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Equilibrium strategies in random-demand procurement auctions with sunk costs

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We address an auction model which captures basic features of balancing markets for electricity. The existence and uniqueness of equilibrium are examined and a method for explicit calculation of bid strategies is presented.

Keywords: auctions; equilibrium; electricity markets; risk management.

1. Introduction

Over the last two decades, a process has been underway worldwide to privatize state enterprises and to liberalize markets for services of infrastructure industries, such as electricity, gas, telecommunications, transport and water. As a rule, privatization and liberalization are effected by the introduction of competitive wholesale markets, organized by an appropriate auction principle. The perspective of game theory provides a suitable framework to analyse their properties. In this work, we focus on multi-unit procurement auctions with random demand and sunk costs which reflect the key structure of real-time electricity trading.

To explain how our models are to range into the framework of auction theory, let us first give a short overview on related auction modelling. The optimal behaviour in auction games and their equilibria have attracted research interests since a long time. The interested reader will find in Klemperer (1999) a comprehensive survey on auction theory including valuable references to the most important work in this field. Moreover, the recent book of Krishna (2002) gives a state-of-the-art summary on basic results in modern auction theory. However, a typical issue of auction games is that minor changes in game rules may lead to another auction type, which requires a modified approach. As a result, auction theory deals with numerous observations, in general, not transferable from one auction format to another. In particular, the efforts in electricity auction modelling have followed various approaches, the more as each country has adopted its own solution when restructuring the electricity industry. Let us recall some common auction mechanisms.

1.1 *Classical auctions*

Consider a situation in which multiple identical objects are for sale. There are many options to arrange the trading: typically, all objects go into the same process, but are not necessarily sold to the same bidder. For instance, in the *Dutch auction*, the auctioneer begins by calling out a price high enough so that no bidder is willing to buy any unit. The price is then lowered until some bidder wishes to buy one unit, which is then sold at this price. Thereafter, the auctioneer continues lowering the price and the same procedure is repeated until all objects are sold. In the *English auction*, the auctioneer

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begins by calling out a low price and then gradually raises it. Each bidder indicates the number of units he/she is willing to buy at that price. As the price raises, the bidders reduce their bid numbers. The auction stops at the first price where the total number of units demanded matches the number of units being sold. All units are sold at this price. Another auction format is referred to as the *Ausubel auction*. Similar to the English auction, the price is continuously raised, but the selling price is effected by a more complicated procedure (see Krishna, 2002). For these auctions, the bids are publicly available (oral-bid auctions), contrary to the sealed-bid auctions, where the agents cannot observe the bid behaviour of their competitors. The sealed-bid counterparts of the Dutch, English and Ausubel auctions are known as the *discriminatory*, the *uniform* and the *Vickery* auctions, respectively. Under the *private value* model, one supposes that the value of the objects to the bidders is modelled by independent random variables. In view of this independence, the competitor's behaviour is not relevant and so a sealed-bid auction is seen to be equivalent to its open-format counterpart. To give the reader an understanding of how the auctions considered in this work differ from classical forms, we follow Klemperer (1999) on sketching a mathematical description to the latter.

EXAMPLE 1 (SINGLE OBJECT, PRIVATE VALUE MODEL) Assume that $N+1$ identical bidders $i = 0, \dots, N$ compete for the possession of a *single* object. The object value V_i to the bidder i is random and $(V_i)_{i=0}^N$ are non-negative, independent and identically distributed with continuous distribution function. The bid behaviour is based only on the own value estimation. Thus, agent i observes the realization υ of the private value V_i to submit the bid $\beta_i = b_i(\upsilon)$, where bid strategy b_i is chosen from

$$\mathcal{S} = \left\{ b \text{ continuous on } [0, \infty[, \text{ strictly increasing, } b(0) = 0, \ E\big(b\big(\max_{j=1}^N V_j\big)\big) < \infty \right\}.$$

The payment $C_i(\beta_0, \dots, \beta_N)$ of each bidder i depends on the bids $\beta_0, \dots, \beta_N$ and is calculated by the same rule for each player. For example, in the Dutch auction, the winner has to pay his/her own bid

$$C_i(\beta_0, \dots, \beta_N) = \begin{cases} \beta_i, & \text{if } \beta_i = \max_{j=0}^N \beta_j, \\ 0, & \text{else.} \end{cases}$$

Consider auction mechanisms where the highest bid always wins the object and choose, without lost of generality, the viewpoint of the agent $i = 0$. If all competitors follow $b \in \mathcal{S}$ and the agent observes realization υ of the private value, then the agent submits a bid maximizing the own expected profit

$$\beta \mapsto G^b(\upsilon, \beta) := E\left(V_0 1_{\{\beta > \max_{j=1}^N b(V_j)\}} - C_0(\beta, b(V_1), \dots, b(V_N)) | V_0 = \upsilon \right).$$

An *equilibrium strategy* $b^* \in \mathcal{S}$ is characterized by the property that if all competitors follow $b^* \in \mathcal{S}$, then the agent's best choice is to choose b^*, too: $G^{b^*}(\upsilon, b^*(\upsilon)) \geqslant G^{b^*}(\upsilon, b(\upsilon))$ for all $\upsilon \in [0, \infty[$ and $b \in \mathcal{S}$.

Note that the only source of randomness in this modelling is effected by the agent's inconclusiveness about the object's value. This circumstance is fully justified if one considers auctioning of a painting. However, for electricity, the object value (production costs) is known, whereas the randomness is induced by the demand uncertainty. Responding to this issue, a realistic description of electricity trading shall go beyond classical formats.

1.2 *Electricity auctions*

Ignoring a huge variety of technical effects (startup costs, ramping constraints, transmission constraints, etc.), a simplified model for electricity auction is described as follows: each producer submits for each hour of the next day schedules consisting of a bid quantity and a bid price for power which the producer is willing to sell *at least* at this price. The *system operator* arranges the bids for each hour in the increasing price order. The *system price* set for the current hour equals to the bid price of the last generator needed to meet the demand. Each producer who is *in merit* (i.e. whose bid price was below or equal to the system price) supplies power and sells it at a price depending on its bid price, electricity demand and auction type. Then, each producer receives a payment, depending on the auction type, as follows: for the case of *discriminatory*, *uniform* and *Vickery* auctions, each participant obtains the own bid price, the system price, and for the those system price, which would be dispatched with the own bid, respectively. Other producers (with bid price above the system price) suffer a loss since they have to pay fixed costs for their idle production units.

Although electricity auctions seem similar to classical ones, there are several specialties which prevent the classical framework to be adapted easily. Besides the minor difference that each electricity auction is a procurement auction (bidders are sellers rather than buyers), the main problem is the value of production to the bidder. In the case the agent produces, the agent earns the auction price less full production costs, otherwise the fixed costs are to be paid. That is, there is no randomness involved in the estimate of the object value, since the cost structure is deterministic. On the other hand, there is a demand uncertainty to be considered (in particular for long-lived bids, see Fabra *et al.*, 2002) since electricity load is not known in advance. Another important issue of all electricity auctions is that there is a price limitation, meaning that the system operator does not accept bids with bid price above a predetermined level, the so-called *price cap*. If the electricity demand exceeds all submitted production capacities, then the system price is set at the price cap and the demand is to be met by additionally running reserves from ancillary services. Our analysis shows that the price cap plays a significant role.

EXAMPLE 2 (UNIFORM AUCTION, SUPPLY FUNCTION MODELS) A number of authors, among them Rudkevich (2003), Baldick *et al.* (2001), Anderson & Philpott and Anderson & Xu (2005), have followed the concept of *supply functions* in their study of uniform auctions for electricity trading. A supply function (see Klemperer & Meyer, 1989) gives the energy amount offered by a generator depending on the sales price. Considering continuous supply functions, one implicitly assumes an infinite number of infinitesimal bids which gives a simplification of the real-world situation. For instance, in Anderson & Xu (2005), the bid behaviour of a specified agent is modeled by a parameterized curve $\{(q(\tau), p(\tau)) \colon 0 \leqslant \tau \leqslant T\}$ in $\mathbb{R}^2$ where $q(\tau)$ stands for energy amount which the producer is willing to sell at price $p(\tau)$. The market is described by an exogenously given mapping $(p, q) \mapsto \psi(p, q)$ describing the probability that the generator is not fully dispatched if it offers an amount of generation q at price p. The profit of the generator producing q (MWh) at price p (USD/MWh) is modelled by a function $(q, p) \mapsto R(q, p)$. The work of Anderson & Xu (2005) deals with calculation of optimal supply function (maximizing generator's profit), given ψ and R.

EXAMPLE 3 (DISCRIMINATORY AUCTION, SINGLE ORAL-BID MODEL) Another model (Hinz, 2003a,b) considers competition of production technologies within discriminatory auction to understand the so-called supply stack. Assuming that the bids are publicly available, the supply stack is modelled by a deterministic function $\mathcal{I}$ where $\mathcal{I}(p)$ stands for the total energy offered at the bid price less or equal to p. Given random demand Q and $N \in \mathbb{N}$ different production technologies with $p_i^{\mathrm{f}} > 0$ fixed, $p_i^{\mathrm{v}} \geqslant 0$ variable production costs ($i = 1, \dots, N$), one compares two strategies: to rent one small unit of the technology $i \in \{1, \dots, N\}$ with capacity $c > 0$ and to submit a schedule at the price p which yields a random gain $G^{\mathcal{I}}(p, i) := c(p - p_i^{\mathrm{v}})1_{\{Q > \mathcal{I}(p)\}} - cp_i^{\mathrm{f}}$ or to be idle which gives a nonrandom gain of zero: $G^{\mathcal{I}}(\text{idle}) = 0$. The equilibrium supply stack $\mathcal{I}^*$ is characterized by the property that a marginal producer (with a pre-specified utility function U) is indifferent about both strategies

$E(U(G^{\mathcal{I}^*}(p,i))) = E(U(G^{\mathcal{I}^*}(\text{idle}))) = U(0)$ for $i = 1, \dots, N$. Under additional assumptions, the existence and uniqueness of $\mathcal{I}^*$ are shown in Hinz (2003b); moreover, Hinz (2003c) illustrates how, in equilibrium, to calculate production capacity allocation along different technologies.

Let us mention some related work. Variations in electricity market design are discussed in Wilson (2002). For a comprehensive overview of economic issues behind real-world electricity auctions, we refer the reader to Fabra *et al.* (2002, 2005) and to the literature cited therein. There has been a lively debate on the merits of the uniform or discriminatory pricing rules in electricity markets, whose discussion is given by Fabra *et al.* (2002, 2005). In conformance with our revenue equivalence theorem (Hinz, 2004), these authors find out that no support can be given to the presumption that by changing the auction format from uniform to discriminatory, a significant improvement in market performance can be achieved. Furthermore, Fabra *et al.* illustrate that energy costs could be less important in practice than the auction vulnerability with respect to collusion. Some auction theorists argue that uniform auctions are more subject to strategic manipulations by large traders than are discriminatory auctions, being actually the reason for the implementation of the discriminatory mechanism in England and Wales in March 2001.

In this work, we discuss an auction class that is designed to match energy demand and supply to continuously maintain equilibrium of the electrical network. To simplify the framework for a comprehensive model analysis, we omit some technical features of electricity production and trading. In our model, each bid is made at a single price, which does not match actual electricity markets: for instance, in Fabra *et al.* (2005), the authors report that in Spain an agent may submit up to 25 price-quantity bids. Moreover, we assume that each unit is dispatched either completely or not at all. This assumption simplifies that in reality a part of the offer may be dispatched. In spite of these limitations, we approximately describe a market consisting of similar players using the same technology with narrow dispatch intervals. The incremental contribution of our work is to analyse a *general* auction class, however, restricted to single bid per agent (giving the opposite simplification than supply function models). Our approach also gives an extension of models from Example 3 to the case of the sealed-bid auctions.

2. An auction model

Let us agree with the following notations. Denote by $M^1(X)$ the set of all probability measures on the σ-algebra $\mathcal{B}(X)$ of Borel measurable subsets of X. The Dirac measure of a point $p \in X$ is denoted ε_p. For $\nu \in M^1(X)$ and $N \in \mathbb{N}$, we write ν^N for the product measure $\nu^N := \nu \otimes \dots \otimes \nu \in M^1(X^N)$. The maximum and the minimum of $a, b \in \mathbb{R}$ are denoted by $a \vee b$ and $a \wedge b$, respectively. The derivative of a function g is denoted by $\dot{g}$ or ∂g, and $\partial^- g$, $\partial^+ g$ stand for its left and the right derivatives, respectively.

Let $Q > 0$ (MWh within 1 h of the next day) be the random electricity demand modelled on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $J =]0, \sup J[$, the interval of all prices accepted by the system operator. With this notation, $\sup J$ stands for the price cap. Suppose that $N + 1 \geqslant 2$ producers $i = 0, \dots, N$ compete at the electricity market. Each agent owns one of the $N + 1$ identical electricity production units of the capacity $c > 0$ (MWh). Bidding at the electricity auction, the agent offers the entire production of c MWh for the corresponding hour at a single bid price p (USD/MWh). Given submitted bids $\vec{x} = (x_0, \dots, x_N) \in J^{\{0,\dots,N\}}$, the probability that the agent $j \in \{0, \dots, N\}$ is in merit depends on the number of bids placed in front of the own bid. Let us explain how it is to be determined. The system operator places in front of agents j bid x_j, all bids x_i with prices $x_i < x_j$ and after x_j those x_i with $x_i > x_j$. For competitors i bidding at the same price $x_i = x_j$, a *sharing rule* has to be applied. The procedure proposed here is to decide randomly giving all bidders the same chance, e.g. to draw lots establishing a priority order of scheduling. Let us formally define such a sharing rule: after receiving the bids $x_0, \ldots, x_N$, the system operator generates the *priority numbers* $z_0, \ldots, z_N \in [0, 1]$ taking a realization of independent uniformly on [0, 1] distributed random variables $(Z_0, \ldots, Z_N) = \vec{Z}$ independent from the future demand Q. The system operator arranges the bids in the increasing price order, and in the case of price coincidence $x_i = x_j$, the bid with the lower priority number is placed first. Since the distribution of priority numbers is continuous, this procedure establishes almost surely a well-defined bid list. To make this concept precise, we introduce the rank $r_j(\vec{x}, \vec{z})$ of the agent j, given the bid prices $\vec{x} := (x_i)_{i=0}^N \in J^{\{0,\ldots,N\}}$ and priority numbers $\vec{z} := (z_i)_{i=0}^N \in [0, 1]^{\{0,\ldots,N\}}$ by

$$r_j(\vec{x}, \vec{z}) := |\{i \in \{0, \ldots, N\}: x_i < x_j\}| + |\{i \in \{0, \ldots, N\}: x_i = x_j, z_i < z_j\}|.$$

Note that the ranks satisfy

$$r_{\rho(i)}(\vec{x}, \vec{z}) := r_i(\vec{x}^\rho, \vec{z}^\rho), \quad \text{for all } i = 0, \ldots, N, \tag{1}$$

for each ρ from the set of all permutations S_{N+1} of $\{0, \ldots, N\}$ acting as

$$\vec{w}^\rho := (w_{\rho(0)}, \ldots, w_{\rho(N)}), \quad \text{for all } \vec{w} = (w_0, \ldots, w_N) \in \mathbb{R}^{\{0,\ldots,N\}}.$$

Suppose that within 1 h, the production unit incurs the fixed costs $p^{\mathrm{f}} \in J$ (USD/MWh), the variable production costs $p^{\mathrm{v}} \in J$ (USD/MWh) and full production costs $p^{\mathrm{fv}} := p^{\mathrm{f}} + p^{\mathrm{v}} \in J$. For submitted bids $\vec{x} \in J^{\{0,\ldots,N\}}$, the gain of the agent j is given by

$$\mathcal{G}_j(\vec{x}) := c(\Pi_j(\vec{x}, Q) - p^{\mathrm{fv}})1_{\{Q > cr_j(\vec{x}, \vec{Z})\}} - cp^{\mathrm{f}}1_{\{Q \leqslant cr_j(\vec{x}, \vec{Z})\}}, \quad j = 0, \ldots, N. \tag{2}$$

Here, the payment prices

$$\Pi_j \colon J^{\{0,\ldots,N\}} \times [0, \infty[\to]0, \sup J] \quad (\vec{x}, q) \mapsto \Pi_j(\vec{x}, q), \quad j = 0, \ldots, N \tag{3}$$

are symmetric, similar to (1):

$$\Pi_{\rho(i)}(\vec{x}, q) = \Pi_i(\vec{x}^\rho, q), \quad \text{for all } \rho \in \mathcal{S}_{N+1}, \ \ q \in [0, \infty[. \tag{4}$$

Given a strictly increasing, convex utility function $U \in C^1(\mathbb{R})$, we define the loss functions $u_0, \ldots, u_N$ by $u_j(\vec{x}) := E(U(\mathcal{G}_j(\vec{x})))$ for all $\vec{x} \in J^{\{0,\ldots,N\}}$ and $j = 0, \ldots, N$ to write the non-cooperative game model of electricity auction as

$$\Gamma := (J^{\{0,\ldots,N\}}, u_0, \ldots, u_N). \tag{5}$$

Note that we cannot use general game-theoretic results to automatically ensure the equilibrium existence since $u_0, \ldots, u_N$ are in general not continuous. For this reason, we work out mild conditions guaranteeing existence and uniqueness of the symmetric equilibrium for mixed strategies. In this context, the price cap seems to be an essential model ingredient. To clarify why the cap is important, consider a discriminatory auction with $J =]0, \infty[$. Here, in the case of possible production capacity shortage $P(Q > cN) > 0$, the bidder infinitely increases his/her own utility by raising the bid $p \uparrow \infty$. So, the probability of being scheduled converges to $P(Q > cN) > 0$, whereas the profit rises as $c(p - p^{\mathrm{fv}}) \uparrow \infty$. That is, for arbitrary bid behaviour of competitors, the agent will not able to find a strategy which maximizes his/her own profit. In other words, if there is no price cap, then in general, a Nash equilibrium does not exists.

In view of (1) and (4), we focus on *symmetric equilibria* $(\nu^*, \ldots, \nu^*)$ of Γ, and call such $\nu^* \in M^1(J)$ *equilibrium strategy*. Since our model is symmetric, it suffices to choose the viewpoint of the first agent to decide, whereas $\nu \in M^1(J)$ is an equilibrium strategy.

For $j \in \{0, \ldots, N\}$, fix some $\rho \in \mathcal{S}_{N+1}$ with $\rho(j) = 0$ to consider for $(p, x) \in J^{\{0,\ldots,N\}}$ the gain $\mathcal{G}_j((p, x)^\rho)$ of the agent j bidding $p \in J$, whereas the competitors $\rho^{-1}(1), \ldots, \rho^{-1}(N)$ submit the bids $x = (x_1, \ldots, x_N) \in J^N$. Define for all $p \in J$ and $x \in J^N$

$$\begin{aligned}
\pi_x(p) &:= \Pi_0((p, x), Q),\\
\mathcal{I}_x(p) &:= cr_0((p, x), \vec{Z}),\\
G_x(p) &:= c(\pi_x(p) - p^{\mathrm{fv}})1_{\{Q > \mathcal{I}_x(p)\}} - cp^{\mathrm{f}}1_{\{Q \leqslant \mathcal{I}_x(p)\}}.
\end{aligned}$$

Obviously, the left and the right limits of $p \mapsto \mathcal{I}_x(p)$ do exist:

$$\mathcal{I}_x^+(p) := \lim_{b \downarrow p} \mathcal{I}_x(b) = c \sum_{i=1}^N 1_{[x_i, \sup J[}(p), \quad \text{for all } p \in J, \tag{6}$$

$$\mathcal{I}_x^-(p) := \lim_{a \uparrow p} \mathcal{I}_x(a) = c \sum_{i=1}^N 1_{]x_i, \sup J[}(p), \quad \text{for all } p \in J. \tag{7}$$

For $p \in J$, we agree to denote by $\mathcal{Q}$, $\pi(p)$, $\mathcal{I}(p)$, $\mathcal{I}^-(p)$, $\mathcal{I}^+(p)$, $G(p)$ the random variables on the measure space $(\Omega \times J^N, \mathcal{F} \otimes \mathcal{B}(J^N))$ which map each $(\omega, x) \in \Omega \times J^N$ to

$$Q(\omega),\ \pi_x(p)(\omega),\ \mathcal{I}_x(p)(\omega),\ \mathcal{I}_x^-(p)(\omega),\ \mathcal{I}_x^+(p)(\omega),\ G_x(p)(\omega),$$

respectively. Define

$$\begin{aligned}
\mathcal{U}(p, \nu) &:= E_{P \otimes \nu^N}(U(G(p))), \quad \text{for all } p \in J,\ \ \nu \in M^1(J),\\
S(\nu) &:= \sup_{y \in J} \mathcal{U}(y, \nu), \qquad \text{for all } \nu \in M^1(J).
\end{aligned} \tag{8}$$

With these notations, we obtain the following characterization of equilibrium strategies:

LEMMA 1 $\nu^* \in M^1(J)$ is an equilibrium strategy if and only if

$$\nu^*(\{p \in J : \mathcal{U}(p, \nu^*) < S(\nu^*)\}) = 0. \tag{9}$$

Proof. Observe that for $p \in J$, $x \in J^N$ and $\rho \in \mathcal{S}_{N+1}$ with $\rho(j) = 0$,

$$\mathcal{G}_j((p, x)^\rho) \text{ and } G_x(p) \text{ follow the same distribution.} \tag{10}$$

Indeed, by (1) and (4), we have

$$\begin{aligned}
\mathcal{G}_j((p, x)^\rho) &= c(\Pi_j((p, x)^\rho, Q) - p^{\mathrm{fv}})1_{\{Q > cr_j((p,x)^\rho, \vec{Z})\}} - cp^{\mathrm{f}}1_{\{Q \leqslant cr_j((p,x)^\rho, \vec{Z})\}}\\
&= c(\Pi_0((p, x), Q) - p^{\mathrm{fv}})1_{\{Q > cr_0((p,x), \vec{Z}^{\rho^{-1}})\}} - cp^{\mathrm{f}}1_{\{Q \leqslant cr_0((p,x), \vec{Z}^{\rho^{-1}})\}}.
\end{aligned}$$

On the other hand, we have

$$G_x(p) = c(\Pi_0((p, x), Q) - p^{\mathrm{fv}})1_{\{Q > cr_0((p,x), \vec{Z})\}} - cp^{\mathrm{f}}1_{\{Q \leqslant cr_0((p,x), \vec{Z})\}}$$

and (10) follows since the distribution of $(Q, \vec{Z}^{\rho^{-1}})$ equals to that of $(Q, \vec{Z})$. Using (10), the expected loss of the agent j following the strategy $\nu \in M^1(J)$, whereas all competitors follow $\nu^* \in M^1(J)$ is

$$\begin{aligned}
\int_J \int_{J^N} E(U(\mathcal{G}_j((p, x)^\rho)))\nu^{*N}(\mathrm{d}x)\nu(\mathrm{d}p) &= \int_J \int_{J^N} E(U(G_x(p)))\nu^{*N}(\mathrm{d}x)\nu(\mathrm{d}p)\\
&= \int_J \mathcal{U}(p, \nu^*)\nu(\mathrm{d}p).
\end{aligned}$$

Now it remains to prove that the equilibrium property of $(\nu^*, \dots, \nu^*)$ for Γ

$$\int_J \mathcal{U}(p, \nu^*)\nu(\mathrm{d}p) \leqslant \int_J \mathcal{U}(p, \nu^*)\nu^*(\mathrm{d}p), \quad \text{for all } \nu \in M^1(J), \tag{11}$$

is equivalent to (9). If $\nu^* \in M^1(J)$ fulfils (11), then $\nu^*(\{p \in J : \mathcal{U}(p, \nu^*) < S(\nu^*)\}) > 0$ is impossible, since otherwise we obtain $\int_J \mathcal{U}(p, \nu^*)\nu^*(\mathrm{d}p) < S(\nu^*)$; hence, there would exist a $z \in J$ with

$$\int_J \mathcal{U}(p, \nu^*)\nu^*(\mathrm{d}p) < \mathcal{U}(z, \nu^*) = \int_J \mathcal{U}(p, \nu^*)\varepsilon_z(\mathrm{d}p)$$

contradicting to (11). On the other hand, if (9) holds, then $\mathcal{U}(p, \nu^*) = S(\nu^*)$ for ν^*-almost all $p \in J$, i.e.

$$\int_J \mathcal{U}(p, \nu^*)\nu^*(\mathrm{d}p) = S(\nu^*). \tag{12}$$

□

The remaining part of our work presents a method to discuss the existence and uniqueness, and to obtain an explicit calculation of equilibrium strategies for a general class of electricity auctions specified below. The idea comes from the following specialty of this class: for equilibrium strategy ν^*, the function $\mathcal{U}(\cdot, \nu^*)$ is always continuous and there exists $\rho \in [0, \sup J[$ with

$$\{p \in J : \mathcal{U}(p, \nu^*) = S(\nu^*)\} = [\rho, \sup J[\cap J. \tag{13}$$

In other words, $\mathcal{U}(\cdot, \nu^*)$ stays at the maximum level once it has been reached, as illustrated in Fig. 1 (which qualitatively shows a typical shape of $\mathcal{U}(\cdot, \nu^*)$, whereas Fig. 2 gives quantitative examples). Thus, the derivative vanishes $\partial\mathcal{U}(p, \nu^*) = 0$ for all $p \in]\rho, \sup J[$. This property yields a differential equation for $p \mapsto \nu^*(]0, p])$ on $]\rho, \sup J[$, giving a framework to discuss the existence, the uniqueness and the explicit form of equilibrium strategies.

EXAMPLE 4 For the case of production capacity surplus $P(Q \leqslant cN) = 1$, it turns out that $\varepsilon_{p^{\mathrm{v}}}$ is an equilibrium strategy for the uniform, discriminatory and Vickery auctions. Let us show this. Assume

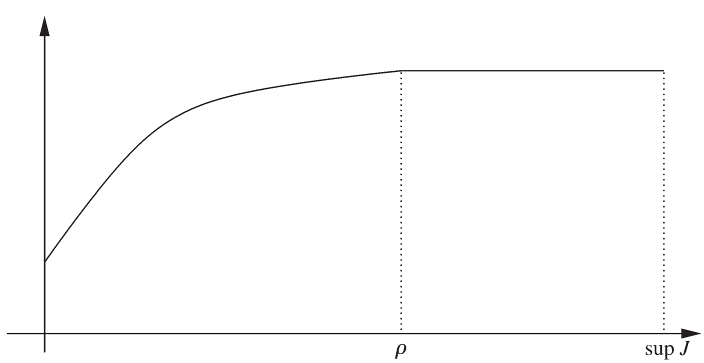

FIG. 1. The function $\mathcal{U}(\cdot, \nu^*)$ in the case ν^* is an equilibrium strategy.

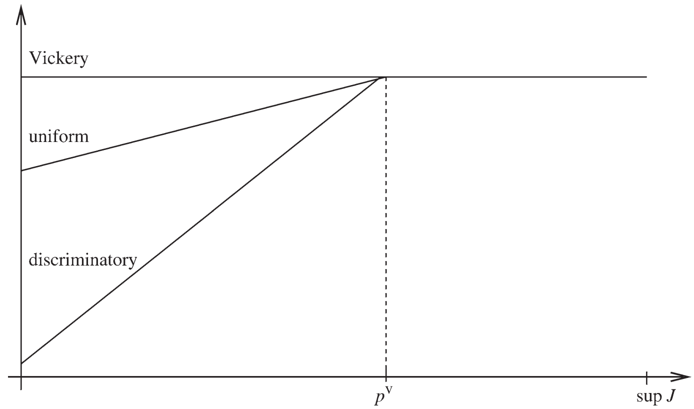

FIG. 2. The functions $\mathcal{U}(\cdot, \varepsilon_{p^{\mathrm{v}}})$ for the case $P(Q \leqslant cN) = 1$ and $U(x) = x$ for all $x \in \mathbb{R}$.

that all competitors bid at p^{v}, then for bidding at $p < p^{\mathrm{v}}$, the agent produces in any case:

$$\mathcal{U}(p, \varepsilon_{p^{\mathrm{v}}}) = U(c(p - p^{\mathrm{fv}})), \quad \text{for the discriminatory auction,}$$

$$\mathcal{U}(p, \varepsilon_{p^{\mathrm{v}}}) = U(c(p - p^{\mathrm{fv}}))P(Q \leqslant c) + U(\underbrace{c(p^{\mathrm{v}} - p^{\mathrm{fv}})}_{-cp^{\mathrm{f}}})P(Q > c), \quad \text{for the uniform auction and}$$

$$\mathcal{U}(p, \varepsilon_{p^{\mathrm{v}}}) = U(\underbrace{c(p^{\mathrm{v}} - p^{\mathrm{fv}})}_{-cp^{\mathrm{f}}}), \quad \text{for the Vickery auction.}$$

Further, if the agent submits the bid at $p = p^{\mathrm{v}}$, then, in case of production, the paid price is p^{v}, thus $\mathcal{U}(p^{\mathrm{v}}, \varepsilon_{p^{\mathrm{v}}}) = U(-cp^{\mathrm{f}})$. If the bid price is $p > p^{\mathrm{v}}$, then the agent never produces: $\mathcal{U}(p^{\mathrm{v}}, \varepsilon_{p^{\mathrm{v}}}) = U(-cp^{\mathrm{f}})$ (for all three auction mechanisms). Thus, Fig. 2 depicts $\mathcal{U}(\cdot, \varepsilon_{p^{\mathrm{v}}})$ and Lemma 1 shows that $\varepsilon_{p^{\mathrm{v}}}$ is, indeed, an equilibrium strategy.

Another interesting consequence of (13) is that $S(\nu^*) = \lim_{p \uparrow \sup J} \mathcal{U}(p, \nu^*)$, where the limit depends neither on the auction type nor on its particular equilibrium, as the explicit expression

$$S(\nu^*) = \lim_{p \uparrow \sup J} \mathcal{U}(p, \nu^*) = U(c(\sup J - p^{\mathrm{fv}}))P(Q > cN) + U(-cp^f)P(Q \leqslant cN) \tag{14}$$

shows. Due to (12), the utility of an energy producer

$$\int_J \mathcal{U}(p, \nu^*)\nu^*(\mathrm{d}p) = S(\nu^*) \tag{15}$$

at the equilibrium is independent of the auction format. This property is used in Hinz (2004) to derive a revenue equivalence theorem for electricity auctions, stating that if the energy producers are not risk-averse $U(r) = r$ for all $r \in \mathbb{R}$, then at the equilibrium, each auction yields the same expected payment for produced electricity.

We shall point out that the price cap significantly influences the auction itself, since the expected utility of electricity producers in (14), (15) depends on sup J. That is, given a cap sup J, the auction will attract (quench) additional production until the number of participants N reaches some level N' with utility of the marginal producer

$$U(c(\sup J - p^{\mathrm{fv}}))P(Q > cN') + U(-cp^{\mathrm{f}})P(Q \leqslant cN')$$

near zero.

Our main application is to show that for positive probability

$$P(\text{exactly } k+1 \text{ generators are on-line}) = P(Q/c \in]k, k+1]) > 0, \quad k = 0, \ldots, N,$$

there exists a unique symmetric auction equilibrium and to present a method for calculating the corresponding bid strategies. To give the reader an intuition why this condition is relevant, we consider on the contrary the case $P(Q > c(N+1)) = 1$ that the price cap is always reached. For the uniform auction, we obtain that each producer is paid $\sup J$ independently on the own bid, showing that each $\nu \in M^1(J)$ gives an equilibrium strategy. Hence, no equilibrium uniqueness is available in this case.

3. Equilibrium strategies

In the present work, we study electricity auctions, where the paid price is increasing

$$\pi_x(a) \leqslant \pi_x(b), \quad \text{for all } x \in J^N,\ a, b \in J \text{ with } a \leqslant b. \tag{16}$$

We also assume that it never falls below the own bid price

$$\pi_x(p)1_{\{Q>\mathcal{I}_x^-(p)\}} \geqslant p1_{\{Q>\mathcal{I}_x^-(p)\}}, \quad \text{for all } p \in J, x \in J^N, \tag{17}$$

and cannot exceed those system price, which would be dispatched without agent's bid

$$\pi_x(p)1_{\{Q>\mathcal{I}_x^-(p)\}} \leqslant \inf\{y \in J\colon \mathcal{I}_x^+(y) \geqslant Q\}1_{\{Q>\mathcal{I}_x^-(p)\}}, \quad \text{for all } p \in J,\ x \in J^N. \tag{18}$$

Further, we assume that the paid price depends Lipschitz-continuously on the own bid price with a global constant $C \in]0, \infty[$

$$|\pi_x(b)(\omega) - \pi_x(a)(\omega)| \leqslant C|b-a|, \quad \text{for all } b, a \in J,\ x \in J^N,\ \omega \in \Omega, \tag{19}$$

and for all $x = (x_1, \ldots, x_N) \in J^N$, $\omega \in \Omega$ fulfils

$$\pi_x(\cdot)(\omega) \text{ is differentiable on } J\backslash\{x_1, \ldots, x_N\}. \tag{20}$$

Note that (17) and (18) together imply for all $a \leqslant b \in J$, $x \in J^N$ that

$$a1_{\{\mathcal{I}_x^-(a)<Q\leqslant\mathcal{I}_x^+(b)\}} \leqslant \pi_x(p)1_{\{\mathcal{I}_x^-(p)<Q\leqslant\mathcal{I}_x^+(p')\}} \leqslant b1_{\{\mathcal{I}_x^-(a)<Q\leqslant\mathcal{I}_x^+(b)\}}, \tag{21}$$

which gives

$$\pi_x(p)1_{\{\mathcal{I}_x^-(p)<Q\leqslant\mathcal{I}_x^+(p)\}} = p1_{\{\mathcal{I}_x^-(p)<Q\leqslant\mathcal{I}_x^+(p)\}}, \quad \text{for all } p \in J,\ x \in J^N. \tag{22}$$

The payment prices for our auctions are for all $p \in J$, $x = (x_1, \ldots, x_N) \in J^N$ and $\omega \in \Omega$ given by

$$\pi_x(p)(\omega) = p \quad \text{(discriminatory auction)},$$

$$\pi_x(p)(\omega) = \inf\left\{y \in J\colon \left(\sum_{i=1}^N 1_{[x_i, \sup J[} + 1_{[p, \sup J[}\right)(y) \geqslant Q(\omega)/c\right\} \quad \text{(uniform auction)},$$

$$\pi_x(p)(\omega) = \inf\left\{y \in J\colon \sum_{i=1}^N 1_{[x_i, \sup J[}(y) \geqslant Q(\omega)/c\right\} \quad \text{(Vickery auction)}.$$

They meet the conditions (16)–(20).

LEMMA 2 Let $\nu \in M^1(J)$.

(i) For $a, b \in J$ with $a < b$ holds: if $\nu([a, b]) = 0$, then $\mathcal{U}(a, \nu) \leqslant \mathcal{U}(b, \nu)$.

(ii) For all $p \in J$, there exist the limits

$$\mathcal{U}^+(p, \nu) := \lim_{y \downarrow p} \mathcal{U}(y, \nu), \quad \mathcal{U}^-(p, \nu) := \lim_{y \uparrow p} \mathcal{U}(y, \nu).$$

(iii) For $p \in J$ holds: if $\nu(\{p\}) = 0$, then $\mathcal{U}(\cdot, \nu)$ is continuous on p.

(iv) $\mathcal{U}(\cdot, \nu)$ is continuous on p^{v}.

(v) If $\mathcal{U}(\cdot, \nu)$ is not continuous on $p \in J \backslash \{p^{\mathrm{v}}\}$, then

$$\mathcal{U}(p, \nu) > \mathcal{U}^+(p, \nu) \quad \text{and} \quad \mathcal{U}^-(p, \nu) > \mathcal{U}(p, \nu), \quad \text{if } p \in]p^{\mathrm{v}}, \sup J[, \tag{23}$$

$$\mathcal{U}(p, \nu) < \mathcal{U}^+(p, \nu) \quad \text{and} \quad \mathcal{U}^-(p, \nu) < \mathcal{U}(p, \nu), \quad \text{if } p \in]0, p^{\mathrm{v}}[. \tag{24}$$

Proof.

(i) For $\omega \in \Omega$, the function $y \mapsto \mathcal{I}_x(y)(\omega)$ is constant on $[a, b]$ for all $x \in (J \backslash [a, b])^N$. Since $\nu^N((J \setminus [a, b])^N) = 1$, the equality $\mathcal{I}(a) = \mathcal{I}(b)$ holds $P \otimes \nu^N$-almost surely. Using (16), we obtain

$$\begin{aligned} U(G(a)) &= U(c(\pi(a) - p^{\mathrm{fv}}))1_{\{\mathcal{Q} > \mathcal{I}(a)\}} + U(-cp^{\mathrm{f}})1_{\{\mathcal{Q} \leqslant \mathcal{I}(a)\}} \\ &\leqslant U(c(\pi(b) - p^{\mathrm{fv}}))1_{\{\mathcal{Q} > \mathcal{I}(b)\}} + U(-cp^{\mathrm{f}})1_{\{\mathcal{Q} \leqslant \mathcal{I}(b)\}} = U(G(b)) \end{aligned}$$

which gives $\mathcal{U}(a, \nu) = E_{P \otimes \nu^N}(U(G(a))) \leqslant E_{P \otimes \nu^N}(U(G(b))) = \mathcal{U}(b, \nu)$.

(ii) For later use of Lebesgue's dominated convergence theorem, we point out that $(U(G(p)))_{p \in J}$ are bounded

$$|U(G(p))| \leqslant |U(c(\sup J - p^{\mathrm{fv}}))| \vee |U(-cp^{\mathrm{f}})|, \quad \text{for all } p \in J. \tag{25}$$

Since for all $x \in J$ and $\omega \in \Omega$, the function $\mathcal{I}_x(\cdot)(\omega)$ takes a finite number of values, we obtain

$$\lim_{y \downarrow p} 1_{\{Q > \mathcal{I}_x(y)\}} = 1_{\{Q > \mathcal{I}_x^+(p)\}}, \quad \lim_{y \uparrow p} 1_{\{Q > \mathcal{I}_x(y)\}} = 1_{\{Q > \mathcal{I}_x^-(p)\}},$$

P-almost surely for each $x \in J^N$. From (19), we have the $P \otimes \nu^N$-almost sure convergence

$$\lim_{y \downarrow p} U(G(y)) = U(c(\pi(p) - p^{\mathrm{fv}}))1_{\{\mathcal{Q} > \mathcal{I}^+(p)\}} + U(-cp^{\mathrm{f}})1_{\{\mathcal{Q} \leqslant \mathcal{I}^+(p)\}}, \tag{26}$$

$$\lim_{y \uparrow p} U(G(y)) = U(c(\pi(p) - p^{\mathrm{fv}}))1_{\{\mathcal{Q} > \mathcal{I}^-(p)\}} + U(-cp^{\mathrm{f}})1_{\{\mathcal{Q} \leqslant \mathcal{I}^-(p)\}}. \tag{27}$$

Thus, (25) and dominated convergence yield the assertion.

(iii) If $x \in (J \backslash \{p\})^N$, then we have with (19) the continuity on p:

$$\lim_{y \to p} U(G_x(y)) = U(c(\pi_x(p) - p^{\mathrm{fv}}))1_{\{Q > \mathcal{I}_x(p)\}} + U(-cp^{\mathrm{f}})1_{\{Q \leqslant \mathcal{I}_x(p)\}} \quad P\text{-almost surely.}$$

Since $\nu^N((J\backslash\{p\})^N) = 1$, we have with (25) the dominated $P \otimes \nu^N$-almost sure convergence in

$$\lim_{y\to p} U(G(y)) = U(c(\pi(p) - p^{\mathrm{fv}}))1_{\{Q>\mathcal{I}(p)\}} + U(-cp^{\mathrm{f}})1_{\{Q\leqslant\mathcal{I}(p)\}},$$

giving $\lim_{y\to p}\mathcal{U}(y,\nu) = \lim_{y\to p} E_{P\otimes\nu^N}(U(G(y))) = E_{P\otimes\nu^N}(U(G(p))) = \mathcal{U}(p,\nu)$.

(iv) For $x \in J^N$ and $a, b \in J$ with $a < b$, we calculate

$$\begin{aligned} U(G_x(b)) - U(G_x(a)) = {} & (U(-cp^{\mathrm{f}}) - U(c(\pi_x(a) - p^{\mathrm{fv}})))1_{\{\mathcal{I}_x(a)<Q\leqslant\mathcal{I}_x(b)\}} \\ & + (U(c(\pi_x(b) - p^{\mathrm{fv}})) - U(c(\pi_x(a) - p^{\mathrm{fv}})))1_{\{Q>\mathcal{I}_x(b)\}}, \end{aligned} \tag{28}$$

which gives with (19) and (22) the right and the left limits

$$\begin{aligned} \lim_{b\downarrow p}[U(G_x(b)) - U(G_x(p))] &= (U(-cp^{\mathrm{f}}) - U(c(\pi_x(p) - p^{\mathrm{fv}})))1_{\{\mathcal{I}_x(p)<Q\leqslant\mathcal{I}_x^+(p)\}} \\ &= (U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))1_{\{\mathcal{I}_x(p)<Q\leqslant\mathcal{I}_x^+(p)\}}, \end{aligned} \tag{29}$$

$$\lim_{a\uparrow p}[U(G_x(p)) - U(G_x(a))] = (U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))1_{\{\mathcal{I}_x^-(p)<Q\leqslant\mathcal{I}_x(p)\}}. \tag{30}$$

If $p = p^{\mathrm{v}}$, then we have $U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})) = 0$, hence for all $x \in J^N$ P-almost surely holds

$$\lim_{b\downarrow p^{\mathrm{v}}}[U(G_x(b)) - U(G_x(p^{\mathrm{v}}))] = 0, \quad \lim_{a\uparrow p^{\mathrm{v}}}[U(G_x(a)) - U(G_x(p^{\mathrm{v}}))] = 0$$

and with (25) we obtain by $\mathcal{U}^+(p^{\mathrm{v}},\nu) = \mathcal{U}(p^{\mathrm{v}},\nu)$, $\mathcal{U}^-(p^{\mathrm{v}},\nu) = \mathcal{U}(p^{\mathrm{v}},\nu)$ the continuity of $\mathcal{U}(\cdot,\nu)$ on p^{v}.

(v) To prove (23), we suppose first that $p > p^{\mathrm{v}}$ and assume that $\mathcal{U}(\cdot,\nu)$ is not right continuous on p. By (29), it follows that

$$\mathcal{U}^+(p,\nu) - \mathcal{U}(p,\nu) = (U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}}))) \int_{J^N} P(\mathcal{I}_x(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x) \tag{31}$$

which, by assumption, is not equal to zero. From $p > p^{\mathrm{v}}$, we have $U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})) < 0$ and consequently the first inequality of (23). Moreover, (31) also yields the strict positivity

$$0 < \int_{J^N} P(\mathcal{I}_x(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x) \leqslant \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x). \tag{32}$$

To show the second inequality of (23), we calculate similar to (31) from (30) that

$$\mathcal{U}(p,\nu) - \mathcal{U}^-(p,\nu) = (U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}}))) \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x(p))\nu^N(\mathrm{d}x). \tag{33}$$

Further, we need for $x \in J^N$ the estimation

$$\begin{aligned} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x(p)) &\geqslant P\left(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x(p), Z_0 > \max_{j=1}^N Z_j\right) \\ &\geqslant P\left(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p) | Z_0 > \max_{j=1}^N Z_j\right) P\left(Z_0 > \max_{j=1}^N Z_j\right) \\ &\geqslant P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p)) \cdot (N+1)^{-1}, \end{aligned} \tag{34}$$

where the second inequality holds due to

$$\mathcal{I}_x(p)1_{\{Z_0>\max_{j=1}^N Z_j\}} = \mathcal{I}_x^+(p)1_{\{Z_0>\max_{j=1}^N Z_j\}}, \quad p \in J, \ \ x \in J^N$$

and in the third inequality, we use $P(Z_0 > \max_{j=1}^N Z_j) = (N+1)^{-1}$ and the independence of $(\mathcal{I}_x^-(p), Q, \mathcal{I}_x^+(p))$ and $\vec{Z}$, see (7) and (6). Combine now (34) with positivity (32) to obtain

$$\int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x(p))\nu^N(\mathrm{d}x) \geqslant \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x)(N+1)^{-1} > 0,$$

showing that (33) is in fact negative, which yields the second inequality of (23).

Now we assume that that $\mathcal{U}(\cdot, \nu)$ is not left continuous on $p > p^{\mathrm{v}}$. Then (33) is not equal to zero, giving the second inequality of (23) and showing the strict positivity

$$0 < \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x(p))\nu^N(\mathrm{d}x) \leqslant \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x). \qquad (35)$$

To show the first inequality, we estimate exactly as in (34)

$$\begin{aligned} P(\mathcal{I}_x(p) < Q \leqslant \mathcal{I}_x^+(p)) &\geqslant P\left(\mathcal{I}_x(p) < Q \leqslant \mathcal{I}_x^+(p), Z_0 < \min_{j=1}^N Z_j\right) \\ &\geqslant P\left(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p) | Z_0 < \min_{j=1}^N Z_j\right) P\left(Z_0 < \min_{j=1}^N Z_j\right) \\ &\geqslant P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p)) \cdot (N+1)^{-1} \end{aligned} \qquad (36)$$

to obtain

$$\int_{J^N} P(\mathcal{I}_x(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x) \geqslant \int_{J^N} P(\mathcal{I}_x^-(p) < Q \leqslant \mathcal{I}_x^+(p))\nu^N(\mathrm{d}x)(N+1)^{-1} > 0.$$

Hence, (31) is in fact negative, which yields the first inequality of (23). The assertion (24) is shown in analogous way to (23). □

LEMMA 3 If ν^* is an equilibrium strategy, then the following holds:

(i) $J \to \mathbb{R}$, $p \mapsto \mathcal{U}(p, \nu^*)$ is continuous.

(ii) If $\nu^*(]0, p]) > 0$, then $\mathcal{U}(p, \nu^*) = S(\nu^*)$.

(iii) If $P(Q \leqslant cN) > 0$, then

$$\nu^*(\{p\}) = 0 \text{ for all } p \in J \backslash \{p^{\mathrm{v}}\}. \qquad (37)$$

(iv) If $P(Q \leqslant cN) > 0$, then $\nu^*(]0, p^{\mathrm{v}}[) = 0$.

Proof.

(i) Suppose on the contrary that $\mathcal{U}(\cdot, \nu^*)$ is discontinuous on $p \in J$, then by (iv) and (iii) of the previous lemma, we have $p \neq p^{\mathrm{v}}$ and $\nu^*(\{p\}) > 0$. By (9), $\nu^*(\{p\}) > 0$ implies that $\mathcal{U}(p, \nu^*) = S(\nu^*)$. By (v) of the previous lemma, the case $p > p^{\mathrm{v}}$ yields a contradiction to (8) by $\mathcal{U}^-(p, \nu^*) > S(\nu^*)$ from (23), whereas the case $p < p^{\mathrm{v}}$ gives a contradiction to (8) by $\mathcal{U}^+(p, \nu^*) > S(\nu^*)$ from (24).

(ii) Since ν^* is an equilibrium strategy, $M := \{y \in J: \mathcal{U}(y, \nu^*) = S(\nu^*)\} \neq \emptyset$ satisfies by (9) $\nu^*(J \backslash M) = 0$. It suffices to prove that

$$M \text{ forms an interval with } \sup M = \sup J, \tag{38}$$

proceeding by the following argument: If $p \in J$ satisfies $\nu^*(]0, p]) > 0$, then $]0, p] \subseteq J \backslash M$ is impossible in view of $\nu^*(J \backslash M) = 0$. Thus, $]0, p] \cap M \neq \emptyset$ and (38) implies that $p \in M$ showing (ii). Let us prove (38), by supposing on the contrary that there exist $z, b \in J$ with $z < b$ and

$$\mathcal{U}(z, \nu^*) = S(\nu^*), \quad \mathcal{U}(b, \nu^*) < S(\nu^*). \tag{39}$$

Since (i) ensures that $\mathcal{U}(\cdot, \nu^*)$ is continuous, the infimum

$$p := \inf\{a \in [z, b]: \mathcal{U}(y, \nu^*) < S(\nu^*) \text{ for all } y \in [a, b]\}$$

satisfies $\mathcal{U}(p, \nu^*) = S(\nu^*)$ such that $\mathcal{U}(y, \nu^*) < S(\nu^*)$ for $y \in]p, b]$, giving with (9) that

$$\nu^*([a, b]) = 0, \qquad \text{for all } a \in]p, b]. \tag{40}$$

Now, (i) of the previous lemma is applied to (40) to conclude that $\mathcal{U}(a, \nu^*) \leqslant \mathcal{U}(b, \nu^*)$ for all $a \in]p, b]$. Passing through the right limit $a \downarrow p$, we obtain with continuity of $\mathcal{U}(\cdot, \nu^*)$ that

$$S(\nu^*) = \mathcal{U}(p, \nu^*) = \lim_{a \downarrow p} \mathcal{U}(a, \nu^*) \leqslant \mathcal{U}(b, \nu^*),$$

contradicting to the inequality in (39).

(iii) If $P(Q \leqslant cN) > 0$, $p \neq p^{\mathrm{v}}$ and $\nu^*(\{p\}) > 0$, then for $p \in J$, we estimate as in (36)

$$\begin{aligned} P \otimes \nu^{*N}(\mathcal{I}(p) < \mathcal{Q} \leqslant \mathcal{I}^+(p)) &\geqslant P \otimes \nu^{*N}(\mathcal{Q} \leqslant cN, \mathcal{I}(p) = 0, \mathcal{I}^+(p) = cN) \\ &\geqslant P \otimes \nu^{*N}(\mathcal{Q} \leqslant cN, \mathcal{I}^-(p) = 0, \mathcal{I}^+(p) = cN)/(N+1) \\ &\geqslant P(Q \leqslant cN)\nu^*(\{p\})^N/(N+1) > 0, \end{aligned}$$

which shows with (31), (32) and (34) that for $p > p^{\mathrm{v}}$, we have a discontinuity (23) and for $p < p^{\mathrm{v}}$ a discontinuity (24) on p contradicting (i).

(iv) To show $\nu^*(]0, p^{\mathrm{v}}[) = 0$, we suppose on the contrary that $\nu^*(]0, p^{\mathrm{v}}[) > 0$, then (37) yields $0 < a < b < p^{\mathrm{v}}$ with

$$\nu^*(]0, a]) > 0, \quad \nu^*(]a, b[) > 0. \tag{41}$$

The first inequality ensures by (ii) that

$$\mathcal{U}(a, \nu^*) = S(\nu^*), \tag{42}$$

whereas the second yields

$$\begin{aligned} P \otimes \nu^{*N}(\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(b)) &\geqslant P \otimes \nu^{*N}(\mathcal{Q} \leqslant cN, \mathcal{I}(a) = 0, \mathcal{I}(b) = cN) \\ &\geqslant \nu^*(]a, b[)^N P(Q \leqslant cN) > 0. \end{aligned} \tag{43}$$

As in (28), we write

$$U(G(b)) - U(G(a)) = T_1(b, a) + T_2(b, a), \tag{44}$$

where

$$T_1(b,a) := (U(-cp^{\mathrm{f}}) - U(c(\pi(a) - p^{\mathrm{fv}})))1_{\{\mathcal{I}(a)<\mathcal{Q}\leqslant\mathcal{I}(b)\}}, \tag{45}$$

$$T_2(b,a) := (U(c(\pi(b) - p^{\mathrm{fv}})) - U(c(\pi(a) - p^{\mathrm{fv}})))1_{\{\mathcal{Q}>\mathcal{I}(b)\}}. \tag{46}$$

$T_2(b,a)$ is non-negative due to $a < b$ and (16). The estimation (21) shows that $T_1(b,a)$ is non-negative

$$T_1(b,a) \geqslant (U(-cp^{\mathrm{f}}) - U(c(b - p^{\mathrm{fv}})))1_{\{\mathcal{I}(a)<\mathcal{Q}\leqslant\mathcal{I}(b)\}} \geqslant 0$$

and is due to $b < p^{\mathrm{v}}$ strictly positive $U(-cp^{\mathrm{f}}) - U(c(b - p^{\mathrm{fv}})) > 0$ on the set $\{\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(b)\}$ of positive (43) measure. That is, the difference (44) is non-negative and strictly positive on a set of positive measure, hence with (42) we obtain a contradiction to (8)

$$\mathcal{U}(a,\nu^*) = E_{P\otimes\nu^{*N}}(U(G(a))) < E_{P\otimes\nu^{*N}}(U(G(b))) = \mathcal{U}(b,\nu^*).$$

□

To formulate the main result, we introduce the notation

$$\alpha_k := P(Q/c \in]k, k+1]), \quad \text{for } k = 0, \dots, N, \tag{47}$$

$$H(q) := \sum_{k=0}^{N-1} \binom{N-1}{k} \alpha_k q^k (1-q)^{N-1-k}, \quad \text{for all } q \in [0,1]. \tag{48}$$

PROPOSITION 1 Suppose $P(Q \leqslant cN) > 0$, then for the distribution function F of an equilibrium strategy ν^*, the following holds: if $p > p^{\mathrm{v}}$ and $F(p) > 0$, then $\partial F(p)$ exists and fulfils

$$E_{P\otimes\nu^{*N}}(\dot U(c(\pi(p) - p^{\mathrm{fv}}))c\dot\pi(p)1_{\{\mathcal{Q}>\mathcal{I}(p)\}}) = \partial F(p)(U(c(p - p^{\mathrm{fv}}) - U(-cp^{\mathrm{f}})))NH(F(p)). \tag{49}$$

Proof. If ν^* is an equilibrium strategy, then F is continuous on $p \in]p^{\mathrm{v}}, \sup J[$ due to (37) and (iii) of Lemma 2. Hence, $F(p) > 0$ ensures that F is positive in a neighbourhood of p. From (ii) of the Lemma 3, we have $\partial\mathcal{U}(p,\nu^*) = 0$. Let us use the notations (45), (46) to derive (49) from

$$0 = \partial^-\mathcal{U}(p,\nu^*) = \lim_{a\uparrow p} E_{P\otimes\nu^{*N}}(T_1(p,a))/(p-a) + \lim_{a\uparrow p} E_{P\otimes\nu^{*N}}(T_2(p,a))/(p-a), \tag{50}$$

$$0 = \partial^+\mathcal{U}(p,\nu^*) = \lim_{b\downarrow p} E_{P\otimes\nu^{*N}}(T_1(b,p))/(b-p) + \lim_{b\downarrow p} E_{P\otimes\nu^{*N}}(T_2(b,p))/(b-p). \tag{51}$$

Consider the first term on the right of (50) for $a \in]p^{\mathrm{v}}, p[$. By (21), $T_1(p,a)$ is bounded

$$E_{P\otimes\nu^{*N}}(T_1(p,a)) \leqslant (U(-cp^{\mathrm{f}}) - U(c(a - p^{\mathrm{fv}})))P\otimes\nu^{*N}(\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(p)), \tag{52}$$

$$E_{P\otimes\nu^{*N}}(T_1(p,a)) \geqslant (U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))P\otimes\nu^{*N}(\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(p)). \tag{53}$$

Moreover, due to $P\otimes\nu^{*N}$-independence of $\mathcal{Q}$ and $(\mathcal{I}(a), \mathcal{I}(p))$, we have

$$P\otimes\nu^{*N}(\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(p)) = \sum_{k=0}^{N-1}\sum_{j=1}^{N-k} P\otimes\nu^{*N}(\mathcal{Q}/c \in]k, k+j]) \cdot P\otimes\nu^{*N}(\mathcal{I}(a) = ck, \mathcal{I}(p) = c(k+j)), \tag{54}$$

where we utilize (37) to obtain

$$\begin{aligned}&P \otimes \nu^{*N}(\mathcal{I}(a) = ck, \mathcal{I}(p) = c(k+j))\\ &\quad = \frac{N!}{k!j!(N-k-j)!}\nu^*(]0,a])^k \nu^*(]a,p])^j \nu^*(]p, \sup J[)^{N-k-j}.\end{aligned}$$

Approaching the point p from the left $a \uparrow p$, we conclude that the limit

$$\lim_{a\uparrow p} \frac{P \otimes \nu^{*N}(\mathcal{I}(a) < \mathcal{Q} \leqslant \mathcal{I}(p))}{p-a}$$

exists and is equal to

$$\partial^- F(p) \sum_{k=0}^{N-1} \frac{N!}{k!(N-k-1)!} P\left(\frac{Q}{c} \in]k, k+1]\right) F(p)^k (1-F(p))^{N-k-1} = \partial^- F(p) N H(F(p)),$$

if and only if $\partial^- F(p)$ exists, since the continuity of F on p ensures that the asymptotic is determined by those summands of (54) where $j = 1$. Finally, we use the bounds (52) and (53) to see that

$$\lim_{a\uparrow p} \frac{E_{P\otimes\nu^{*N}}(T_1(p,a))}{p-a} = \partial^- F(p)(U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))NH(F(p)), \tag{55}$$

where both sides exist simultaneously. Now, we turn to the second term on the right of (50). Here, (37) ensures that $\nu^{*N}((J\backslash\{p\})^N) = 1$ giving with (19) and (20) the bounded $P \otimes \nu^{*N}$-almost sure convergence in

$$\lim_{a\uparrow p} \frac{T_2(p,a)}{p-a} = c\dot{U}(c(\pi(p) - p^{\mathrm{fv}}))\dot{\pi}(p)1_{\{\mathcal{Q}>\mathcal{I}(p)\}}.$$

Thus,

$$\lim_{a\uparrow p} \frac{E_{P\otimes\nu^{*N}}(T_2(p,a))}{p-a} = E_{P\otimes\nu^{*N}}(\dot{U}(c(\pi(p) - p^{\mathrm{fv}}))c\dot{\pi}(p)1_{\{\mathcal{Q}>\mathcal{I}(p)\}}) \tag{56}$$

is fulfilled. Because of (50), the sum of (55) and (56) vanishes, showing that

$$E_{P\otimes\nu^{*N}}(\dot{U}(c(\pi(p) - p^{\mathrm{fv}}))c\dot{\pi}(p)1_{\{\mathcal{Q}>\mathcal{I}(p)\}}) = -\partial^- F(p)(U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))NH(F(p)),$$

in particular, $\partial^- F(p)$ exists. The same derivation is repeated for (51) to obtain

$$E_{P\otimes\nu^{*N}}(\dot{U}(c(\pi(p) - p^{\mathrm{fv}}))c\dot{\pi}(p)1_{\{\mathcal{Q}>\mathcal{I}^+(p)\}}) = -\partial^+ F(p)(U(-cp^{\mathrm{f}}) - U(c(p - p^{\mathrm{fv}})))NH(F(p)).$$

Finally, we use $\mathcal{I}(p) = \mathcal{I}^+(p)$ resulting from (37) to obtain (49). □

4. Application

PROPOSITION 2 For discriminatory, uniform and Vickery auctions, the following holds: if $\alpha_k > 0$ for all $k = 0, \ldots, N$, then there exists a unique equilibrium strategy.

Proof. The requirement of the Proposition 1 is fulfilled as

$$P(Q \leqslant cN) = \sum_{k=0}^{N-1} \alpha_k > 0.$$

Consider the uniform auction. First, we show uniqueness. Introduce the bid price $p_x(k)$ of the k-th generator within a list scheduled from the bids $x \in J^N$ as

$$p_x(k) = \inf\{y \in J : ck \leqslant \mathcal{I}_x(y)\}, \quad \text{for } k = 0, \dots, N.$$

With this notation, the payment price for the uniform auction is written as

$$\pi_x(p) = \sum_{k=0}^{N} 1_{\{\mathcal{Q}/c \in]k,k+1]\}} (p_x(k) 1_{\{\mathcal{I}_x(p) < ck\}} + p 1_{\{\mathcal{I}_x(p) = ck\}} + p_x(k+1) 1_{\{\mathcal{I}(p) > ck\}})$$

for all $x \in J^N$ and $p \in J$. We have therefore

$$\dot{\pi}_x(p) = \sum_{k=0}^{N} 1_{\{\mathcal{I}_x(p)=k\}} 1_{\{Q/c \in]k,k+1]\}}, \quad x = (x_1, \dots, x_n) \in J^N, \ p \in J \backslash \{x_1, \dots, x_N\}. \tag{57}$$

Now, we use (57) to calculate for $p > p^{\mathrm{v}}$

$$\dot{U}(c(\pi(p) - p^{\mathrm{fv}})) c \dot{\pi}(p) 1_{\{\mathcal{Q} > \mathcal{I}(p)\}} = c \dot{U}(c(p - p^{\mathrm{fv}})) \sum_{k=0}^{N} 1_{\{\mathcal{I}(p)=k\}} 1_{\{\mathcal{Q}/c \in]k,k+1]\}}$$

and so for all $p > p^{\mathrm{v}}$ the left side of (49) is

$$\begin{aligned} &E_{P \otimes \nu^{*N}} (\dot{U}(c(\pi(p) - p^{\mathrm{fv}})) c \dot{\pi}(p) 1_{\{\mathcal{Q} > \mathcal{I}(p)\}}) \\ &\quad = c \dot{U}(c(p - p^{\mathrm{fv}})) \sum_{k=0}^{N} P \otimes \nu^{*N}(\mathcal{I}(p) = k) \cdot P \otimes \nu^{*N}(\mathcal{Q}/c \in]k, k+1]) \\ &\quad = c \dot{U}(c(p - p^{\mathrm{fv}})) L(F(p)), \end{aligned} \tag{58}$$

where $F(p) = \nu^*(]0, p])$ for $p \in J$ and

$$L(q) = \sum_{k=0}^{N} \binom{N}{k} \alpha_k q^k (1-q)^{N-k}, \quad q \in [0, 1]. \tag{59}$$

Rewrite now (49) using (58) as

$$\partial F(p) = \frac{c \dot{U}(c(p - p^{\mathrm{fv}}))}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})} \frac{L(F(p))}{N H(F(p))}, \quad \text{for } p > p^{\mathrm{v}}, \ F(p) > 0, \tag{60}$$

where the positivity of the denominator is ensured by $p > p^{\mathrm{v}}$ and by

$$1 \geqslant H(q) \geqslant \min_{k=0}^{N-1} \alpha_k > 0, \quad \text{for all } q \in [0, 1]. \tag{61}$$

Similarly, we obtain the bound

$$m := \min_{q \in [0,1]} \frac{L(q)}{N H(q)} \geqslant \frac{\min_{k=0}^{N} \alpha_k}{N \cdot 1} > 0. \tag{62}$$

Define now

$$f\colon]p^{\mathrm{v}}, \sup J] \times [0,1] \to \mathbb{R}, \quad (p,q) \mapsto \frac{c\dot{U}(c(p-p^{\mathrm{fv}}))}{(U(c(p-p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}}))} \frac{L(q)}{NH(q)},$$

to deduce from (60) that on $\{p > p^{\mathrm{v}} \colon F(p) > 0\}$, F is a solution to

$$F(\sup J) = 1, \quad \partial F(p) = f(p, F(p)), \tag{63}$$

which is unique since the partial derivative $(p,q) \mapsto \partial_{(0,1)} f(p,q)$ is continuous ensuring the Lipschitz condition. This gives the uniqueness of the equilibrium strategy. Let us show the existence. Consider the global solution $\tilde{F}\colon [\rho, \sup J] \to [0,1]$ to

$$\tilde{F}(\sup J) = 1, \quad \partial \tilde{F}(p) = f(p, \tilde{F}(p)).$$

Then $\tilde{F}(\rho) = 0$, as the following estimation for $p \in [\rho, \sup J]$ shows:

$$\begin{aligned}
\tilde{F}(p) &= -\int_p^{\sup J} f(s, \tilde{F}(s))\,\mathrm{d}s + 1 \\
&\leqslant -\int_p^{\sup J} \frac{c\dot{U}(c(s-p^{\mathrm{fv}}))}{(U(c(s-p^{\mathrm{fv}})) - U(-cs^{\mathrm{f}}))} m\,\mathrm{d}s + 1 \\
&\leqslant m \ln\left(\frac{U(c(p-p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}{U(c(\sup J - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}\right) + 1.
\end{aligned}$$

Given $\tilde{F}$, the distribution ν^* on J is well defined by

$$\nu^*(]0,p]) = \begin{cases} 0, & \text{if } p \leqslant \rho, \\ \tilde{F}(p), & \text{if } p > \rho, \end{cases}$$

since $\tilde{F}$ is continuous and non-decreasing. To show the equilibrium property (9) of ν^*, it suffices to prove that

$$\mathcal{U}(\cdot, \nu^*) \text{ is constant on } [\rho, \sup J[, \tag{64}$$

because from (i) of the Lemma 2, we have $\mathcal{U}(p,\nu^*) \leqslant \mathcal{U}(\rho,\nu^*)$ for all $p \leqslant \rho$ and combine this estimate with (64) to obtain $\mathcal{U}(\rho,\nu^*) = S(\nu^*)$ giving the equilibrium property (9):

$$\{p \in J \colon \mathcal{U}(p,\nu^*) < S(\nu^*)\} \subseteq]0,\rho] \Rightarrow \nu^*(\{p \in J \colon \mathcal{U}(p,\nu^*) < S(\nu^*)\}) \leqslant \nu^*(]0,\rho]) = 0.$$

The assertion (64) is verified by calculating the derivative $\partial\mathcal{U}(p,\nu^*)$ exactly as in the proof of the Proposition 1, we thus obtain

$$\partial\mathcal{U}(p,\nu^*) = -c\dot{U}(c(p-p^{\mathrm{fv}}))L(\tilde{F}(p)) + \partial\tilde{F}(p)(U(c(p-p^{\mathrm{fv}}) - U(-cp^{\mathrm{f}})))H(\tilde{F}(p)) = 0,$$

for all $p \in]\rho, \sup J[$, since $\tilde{F}$ solves by construction (63) on $]\rho, \sup J[$.

For the discriminatory auction, we repeat the above argumentation, replacing (57) by

$$\dot{\pi}_x(p) = 1, \quad \text{for all } x \in J^N, \ p \in J,$$

in this case, we have

$$L(q) := \sum_{k=0}^{N} \binom{N}{k} \left(1 - \sum_{j=0}^{k-1} \alpha_j\right) q^k (1-q)^{N-k}, \quad m \geqslant \frac{(1 - \sum_{j=0}^{N-1} \alpha_j)}{N} \geqslant \frac{\alpha_N}{N} > 0. \tag{65}$$

For the Vickery auction, the derivative is

$$\dot{\pi}_x(p) = 0, \quad \text{for all } x \in J^N, \ p \in J.$$

Therefore, the left side of (49) vanishes and the differential equation (63) in this case is

$$F(\sup J) = 1, \quad \partial F(p) = 0.$$

That is, if F is the distribution function of an equilibrium strategy, then $F(p) = 1$ for all $p \in]p^{\mathrm{v}}, \sup J[$. On the other hand, (iv) of Lemma 3 shows that $F(p) = 0$ for all $p \in]0, p^{\mathrm{v}}[$, hence, $F = 1_{[p^{\mathrm{v}}, \sup J[}$, implying that there is a single candidate $\varepsilon_{p^{\mathrm{v}}}$ for the equilibrium strategy. A direct calculation shows that $\mathcal{U}(\cdot, \varepsilon_{p^{\mathrm{v}}})$ is constant

$$\mathcal{U}(p, \varepsilon_{p^{\mathrm{v}}}) = U(-cp^{\mathrm{f}})P(Q \leqslant cN) + U(c(\sup J - p^{\mathrm{fv}}))P(Q/c > N), \quad \text{for all } p \in J,$$

which proves the equilibrium property (9) of $\varepsilon_{p^{\mathrm{v}}}$. □

5. Case study: symmetric equilibrium in a two-agent market

Let us illustrate[1] the use of the differential equation (63) calculating equilibrium strategies for our auction formats in the case of two competing energy producers

$$N = 1, \quad \alpha_0 > 0, \ \alpha_1 > 0. \tag{66}$$

The conditions of Proposition 1 are fulfilled and we have $H(q) = \alpha_0$ for all $q \in [0, 1]$. According to (59), we obtain for the uniform auction

$$L(q) = \alpha_0(1 - q) + \alpha_1 q, \quad \text{for all } q \in [0, 1].$$

Hence, (60) yields with $\alpha = \alpha_1/\alpha_0$

$$\partial F(p) = \frac{\dot{U}(c(p - p^{\mathrm{fv}}))c}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}(1 - F(p)(1 - \alpha)), \quad \text{for } p > p^{\mathrm{v}}, \ F(p) > 0.$$

Solving this equation with initial condition $F(\sup J) = 1$, we obtain for $\alpha \neq 1$ the distribution of the unique equilibrium strategy for the uniform auction as

$$F(p) = \begin{cases} \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{U(c(\sup J - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}\right)^{1-\alpha}\right) \vee 0, & \text{for } p \in]p^{\mathrm{v}}, \sup J[, \\ 0, & \text{for } p \in]0, p^{\mathrm{v}}], \end{cases} \tag{67}$$

[1]The author thanks anonymous referees for suggestions on visual illustration.

and for $\alpha = 1$ as

$$F(p) = \begin{cases} \left(1 - \ln\left(\frac{U(c(\sup J - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}\right)\right) \vee 0, & \text{for } p \in\,]p^{\mathrm{v}}, \sup J[, \\ 0, & \text{for } p \in\,]0, p^{\mathrm{v}}]. \end{cases} \tag{68}$$

Now, we turn to the discriminatory auction. According to (65), we have

$$L(q) = (1 - q) + (1 - \alpha_0)q, \quad \text{for all } q \in [0, 1].$$

Hence, (60) yields

$$\partial F(p) = \frac{\dot{U}(c(p - p^{\mathrm{fv}}))c}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}(1/\alpha_0 - F(p)), \quad \text{for } p > p^{\mathrm{v}}, F(p) > 0.$$

Thus, the distribution of the unique equilibrium strategy for the discriminatory auction is

$$F(p) = \begin{cases} \left(\frac{1}{\alpha_0} - \frac{1-\alpha_0}{\alpha_0}\frac{U(c(\sup J - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}{U(c(p - p^{\mathrm{fv}})) - U(-cp^{\mathrm{f}})}\right) \vee 0, & \text{for } p \in\,]p^{\mathrm{v}}, \sup J[, \\ 0, & \text{for } p \in\,]0, p^{\mathrm{v}}]. \end{cases} \tag{69}$$

For the Vickery auction, the distribution function of the unique equilibrium strategy is $F(p) = 1_{[p^{\mathrm{v}}, \sup J[}(p)$ for all $p \in J$, as explained in the proof of the Proposition 2.

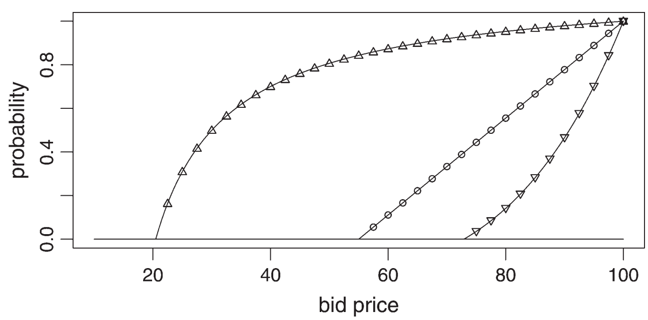

FIG. 3. Strategies (67) for $U\colon x \mapsto x$ and $\alpha = 1/6$, 2 and 6, denoted by $\triangle$, $\circ$ and $\triangledown$, respectively.

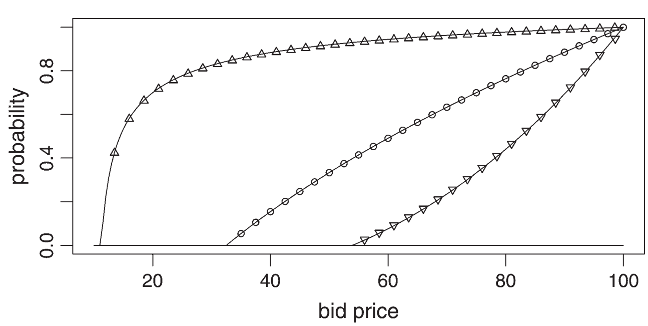

FIG. 4. Strategies (67) for $U\colon x \mapsto \sqrt{x + cp^{\mathrm{f}}}$ and $\alpha = 1/6$, 2 and 6, denoted by $\triangle$, $\circ$ and $\triangledown$, respectively.

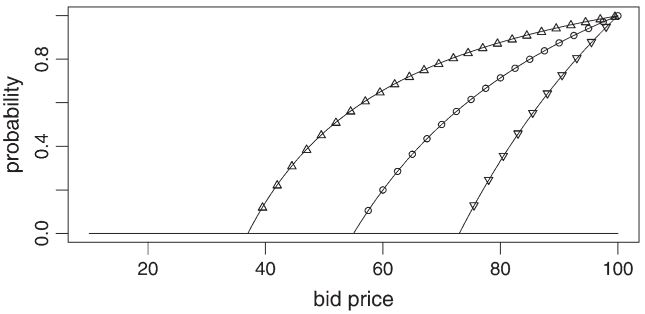

FIG. 5. Strategies (69) for $U\colon x \mapsto x$ and $\alpha_0 = 0.7$, 0.5 and 0.3, denoted by $\triangle$, $\circ$ and $\triangledown$, respectively.

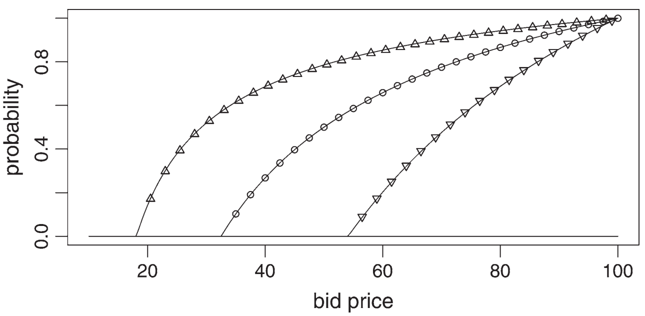

FIG. 6. Strategies (69) for $U\colon x \mapsto \sqrt{x + cp^{\mathrm{f}}}$ and $\alpha_0 = 0.7$, 0.5 and 0.3, denoted by $\triangle$, $\circ$ and $\triangledown$, respectively.

To give the reader a visual interpretation of equilibrium in uniform and discriminatory auction, we finally include a numerical illustration of expressions (67) and (69) choosing the price interval $J :=]0, 100[$, capacity $c := 1$ and production costs $p^{\mathrm{f}} := 15$, $p^{\mathrm{v}} := 10$.

Let us start with uniform auction assuming first $U(x) = x$ for all $x \in \mathbb{R}$, which means that the agents are not risk averse. According to (67) and (68), the unique equilibrium strategy depends on $\alpha = \alpha_1/\alpha_0$ being the probability that two generators are needed to cover the demand divided by the probability that one generator is sufficient. If this fraction is low (small α), then the equilibrium bids are concentrated at lower prices. This is seen from the upper curve in Fig. 3 (marked by $\triangle$) which shows the shape of (67) for $\alpha = 1/6$. In the opposite case (for $\alpha = 6$), bids are to place at higher prices, as one observes from the lower curve in Fig. 3, labelled by $\triangledown$. For $\alpha = 2$, we find equidistant bid distribution, marked by $\circ$ in this picture. The effect of risk aversion is illustrated in Fig. 4, where the same situation is plotted for the case of utility function $U(x) = \sqrt{x + cp^{\mathrm{f}}}$ for all $x \in]-cp^{\mathrm{f}}, \infty[$. As expected, risk-averse agents bid less aggressive, so we recognize in this picture the strategies from Fig. 3, shifted towards lower prices.

Now, we turn to the discriminatory auction. Here, (69) shows that the probability α_0 is essential. Note that lowering α_0, we automatically increase $1-\alpha_0$, which is the probability to be scheduled despite being the most expansive bidder. Hence, the lower is α_0, the more attractive are higher bid prices. This intuition is validated by Fig. 5, where, in the case of not risk-averse agents, we plot (69) for $\alpha_0 = 0.3$, 0.5 and 0.7, labelled by $\triangledown$, $\circ$ and $\triangle$, respectively. The last picture, Fig. 6, depicts the same situation for utility function $U(x) = \sqrt{x + cp^{\mathrm{f}}}$ for all $x \in]-cp^{\mathrm{f}}, \infty[$. Here, we observe the same effect of the risk aversion as in the case of uniform auction.

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