$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$. So $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}t + \begin{bmatrix} b \\ d \end{bmatrix}u\right)$, M maps the first (positive) quadrant Q to MQ the sector bounded by the two half-lines generated by the two (positive) columns of M: $Q \supset MQ$. We let U denote the unit circle in the plane and $K = Q \cap U$. K is a line segment. We define $f : K \to K$ by $f(z) = Mz / \|Mz\|$. Then f is a continuous function. So f has a fixed point $p : f(p) = p$ ([1, Theorem 9.1 p. 60]). This means that $Mp = \|Mp\| p$, so p is an eigenvector of M.

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99.31 A generalisation of Proizvolov's identity

Theorem: Take a sequence of real numbers $c_1 \leqslant c_2 \leqslant \dots \leqslant c_{2n}$ and split it into non-decreasing and non-increasing sequences of length n

$$a_1 = c_{j_1} \leqslant a_2 = c_{j_2} \leqslant \dots \leqslant a_n = c_{j_n}$$

and

$$b_1 = c_{k_1} \geqslant b_2 = c_{k_2} \geqslant \dots \geqslant b_n = c_{k_n}$$

with

$$\{j_1, \dots, j_n\} \cup \{k_1, \dots, k_n\} = \{1, \dots, 2n\}.$$

[*This is equivalent to choosing any n elements of the sequence, putting them in non-decreasing order and the other n elements in non-increasing order*.]

Then the sum of the distances $|a_\ell - b_\ell|$ is independent of the chosen split and equals the difference between the sums of the upper and lower halves of the given sequence, that is

$$\sum_{\ell=1}^{n} |a_\ell - b_\ell| = \sum_{i=n+1}^{2n} c_i - \sum_{i=1}^{n} c_i.$$

The original identity, proposed by Proizvolov as a problem in the 1985 USSR Mathematical Olympiad, refers only to the integer sequence $1 < 2 < \dots < 2n$ and provides the constant distance sum $\sum_{i=n+1}^{2n} i - \sum_{i=1}^{n} i = n^2$ [1, pp. 52, 69-71], [2, pp. 66-69] and [3]. We found no trace of a generalisation in the literature.

Proof: There is no pair (a_ℓ, b_ℓ) where both elements are (strictly) smaller than c_{n+1} or greater than c_n. Otherwise, there would be $\ell + (n + 1 - \ell) = n + 1$ numbers smaller than c_{n+1} or greater than c_n in the sequence, a contradiction.

In the case $c_n < c_{n+1}$, each distance $|a_\ell - b_\ell|$ is thus equal to $c_m - c_p$ for different m, p with $m > n \geqslant p$ and the assertion is proven.

We now turn to the case $c_n = c_{n+1} = c$. We can suppose without loss of generality that the terms c_i equal to c appear with increasing indices $j_.$ in the sequence (a_ℓ) and with greater and decreasing indices $k_.$ in (b_ℓ) (they may appear in only one sequence). Replace now for $i > n$ the terms c_i equal to c with $c + \varepsilon$, where $\varepsilon > 0$ is small enough to leave the order $c_1 \leqslant \ldots \leqslant c_{2n}$ unchanged. The orders $a_1 \leqslant \ldots \leqslant a_n$ and $b_1 \geqslant \ldots \geqslant b_n$ also remain the same. The sum of the distances $|a_\ell - b_\ell|$ changes by no more than $\pm n\varepsilon$ and is now, as the new c_{n+1} is greater than c_n, the difference between the sums of the upper and lower halves of the new sequence (c_i), which surpasses the old difference by no more than $n\varepsilon$. Letting ε tend to 0^+ proves the result.

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99.32 On the square wave series

The trigonometric series

$$f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots \qquad (1)$$

converges in $(0, \pi)$ by Dirichlet's test since the reciprocals form a positive null sequence and the sum of the first n odd sines (that is, $\sin^2 nx / \sin x$) is bounded.

In this note we will show that $f(x) = \pi / 4$ in $(0, \pi)$ without the use of Fourier series, thus demonstrating a square wave pattern by periodicity.

Note in passing that the derivative series of unattenuated cosine terms diverges despite its convergence 'in the mean'.