Transport in Porous Media **54:** 265–271, 2004.

The Free Convection Boundary-Layer Flow Induced in a Fluid Saturated Porous Medium by a Non-Isothermal Vertical Cylinder Approaches the Shape of Schlichting's Round Jet as the Cylinder Radius Tends to Zero

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(Received: 24 July 2002)

Abstract. The title statement is proven for a circular cylinder whose surface temperature (above that of the ambient fluid) varies inversely proportional with the axial distance from the leading edge.

Key words: boundary layer, free convection, porous medium, round jet, vertical cylinder.

1. Introduction

Since the pioneering work of Minkowycz and Cheng (1976), the free convection from a vertical cylinder with a power-law surface temperature distribution $T_s(x) = T_\infty + A \cdot x^n$ and embedded in a fluid saturated porous medium has attracted considerable interest. In the paper of Minkowycz and Cheng (1976), the boundary layer approximation of the governing equations were established, numerical results for various values of n between 0 and 1 were reported and comparisons between the local similarity and local nonsimilarity methods were given. A detailed study of the isothermal case $n = 0$ using a nonlocal marching method and an asymptotic analysis for large values of x has been performed by Merkin (1986). Extensions of the problem for the case of mixed convection were presented by Merkin and Pop (1987), for non-Darcy free convection by Hossain and Nakayama (1993) and for non-Darcy mixed convection by Kumari and Nath (1989), respectively. More recently, Bassom and Rees (1996) have extended the work of Merkin (1986) to a range of values on the power-law exponent n. These authors found that the asymptotic flow field far from the leading edge of the cylinder takes a multilayer structure for $n < 1$. This multilayer structure still persists for $n > 1$ close to the leading edge, while far downstream a simple single layer is present.

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The present paper considers the free convection from a vertical cylinder with inversely-linear surface temperature distribution ($n = -1$) embedded in a fluid saturated porous medium and shows that in the limiting case of the vanishing cylinder radius a the corresponding flow field can be overlapped on the velocity field of Schlichting's celebrated round jet in a clear viscous fluid.

2. Basic Equations

We consider the steady free convection boundary-layer from a vertical impermeable cylinder of radius a embedded in a fluid saturated porous medium of ambient temperature T_∞. The cylinder is heated and its axial symmetric surface temperature T_s, which everywhere exceeds the ambient temperature T_∞ of the fluid, is prescribed (see below). Under these conditions, over the cylinder an ascending free convection boundary layer flow will be formed (see Figure 1). Assuming the Boussinesq approximation is valid, the basic boundary-layer equations are (Minkowycz and Cheng, 1976):

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{1}$$

$$u = \frac{g\beta K}{\upsilon}(T - T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{3}$$

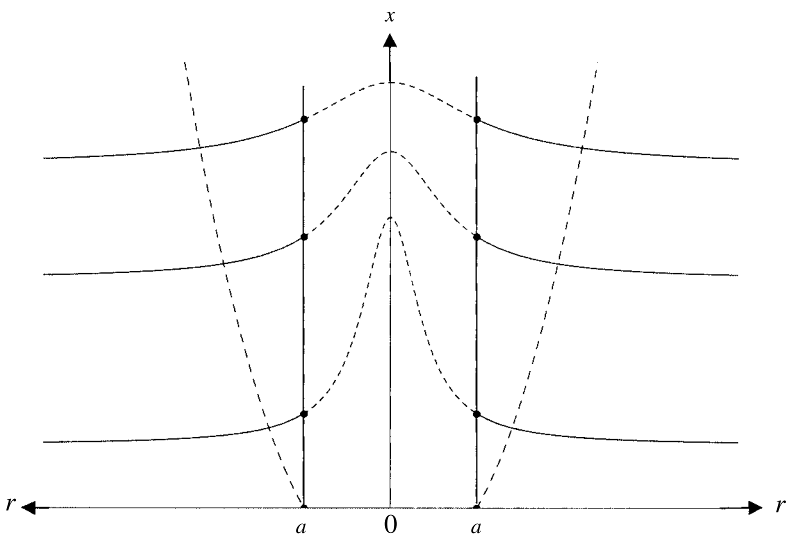

Figure 1. Schematics of the flow domain around the cylinder of radius a (full lines) and coordinate system. In the limiting case $a \to 0$ of an infinitesimally thin cylinder (and a suitable surface temperature distribution), the occurrence of a jet-like velocity field is expected.

Here $x \geqslant 0$ and $r \geqslant 0$ are the axial and radial coordinates, respectively, u and v the velocity components along x and r axes, T is the fluid temperature, K the permeability of the porous medium, g the acceleration due to gravity, and α, β and $\upsilon = \mu/\rho$ are the effective thermal diffusivity, thermal expansion coefficient and kinematic viscosity, respectively. These equations are to be solved subject to the boundary conditions

$$T = T_\infty + T_0 \cdot \left(\frac{x}{L}\right)^n \equiv T_s(x) \quad \text{and} \quad v = 0 \quad \text{on } r = a,$$
$$T \to T_\infty \quad \text{as } y \to \infty \tag{4}$$

where L is a reference length which represents the axial distance x at which $T_s(x)$ takes a prescribed value $T_s(L) = T_\infty + T_0$, where $T_0 > 0$ will be assumed. The major part of the present paper is concerned with the special case $n = -1$ of the inversely-linear surface temperature distribution.

After the introduction of Stokes' stream function ψ and the dimensionless quantities X, Y, ϕ and Ra defined by the expressions

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}, \qquad Ra = \frac{g\beta K T_0 L}{\alpha \upsilon}, \qquad x = LX,$$
$$r = LY, \qquad \psi = \alpha L \phi \tag{5}$$

the dimensional temperature and velocity fields will be given by

$$T = T_\infty + \frac{T_0}{Ra}\frac{1}{Y}\frac{\partial \phi}{\partial Y}, \qquad u = \frac{\alpha}{L}\frac{1}{Y}\frac{\partial \phi}{\partial Y}, \qquad v = -\frac{\alpha}{L}\frac{1}{Y}\frac{\partial \phi}{\partial X} \tag{6}$$

and the problem (1)–(4) reduces to solve equation

$$Y^2 \frac{\partial^3 \phi}{\partial Y^3} - \left(1 - \frac{\partial \phi}{\partial X}\right)\left(Y\frac{\partial^2 \phi}{\partial Y^2} - \frac{\partial \phi}{\partial Y}\right) - Y\frac{\partial \phi}{\partial Y}\frac{\partial^2 \phi}{\partial X \partial Y} = 0 \tag{7}$$

along with the boundary conditions

$$\frac{1}{Y}\frac{\partial \phi}{\partial Y} = Ra \cdot X^n \quad \text{and} \quad \frac{\partial \phi}{\partial X} = 0 \quad \text{on } Y = \frac{a}{L}, \qquad \frac{1}{Y}\frac{\partial \phi}{\partial Y} \to 0$$
$$\text{as } Y \to \infty \tag{8}$$

Thus, the axial velocity profile u measured in units of α/L and the temperature difference $T - T_\infty$ measured in units of T_0/Ra coincide:

$$\frac{u}{\alpha/L} = \frac{T - T_\infty}{T_0/Ra} = \frac{1}{Y}\frac{\partial \phi}{\partial Y} \tag{9}$$

3. Similarity Solutions

We start with the similarity transformation

$$\phi = 2Xf(\eta), \quad \eta = Ra \cdot \left(\frac{Y}{2}\right)^2 X^{n-1} \tag{10}$$

which yields for the temperature and velocity fields (6) the expressions

$$\begin{aligned} T &= T_\infty + T_0 X^n f'(\eta), \qquad u = \frac{\alpha}{L} Ra \cdot X^n f'(\eta), \\ v &= -\frac{2\alpha}{LY}[f(\eta) + (n-1)\eta f'(\eta)] \end{aligned} \tag{11}$$

Here the prime denotes differentiation with respect to the similarity variable η whose variation range is

$$Ra \cdot \left(\frac{a}{2L}\right)^2 X^{n-1} \equiv \eta_0 \leqslant \eta \leqslant \infty \tag{12}$$

and the function $f(\eta)$ satisfies the ordinary differential equation

$$\eta f''' + (1 + f)f'' - nf'^2 = 0 \tag{13}$$

along with the boundary conditions

$$f(\eta_0) = (1-n)\eta_0, \qquad f'(\eta_0) = 1, \qquad f'(\infty) = 0 \tag{14}$$

For the present paper it is important to notice that Equation (13) may be transcribed in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(\eta f' + \frac{1}{2}f^2 - f\right) = (1+n)f'^2 \tag{15}$$

As we see, the transformation (10) leads to the exact similarity equation (13) subject to the boundary conditions (14), among them the first one depends in general on the axial coordinate x. Thus, we are faced here for any $n \neq 1$ (and $a \neq 0$) with an exact local similarity problem, where the local feature occurs only in one of the boundary conditions (but not in the differential equation). In the case $n = +1$, η_0 is independent of x and the problem becomes fully similar. It also becomes fully similar for any n in the limiting case of an infinitesimally thin cylinder, $a \to 0$, where $\eta_0 \to 0$.

The present form of the exact local similarity implies that for any specified axial distance x from the leading edge (i.e., for any specified value of η_0), that value of $f''(\eta_0)$ has to be sought for which the boundary value problem (13) and (14) becomes a well-posed initial value problem whose solution satisfies the additional condition $f'(\infty) = 0$. Once the quantity $f''(\eta_0)$ has been found (e.g., by the familiar shooting method), the corresponding temperature and velocity fields will be obtained (numerically) according to Equations (11). The dimensionless surface heat flux as specified by the local Nusselt number Nu_x is then given by the expression

$$Nu_x = -\frac{aRa}{2L} f''(\eta_0) X^n \tag{16}$$

In the special case $n = -1$ the above reduction to an initial value problem (with the additional condition $f'(\infty) = 0$) becomes even much simpler, since for $n = -1$ the right-hand side of Equation (15) vanishes and thus from this equation two first integrals emerge:

$$\eta f'' + f \cdot f' = C_1 \qquad \eta f' + \tfrac{1}{2} f^2 - f = C_1 \eta + C_2 \tag{17}$$

where C_1 and C_2 are integration constants which depend in general on η_0. The first two boundary conditions (14) imply $C_1 = \eta_0 \cdot [2 + f''(\eta_0)]$ and $C_2 = -\eta_0 \cdot [1 + \eta_0 \cdot f''(\eta_0)]$ and thus we obtain the first order Riccati equation

$$\eta f' = -\tfrac{1}{2} f^2 + f + \eta_0 [2 + f''(\eta_0)] \cdot \eta - \eta_0 [1 + \eta_0 f''(\eta_0)] \tag{18}$$

A comprehensive discussion of the local similarity solutions corresponding to arbitrary values of n and η_0 is out of the scope of this paper. As specified in the title, our present concern is the limiting case of an infinitesimally thin cylinder, $a \to 0$, with an inversely linear ($n = -1$) surface temperature distribution $T_s(x)$. In this case $\eta_0 = C_1 = C_2 = 0$ and the exact local similarity problem reduces to an exact full similarity problem. The corresponding solution of Equation (18) is easily found. It is

$$f = \frac{\eta}{1 + \eta/2}, \quad \eta = Ra \cdot \left(\frac{r}{2x}\right)^2 \tag{19}$$

Accordingly

$$T = T_\infty + \frac{L}{x} \frac{T_0}{(1 + \eta/2)^2} \tag{20}$$

and

$$u = \frac{\alpha}{x} \frac{Ra}{(1 + \eta/2)^2}, \qquad v = \frac{\alpha Ra}{2} \frac{r}{x^2} \frac{1 - \eta/2}{(1 + \eta/2)^2} \tag{21}$$

4. Discussion and Conclusions

Comparing expressions (21) of u and v to the corresponding velocity components of Schlichting's round jet (Schlichting, 1933; Schlichting and Gersten, 1997), it turns out immediately that the functional dependence of these two velocity fields on x and r is exactly the same. As an illustration, in Figure 2 the jet-like profiles

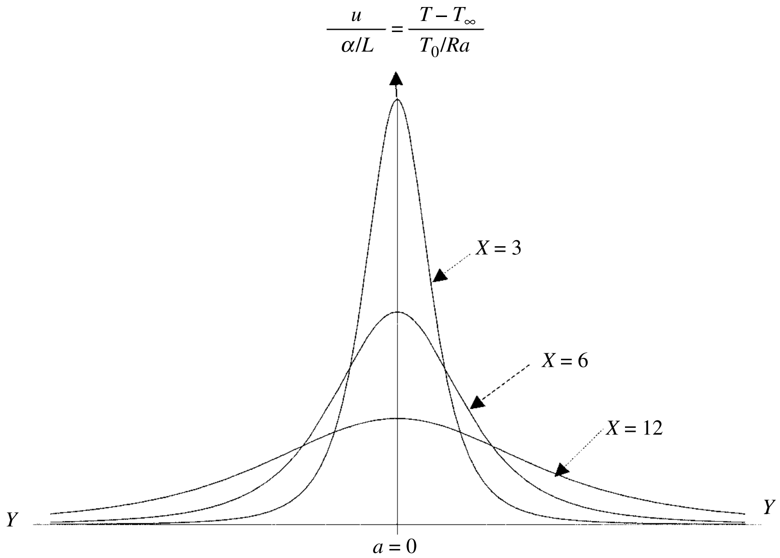

Figure 2. Dimensionless downstream velocity (and temperature) profiles as a function of the (dimensionless) radial coordinate *Y* for different axial distances *X*, plotted for $Ra = 100$.

of the downstream velocity component of our free convection flow, as given by the first equation of (21), are shown for different axial distances from the leading edge. The momentum-spreading with increasing *x*, which is a characteristic feature of free jets, is also clearly seen in Figure 2. According to Equation (9), these velocity profiles measured in units of α/L also coincide with those of the dimensionless temperature field $(T - T_\infty)/(T_0/Ra)$. Thus, while Schlichting's round jet is isothermal, the present free convection boundary layer flow is associated with a thermal boundary layer which is congruent with the downstream velocity. The corresponding Nusselt number as given by Equation (16) is vanishing for any $x \neq 0$ as $a \to 0$. This means, that in our free convection flow from an infinitesimally thin cylinder ("wire") heat is transferred to the fluid only at the leading edge, $x = 0$.

In order to achieve a full overlap of the velocity field the free convection flow and Schlichting's round jet, in addition to their identical functional dependences on *x* and *r*, also certain gauge conditions must be fulfilled. There are two such conditions. One of them is physically evident: we must require that the momentum flux

$$\dot{J} = 2\pi\rho \int_0^\infty u^2 r \, \mathrm{d}r \tag{22}$$

which is now a conserved quantity, has the same value for the two flow fields. For the free convection flow this implies to chose the reference length *L* such that the Darcy–Rayleigh number defined under Equations (5) takes the value

$$Ra = \frac{3\dot{J}}{8\pi\rho\alpha^2} \tag{23}$$

The second gauge condition results by a simple inspection of the corresponding equations (obtained after the expression (23) of *Ra* has been substituted in Equations (21)). This second condition requires to consider the round jet of such a clear fluid whose kinematic viscosity υ equals the effective thermal diffusivity α of the fluid saturated porous medium in which our free convection flow has been formed.

We may therefore conclude that the velocity field of the free convection boundary layer flow from an infinitesimally thin vertical cylinder (wire) with inversely linear temperature distribution and embedded in a fluid saturated porous medium, can be mapped on the far field of Schlichtling's round jet formed in a clear viscous fluid.

Finally, it is worth noticing that a similar mapping of Schlichtling's round jet on a boundary layer flow induced by a stretching wire is also possible (Magyari and Keller, 2001). In this case, already the gauge condition of the coinciding momentum fluxes suffices for a full overlap of the two velocity fields (Magyari and Keller, 2001).

References

Bassom, A. P. and Rees, D. A. S.: 1996, Free convection from a heated vertical cylinder embedded in a fluid saturated porous medium, *Acta Mech.* **116**, 139–151.

Hossain, M. A. and Nakayama, A.: 1993, Non-Darcy free convection along a vertical cylinder embedded in a porous medium with surface mass flux, *Int. J. Heat Fluid Flow* **14**, 385–390.

Kumari, M. and Nath, G.: 1989, Non-Darcy mixed convection boundary layer flow on a vertical cylinder in a saturated porous medium with surface mass flux, *Int. J. Heat Mass Transfer* **32**, 183–187.

Magyari, E. and Keller, B.: 2001, The free jets as boundary layer flows induced by continuous stretching surfaces, *Heat and Mass Transfer* **38**, 111–1114.

Merkin, J. H.: 1986, Free convection from a vertical cylinder embedded in a saturated porous medium, *Acta Mech*. **62**, 19–28.

Merkin, J. H. and Pop, I.: 1987, Mixed convection boundary layer flow on a vertical cylinder embedded in a porous medium, *Acta Mech*. **66**, 251–262.

Minkowycz, W. J. and Cheng, P.: 1976, Free convection about a vertical cylinder embedded in a porous medium, *Int. J. Heat Mass Transfer* **19**, 805–813.

Schlichting, H.: 1933, Laminare Strahlausbreitung, *J. Appl. Math. Mech. (ZAMM)* **13**, 260–263.

Schlichting, H. and Gersten, K.: 1997, *Grenzschicht-Theorie*, 9th edn., Springer, Berlin. (1933), Laminare Strahlausbreitung, *J. Appl. Math. Mech. (ZAMM)* **13**, 260–263.