$$

The boundary term in (2.8) vanishes since the convective velocity is parallel to the boundary. The boundary term in (2.9) does not vanish, in general, and produces a boundary condition. However, as we will see below, it vanishes for the convective Lagrangian (2.6) due to the special form of the internal energy.

Theorem 2.2 *The Euler–Poincaré equations associated to ℓ_{conv} produce the Euler equations in convective description*

$$\begin{cases} \bar{\varrho}(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2\,\mathrm{Div}_C\left(\dfrac{\partial \mathcal{E}}{\partial C}\bar{\varrho}\right), \\ \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0, \quad \mathcal{V} \| \partial\mathcal{B}, \end{cases} \tag{2.11}$$

where the right hand side is related to the spatial pressure p by the formula

$$2\frac{\partial \mathcal{E}}{\partial C}\bar{\varrho} = -(p \circ \eta)\mu(C)C^{\sharp}, \quad \textit{and thus} \quad 2\,\mathrm{Div}_C\left(\frac{\partial \mathcal{E}}{\partial C}\bar{\varrho}\right) = -\,\mathrm{grad}_C(p \circ \eta)\mu(C),$$

where $C^{\sharp} \in S^2(\mathcal{D})$ is the cometric and grad_C *denotes the gradient relative to C.*

Proof The functional derivatives are given by

$$\frac{\delta \ell_{\mathrm{conv}}}{\delta \mathcal{V}} = \mathcal{V}^{\flat_C}\bar{\varrho}, \qquad \frac{\delta \ell_{\mathrm{conv}}}{\delta C} = \frac{1}{2}\mathcal{V} \otimes \mathcal{V}\bar{\varrho} - \frac{\partial \mathcal{E}}{\partial C}\bar{\varrho}.$$

The motion equations (2.11) are obtained by replacing these functional derivatives in the Euler–Poincaré equations (2.10) and making use of the following formulas:

$$\partial_t\big(\mathcal{V}^{\flat_C}\big) = (\partial_t \mathcal{V})^{\flat_C} + \boldsymbol{\pounds}_{\mathcal{V}}\big(\mathcal{V}^{\flat_C}\big),$$
$$\big(\mathrm{Div}_C(\mathcal{V} \otimes \mathcal{V}\bar{\varrho})\big)^{\flat_C} = \bar{\varrho}\nabla_{\mathcal{V}}\mathcal{V}^{\flat_C} + \boldsymbol{\pounds}_{\mathcal{V}}\big(\bar{\varrho}\mathcal{V}^{\flat_C}\big) - \bar{\varrho}\boldsymbol{\pounds}_{\mathcal{V}}\mathcal{V}^{\flat_C}.$$

To show that the boundary term in (2.9) vanishes as well as to prove the relation with the pressure, we recall that $\mathcal{E}(C, \bar{\varrho}) = e(\bar{\varrho}/\mu(C))$. Using the formulas $\delta(\mu(C)) = \frac{1}{2}\mu(C)\,\mathrm{Tr}_C(\delta C)$ and $\bar{\varrho} = \eta^*\bar{\rho} = \eta^*(\rho\mu(g)) = (\rho \circ \eta)\mu(C)$, we obtain the desired relation $2\bar{\varrho}\partial\mathcal{E}/\partial C = -(p \circ \eta)\mu(C)C^{\sharp}$. □

Hypothesis on the Lagrangian Throughout this paper we assume that the Lagrangian admits functional derivatives in the following sense. For any variable θ (a tensor field or a tensor field density) on which the Lagrangian depends, we can write

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \ell(\theta_\varepsilon) = \int_{\mathcal{D}} \frac{\delta \ell}{\delta \theta} : \delta\theta,$$

where the symbol: means contraction on all indices. This hypothesis may not be verified if the Lagrangian depends on derivatives of θ.

Remark 2.3 (Ideal homogeneous incompressible fluids) Recall that the convective velocity $\mathcal{V}_t$ is expressed in terms of the spatial velocity $\mathbf{v}_t$ by $\mathcal{V}_t = \eta_t^*\mathbf{v}_t$ which implies that $\mathrm{div}\,\mathcal{V}_t = 0$. The convective Lagrangian has the expression

$$\ell_{\mathrm{conv}}(\mathcal{V}, C) = \frac{1}{2}\int_{\mathcal{D}} C(\mathcal{V}, \mathcal{V})\mu(g). \tag{2.12}$$

As in Remark 2.1 there are two choices for the dual: $\Omega^1_{\delta,\parallel}(\mathcal{D})$ and $\Omega^2_{\mathrm{ex}}(\mathcal{D})$. Proceeding as in the compressible barotropic case, but recomputing the functional derivatives relative to the first pairing giving the formal duals $\Omega^1_{\delta,\parallel}(\mathcal{D})$, we get the analogue of (2.10)

$$\mathbb{P}\left(\partial_t \frac{\delta \ell_{\mathrm{conv}}}{\delta \mathcal{V}} - \pounds_{\mathcal{V}}\left(\frac{\delta \ell_{\mathrm{conv}}}{\delta \mathcal{V}}\right) + 2\,\mathrm{Div}_C\left(\frac{\delta \ell_{\mathrm{conv}}}{\delta C}\right)^{\flat_C}\right) = 0 \quad \text{on } \mathcal{D}, \tag{2.13}$$

where $\mathbb{P} : \Omega^1(\mathcal{D}) \to \Omega^1_{\delta,\parallel}(\mathcal{D})$ is the orthogonal Hodge projector using the metric g. For the convective Lagrangian (2.12) using

$$\frac{\delta \ell_{\mathrm{conv}}}{\delta \mathcal{V}} = \mathbb{P}\big(\mathcal{V}^{\flat_C}\big) \quad \text{and} \quad \frac{\delta \ell_{\mathrm{conv}}}{\delta C} = \frac{1}{2}\mathcal{V} \otimes \mathcal{V}$$

we get the equations of motion

$$\partial_t \mathbb{P}\big(\mathcal{V}^{\flat_C}\big) = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0.$$

An analogous computation using the second pairing yielding the formal dual $\Omega^2(\mathcal{D})$ gives the equations of motion

$$\partial_t \Omega = 0 \quad \text{and} \quad \partial_t C - \pounds_{\mathcal{V}} C = 0,$$

where $\Omega := \mathbf{d}\mathcal{V}^{\flat_C}$ is convective vorticity. Note that the two systems are equivalent, as expected.

The same equations could have been obtained in a different manner, namely, the first equation on both systems is simply Noether's Theorem, i.e., conservation of the momentum mapping for the right action of the diffeomorphism group on its cotangent bundle, as a direct verification shows.

3 Lagrangian Reduction in Nonlinear Elasticity

We consider elastic bodies whose configuration at each time can be described by an embedding $\eta : \mathcal{B} \to \mathcal{S}$, where $\mathcal{B}$ is a compact n-dimensional submanifold of $\mathcal{S} = \mathbb{R}^n$, with smooth boundary. We denote by $\mathrm{Emb}(\mathcal{B}, \mathcal{S})$ the manifold of all embeddings of $\mathcal{B}$ into $\mathcal{S}$. A motion of a body is thus a curve $t \mapsto \eta_t$ in $\mathrm{Emb}(\mathcal{B}, \mathcal{S})$.

The Lagrangian of classical nonlinear elasticity in material representation is given by

$$\begin{aligned} L(V_\eta, \bar{\varrho}, g, G) = {} & \frac{1}{2}\int_{\mathcal{B}} g\big(\eta(X)\big)\big(V_\eta(X), V_\eta(X)\big)\bar{\varrho}(X) \\ & - \int_{\mathcal{B}} W\big(g\big(\eta(X)\big), T_X\eta, G(X)\big)\bar{\varrho}(X). \end{aligned} \tag{3.1}$$

In this expression, $V_\eta \in T_\eta \mathrm{Emb}(\mathcal{B}, \mathcal{S})$, the tangent space to $\mathrm{Emb}(\mathcal{B}, \mathcal{S})$ at η, represents the *material velocity*. As for fluids, $\bar{\varrho} \in |\Omega^n(\mathcal{D})|$ is the *mass density in the reference configuration* and $g \in S_2(\mathcal{S})$ is a Riemannian metric on $\mathcal{S}$. For elasticity, we also need an additional Riemannian metric $G \in S_2(\mathcal{B})$ on $\mathcal{B}$. By the hypothesis of material frame indifference, the *material stored energy function* W is assumed to be invariant under the transformations

$$(\eta, g) \mapsto (\psi \circ \eta, \psi_* g), \quad \psi \in \mathrm{Diff}(\mathcal{S}).$$

By definition, this means that for each $\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ and all diffeomorphisms $\psi : \eta(\mathcal{B}) \to \mathcal{B}$ we have

$$W\big(\psi_* g\big(\psi\big(\eta(X)\big)\big), T_{\eta(X)}\psi \circ T_X\eta, G(X)\big) = W\big(g\big(\eta(X)\big), T_X\eta, G(X)\big).$$

In particular, we can define the *convective stored energy* $\mathcal{W}$ by

$$\mathcal{W}\big(C(X), G(X)\big) := W\big(\eta^* g(X), \mathbf{I}, G(X)\big) = W\big(g\big(\eta(X)\big), T_X\eta, G(X)\big). \tag{3.2}$$

Material and Convective Tensors The *Cauchy stress tensor* $\boldsymbol{\sigma} \in S^2(\eta(\mathcal{B}))$ is related to the stored energy function by the *Doyle–Ericksen formula*

$$\boldsymbol{\sigma} = 2\rho \frac{\partial W}{\partial g},$$

where we recall that $\bar{\rho} = \rho\mu(g)$. This relation can be obtained by the axioms for constitutive theory; see Marsden and Hughes (1983), Sect. 3.2. By pulling back $\boldsymbol{\sigma}$ to $\mathcal{B}$ we obtain the *convected stress tensor* $\boldsymbol{\Sigma} := \eta^*\boldsymbol{\sigma} \in S^2(\mathcal{B})$, related to the convective stored energy function by

$$\boldsymbol{\Sigma} = 2\mathcal{R}\frac{\partial \mathcal{W}}{\partial C}, \tag{3.3}$$

where $\mathcal{R}$ is the *convected mass density* defined by the equality $\bar{\varrho} = \mathcal{R}\mu(C)$, i.e., $\mathcal{R} \circ \eta = \rho$. The *first Piola–Kirchhoff* tensor is the two-point tensor over η defined by

$$\mathbf{P}(\alpha_X, \beta_x) := J_\eta(X)\boldsymbol{\sigma}\big(\eta(X)\big)\big(T^*\eta^{-1}(\alpha_X), \beta_x\big), \quad x = \eta(X),$$

where $\alpha_X \in T^*\mathcal{B}$, $\beta_x \in T^*\mathcal{S}$, J_η is the Jacobian of η relative to the metrics g and G, i.e., $\eta^*\mu(g) = J_\eta\mu(G)$. We thus have the relations

$$\mathbf{P}(\alpha_X, \beta_x)\mu(G) = \boldsymbol{\sigma}\big(\eta(X)\big)\big(T^*\eta^{-1}(\alpha_X), \beta_x\big)\mu(C) = \boldsymbol{\Sigma}(X)\big(\alpha_X, T^*\eta(\beta_x)\big)\mu(C)$$

and the Doyle–Ericksen relation reads

$$\mathbf{P} = 2\varrho\left(\frac{\partial W}{\partial T\eta}\right)^{\sharp_g},$$

where $\sharp_g$ denotes the index raising operator associated to the Riemannian metric g.

Boundary Conditions We can consider two types of boundary condition (see Marsden and Hughes 1983). For *pure displacement boundary conditions*, the configuration

$\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ is prescribed on the boundary $\partial\mathcal{B}$:

$$\eta|_{\partial\mathcal{B}} = \tilde{\eta}. \tag{3.4}$$

For *traction boundary condition*, the traction $\mathbf{P}\cdot\mathbf{N}_C$ is prescribed on the boundary:

$$\mathbf{P}\cdot\mathbf{N}_C|_{\partial\mathcal{B}} = \tilde{\boldsymbol{\tau}}, \tag{3.5}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff tensor and $\mathbf{N}_C$ is the normal to the boundary relative to the Cauchy–Green tensor. Below we will treat only the case $\tilde{\boldsymbol{\tau}} = 0$; the case $\tilde{\boldsymbol{\tau}} \neq 0$ requires the addition of another term in the Lagrangian.

One can also consider *mixed boundary conditions* by imposing (3.4) on $\partial_d\mathcal{B}$ and (3.5) on $\partial_{\boldsymbol{\tau}}\mathcal{B}$, where

$$\overline{\partial_d\mathcal{B} \cup \partial_{\boldsymbol{\tau}}\mathcal{B}} = \partial\mathcal{B}, \qquad \partial_d\mathcal{B} \cap \partial_{\boldsymbol{\tau}}\mathcal{B} = \varnothing.$$

We call $\eta|_{\partial_d\mathcal{B}} = \tilde{\eta}$ the *essential boundary condition* and build it directly into the configuration space, defined to be

$$\mathcal{C} := \left\{\eta \in \mathrm{Emb}(\mathcal{B}, \mathcal{S}) \mid \eta|_{\partial_d\mathcal{B}} = \tilde{\eta}\right\}.$$

3.1 Convective Representation

Using material frame indifference of the stored energy function, one notes that the material Lagrangian depends on the Lagrangian variables only through the convective quantities

$$(\mathcal{V}, \bar{\varrho}, C, G) := \left(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\right) \in \mathfrak{X}(\mathcal{B}) \times \left|\Omega^n(\mathcal{B})\right| \times S_2(\mathcal{B}) \times S_2(\mathcal{B}),$$

where $T\eta^{-1} : T(\eta(\mathcal{B})) \mapsto T\mathcal{B}$ denotes the tangent map of the diffeomorphism $\eta^{-1} : \eta(\mathcal{B}) \to \mathcal{B}$. Note that for elasticity, contrary to the case of fluids, the convective velocity is not tangent to the boundary. In terms of these variables, the Lagrangian reads

$$\ell_{\mathrm{conv}}(\mathcal{V}, \bar{\varrho}, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{W}(C, G)\bar{\varrho}.$$

The essential boundary condition on the convective velocity reads $\mathcal{V}|_{\partial_d\mathcal{B}} = 0$. We will denote by $\mathfrak{X}_0(\mathcal{B})$ the corresponding space of vector fields.

Reduced Variations In order to write the Euler–Lagrange equations in terms of the variables $(\mathcal{V}, \bar{\varrho}, C, G)$ we first need the expression of the reduced variations. Note that this is not a particular case of the Euler–Poincaré reduction theorem since the Lagrangian is not defined on the tangent bundle of the symmetry group. Therefore we need to compute the variations by hand. Let $\eta_\varepsilon \in \mathcal{C} \subset \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ be a variation of the embedding $\eta_0 := \eta$. Viewed as a vector in $T_\mathcal{V}\mathfrak{X}_0(\mathcal{B})$, the variation of $\delta\mathcal{V}$ is

$$\begin{aligned}\delta\mathcal{V} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} T\eta_\varepsilon^{-1} \circ \dot{\eta}_\varepsilon = -TT\eta^{-1} \circ T\delta\eta \circ T\eta^{-1} \circ \dot{\eta} + TT\eta^{-1} \circ \delta\dot{\eta} \\ &= -TT\eta^{-1} \circ T\delta\eta \circ T\eta^{-1} \circ \dot{\eta} + \frac{\mathrm{d}}{\mathrm{d}t}\big(T\eta^{-1} \circ \delta\eta\big) + TT\eta^{-1} \circ T\dot{\eta} \circ T\eta^{-1} \circ \delta\eta \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\zeta + T\mathcal{V} \circ \zeta - T\zeta \circ \mathcal{V},\end{aligned}$$

where we defined $\zeta := T\eta^{-1} \circ \delta\eta \in \mathfrak{X}_0(\mathcal{B})$. Thus, as an element in $\mathfrak{X}_0(\mathcal{B})$, the variation $\delta\mathcal{V}$ of the convective velocity reads

$$\delta\mathcal{V} = \dot{\zeta} - [\mathcal{V}, \zeta]. \tag{3.6}$$

The variation δC of the Cauchy–Green tensor reads

$$\delta C = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \eta_\varepsilon^* g = \eta^* \pounds_{\delta\eta\circ\eta^{-1}} g = \pounds_{T\eta^{-1}\circ\delta\eta}\eta^* g = \pounds_\zeta C. \tag{3.7}$$

Note that the constrained variations are identical to those of the convective representation of fluids. However, as we have seen, the computation of these variations is more involved since η is in this case an embedding and not a diffeomorphism.

Reduced Convective Euler–Lagrange Equations By reduction of the variational principle associated to the Euler–Lagrange equations on $T\mathcal{C}$, the convective equations of motion are given by the stationarity condition

$$\delta \int_{t_0}^{t_1} \ell_{\mathrm{conv}}(\mathcal{V}, \bar{\varrho}, C, G)\,\mathrm{d}t = 0, \tag{3.8}$$

relative to the constrained variations (3.6) and (3.7). The computation of the stationarity condition is similar to that for fluids and we get the relations (2.7), (2.8), and (2.9), replacing $\mathcal{D}$ by $\mathcal{B}$. The only difference is that now $\mathcal{V}$ and ζ are not tangent to the boundary and vanish on the subset $\partial_d\mathcal{B}$ of the boundary. In this case, we get the equations

$$\partial_t \frac{\delta\ell_{\mathrm{conv}}}{\delta\mathcal{V}} - \pounds_\mathcal{V}\left(\frac{\delta\ell_{\mathrm{conv}}}{\delta\mathcal{V}}\right) = -2\,\mathrm{Div}_C\left(\frac{\delta\ell_{\mathrm{conv}}}{\delta C}\right)^{\flat_C} \quad \text{on } \mathcal{B} \tag{3.9}$$

and

$$\frac{\delta\ell_{\mathrm{conv}}}{\delta\mathcal{V}} C(\mathcal{V}, \mathbf{N}_C) = 2\mathbf{i}_{\mathbf{N}_C}\left(\frac{\delta\ell_{\mathrm{conv}}}{\delta C}\right)^{\flat_C} \quad \text{on } \partial_{\boldsymbol{\tau}}\mathcal{B}. \tag{3.10}$$

Applying these relations to the Lagrangian ℓ_{conv} of elasticity, we obtain the following result.

Theorem 3.1 *The reduced Euler–Lagrange equations associated to ℓ_{conv} produce the equations for elasticity in the convective description*

$$\begin{cases} \bar{\varrho}(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2\,\mathrm{Div}_C\left(\dfrac{\partial \mathcal{W}}{\partial C}\bar{\varrho}\right), \\ \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0, \end{cases}$$

with boundary conditions

$$\mathcal{V}|_{\partial_d \mathcal{B}} = 0, \qquad \boldsymbol{\Sigma} \cdot \mathbf{N}_C^{\flat_C}\big|_{\partial_\tau \mathcal{B}} = 0,$$

where ∇ is the Levi-Civita covariant derivative associated to the Cauchy–Green tensor C. The first equation can equivalently be written as

$$\mathcal{R}(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = \mathrm{Div}_C(\boldsymbol{\Sigma}),$$

where $\boldsymbol{\Sigma}$ is the convected stress tensor.

Proof The motion equation is obtained as in Theorem 2.2, by using the expression of the functional derivatives

$$\frac{\delta \ell_{\text{conv}}}{\delta \mathcal{V}} = \mathcal{V}^{\flat_C}\bar{\varrho}, \qquad \frac{\delta \ell_{\text{conv}}}{\delta C} = \frac{1}{2}\mathcal{V} \otimes \mathcal{V}\bar{\varrho} - \frac{\partial \mathcal{W}}{\partial C}\bar{\varrho}.$$

To obtain the boundary condition, we insert the functional derivative in (3.10) and use the Doyle–Ericksen formula (3.3). After a remarkable cancellation, we get the desired boundary condition. To obtain the last equation, we recall the relation $\bar{\varrho} = \mathcal{R}\mu(C)$. □

3.2 Spatial Representation

We now search conditions under which the material Lagrangian (3.1) is invariant under the *right* action of $\mathrm{Diff}(\mathcal{B})$ given by

$$(V_\eta, \bar{\varrho}, g, G) \mapsto \big(V_\eta \circ \varphi, \varphi^*\bar{\varrho}, g, \varphi^* G\big),$$

for all diffeomorphisms $\varphi \in \mathrm{Diff}(\mathcal{B})$. The kinetic energy part of L is clearly right-invariant. Note that a sufficient condition to have right invariance of the internal energy term is that

$$W\big(g\big(\eta(\varphi(X))\big), T_X(\eta \circ \varphi), \varphi^* G(X)\big) = \big(W\big(g\big(\eta(\_)\big), T\_\eta, G(\_)\big) \circ \varphi\big)(X),$$

for all $\varphi \in \mathrm{Diff}(\mathcal{B})$. This is equivalent to

$$\mathcal{W}\big(\varphi^* C, \varphi^* G\big) = \mathcal{W}(C, G) \circ \varphi, \tag{3.11}$$

for all $\varphi \in \mathrm{Diff}(\mathcal{B})$ by the defining relation (3.2) for $\mathcal{W}$. This condition on the stored energy function is called *material covariance* (as defined in Marsden and Hughes 1983, Definition 3.3.4) and implies that the material is isotropic (see Marsden and Hughes 1983, Proposition 3.5.7).

We shall assume material covariance when working in the spatial representation.

The Spatial Lagrangian Under the hypothesis of material covariance, the material Lagrangian depends on the variables V_η, $\bar{\varrho}$, and G only through the spatial quantities

$$\mathbf{v} := V_\eta \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \qquad \bar{\rho} := \eta_* \bar{\varrho} \in \left|\Omega^n(D_\Sigma)\right|, \qquad c := \eta_* G \in \mathcal{S}_2(D_\Sigma),$$

where $\Sigma = \eta(\partial\mathcal{B})$ is the boundary of the *current configuration* $D_\Sigma := \eta(\mathcal{B}) \subset \mathcal{S}$.

For the kinetic energy, a change of variables yields

$$\begin{aligned}\int_{\mathcal{B}} g\big(\eta(X)\big)\big(V_\eta(X), V_\eta(X)\big)\bar{\varrho}(X) &= \int_{\eta(\mathcal{B})} g(x)\big(V_\eta\big(\eta^{-1}(x)\big), V_\eta\big(\eta^{-1}(x)\big)\big)\eta_*\bar{\varrho}(x)\\ &= \int_{D_\Sigma} g(x)\big(\mathbf{v}(x), \mathbf{v}(x)\big)\bar{\rho}(x).\end{aligned}$$

It is important to notice that the current boundary Σ is itself a variable and that the above defined spatial quantities are defined on the current configuration.

For the internal energy, by material covariance, we can define a function w_Σ associated to the current configuration D_Σ, by

$$w_\Sigma(c, g) := \mathcal{W}\big(\eta^* g, \eta^* c\big) \circ \eta^{-1},$$

where η is a parametrization of D_Σ, that is, $\eta : \mathcal{B} \to \mathcal{S}$ is an embedding such that $\eta(\mathcal{B}) = D_\Sigma$. The right hand side does not depend on the embedding η since, if one takes another parametrization $\bar{\eta}$, then $\bar{\eta} = \eta \circ \varphi$ for some $\varphi \in \mathcal{B}$ and hence we clearly have

$$\mathcal{W}\big(\bar{\eta}^* g, \bar{\eta}^* c\big) \circ \bar{\eta}^{-1} = \mathcal{W}\big(\eta^* g, \eta^* c\big) \circ \eta^{-1}.$$

Note that the functions w_Σ, $\mathcal{W}$, and W are related by the formula

$$\big(w_\Sigma(c, g) \circ \eta\big)(X) = \mathcal{W}\big(\eta^* g(X), \eta^* c(X)\big) = W\big(g\big(\eta(X)\big), T_X\eta, \eta^* c(X)\big).$$

In spatial formulation, the Doyle–Ericksen formula (3.3) can be written as

$$\boldsymbol{\sigma} = 2\rho \frac{\partial w_\Sigma}{\partial g}, \tag{3.12}$$

where $\boldsymbol{\sigma} \in S^2(D_\Sigma)$ is the Cauchy stress tensor. By a change of variable and using (3.2) we can write

$$\begin{aligned}&\int_{\mathcal{B}} W\big(g\big(\eta(X)\big), T_X\eta, G(X)\big)\bar{\varrho}(X)\\ &= \int_{\eta(\mathcal{B})} \big(W\big(g\big(\eta(\_)\big), T\_\eta, G(\_)\big) \circ \eta^{-1}\big)(x)(\eta_*\bar{\varrho})(x)\\ &= \int_{D_\Sigma} \big(\mathcal{W}\big(\eta^* g(\_), G(\_)\big) \circ \eta^{-1}\big)(x)\bar{\rho}(x)\\ &= \int_{D_\Sigma} w_\Sigma\big(c(x), g(x)\big)\bar{\rho}(x)\end{aligned}$$

and we get the Lagrangian in spatial representation

$$\ell_{\text{spat}}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} w_\Sigma(c, g)\bar{\rho}, \tag{3.13}$$

recalling that the variables are defined on the current configuration D_Σ and that Σ is a variable.

Reduced Variations The computations below will be done using the following observation. Assume that $\eta_\varepsilon \in \operatorname{Emb}(\mathcal{B}, \mathcal{S})$ is a small perturbation of $\eta \in \operatorname{Emb}(\mathcal{B}, \mathcal{S})$, that is, $\eta_\varepsilon \to \eta$ as $\varepsilon \to 0$ in the Whitney C^∞-topology of $\operatorname{Emb}(\mathcal{B}, \mathcal{S})$. Let $\Sigma_\varepsilon := \eta_\varepsilon(\partial\mathcal{B})$ be the boundary of the submanifold $\eta_\varepsilon(\mathcal{B})$. Consider a tubular neighborhood N_δ of the submanifold Σ with radius δ. By uniform convergence, for any $\delta > 0$, we have the inclusion $\Sigma_\varepsilon \subset N_\delta$ for all ε sufficiently small.

Let $x \in \operatorname{int} D_\Sigma$ and take δ strictly smaller than half of the distance from x to Σ. Thus the open ball B_x centered at x of radius strictly less than half of this distance is contained in $\operatorname{int} D_{\Sigma_\varepsilon}$ for all ε sufficiently small.

Using this observation, the computations below can be made in an appropriate open ball B_x containing $x \in \operatorname{int} D_\Sigma$. Then the formulas are extended to D_Σ by continuity.

Let $\eta_\varepsilon \in \operatorname{Emb}(\mathcal{B}, \mathcal{S})$ be a variation of the embedding η. Viewed as a vector in $T_{\mathbf{v}}\mathfrak{X}(D_\Sigma)$, the variation of $\mathbf{v}$ is

$$\begin{aligned}\delta\mathbf{v} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \dot{\eta}_\varepsilon \circ \eta_\varepsilon^{-1} = \delta\dot{\eta} \circ \eta^{-1} - T\dot{\eta} \circ T\eta^{-1} \circ \delta\eta \circ \eta^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\delta\eta \circ \eta^{-1}\big) + T\delta\eta \circ T\eta^{-1} \circ \dot{\eta} \circ \eta^{-1} - T\dot{\eta} \circ T\eta^{-1} \circ \delta\eta \circ \eta^{-1} \\ &= \dot{\xi} + T\xi \circ \mathbf{v} - T\mathbf{v} \circ \xi,\end{aligned}$$

where $\xi := \delta\eta \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma)$ is an arbitrary curve with vanishing endpoints. Thus, as an element in $\mathfrak{X}(D_\Sigma)$, the variation of the spatial velocity is

$$\delta\mathbf{v} = \dot{\xi} + [\mathbf{v}, \xi]. \tag{3.14}$$

Since the only part of ξ contributing to the motion of the boundary Σ is its normal part, we define for any $x \in \Sigma$,

$$\delta\Sigma(x) := g(x)\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \eta_\varepsilon\big(\eta^{-1}(x)\big), \mathbf{n}_g(x)\right), \tag{3.15}$$

where $\mathbf{n}_g$ is the outward-pointing unit normal vector field relative to g. Therefore,

$$\delta\Sigma = g(\xi, \mathbf{n}). \tag{3.16}$$

The variation of $\bar{\rho}$ is computed in the following way:

$$\delta\bar{\rho} := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\bar{\varrho} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\eta^*\eta_*\bar{\varrho} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left(\eta\circ\eta_\varepsilon^{-1}\right)^*\bar{\rho}$$
$$= -\boldsymbol{\pounds}_\xi\bar{\rho} \tag{3.17}$$

because

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \eta\circ\eta_\varepsilon^{-1} = -T\eta\circ T\eta^{-1}\circ\delta\eta\circ\eta^{-1} = -\delta\eta\circ\eta^{-1} = -\xi.$$

Similarly, the variation of c is

$$\delta c := \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*G = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} (\eta_\varepsilon)_*\eta^*\eta_*G = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left(\eta\circ\eta_\varepsilon^{-1}\right)^*c_\Sigma = -\boldsymbol{\pounds}_\xi c. \tag{3.18}$$

These computations are also valid on the submanifold $\mathcal{C} := \{\eta\in\mathrm{Emb}(\mathcal{B},\mathcal{S}) \mid \eta|_{\partial_d\mathcal{B}} = \tilde{\eta}\}$ associated to the essential boundary condition. In this case, the subset $\Sigma_d := \tilde{\eta}(\partial_d\mathcal{B})$ of the current boundary $\Sigma = \eta(\partial\mathcal{B})$ is fixed. Thus, in the variations above, ξ is an arbitrary curve in $\mathfrak{X}_0(D_\Sigma) := \{\xi\in\mathfrak{X}(D_\Sigma) \mid \xi|_{\Sigma_d} = 0\}$, with vanishing endpoints. We will use the notation $\Sigma_{\boldsymbol{\tau}} := \eta(\partial_{\boldsymbol{\tau}}\mathcal{B})$ for the part of the current boundary that is allowed to move.

Spatial Reduced Euler–Lagrange Equations By reduction of the variational principle associated to the Euler–Lagrange equations on $T\mathcal{C}$, the spatial equations of motion are given by the stationarity condition

$$\delta\int_{t_0}^{t_1} \ell_{\mathrm{spat}}(\Sigma,\mathbf{v},\bar{\rho},g,c)\,\mathrm{d}t = 0,$$

relative to the constrained variations (3.14), (3.15), (3.17), and (3.18). In our computation below, we need take into account the fact that the current boundary Σ is time dependent. We will make use of the following consequence of the transport formula and divergence theorem. Let η_ε be a flow on D_Σ and let $f_\varepsilon\in\mathcal{F}(\mathcal{S})$ be an ε-dependent function with $f_0 = f$. Then we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\int_{\eta_\varepsilon(D_\Sigma)} f_\varepsilon\mu(g) = \int_{D_\Sigma}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} f_\varepsilon\mu(g) + \int_\Sigma fg(\xi,\mathbf{n}_g)\gamma(g), \tag{3.19}$$

where $\xi := \mathrm{d}\eta_\varepsilon/\mathrm{d}\varepsilon|_{\varepsilon=0}$, $\gamma(g)$ is the boundary volume induced by g, and $\mathbf{n}_g$ is the outward-pointing unit normal vector field along the boundary, associated to g. We first consider the variation $\delta\mathbf{v} = \dot{\xi} + [\mathbf{v},\xi]$. Using (3.19) and the fact that ξ vanishes at the endpoints, we have

$$\int_{t_0}^{t_1}\int_{D_\Sigma}\frac{\delta\ell_{\mathrm{spat}}}{\delta\mathbf{v}}\cdot\dot{\xi} = -\int_{t_0}^{t_1}\left(\int_{D_\Sigma}\partial_t\frac{\delta\ell_{\mathrm{spat}}}{\delta\mathbf{v}}\cdot\xi + \int_\Sigma\frac{\widetilde{\delta\ell_{\mathrm{spat}}}}{\delta\mathbf{v}}\cdot\xi g(\mathbf{v},\mathbf{n})\gamma(g)\right),$$

where the symbol $\tilde{}$ on a tensor density denotes the tensor obtained by removing the Riemannian volume. On the other hand, by the divergence theorem, we get

$$\int_{D_\Sigma}\frac{\delta\ell_{\mathrm{spat}}}{\delta\mathbf{v}}\cdot\boldsymbol{\pounds}_\mathbf{v}\xi = -\int_{D_\Sigma}\left(\boldsymbol{\pounds}_\mathbf{v}\frac{\delta\ell_{\mathrm{spat}}}{\delta\mathbf{v}}\right)\cdot\xi + \int_\Sigma\frac{\widetilde{\delta\ell_{\mathrm{spat}}}}{\delta\mathbf{v}}\cdot\xi g(\mathbf{v},\mathbf{n})\gamma(g).$$

Using these two equalities, formula (2.9), and the expressions of the constrained variations $\delta\mathbf{v}$, $\delta\bar{\rho}$, δc, $\delta\Sigma$, we compute

$$\begin{aligned}
&\delta\int_{t_0}^{t_1}\ell_{\text{spat}}(\Sigma,\mathbf{v},\bar{\rho},g,c)\,\mathrm{d}t\\
&\quad=\int_{t_0}^{t_1}\int_{D_\Sigma}\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}\cdot\delta\mathbf{v}+\int_{D_\Sigma}\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}\delta\bar{\rho}+\int_{D_\Sigma}\frac{\delta\ell_{\text{spat}}}{\delta c}:\delta c+\int_{\Sigma}\frac{\delta\ell_{\text{spat}}}{\delta\Sigma}\delta\Sigma\\
&\quad=\int_{t_0}^{t_1}\int_{D_\Sigma}\left(-\partial_t\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}-\pounds_{\mathbf{v}}\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}+\bar{\rho}\,\mathbf{d}\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}+2\,\mathrm{Div}_c\left(\frac{\delta\ell_{\text{spat}}}{\delta c}\right)^{\flat_c}\right)\cdot\xi\\
&\qquad+\int_{t_0}^{t_1}\int_{\Sigma}\left(\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta\Sigma}-\rho\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}\right)g(\xi,\mathbf{n}_g)\gamma(g)-2\int_{\Sigma}\left(\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta c}\right)^{\flat_c}(\xi,\mathbf{n}_c)\gamma(c),
\end{aligned}$$

where $\mathbf{n}_c$ denotes the outward-pointing unit normal vector field relative to the metric c. Since ξ is an arbitrary curve in $\mathfrak{X}_0(D_\Sigma)$, we get the stationarity condition

$$\partial_t\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}+\pounds_{\mathbf{v}}\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}=\bar{\rho}\,\mathbf{d}\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}+2\,\mathrm{Div}_c\left(\frac{\delta\ell_{\text{spat}}}{\delta c}\right)^{\flat_c}\quad\text{on } D_\Sigma \tag{3.20}$$

and

$$\left(\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta\Sigma}-\rho\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}\right)(\mathbf{n}_g)^{\flat_g}\gamma(g)-2\mathbf{i}_{\mathbf{n}_c}\left(\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta c}\right)^{\flat_c}\gamma(c)=0\quad\text{on } \Sigma_{\boldsymbol{\tau}}. \tag{3.21}$$

Applying these relations to the Lagrangian of elasticity we get the following result.

Theorem 3.2 *The reduced Euler–Lagrange equations of an isotropic material associated to ℓ_{spat} produce the equations for nonlinear elasticity in spatial formulation*

$$\begin{cases}\rho(\partial_t\mathbf{v}+\nabla_{\mathbf{v}}\mathbf{v})=\mathrm{Div}_g(\boldsymbol{\sigma}),\\ \partial_t c+\pounds_{\mathbf{v}}c=0,\qquad \partial_t\bar{\rho}+\pounds_{\mathbf{v}}\bar{\rho}=0\end{cases} \tag{3.22}$$

with boundary conditions and boundary movement

$$\mathbf{v}|_{\Sigma_d}=0,\qquad \boldsymbol{\sigma}\cdot\mathbf{n}_g|_{T\Sigma_{\boldsymbol{\tau}}}=0,\qquad \partial_t\Sigma=g(\mathbf{v},\mathbf{n}_g),$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor related to w_Σ by the Doyle–Ericksen formula (3.12).

Note the difference between the two boundary conditions. The first one requires that $\mathbf{v}(x)=0$ for all $x\in\Sigma_d$. The second one means that $\boldsymbol{\sigma}(x)((\mathbf{n}_g)^{\flat_g}(x),\alpha_x)=0$ for all $\alpha\in T_x^*\Sigma_{\boldsymbol{\tau}}$.

Proof The functional derivatives of the Lagrangian (3.13) are

$$\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}}=\mathbf{v}^{\flat_g}\bar{\rho},\qquad \frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}=\frac{1}{2}g(\mathbf{v},\mathbf{v})-w_\Sigma(c,g),\qquad \frac{\delta\ell_{\text{spat}}}{\delta c}=-\frac{\partial w_\Sigma}{\partial c}\bar{\rho}, \tag{3.23}$$

where $\flat_g$ is the index lowering operator associated to the metric g on $\mathcal{S}$. Inserting these expressions in (3.20) and using the relations $\pounds_{\mathbf{v}}\mathbf{v}^{\flat_g} = \nabla_{\mathbf{v}}\mathbf{v}^{\flat_g} + \frac{1}{2}\mathbf{d}(g(\mathbf{v},\mathbf{v}))$ and $\partial_t\bar{\rho} + \pounds_{\mathbf{v}}\bar{\rho} = 0$, we get

$$(\partial_t \mathbf{v} + \nabla_{\mathbf{v}}\mathbf{v})^{\flat_g}\bar{\rho} = -2\,\mathrm{Div}_c\left(\bar{\rho}\frac{\partial w_\Sigma}{\partial c}\right)^{\flat_c} - \bar{\rho}\,\mathbf{d}\big(w_\Sigma(c,g)\big). \tag{3.24}$$

Equation (3.19) yields then

$$\frac{\delta \ell_{\mathrm{spat}}}{\delta \Sigma} = \left(\rho\frac{1}{2}g(\mathbf{v},\mathbf{v}) - \rho w_\Sigma(c,g)\right)\gamma(g),$$

as a density on Σ. Therefore, the first two terms in (3.21) simplify to

$$\mathbf{i}_{\mathbf{n}_c}\left(\frac{\partial w_\Sigma}{\partial c}\right)^{\flat_c}\bar{\rho} = 0 \quad \text{on } \Sigma_{\boldsymbol{\tau}}. \tag{3.25}$$

We shall transform the right hand sides of (3.24) and (3.25) using the isotropy of the material, that is, condition (3.11). Suppose that $\varphi_t \in \mathrm{Diff}(\mathcal{B})$ fixes $\partial_d\mathcal{B}$ and is the flow of the vector field $\mathbf{U} \in \mathfrak{X}_{\parallel}(\mathcal{B})$. Taking the time derivative in (3.11) yields the identity

$$\mathbf{d}\big(\mathcal{W}(C,G)\big)\cdot\mathbf{U} = \frac{\partial\mathcal{W}}{\partial C}(C,G):\pounds_{\mathbf{U}}C + \frac{\partial\mathcal{W}}{\partial G}(C,G):\pounds_{\mathbf{U}}G.$$

Integrating, we get

$$\int_{\mathcal{B}}\bar{\varrho}\,\mathbf{d}\big(\mathcal{W}(C,G)\big)\cdot\mathbf{U} = \int_{\mathcal{B}}\bar{\varrho}\frac{\partial\mathcal{W}}{\partial C}(C,G):\pounds_{\mathbf{U}}C + \int_{\mathcal{B}}\bar{\varrho}\frac{\partial\mathcal{W}}{\partial G}(C,G):\pounds_{\mathbf{U}}G.$$

Using formula (2.9), the relation above becomes

$$\begin{aligned}\int_{\mathcal{B}}\bar{\varrho}\,\mathbf{d}\big(\mathcal{W}(C,G)\big)\cdot\mathbf{U} = {}& -2\int_{\mathcal{B}}\mathrm{Div}_C\left(\bar{\varrho}\frac{\partial\mathcal{W}}{\partial C}\right)^{\flat_C}\cdot\mathbf{U} - 2\int_{\mathcal{B}}\mathrm{Div}_G\left(\bar{\varrho}\frac{\partial\mathcal{W}}{\partial G}\right)^{\flat_G}\cdot\mathbf{U}\\ &+2\int_{\partial\mathcal{B}}\left(\frac{\partial\mathcal{W}}{\partial C}\right)^{\flat_C}(\mathbf{N}_C,\mathbf{U})\gamma(C)\\ &+2\int_{\partial\mathcal{B}}\left(\frac{\partial\mathcal{W}}{\partial G}\right)^{\flat_G}(\mathbf{N}_G,\mathbf{U})\gamma(G)\end{aligned}$$

for all $\mathbf{U}$. Thus we get

$$\bar{\varrho}\,\mathbf{d}\big(\mathcal{W}(C,G)\big) = -2\,\mathrm{Div}_C\left(\bar{\varrho}\frac{\partial\mathcal{W}}{\partial C}\right)^{\flat_C} - 2\,\mathrm{Div}_G\left(\bar{\varrho}\frac{\partial\mathcal{W}}{\partial G}\right)^{\flat_G}, \quad \text{on } \mathcal{B}$$

and

$$\mathbf{i}_{\mathbf{N}_C}\left(\frac{\partial\mathcal{W}}{\partial C}\right)^{\flat_C} = -\mathbf{i}_{\mathbf{N}_G}\left(\frac{\partial\mathcal{W}}{\partial G}\right)^{\flat_G},$$

as one-forms on $\partial_{\boldsymbol{\tau}}\mathcal{B}$. Taking the push forward of these identities by $\eta \in \mathcal{C}$ we get the formulas

$$\bar{\rho}\,\mathbf{d}\big(w_\Sigma(g,c)\big) = -2\,\mathrm{Div}_g\left(\bar{\rho}\frac{\partial w_\Sigma}{\partial g}\right)^{\flat_g} - 2\,\mathrm{Div}_c\left(\bar{\rho}\frac{\partial w_\Sigma}{\partial c}\right)^{\flat_c}, \quad \text{on } D_\Sigma$$

and

$$\mathbf{i}_{\mathbf{n}_g}\left(\frac{\partial w_\Sigma}{\partial g}\right)^{\flat_g} = -\mathbf{i}_{\mathbf{n}_c}\left(\frac{\partial w_\Sigma}{\partial c}\right)^{\flat_c},$$

as one-forms on $\Sigma_{\boldsymbol{\tau}}$. Thus (3.24) and (3.25) produce the desired equations of motion and boundary conditions. □

4 Free Boundary Hydrodynamics

The Hamiltonian description of free boundary fluids has been studied in Lewis et al. (1986), Mazer and Ratiu (1989). The goal of this section is to carry out Lagrangian reduction for free boundary fluids and deduce the equations of motion and their associated constrained variational principles in convective and spatial representation.

The kinematic description of a free boundary fluid is the same as that for elasticity. The motion is described by a curve η in the space $\mathrm{Emb}(\mathcal{B},\mathcal{S})$ of embeddings. The material Lagrangian of a barotropic free boundary fluid is

$$L_{(\bar{\varrho},g)}(V_\eta) = \frac{1}{2}\int_{\mathcal{B}} g(V_\eta, V_\eta)\bar{\varrho} - \int_{\mathcal{B}} E\big(\bar{\varrho}(X), g\big(\eta(X)\big), T_X\eta\big)\bar{\varrho} - \tau\int_{\partial\mathcal{B}} \gamma\big(\eta^* g\big), \tag{4.1}$$

where E is the Lagrangian internal energy, related to the spatial energy e as in (2.3) and τ is a constant. The third term is proportional to the area of the current configuration and represents the potential energy associated with surface tension.

4.1 Convective Representation

The Lagrangian is invariant under the left action of the diffeomorphism group $\mathrm{Diff}(\mathcal{S})$ and produces the convective expression

$$\ell_{\mathrm{conv}}(\mathcal{V},\bar{\varrho},C) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V},\mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{E}(\bar{\varrho},C)\bar{\varrho} - \tau\int_{\partial\mathcal{B}}\gamma(C).$$

The computation of the reduced Euler–Lagrange equation goes exactly as for elasticity, except that one needs to take into account the surface integral, whose variation is

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_{\partial\mathcal{B}}\gamma\big(\eta_\varepsilon^* g\big) = \int_{\partial\mathcal{B}}\kappa_C C(\zeta,\mathbf{N}_C)\gamma(C),$$

where κ_C denotes the mean curvature of $\partial\mathcal{B}$ relative to C. This formula is an analogue of the transport theorem restricted to the boundary. Thus, for a general convective

Lagrangian whose boundary term is given as above, formula (3.10) is replaced by

$$\frac{\delta \ell_{\mathrm{conv}}}{\delta \mathcal{V}} C(\mathcal{V}, \mathbf{N}) + \tau \kappa_C \gamma(C)(\mathbf{N}_C)^{\flat_C} = 2\mathbf{i}_{\mathbf{N}_C}\left(\frac{\delta \ell_{\mathrm{conv}}}{\delta C}\right)^{\flat_C} \quad \text{on } \partial\mathcal{B}. \tag{4.2}$$

In a similar way as in Theorems 2.2 and 3.1, we obtain the following result.

Theorem 4.1 *The reduced Euler–Lagrange equations associated to ℓ_{conv} produce the equations for free boundary hydrodynamics in the convective description*

$$\begin{cases} \bar{\varrho}(\partial_t \mathcal{V} + \nabla_{\mathcal{V}} \mathcal{V}) = 2\,\mathrm{Div}_C\left(\dfrac{\partial \mathcal{E}}{\partial C}\bar{\varrho}\right), \\ \partial_t C - \boldsymbol{\pounds}_{\mathcal{V}} C = 0, \end{cases}$$

with the boundary condition

$$p \circ \eta|_{\partial\mathcal{B}} = \tau \kappa_C,$$

where κ_C is the mean curvature of $\partial\mathcal{B}$ relative to C and p is the spatial pressure. In terms of p, the right hand side of the motion equation reads $-\operatorname{grad}_C(p \circ \eta)\mu(C)$.

4.2 Spatial Representation

By a change of variables, one observes that the material Lagrangian (4.1) is invariant under the right action of $\mathrm{Diff}(\mathcal{B})$ given by

$$(V_\eta, \bar{\varrho}, g) \mapsto \big(V_\eta \circ \varphi, \varphi^* \bar{\varrho}, g\big), \quad \varphi \in \mathrm{Diff}(\mathcal{B}).$$

Using the same notations as in Sect. 3.2, this leads to the spatial expression

$$\ell_{\mathrm{spat}}(\Sigma, \mathbf{v}, \bar{\rho}, g) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} e(\rho)\bar{\rho} - \tau \int_\Sigma \gamma(g). \tag{4.3}$$

In order to take into account the boundary term, we will use a slight reformulation of the formula above, obtained by a change of variables. Given a path $\varepsilon \mapsto \eta_\varepsilon \in \mathrm{Diff}(D_\Sigma)$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_{\eta_\varepsilon(\Sigma)} \gamma(g) = \int_\Sigma \kappa_g g(\xi, \mathbf{n}_g)\gamma(g), \tag{4.4}$$

where $\mathbf{n}_g$ is the outward-pointing unit normal vector field relative to g, ξ is the tangent vector to η_ε at $\varepsilon = 0$, and κ_g is the mean curvature of Σ in the metric g. Adding this contribution to the boundary condition (3.21) (and eliminating the variable c), we get the relation

$$\left(\frac{\delta \widetilde{\ell_{\mathrm{spat}}}}{\delta \Sigma} - \rho \frac{\delta \ell_{\mathrm{spat}}}{\delta \bar{\rho}}\right)(\mathbf{n}_g)^{\flat_g}\gamma(g) = \tau \kappa_g (\mathbf{n}_g)^{\flat_g}\gamma(g) \quad \text{on } \Sigma, \tag{4.5}$$

where $\ell_{\mathrm{conv}} = \ell(\Sigma, \mathbf{v}, \bar{\rho}, g)$ is an arbitrary Lagrangian whose boundary term is given as above. This leads to the boundary condition

$$\frac{\delta \widetilde{\ell_{\mathrm{spat}}}}{\delta \Sigma} - \rho \frac{\delta \ell_{\mathrm{spat}}}{\delta \bar{\rho}} = \tau \kappa_g .$$

In the particular case of the Lagrangian (4.3) we get the following result.

Theorem 4.2 *The reduced Euler–Lagrange equations associated to ℓ_{spat} produces the equation for free boundary hydrodynamics*,

$$\begin{cases} \rho(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}) = -\operatorname{grad}_g p \\ \partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0 \end{cases} \quad \text{on } \Sigma \tag{4.6}$$

with the boundary condition and boundary movement

$$p|_\Sigma = \tau \kappa_g, \qquad \partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g).$$

5 Lagrangian Reduction for General Continua

We consider the general form of the Lagrangian for elasticity in material representation given by

$$\begin{aligned} L(V_\eta, \bar{\varrho}, g, G) = {} & \frac{1}{2} \int_{\mathcal{B}} g\big(\eta(X)\big)\big(V_\eta(X), V_\eta(X)\big) \bar{\varrho}(X) \\ & - \int_{\mathcal{B}} U\big(g\big(\eta(X)\big), T_X \eta, G(X), \bar{\varrho}(X)\big) - \tau \int_{\partial \mathcal{B}} \gamma\big(\eta^* g\big), \end{aligned} \tag{5.1}$$

where U is a density on $\mathcal{B}$. Note that this form is more general than the Lagrangian for free boundary fluids and for elastic materials considered before.

The hypothesis of material frame indifference assumes that U has the following invariance property: for all $\eta \in \operatorname{Emb}(\mathcal{B}, \mathcal{S})$ and all diffeomorphisms $\psi : \eta(\mathcal{B}) \to \mathcal{B}$ we have

$$\begin{aligned} & U\Big(\psi_* g\big(\psi\big(\eta(X)\big)\big), T_{\eta(X)} \psi \circ T_X \eta, G(X), \bar{\varrho}(X)\Big) \\ & \quad = U\big(g\big(\eta(X)\big), T_X \eta, G(X), \bar{\varrho}(X)\big). \end{aligned}$$

Therefore, we can define the convective energy density $\mathcal{U}$ by

$$\begin{aligned} \mathcal{U}\big(C(X), G(X), \bar{\varrho}(X)\big) & := U\big(\eta^* g(X), \mathbf{I}, G(X), \bar{\varrho}(X)\big) \\ & = U\big(g\big(\eta(X)\big), T_X \eta, G(X), \bar{\varrho}(X)\big). \end{aligned} \tag{5.2}$$

5.1 Convective Representation

Using the covariance assumption on the energy density U, one notes that the material Lagrangian depends on the Lagrangian variables only through the convective quantities

$$(\mathcal{V}, \bar{\varrho}, C, G) := \big(T\eta^{-1} \circ V_\eta, \bar{\varrho}, \eta^* g, G\big) \in \mathfrak{X}(\mathcal{B}) \times \big|\Omega^n(\mathcal{B})\big| \times S_2(\mathcal{B}) \times S_2(\mathcal{B}).$$

In terms of these variables, the Lagrangian is

$$\ell_{\text{conv}}(\mathcal{V}, \bar{\varrho}, C, G) = \frac{1}{2}\int_{\mathcal{B}} C(\mathcal{V}, \mathcal{V})\bar{\varrho} - \int_{\mathcal{B}} \mathcal{U}(C, G, \bar{\varrho}) - \tau \int_{\partial\mathcal{B}} \gamma(C).$$

The essential boundary condition on the convective velocity is $\mathcal{V}|_{\partial_d\mathcal{B}} = 0$. We will denote by $\mathfrak{X}_0(\mathcal{B})$ the corresponding space of vector fields. The convective equations of motion are derived in the same way as in Theorem 3.1, so we get

$$\begin{cases} \bar{\varrho}(\partial_t \mathcal{V} + \nabla_{\mathcal{V}}\mathcal{V}) = 2\,\text{Div}_C\left(\dfrac{\partial \mathcal{U}}{\partial C}\right), \\ \partial_t C - \pounds_{\mathcal{V}} C = 0, \end{cases} \tag{5.3}$$

with boundary conditions

$$\mathcal{V}|_{\partial_d\mathcal{B}} = 0, \qquad \left(2\frac{\partial \mathcal{U}}{\partial C} \cdot \mathbf{N}_C^{\flat_C} + \tau\kappa_C \gamma(C)\mathbf{N}_C^{\flat_C}\right)\bigg|_{\partial_\tau\mathcal{B}} = 0, \tag{5.4}$$

where ∇ is the Levi-Civita covariant derivative associated to the Cauchy–Green tensor C.

5.2 Spatial Representation

We now search conditions under which the material Lagrangian (5.1) is invariant under the *right* action of $\text{Diff}(\mathcal{B})$ given by

$$(V_\eta, \bar{\varrho}, g, G) \mapsto \big(V_\eta \circ \varphi, \varphi^*\bar{\varrho}, g, \varphi^* G\big),$$

for all diffeomorphisms $\varphi \in \text{Diff}(\mathcal{B})$. The kinetic energy and the boundary term on the potential energy are clearly right-invariant. For right invariance of the first term in the potential energy it suffices that

$$\begin{aligned} &U\big(g\big(\eta\big(\varphi(X)\big)\big), T_X(\eta \circ \varphi), \varphi^* G(X), \varphi^*\bar{\varrho}(X)\big) \\ &\quad = \varphi^*\big(U\big(g\big(\eta(\_)\big), T\_\eta, G(\_), \bar{\varrho}(\_)\big)\big)(X), \end{aligned}$$

for all $\varphi \in \text{Diff}(\mathcal{B})$. This is equivalent to

$$\mathcal{U}\big(\varphi^* C, \varphi^* G, \varphi^*\bar{\varrho}\big) = \varphi^*\mathcal{U}(C, G, \bar{\varrho}), \tag{5.5}$$

for all $\varphi \in \text{Diff}(\mathcal{B})$ by the defining relation (5.2) for $\mathcal{U}$.

Isotropy and Covariance An internal energy density satisfying (5.5) is said to be *materially covariant*.

Note that this definition is more general than the usual one given in Marsden and Hughes (1983, Definition 3.3.4) and recovers it if $\mathcal{U}(C, G, \bar{\varrho}) = \mathcal{W}(C, G)\bar{\varrho}$ for some function $\mathcal{W}$. In this section we shall extend the notions of isotropy and material covariance to continua described by the material Lagrangian (5.1). As we shall see, this will include both free boundary fluids and classical nonlinear elasticity.

Define *material symmetry* at $X \in \mathcal{B}$ to be a linear isometry $\boldsymbol{\lambda} : T_X\mathcal{B} \to T_X\mathcal{B}$ relative to $G(X)$ such that

$$\boldsymbol{\lambda}^*\big[\mathcal{U}\big(C(X), G(X), \bar{\varrho}(X)\big)\big] = \mathcal{U}\big(\big(\boldsymbol{\lambda}^*C\big)(X), G(X), \big(\boldsymbol{\lambda}^*\bar{\varrho}\big)(X)\big).$$

The material is called *isotropic* if, for each $X \in \mathcal{B}$, all proper rotations in $T_X\mathcal{B}$ are material symmetries at X. With these definitions, a direct verification shows that the proofs in Marsden and Hughes (1983, Proposition 3.5.7 and Corollary 3.5.11) go through which shows that *isotropy is equivalent to material covariance*.

From now on, we shall assume material covariance (5.5).

The Spatial Lagrangian Under the invariance hypothesis (5.5), proceeding as in Sect. 3.2, we see that the material Lagrangian induces the spatial Lagrangian

$$\ell_{\text{spat}}(\Sigma, \mathbf{v}, \bar{\rho}, g, c) = \frac{1}{2}\int_{D_\Sigma} g(\mathbf{v}, \mathbf{v})\bar{\rho} - \int_{D_\Sigma} u_\Sigma(c, g, \bar{\rho}) - \tau\int_\Sigma \gamma(g), \tag{5.6}$$

where

$$u_\Sigma(c, g, \bar{\rho}) := \eta_*\big(\mathcal{U}\big(\eta^*g, \eta^*c, \eta^*\bar{\rho}\big)\big),$$

η is a parametrization of D_Σ, that is, $\eta \in \mathcal{C} = \text{Emb}(\mathcal{B}, \mathcal{S})$ such that $\eta(\mathcal{B}) = D_\Sigma$, $\Sigma = \eta(\partial\mathcal{B})$, and the spatial variables are

$$\mathbf{v} := V_\eta \circ \eta^{-1} \in \mathfrak{X}(D_\Sigma), \qquad \bar{\rho} := \eta_*\bar{\varrho} \in \mathcal{F}(D_\Sigma)^*, \qquad c := \eta_*G \in \mathcal{S}_2(D_\Sigma).$$

By Lagrangian reduction on $T\mathcal{C}$, the spatial equations of motion are given by the stationarity condition

$$\delta\int_{t_0}^{t_1} \ell_{\text{spat}}(\Sigma, \mathbf{v}, \bar{\rho}, g, c)\,\mathrm{d}t = 0,$$

relative to the constrained variations (3.14), (3.15), (3.17), and (3.18). We get

$$\partial_t\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}} + \boldsymbol{\pounds}_{\mathbf{v}}\frac{\delta\ell_{\text{spat}}}{\delta\mathbf{v}} = \bar{\rho}\,\mathbf{d}\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}} + 2\,\text{Div}_c\bigg(\frac{\delta\ell_{\text{spat}}}{\delta c}\bigg)^{\flat_c}, \quad \text{on } D_\Sigma \tag{5.7}$$

and

$$\bigg(\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta\Sigma} - \rho\frac{\delta\ell_{\text{spat}}}{\delta\bar{\rho}}\bigg)(\mathbf{n}_g)^{\flat_g}\gamma(g) - 2\mathbf{i}_{\mathbf{n}_c}\frac{\delta\widetilde{\ell_{\text{spat}}}}{\delta c}\gamma(c) = 0 \quad \text{on } \Sigma_{\boldsymbol{\tau}}. \tag{5.8}$$

Applying these relations to the Lagrangian (5.6) we get the following result.

Theorem 5.1 *The reduced Euler–Lagrange equations associated to ℓ_{spat} produce the equations for nonlinear elasticity in spatial formulation*

$$\begin{cases} \bar{\rho}(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}) = 2\,\mathrm{Div}_g\left(\dfrac{\partial u_\Sigma}{\partial g}\right), \\ \partial_t c + \pounds_{\mathbf{v}} c = 0, \qquad \partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0 \end{cases} \tag{5.9}$$

with boundary conditions and boundary movement

$$\mathbf{v}|_{\Sigma_d} = 0, \qquad \left(\tau \kappa_g (\mathbf{n}_g)^{\flat_g} + 2\mathbf{i}_{\mathbf{n}_g}\left(\frac{\partial u_\Sigma}{\partial g}\right)^{\flat_g}\right)\bigg|_{T\Sigma_\tau} = 0, \qquad \partial_t \Sigma = g(\mathbf{v}, \mathbf{n}_g).$$

The second boundary condition says that the total sum of forces exerted on the free boundary Σ_τ is zero: it is the sum of the surface tension force and of the internal traction force.

Proof The functional derivatives of the Lagrangian (5.6) are

$$\frac{\delta \ell_{\text{spat}}}{\delta \mathbf{v}} = \mathbf{v}^{\flat_g} \bar{\rho}, \qquad \frac{\delta \ell_{\text{spat}}}{\delta \bar{\rho}} = \frac{1}{2} g(\mathbf{v}, \mathbf{v}) - \frac{\partial u_\Sigma}{\partial \bar{\rho}}, \qquad \frac{\delta \ell_{\text{spat}}}{\delta c} = -\frac{\partial u_\Sigma}{\partial c}, \tag{5.10}$$

where $\flat_g$ is associated to the metric g on $\mathcal{S}$. Inserting these expressions in (5.7) and using the relations $\pounds_{\mathbf{v}} \mathbf{v}^{\flat_g} = \nabla_{\mathbf{v}} \mathbf{v}^{\flat_g} + \frac{1}{2}\mathbf{d}(g(\mathbf{v}, \mathbf{v}))$ and $\partial_t \bar{\rho} + \pounds_{\mathbf{v}} \bar{\rho} = 0$, we get

$$(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v})^{\flat_g} \bar{\rho} = -2\,\mathrm{Div}_c\left(\frac{\partial u_\Sigma}{\partial c}\right)^{\flat_c} - \bar{\rho}\,\mathbf{d}\frac{\partial u_\Sigma}{\partial \bar{\rho}}. \tag{5.11}$$

Equations (3.19) and (4.4) imply

$$\frac{\delta \ell_{\text{spat}}}{\delta \Sigma} = \left(\rho \frac{1}{2} g(\mathbf{v}, \mathbf{v}) - \tau \kappa_g\right) \gamma(g) - u_\Sigma(c, g, \bar{\rho}),$$

as a density on Σ. Therefore, the first two terms in (5.8) simplify and we get

$$-2\mathbf{i}_{\mathbf{n}_c}\left(\frac{\partial u_\Sigma}{\partial c}\right)^{\flat_c} = \left(\rho \frac{\partial u_\Sigma}{\partial \bar{\rho}} - \tau \kappa_g\right)(\mathbf{n}_g)^{\flat_g} \gamma(g) - u_\Sigma(c, g, \bar{\rho})(\mathbf{n}_g)^{\flat_g} \quad \text{on } \Sigma_\tau. \tag{5.12}$$

We shall transform the right hand side of (5.11) and (5.12) using the isotropy of the material, that is, condition (5.5). Suppose that $\varphi_t \in \mathrm{Diff}(\mathcal{B})$ fixes $\partial_d \mathcal{B}$ and is the flow of the vector field $\mathbf{U} \in \mathfrak{X}_\parallel(\mathcal{B})$. Taking the time derivative in (3.11) yields the identity

$$\int_{\mathcal{B}} \pounds_{\mathbf{U}}\big(\mathcal{U}(C, G, \bar{\varrho})\big) = \int_{\mathcal{B}} \frac{\partial \mathcal{U}}{\partial C}(C, G, \bar{\varrho}) : \pounds_{\mathbf{U}} C + \int_{\mathcal{B}} \frac{\partial \mathcal{U}}{\partial G}(C, G, \bar{\varrho}) : \pounds_{\mathbf{U}} G + \int_{\mathcal{B}} \frac{\partial \mathcal{U}}{\partial \bar{\varrho}} \pounds_{\mathbf{U}} \bar{\varrho}.$$

Using formula (2.9), the relation above becomes

$$\begin{aligned}
&\int_{\partial\mathcal{B}} \mathcal{U}(C,G,\bar{\varrho})C(\mathbf{U},\mathbf{N}_C) \\
&\quad = -2\int_{\mathcal{B}} \operatorname{Div}_C\left(\frac{\partial\mathcal{U}}{\partial C}\right)^{\flat_C}\cdot\mathbf{U} - 2\int_{\mathcal{B}} \operatorname{Div}_G\left(\frac{\partial\mathcal{U}}{\partial G}\right)^{\flat_G}\cdot\mathbf{U} - \int_{\mathcal{B}} \bar{\varrho}\,\mathbf{d}\frac{\partial\mathcal{U}}{\partial\bar{\varrho}}\cdot\mathbf{U} \\
&\qquad + 2\int_{\partial\mathcal{B}} \left(\frac{\partial\mathcal{U}}{\partial C}\right)^{\flat_C}(\mathbf{N}_C,\mathbf{U})\gamma(C) + 2\int_{\partial\mathcal{B}} \left(\frac{\partial\mathcal{U}}{\partial G}\right)^{\flat_G}(\mathbf{N}_G,\mathbf{U})\gamma(G) \\
&\qquad + \int_{\partial\mathcal{B}} \bar{\varrho}\frac{\partial\mathcal{U}}{\partial\bar{\varrho}}C(\mathbf{U},\mathbf{N}_C)
\end{aligned}$$

for all $\mathbf{U}\in\mathfrak{X}_{\parallel}(\mathcal{B})$. Thus we get

$$2\operatorname{Div}_C\left(\frac{\partial\mathcal{U}}{\partial C}\right)^{\flat_C} + 2\operatorname{Div}_G\left(\frac{\partial\mathcal{U}}{\partial G}\right)^{\flat_G} + \bar{\varrho}\,\mathbf{d}\frac{\partial\mathcal{U}}{\partial\bar{\varrho}} = 0, \quad \text{on } \mathcal{B}$$

and

$$\mathcal{U}(C,G,\bar{\varrho})\mathbf{N}_C^{\flat_C} = 2\mathbf{i}_{\mathbf{N}_C}\left(\frac{\partial\mathcal{U}}{\partial C}\right)^{\flat_C} + 2\mathbf{i}_{\mathbf{N}_G}\left(\frac{\partial\mathcal{U}}{\partial G}\right)^{\flat_G} + \bar{\varrho}\frac{\partial\mathcal{U}}{\partial\bar{\varrho}}\mathbf{N}_C^{\flat_C},$$

as one-forms on $\partial_{\boldsymbol{\tau}}\mathcal{B}$. Taking the push forward of these identities by $\eta\in\mathcal{C}$ we get the formulas

$$2\operatorname{Div}_g\left(\frac{\partial u_\Sigma}{\partial g}\right)^{\flat_g} + 2\operatorname{Div}_c\left(\frac{\partial u_\Sigma}{\partial c}\right)^{\flat_c} + \bar{\rho}\,\mathbf{d}\frac{\partial u_\Sigma}{\partial\bar{\rho}} = 0, \quad \text{on } D_\Sigma$$

and

$$u_\Sigma(g,c,\bar{\rho})(\mathbf{n}_g)^{\flat_g} = 2\mathbf{i}_{\mathbf{n}_g}\left(\frac{\partial u_\Sigma}{\partial g}\right)^{\flat_g} + 2\mathbf{i}_{\mathbf{n}_c}\left(\frac{\partial u_\Sigma}{\partial c}\right)^{\flat_c} + \bar{\rho}\frac{\partial u_\Sigma}{\partial\bar{\rho}}(\mathbf{n}_g)^{\flat_g},$$

as a one-forms on $\Sigma_{\boldsymbol{\tau}}$. Thus (5.11) and (5.12) produce the desired equations of motion and boundary conditions. □

5.3 Kelvin–Noether Theorems

We shall derive the Kelvin–Noether theorems for the spatial and convective representations of the elastic materials given by the general Lagrangian (5.1). We begin with the spatial representation.

Given a solution $\mathbf{v}$ of the system (5.1), let γ be a loop in $\mathcal{B}$ and $\gamma_t := \gamma\circ\eta_t$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t}\mathbf{v}^{\flat_g} &= \oint_{\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\eta_t^*\mathbf{v}^{\flat_g} = \oint_{\gamma}\eta_t^*\left(\partial_t\mathbf{v}^{\flat_g} + \boldsymbol{\pounds}_{\mathbf{v}}\mathbf{v}^{\flat_g}\right) \\
&= \oint_{\gamma_t}\left(\partial_t\mathbf{v}^{\flat_g} + \nabla_{\mathbf{v}}\mathbf{v}^{\flat_g}\right) = 2\oint_{\gamma_t}\left(\frac{1}{\bar{\rho}}\operatorname{Div}_g\left(\frac{\partial u_\Sigma}{\partial g}\right)\right)^{\flat_g},
\end{aligned}$$

where we have used the formula $\pounds_{\mathbf{v}} \mathbf{v}^{\flat_g} = \nabla_{\mathbf{v}} \mathbf{v}^{\flat_g} + \frac{1}{2} \mathbf{d} g(\mathbf{v}, \mathbf{v})$. Thus the Kelvin circulation theorem for the spatial representation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \mathbf{v}^{\flat_g} = 2 \oint_{\gamma_t} \left(\frac{1}{\bar{\rho}} \operatorname{Div}_g \left(\frac{\partial u_\Sigma}{\partial g} \right) \right)^{\flat_g}. \tag{5.13}$$

To find a Kelvin circulation theorem for the convective representation, we pick a loop γ in $\mathcal{B}$ and a solution $\mathcal{V}$ of (5.3) and we recall that $\partial_t (\mathcal{V}^{\flat_C}) = (\partial_t \mathcal{V})^{\flat_C} + \pounds_{\mathcal{V}} (\mathcal{V}^{\flat_C})$. We thus obtain the Kelvin–Noether theorem,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \oint_\gamma \mathcal{V}^{\flat_C} &= \oint_\gamma \left((\partial_t \mathcal{V})^{\flat_C} + \pounds_{\mathcal{V}} \left(\mathcal{V}^{\flat_C} \right) \right) = \oint_\gamma \left((\partial_t \mathcal{V})^{\flat_C} + \nabla_{\mathcal{V}} \mathcal{V}^{\flat_C} \right) \\ &= 2 \oint_\gamma \frac{1}{\bar{\varrho}} \operatorname{Div}_C \left(\frac{\partial \mathcal{U}}{\partial C} \right)^{\flat_C}. \end{aligned} \tag{5.14}$$

5.4 Recovering the Classical Cases

So far we have used three type of internal energy densities in material, convective, and spatial representations. In this section we have considered the most general form of internal energy densities $U(g, T\eta, G, \bar{\varrho})$. The internal energy density is assumed to satisfy the *material frame indifference axiom* which states that U is *left* invariant under the spatial diffeomorphism group, that is,

$$\begin{aligned} &U\left(\psi_* g\left(\psi\left(\eta(X)\right)\right), T_{\eta(X)}\psi \circ T_X \eta, G(X), \bar{\varrho}(X)\right) \\ &\quad = U\left(g\left(\eta(X)\right), T_X \eta, G(X), \bar{\varrho}(X)\right), \end{aligned}$$

for all $\psi \in \operatorname{Diff}(\mathcal{S})$. This is always assumed in continuum mechanics and only such internal energy densities have physical meaning. Material frame indifference allows the definition of the internal energy density $\mathcal{U}$ in convective representation, namely

$$\mathcal{U}(G, C, \bar{\varrho}) := U(g \circ \eta, T\eta, G, \bar{\varrho}), \quad C := \eta^* g.$$

On the other hand, *right* invariance under the particle relabeling group $\operatorname{Diff}(\mathcal{B})$ does not hold, in general. If it does, that is,

$$\begin{aligned} &U\left(g\left(\eta\left(\varphi(X)\right)\right), T_X(\eta \circ \varphi), \varphi^* G(X), \varphi^* \bar{\varrho}(X)\right) \\ &\quad = \varphi^*\left(U\left(g\left(\eta(\_)\right), T\_\eta, G(\_), \bar{\varrho}(\_)\right)\right)(X), \end{aligned}$$

for all $\varphi \in \operatorname{Diff}(\mathcal{B})$, then, as we have seen, the material is isotropic. Isotropic materials admit a description in spatial representation since the internal energy density U induces the spatial internal energy density by

$$u_\Sigma(c, g, \bar{\rho}) := \eta_*\left(\mathcal{U}\left(\eta^* g, \eta^* c, \eta^* \bar{\rho}\right)\right), \quad c := \eta_* G, \ \bar{\rho} = \eta_* \bar{\varrho}.$$

Two important special cases appear in continuum mechanics.

Classical Elasticity Classical nonlinear elasticity assumes that U is linear in $\bar{\varrho}$, that is, $U(g, T\eta, G, \bar{\varrho}) = W(g, T\eta, G)\bar{\varrho}$. The convective internal energy density is

$$\mathcal{U}(G, C, \bar{\varrho}) = \mathcal{W}(G, C)\bar{\varrho}$$

The second boundary condition (5.4) becomes $\boldsymbol{\Sigma} \cdot \mathbf{N}_C^{\flat_C}|_{\partial_\tau \mathcal{B}} = 0$ recovering the form in Theorem 3.1. We now obtain Kelvin's circulation theorem using the general expression (5.14). We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_\gamma \mathcal{V}^{\flat_C} &= 2\oint_\gamma \frac{1}{\bar{\varrho}}\operatorname{Div}_C\left(\frac{\partial \mathcal{U}}{\partial C}\right)^{\flat_C} = \oint_\gamma \frac{1}{\bar{\varrho}}\operatorname{Div}_C\left(2\frac{\partial \mathcal{W}}{\partial C}\mathcal{R}\right)^{\flat_C}\mu(C) \\ &= \oint_\gamma \frac{1}{\mathcal{R}}\operatorname{Div}_C(\boldsymbol{\Sigma})^{\flat_C},\end{aligned}$$

by the Doyle–Ericksen formula (3.3) and the relation $\bar{\varrho} = \mathcal{R}\mu(C)$.

The material covariance condition (5.5) becomes in this case

$$W\big(g\big(\eta(\varphi(X))\big), T_X(\eta\circ\varphi), \varphi^* G(X)\big) = \big(W\big(g\big(\eta(\_)\big), T\_\eta, G(\_)\big)\circ\varphi\big)(X),$$

for all $\varphi \in \operatorname{Diff}(\mathcal{B})$ which recovers the classical definition from elasticity (see Marsden and Hughes 1983, Definition 3.3.4) and hence the material is isotropic in the usual sense. For such isotropic materials one can define the spatial energy density by

$$u_\Sigma(g, c, \bar{\rho}) := w_\Sigma(g, c)\bar{\rho}$$

and hence, if $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, we have

$$\frac{\partial u_\Sigma}{\partial g} = \frac{\partial w_\Sigma}{\partial g}\bar{\rho} = \frac{1}{2}\boldsymbol{\sigma}\mu(g)$$

by the Doyle–Ericksen formula.

The second boundary condition in Theorem 5.1 reduces to $\boldsymbol{\sigma} \cdot \mathbf{n}_g|_{T\Sigma_\tau} = 0$, which coincides with the second boundary condition in Theorem 3.2 for classical elasticity. Kelvin's circulation theorem (5.13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t} \mathbf{v}^{\flat_g} = \oint_{\gamma_t}\frac{1}{\rho}\operatorname{Div}_g(\boldsymbol{\sigma})^{\flat_g}.$$

Note that the general approach considered here allows for surface tension in an elastic material in which case the conditions at the free boundary are

$$\big(\tau\kappa_C \mathbf{N}^{\flat_C} + \boldsymbol{\Sigma}\cdot\mathbf{N}^{\flat_C}\big)\big|_{\partial_\tau\mathcal{B}} = 0, \quad \text{respectively,} \quad \big(\tau\kappa_g \mathbf{n}^{\flat_g} + \boldsymbol{\sigma}\cdot\mathbf{n}^{\flat_g}\big)\big|_{T\Sigma_\tau} = 0.$$

Hydrodynamics Fluids are defined by requiring that the internal energy density depend only on the Jacobian of the deformation η. This is equivalent to the condition

$$U(g, T\eta, G, \bar{\varrho}) = E(g, T\eta, \bar{\varrho})\bar{\varrho} = e\left(\frac{\bar{\varrho}}{\mu(\eta^* g)}\right)\bar{\varrho},$$

where e is the internal energy density in spatial representation. Indeed, if this condition holds then U depends only on the Jacobian $\frac{\bar{\varrho}}{\mu(\eta^* g)} = \frac{\bar{\varrho}}{\eta^*\mu(g)}$ of the configuration η relative to the volume $\mu(g)$ defined by the spatial metric g and the material mass density $\bar{\varrho}$.

The internal energy density in convective representation is

$$\mathcal{U}(G, C, \bar{\varrho}) = \mathcal{E}(C, \bar{\varrho})\bar{\varrho} = e\left(\frac{\bar{\varrho}}{\mu(C)}\right)\bar{\varrho}, \quad C := \eta^* g.$$

The second boundary condition (5.4) becomes $p \circ \eta|_{\partial\mathcal{B}} = \tau\kappa_C$; it appears in Theorem 4.1.

A straightforward verification shows that fluids are isotropic materials and hence they have a spatial representation. The spatial internal energy density is

$$u_\Sigma(g, c, \bar{\rho}) := e\left(\frac{\bar{\rho}}{\mu(g)}\right)\bar{\rho}, \quad c := \eta_* G, \ \bar{\rho} = \eta_* \bar{\varrho}.$$

We have

$$\frac{\partial u_\Sigma}{\partial g} = -\frac{1}{2} p\mu(g) g^\sharp, \quad p = \rho^2 \frac{\partial e}{\partial \rho}, \ \rho = \frac{\bar{\rho}}{\mu(g)}$$

which recovers the definition of pressure in fluid dynamics.

The second boundary condition in Theorem 5.1 reduces to $p|_\Sigma = \tau\kappa_g$, which coincides with the second boundary condition in Theorem 4.2 for free boundary fluids. The Kelvin circulation theorems for both the convective and spatial representations state that the circulation of the convective or spatial vector fields is zero:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_\gamma \mathcal{V}^{\flat_C} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \mathbf{v}^{\flat_g} = 0.$$

The fluids we considered here are barotropic, that is, the internal energy density e depends only on ρ. Had the fluid been isentropic, that is, had e depended also on entropy, the right hand sides in the Kelvin circulation theorems would not vanish; see (Holm et al. 1998).

Acknowledgements F. Gay-Balmaz was partially supported by a Swiss NSF postdoctoral fellowship. J.E. Marsden was partially supported by the California Institute of Technology. T.S. Ratiu was partially supported by Swiss NSF grant 200021-140238 and by the government grant of the Russian Federation for support of research projects implemented by leading scientists, Lomonosov Moscow State University under the agreement No. 11.G34.31.0054.

We thank D.D. Holm for helpful discussions.

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