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PREDICTION ERROR OF THE CHAIN LADDER RESERVING METHOD APPLIED TO CORRELATED RUN-OFF TRIANGLES

By M. Merz and M. V. Wüthrich

ABSTRACT

In Buchwalder *et al.* (2006) we revisited Mack's (1993) and Murphy's (1994) estimates for the mean square error of prediction (MSEP) of the chain ladder claims reserving method. This was done using a time series model for the chain ladder method. In this paper we extend the time series model to determine an estimate for the MSEP of a portfolio of N correlated run-off triangles. This estimate differs in the special case $N = 2$ from the estimate given by Braun (2004). We discuss the differences between the estimates.

KEYWORDS

Claims Reserving; Chain Ladder; Mean Square Error of Prediction; Prediction Error; Correlated Business Lines; Run Off

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1. Motivation

There are several different stochastic models which justify the chain ladder algorithm for claims reserving. The most popular ones are Mack's (1993) chain ladder model, the overdispersed Poisson model (see Renshaw & Verrall, 1998; Verrall, 2000) and Bayesian models (see, e.g., Gisler, 2006). All these different models give the chain ladder estimate for the claims reserves, however (due to their differences) they lead to different estimators (and values) for prediction errors.

In this paper we consider the claims reserving problem for several correlated claims reserving triangles. We study this multivariate claims reserving problem within the framework of a multivariate time series model for the chain ladder method. This generalises the univariate time series model for the chain ladder model studied in Buchwalder *et al.* (BBMW) (2006). We have seen in Buchwalder *et al.* (2006) that there are different approaches for estimating the mean square error of prediction (MSEP) in Mack's (1993) univariate chain ladder model (unconditional and conditional approaches). Depending on the approach chosen, one obtains different estimators for the MSEP. In BBMW (2006) we applied the conditional approach, which has led to estimators for the MSEP which differ from Mack's (1993) results. Moreover, we saw that Mack's (1993) formula is a linear approximation from below to BBMW's formula (2006). In most practical examples the resulting estimates (Mack (1993) and BBMW (2006)) are very close to each other, which means that it is sufficient to consider the first terms in the Taylor expansions to get the range of possible estimation errors (see, also, Wüthrich *et al.*, 2006).

The aim of the present article is to extend the MSEP estimate from the univariate chain ladder model to the situation of several correlated run-off triangles. As a result of our extension, we obtain an estimate for the MSEP for aggregated claims reserves of several correlated run-off triangles. Our MSEP formula is compared to Braun's (2004) MSEP result, which is the bivariate extension of Mack's formula (see Section 5).

Our studies are motivated by the fact that, in practice, it is quite natural to subdivide a non-life run-off portfolio into several subportfolios, such that each subportfolio satisfies certain homogeneity properties (in our case the chain ladder assumptions). The total reserves are then obtained by the aggregation of the reserves from the single subportfolios. The calculation of the resulting MSEP of the total portfolio is then quite sophisticated if the subportfolios are correlated. In this work we treat such questions precisely. Hence, our work is one step towards the aggregation of different subportfolios. However, we should also remark that, in practice, one often uses different reserving methods for different subportfolios. It remains a challenging open problem to estimate an overall MSEP for aggregated subportfolios (if we use different claims reserving methods in the subportfolios).

An alternative idea for calculating aggregated reserves and their uncertainties is that one only calculates the reserves and their uncertainties on the aggregrated run-off triangle; but one should pay attention to the fact that, if the subportfolios satisfy the chain ladder assumptions, then the aggregated run-off triangle does not necessarily satisfy the chain ladder assumptions (c.f. Anje, 1994). Henceforth, this is not a promising solution to the claims reserving problem on aggregated subportfolios.

2. Claims Development Triangles

Assume that we have $N \geq 1$ run-off triangles. Figure 1 shows the structure of the run-off triangles (claims development data).

For simplicity, we assume that the number of accident years is equal to the number of observed development periods, i.e. $I = J$. In these triangles the variables:

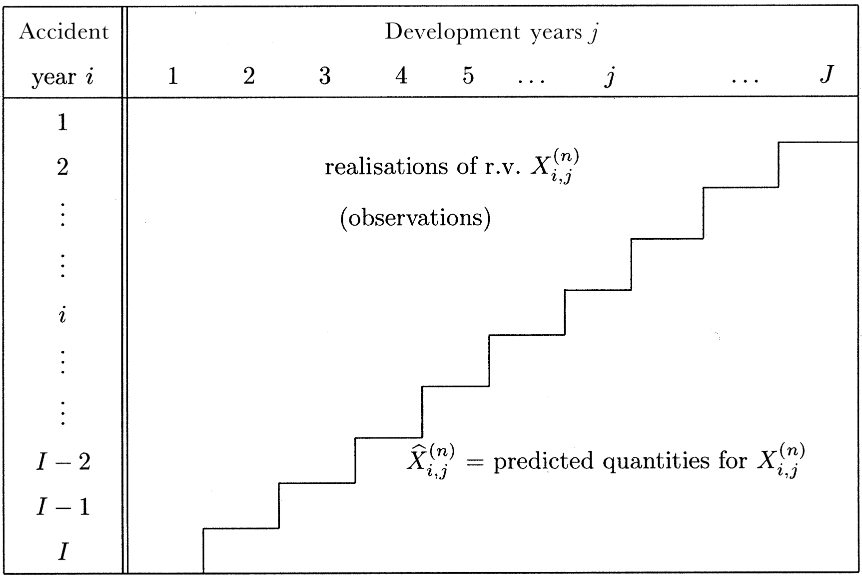

Figure 1. The structure of the run-off triangles

n,	$1 \le n \le N$,	refer to subportfolios (triangles)
i,	$1 \le i \le I$,	refer to accident years (rows)
j,	$1 \le j \le J$,	refer to development years (columns).

Usually, at time I, we have observations:

$$\mathcal{D}_I^{(n)} = \left\{ X_{i,j}^{(n)} \,\middle|\, 1 \le i \le I \quad \text{and} \quad 1 \le j \le I - i + 1 \right\} \tag{2.1}$$

for all N subportfolios. This means that at time I (calendar year) we have observations $\mathcal{D}_I^{(n)}$ ($n = 1, \dots, N$), and we need to predict the random variables in its complement:

$$\mathcal{D}_I^c = \bigcup_{n=1}^{N} \left\{ X_{i,j}^{(n)} \,\middle|\, 1 \le i \le I \quad \text{and} \quad I - i + 1 < j \right\}. \tag{2.2}$$

Here the entries $X_{i,j}^{(n)}$ denote incremental quantities, and may be interpreted as: (a) on a paid claims basis; (b) on an occurred claims basis; or (c) number of newly reported claims. Cumulative quantities are denoted by:

$$C_{i,j}^{(n)} = \sum_{l=1}^{j} X_{i,l}^{(n)}. \tag{2.3}$$

In the following, to simplify our language, $X_{i,j}^{(n)}$ and $C_{i,j}^{(n)}$ always refer to payments.

3. Chain Ladder Estimate and MSEP

The chain ladder method is based on cumulative quantities $C_{i,j}^{(n)}$. Increments $X_{i,j}^{(n)}$ can easily be derived from $X_{i,j}^{(n)} = C_{i,j}^{(n)} - C_{i,j-1}^{(n)}$.

3.1 *Chain Ladder Algorithm*

Often the chain ladder method is understood as a purely mechanical algorithm to estimate claims reserves. For triangle $n \in \{1, \ldots, N\}$, the algorithmic definition of the chain ladder method at time I (i.e. given the observations $\mathcal{D}_I^{(n)}$) reads as follows:

(1) There are constants $f_l^{(n)}$ $(l = 1, \ldots, J-1)$ so that for all i and $j > I - i + 1$:

$$\widehat{C}_{i,j}^{(n)} = C_{i,I-i+1}^{(n)} \cdot f_{I-i+1}^{(n)} \cdot f_{I-i+2}^{(n)} \cdot \ldots \cdot f_{j-1}^{(n)} \tag{3.1}$$

is an appropriate predictor for $C_{i,j}^{(n)}$.

(2) The chain ladder factors $f_l^{(n)}$ (age-to-age factors) are estimated by:

$$\widehat{f}_l^{(n)} = \frac{\sum_{i=1}^{I-l} C_{i,l+1}^{(n)}}{S_l^{(n)}} = \sum_{i=1}^{I-l} \frac{C_{i,l}^{(n)}}{S_l^{(n)}} \cdot \frac{C_{i,l+1}^{(n)}}{C_{i,l}^{(n)}} \tag{3.2}$$

where:

$$S_l^{(n)} = \sum_{i=1}^{I-l} C_{i,l}^{(n)}. \tag{3.3}$$

It is to Mack's (1993) credit that he first gave a stochastic model which satisfies the chain ladder algorithm, and for which he estimated the MSEP.

3.2 *The Chain Ladder Consistent Time Series Model for Several Correlated Run-Off Triangles*

Braun (2004) extended Mack's (1993) method to the multivariate case, in order to estimate the MSEP of the chain ladder method for several subportfolios simultaneously. In this paper we choose a different route, and enlarge our time series framework to N correlated development triangles. Within this framework, we derive an estimate for the MSEP according to the conditional approach described in Buchwalder *et al.* (2006). The resulting formula differs from Braun's formula (2004). The discussion in Section 5 highlights the differences.

Model assumptions 3.1 (Multivariate chain ladder time series model)

There exist constants $f_l^{(n)}, \sigma_l^{(n)} > 0$ such that, for all $1 \le n \le N$, $1 \le i \le I$ and $2 \le j \le J$, we have:

(T1) $C_{i,j}^{(n)} = f_{j-1}^{(n)} \cdot C_{i,j-1}^{(n)} + \sigma_{j-1}^{(n)} \cdot \sqrt{C_{i,j-1}^{(n)}} \cdot \varepsilon_{i,j}^{(n)}$;

with:

(T2) different accident years i are independent, and $\varepsilon_{i,j}^{(n)}$ and $\varepsilon_{k,l}^{(m)}$ are independent if $i \neq k$ or $j \neq l$;

(T3) $\mathrm{E}\big[\varepsilon_{i,j}^{(n)}\big] = 0$ and $\mathrm{Var}\big(\varepsilon_{i,j}^{(n)}\big) = \mathrm{E}\Big[\big(\varepsilon_{i,j}^{(n)}\big)^2\Big] = 1$; and

(T4) $\mathrm{Cor}\big(\varepsilon_{i,j}^{(n)}, \varepsilon_{i,j}^{(m)}\big) = \mathrm{E}\big[\varepsilon_{i,j}^{(n)} \cdot \varepsilon_{i,j}^{(m)}\big] = \rho_{j-1}^{(n,m)} \in [-1, 1]$.

For more technical details we refer to Wüthrich *et al.* (2006).

Corollary 3.2. Under model assumptions 3.1, $\{C_{i,j}^{(n)}\}_{j \ge 1}$ are Markov chains, with:

(P1) different accident years of one or of distinct triangles are independent;

(P2) $\mathrm{E}\big[C_{i,j}^{(n)} \,\big|\, C_{i,j-1}^{(n)}\big] = f_{j-1}^{(n)} \cdot C_{i,j-1}^{(n)}$;

(P3) $\mathrm{Var}\big[C_{i,j}^{(n)} \,\big|\, C_{i,j-1}^{(n)}\big] = \big(\sigma_{j-1}^{(n)}\big)^2 \cdot C_{i,j-1}^{(n)}$; and

(P4) $\mathrm{Cov}\big[C_{i,j}^{(n)}, C_{k,j}^{(m)} \,\big|\, C_{i,j-1}^{(n)}, C_{k,j-1}^{(m)}\big] = \sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)} \sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \cdot \rho_{j-1}^{(n,m)} \cdot 1_{\{i=k\}}$.

Observe that (P1) – (P3) are the classical Mack (1993) properties extended to several run-off triangles.

Lemma 3.3. Under model assumptions 3.1, we have that $\widehat{f}_l^{(n)}$ (given in (3.2)) is (conditionally) unbiased for $f_l^{(n)}$. Moreover, $\widehat{f}_l^{(n)}$ and $\widehat{f}_k^{(m)}$ are uncorrelated for $l \neq k$, and

$$\widehat{C}_{i,J}^{(n)} = C_{i,I-i+1}^{(n)} \cdot \widehat{f}_{I-i+1}^{(n)} \cdot \widehat{f}_{I-i+2}^{(n)} \cdot \ldots \cdot \widehat{f}_{J-1}^{(n)} \tag{3.4}$$

is an (conditionally) unbiased estimator for $\mathrm{E}\big[C_{i,J}^{(n)} \,\big|\, C_{i,I-i+1}^{(n)}\big]$.

Proof. The proof is similar to the one for Theorems 1 and 2 in Mack (1993).

3.3 *Conditional Mean Square Error of Prediction*

We define the set of total information up to time I by:

$$\mathcal{D}_I = \bigcup_{n=1}^{N} \mathcal{D}_I^{(n)}. \tag{3.5}$$

Lemma 3.3 gives unbiased predictors for the ultimate claims. Our goal is to study the conditional mean square error of these predictors. The conditional MSEP is defined as follows (see, e.g., Mack (1993), Section 3):

Definition 3.4 (Conditional mean square error of prediction)

$$\mathrm{MSEP}\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)=\mathrm{E}\left[\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}C_{i,J}^{(n)}-\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)^{2}\Bigg|\mathcal{D}_I\right]. \quad (3.6)$$

Observe that the predictors $\widehat{C}_{i,J}^{(n)}$ are $\mathcal{D}_I$-measurable, hence a constant in the conditional expectation considered in (3.6). Therefore the conditional MSEP decouples into the conditional process error and the conditional estimation error (see also Mack (1993), p217). Notice that this is different from unconditional MSEP considerations (see, e.g., England & Verrall, 2002). Here we have the following (strict) equality for the conditional MSEP:

$$\mathrm{MSEP}\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)=\mathrm{Var}\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}C_{i,J}^{(n)}\Bigg|\mathcal{D}_I\right)+\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}C_{i,J}^{(n)}\Bigg|\mathcal{D}_I\right]\right)^{2}. \quad (3.7)$$

The first term on the right-hand side of (3.7) is the conditional process variance and the second term on the right-hand side of (3.7) is the conditional estimation error. In Section 4 we derive estimators for these two terms, which give Result 3.5.

Result 3.5. For N correlated run-off triangles we have the following estimator for the conditional MSEP for the ultimate loss of aggregated subportfolios and accident years:

$$\widehat{\mathrm{MSEP}}\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)=\sum_{i=I+2-J}^{I}\widehat{\mathrm{MSEP}}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)+2\cdot\sum_{I+2-J\le i<k\le I}\ \sum_{1\le n,m\le N}C_{i,I-i+1}^{(n)}\cdot\widehat{C}_{k,I-i+1}^{(m)}\cdot\Delta_{i,J}^{(m,n)} \quad (3.8)$$

where the conditional MSEP for the ultimate loss of aggregated subportfolios for a single accident year $i\in\{I+2-J,\ldots,I\}$ is given by:

$$\widehat{\text{MSEP}}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)=\sum_{n=1}^{N}\widehat{\text{MSEP}}\left(\widehat{C}_{i,J}^{(n)}\right) + 2\cdot\sum_{1\le n<m\le N}\left(\Gamma_{i,J}^{(n,m)}+C_{i,I-i+1}^{(n)}\cdot C_{i,I-i+1}^{(m)}\cdot\Delta_{i,J}^{(n,m)}\right) \quad (3.9)$$

and where the conditional MSEP for the ultimate loss for a single accident year $i\in\{I+2-J,\ldots,I\}$ and a single subportfolio $n\in\{1,\ldots,N\}$ is given by:

$$\widehat{\text{MSEP}}\left(\widehat{C}_{i,J}^{(n)}\right)=\Gamma_{i,J}^{(n,n)}+\left(C_{i,I-i+1}^{(n)}\right)^2\cdot\Delta_{i,J}^{(n,n)} \quad (3.10)$$

with:

$$\Gamma_{i,J}^{(n,m)}=\widehat{C}_{i,J}^{(n)}\cdot\widehat{C}_{i,J}^{(m)}\cdot\sum_{l=I-i+1}^{J-1}\frac{\widehat{\sigma}_l^{(n)}\cdot\widehat{\sigma}_l^{(m)}\cdot\widehat{\rho}_l^{(n,m)}}{\sqrt{\widehat{C}_{i,l}^{(n)}\cdot\widehat{C}_{i,l}^{(m)}\cdot\widehat{f}_l^{(n)}\cdot\widehat{f}_l^{(m)}}} \quad (3.11)$$

$$\Delta_{i,J}^{(n,m)}=\prod_{l=I-i+1}^{J-1}\left(\widehat{f}_l^{(n)}\widehat{f}_l^{(m)}+\widehat{\rho}_l^{(n,m)}\frac{\widehat{\sigma}_l^{(n)}}{S_l^{(n)}}\frac{\widehat{\sigma}_l^{(m)}}{S_l^{(m)}}\sum_{i=1}^{I-l}\sqrt{C_{i,l}^{(n)}C_{i,l}^{(m)}}\right)-\prod_{l=I-i+1}^{J-1}\widehat{f}_l^{(n)}\widehat{f}_l^{(m)}. \quad (3.12)$$

The estimators $\widehat{\sigma}_l^{(n)}$ and $\widehat{\rho}_l^{(n,m)}$ for the variance parameters $\sigma_l^{(n)}$ and the correlation parameters $\rho_l^{(n,m)}$ are provided in the next subsection. Formulae (3.9) and (3.10) are derived in Result 4.2 and formula (3.8) is derived in Result 4.3.

Remarks 3.6

— From (P4) we obtain:

$$\rho_{j-1}^{(n,m)}=\frac{\text{Cov}\left[\frac{C_{i,j}^{(n)}}{\sqrt{C_{i,j-1}^{(n)}}},\frac{C_{i,j}^{(m)}}{\sqrt{C_{i,j-1}^{(m)}}}\,\middle|\,C_{i,j-1}^{(n)},C_{i,j-1}^{(m)}\right]}{\sigma_{j-1}^{(n)}\cdot\sigma_{j-1}^{(m)}}. \quad (3.13)$$

This means that the correlations between the individual development factors are:

$$\text{Cor}\left[\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}},\frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}}\,\middle|\,C_{i,j-1}^{(n)},C_{i,j-1}^{(m)}\right]=\rho_{j-1}^{(n,m)}. \quad (3.14)$$

This differs slightly from Braun's (2004) covariance notations. Indeed, the $\rho_{j-1}^{(n,m),B}$ used in Braun (2004) is not a correlation, but:

$$\rho_{j-1}^{(n,m),B} = \rho_{j-1}^{(n,m)} \cdot \sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)}. \tag{3.15}$$

Otherwise, the model assumptions in Braun (2004) are (P1) – (P4).

— Our estimate in Result 3.5 differs from the one given in Braun (2004). The differences are analysed in Section 5.

3.4 *Estimation of Model Parameters*

Our extended time series model specified by model assumptions 3.1 has the parameters:

$$f_1^{(n)}, \ldots, f_{J-1}^{(n)} \quad \text{and} \quad \sigma_1^{(n)}, \ldots, \sigma_{J-1}^{(n)} \quad \text{and} \quad \rho_1^{(n,m)}, \ldots, \rho_{J-1}^{(n,m)} \tag{3.16}$$

for $1 \le n \le m \le N$, which we have to estimate. To estimate the chain ladder factors $f_l^{(n)}$, we use the standard formulae (3.2) - (3.3). The appropriate (unbiased) estimator for the variance $\left(\sigma_{j-1}^{(n)}\right)^2$, $2 \le j \le J$, is given by (see Mack (1993), p217):

$$\left(\widehat{\sigma}_{j-1}^{(n)}\right)^2 = \frac{1}{I-j} \cdot \sum_{i=1}^{I-j+1} C_{i,j-1}^{(n)} \cdot \left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} - \widehat{f}_{j-1}^{(n)}\right)^2. \tag{3.17}$$

In fact, the proof of the unbiasedness is a straightforward calculation using the model assumptions. There remains to give an estimator for the correlation coefficient $\rho_{j-1}^{(n,m)}$. If $\sigma_{j-1}^{(n)}$ and $\sigma_{j-1}^{(m)}$ are known, then the following estimator is an unbiased estimator for $\rho_{j-1}^{(n,m)}$:

$$\widetilde{\rho}_{j-1}^{(n,m)} = c_j \cdot \sum_{i=1}^{I-j+1} \frac{\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}}}{\sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)}} \cdot \left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} - \widehat{f}_{j-1}^{(n)}\right) \cdot \left(\frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} - \widehat{f}_{j-1}^{(m)}\right) \tag{3.18}$$

where:

$$c_j = \frac{1}{I-j-1+w_j^2} \quad \text{and} \quad w_j^2 = \frac{\left(\sum_{i=1}^{I-j+1}\left(C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}\right)^{1/2}\right)^2}{S_{j-1}^{(n)} \cdot S_{j-1}^{(m)}}. \tag{3.19}$$

For the proof of the unbiasedness we refer to the Appendix.

Hence, a sound estimator for the correlation coefficient $\rho_{j-1}^{(n,m)}$ is provided by ($2 \le j \le J$):

$$\widehat{\rho}_{j-1}^{(n,m)} = c_j \cdot \sum_{i=1}^{I-j+1} \frac{\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}}}{\widehat{\sigma}_{j-1}^{(n)} \cdot \widehat{\sigma}_{j-1}^{(m)}} \cdot \left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} - \widehat{f}_{j-1}^{(n)} \right) \cdot \left(\frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} - \widehat{f}_{j-1}^{(m)} \right) \tag{3.20}$$

for $n \neq m$ and $\widehat{\rho}_{j-1}^{(n,n)} = 1$ for $n = m$.

Remark 3.7. Note that, in general, we are not able to estimate $\sigma_{J-1}^{(n)}$ and $\rho_{J-1}^{(n,m)}$ from the data and from (3.17) and (3.20). This is due to the lack of data in the tails. This means that the tail parameters (and tail chain ladder factors) cannot be estimated appropriately from data. There is a whole philosophy about estimating tail parameters. We do not want to enter this discussion here, but we simply extrapolate the last parameters by exponentially decreasing series (see Mack (1993) p217 and formulae (6.1) - (6.2)).

4. Derivation of the MSEP Estimate

In this section we derive the estimators for the conditional MSEP which are given in Result 3.5.

4.1 *MSEP for Single Accident Years*

As usual, the conditional MSEP of the sum $\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)}$ can be split into two parts: (a) the stochastic error (conditional process variance); and (b) the conditional estimation error (c.f. Mack, 1993, Section 3). We assume, in this section, that $j + i > I + 1$, hence we consider:

$$\begin{aligned} \mathrm{MSEP}\left(\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)}\right) &= \mathrm{E}\left[\left(\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)} - \left(C_{i,j}^{(n)} + C_{i,j}^{(m)}\right)\right)^2 \,\middle|\, \mathcal{D}_I\right] \\ &= \underbrace{\mathrm{E}\left[\left(C_{i,j}^{(n)} + C_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(n)} + C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right)^2 \,\middle|\, \mathcal{D}_I\right]}_{= \mathrm{Var}\left(C_{i,j}^{(n)} + C_{i,j}^{(m)} \mid \mathcal{D}_I\right)} \\ &\quad + \left(\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(n)} + C_{i,j}^{(m)} \mid \mathcal{D}_I\right]\right)^2. \end{aligned} \tag{4.1}$$

Note that, because of the $\mathcal{D}_I$-measurability of $\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)}$, we have an exact equality for the conditional MSEP in (4.1).

The conditional process variance $\mathrm{Var}\left(C_{i,j}^{(n)} + C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right)$ originates from the stochastic movement of the processes, whereas the conditional estimation error reflects the uncertainty in the estimation of the expectations (mean values). We derive estimates for both the process variance and the estimation error for N correlated run-off triangles. For a quick reference to the relevant formulae for the different error terms, see Table 1.

Table 1. Estimators of conditional process variance, conditional estimation error and conditional MSEP for N correlated run off triangles

		N correlated triangles
Single accident years	Conditional process variance	(4.7)
	Conditional estimation error	(4.22)
	Conditional MSEP	(4.23)
Aggregated a.y.	Conditional MSEP	(4.32)

4.1.1 *Conditional process variance for a single accident year*

The derivation of an estimator for the conditional process variance is straightforward, because all terms can be calculated explicitly. For the univariate case we refer to Mack (1993, Theorem 3) or England & Verrall (2002, Appendix 1).

For the conditional process variance in formula (4.1), we obtain:

$$\mathrm{Var}\left[C_{i,j}^{(n)} + C_{i,j}^{(m)} \mid \mathcal{D}_I\right] = \mathrm{Var}\left[C_{i,j}^{(n)} \mid \mathcal{D}_I\right] + \mathrm{Var}\left[C_{i,j}^{(m)} \mid \mathcal{D}_I\right] + 2 \cdot \mathrm{Cov}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} \mid \mathcal{D}_I\right] \tag{4.2}$$

for $1 \le i \le I$, $I - i + 1 < j \le J$ and $1 \le n, m \le N$.

Using (P2) and (P4) we obtain the recursion:

$$\begin{aligned}\mathrm{Cov}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} | \mathcal{D}_I\right] &= \mathrm{Cov}\left(\mathrm{E}\left[C_{i,j}^{(n)} \mid C_{i,j-1}^{(n)}\right], \mathrm{E}\left[C_{i,j}^{(m)} \mid C_{i,j-1}^{(m)}\right] \mid \mathcal{D}_I\right) \\ &\quad + \mathrm{E}\left[\mathrm{Cov}\left(C_{i,j}^{(n)}, C_{i,j}^{(m)} \mid C_{i,j-1}^{(n)}, C_{i,j-1}^{(m)}\right) \mid \mathcal{D}_I\right] \\ &= f_{j-1}^{(n)} \cdot f_{j-1}^{(m)} \cdot \mathrm{Cov}\left[C_{i,j-1}^{(n)}, C_{i,j-1}^{(m)} \mid \mathcal{D}_I\right] \\ &\quad + \sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)} \cdot \mathrm{E}\left[\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \,\middle|\, \mathcal{D}_I\right] \cdot \rho_{j-1}^{(n,m)}\end{aligned} \tag{4.3}$$

for the conditional covariance term in formula (4.2). From this we deduce recursively the estimator:

$$\begin{aligned}\widehat{\mathrm{Cov}}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} \mid \mathcal{D}_I\right] &= \widehat{f}_{j-1}^{(n)} \cdot \widehat{f}_{j-1}^{(m)} \cdot \widehat{\mathrm{Cov}}\left[C_{i,j-1}^{(n)}, C_{i,j-1}^{(m)} \mid \mathcal{D}_I\right] \\ &\quad + \widehat{\sigma}_{j-1}^{(n)} \cdot \widehat{\sigma}_{j-1}^{(m)} \cdot \sqrt{\widehat{C}_{i,j-1}^{(n)} \cdot \widehat{C}_{i,j-1}^{(m)}} \cdot \widehat{\rho}_{j-1}^{(n,m)}\end{aligned} \tag{4.4}$$

for the conditional covariance term $\mathrm{Cov}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} | \mathcal{D}_I\right]$ (see also Remarks 4.1).

With the definition:

$$\Gamma_{i,j}^{(n,m)} = \widehat{\mathrm{Cov}}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} \mid \mathcal{D}_I\right] \tag{4.5}$$

this can be rewritten in a recursive form:

$$\Gamma_{i,j}^{(n,m)} = \widehat{f}_{j-1}^{(n)} \cdot \widehat{f}_{j-1}^{(m)} \cdot \Gamma_{i,j-1}^{(n,m)} + \widehat{\sigma}_{j-1}^{(n)} \cdot \widehat{\sigma}_{j-1}^{(m)} \cdot \sqrt{\widehat{C}_{i,j-1}^{(n)} \cdot \widehat{C}_{i,j-1}^{(m)}} \cdot \widehat{\rho}_{j-1}^{(n,m)} \tag{4.6}$$

with $\Gamma_{i,I-i+1}^{(n,m)} = 0$, or, in an explicit form, this gives (3.11).

Henceforth, putting all estimates together, we obtain from (4.2) and (4.6) the following recursion for the estimator of the process variance of N correlated triangles:

$$\begin{aligned}\widehat{\mathrm{Var}}\left[\sum_{n=1}^{N} C_{i,j}^{(n)} \middle| \mathcal{D}_I\right] &= \sum_{n=1}^{N} \widehat{\mathrm{Var}}\left[C_{i,j}^{(n)} \middle| \mathcal{D}_I\right] + 2 \cdot \sum_{1 \le n < m \le N} \widehat{\mathrm{Cov}}\left[C_{i,j}^{(n)}, C_{i,j}^{(m)} \middle| \mathcal{D}_I\right] \\ &= \sum_{n=1}^{N} \Gamma_{i,j}^{(n,n)} + 2 \sum_{1 \le n < m \le N} \Gamma_{i,j}^{(n,m)}. \end{aligned} \tag{4.7}$$

Remarks 4.1

— Notice that, for one single run-off triangle (univariate case $N = 1$), we have exactly the well-known estimator for the process variance (see, e.g., England & Verrall, 2002, Section 7.5.4).

— The estimator for $\mathrm{E}\left[\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \middle| \mathcal{D}_I\right]$ given in (4.4) is not unbiased. Indeed:

$$\begin{aligned}&\mathrm{E}\left[\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \middle| \mathcal{D}_I\right] \\ &\quad = \mathrm{Cov}\left[\sqrt{C_{i,j-1}^{(n)}}, \sqrt{C_{i,j-1}^{(m)}} \middle| \mathcal{D}_I\right] + \mathrm{E}\left[\sqrt{C_{i,j-1}^{(n)}} \middle| \mathcal{D}_I\right] \cdot \mathrm{E}\left[\sqrt{C_{i,j-1}^{(m)}} \middle| \mathcal{D}_I\right]. \end{aligned} \tag{4.8}$$

We could replace the estimate in (4.4) by an upper bound for the term in (4.8) (using Jensen's inequality):

$$\begin{aligned}\mathrm{E}\left[\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \middle| \mathcal{D}_I\right] &\le \mathrm{E}\left[C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)} \middle| \mathcal{D}_I\right]^{1/2} \\ &= \left(\mathrm{Cov}\left[C_{i,j-1}^{(n)}, C_{i,j-1}^{(m)} \middle| \mathcal{D}_I\right] + \mathrm{E}\left[C_{i,j-1}^{(n)} \middle| \mathcal{D}_I\right] \cdot \mathrm{E}\left[C_{i,j-1}^{(m)} \middle| \mathcal{D}_I\right]\right)^{1/2}. \end{aligned} \tag{4.9}$$

Hence, in that case the recursion (4.6) is replaced by:

$$\widetilde{\Gamma}_{i,j}^{(n,m)} = \widehat{f}_{j-1}^{(n)} \cdot \widehat{f}_{j-1}^{(m)} \cdot \widetilde{\Gamma}_{i,j-1}^{(n,m)} + \widehat{\sigma}_{j-1}^{(n)} \cdot \widehat{\sigma}_{j-1}^{(m)} \cdot \sqrt{\widetilde{\Gamma}_{i,j-1}^{(n,m)} + \widehat{C}_{i,j-1}^{(n)} \cdot \widehat{C}_{i,j-1}^{(m)}} \cdot \widehat{\rho}_{j-1}^{(n,m)} \tag{4.10}$$

with $\widetilde{\Gamma}_{i,I-i+1}^{(n,m)} = 0$, and we obtain an estimation for an upper bound on the conditional process variance. Observe that this difficulty is not further commented on in Braun (2004).

4.1.2 *Conditional estimation error for a single accident year*

The conditional estimation error for two aggregated run-off triangles n and m is given by the last term on the right-hand side of formula (4.1). Before we derive an estimator for this term, we discuss the difficulties in the derivation of an estimator for the conditional estimation error (this is similar to Buchwalder *et al.*, 2006). If we have one single run-off triangle, the conditional estimation error for $j > I - i + 1$ is given by (see formula (4.11) in Buchwalder *et al.*, 2006):

$$\left(\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[C_{i,j}^{(n)} \,\middle|\, \mathcal{D}_I\right]\right)^2 = \left(C_{i,I-i+1}^{(n)}\right)^2 \cdot \left(\widehat{f}_{I-i+1}^{(n)} \cdot \ldots \cdot \widehat{f}_{j-1}^{(n)} - f_{I-i+1}^{(n)} \cdot \ldots \cdot f_{j-1}^{(n)}\right)^2. \qquad (4.11)$$

Observe that the realisations $\widehat{f}_l^{(n)}$ are known at time $t = I$, but (of course) the true chain ladder factors $f_l^{(n)}$ are unknown (otherwise we would not need to estimate them). Hence, the right-hand side of (4.11) cannot be calculated explicitly, and needs to be estimated using an appropriate technique. The chain ladder model allows for different approaches to estimate the right-hand side of (4.11), and most of them try to estimate the possible fluctuations of the estimators $\widehat{f}_l^{(n)}$ around the true values $f_l^{(n)}$. Such methods involve resampling techniques (e.g. non-parametric bootstrap methods (see England & Verrall, 2002), closed analytical techniques (see, e.g., Mack, 1993), upper and lower bounds (see, e.g., Wüthrich *et al.*, 2006), etc.). In the present work, we derive a closed analytical estimate which is based on the conditional approach described in Buchwalder *et al.* (2006). This closed analytical estimate can also be interpreted as a parametric bootstrap method, for which we can calculate the resulting estimators exactly. Henceforth, we use the terminology 'resampling', though we are able to calculate all terms in a closed form.

For the estimation error in formula (4.1) we obtain the following decomposition ($i + j > I + 1$):

$$\begin{aligned}
&\left(\widehat{C}_{i,j}^{(n)} + \widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(n)} + C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right)^2 \\
&\quad = \left(\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[C_{i,j}^{(n)} \,\middle|\, \mathcal{D}_I\right]\right)^2 + \left(\widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right)^2 \\
&\qquad + 2 \cdot \left(\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[C_{i,j}^{(n)} \,\middle|\, \mathcal{D}_I\right]\right) \cdot \left(\widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right).
\end{aligned} \qquad (4.12)$$

Using (P2) from Corollary 3.2 we have the representation:

$$\begin{aligned}
&\left(\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[C_{i,j}^{(n)} \,\middle|\, \mathcal{D}_I\right]\right) \cdot \left(\widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right) \\
&\quad = C_{i,I-i+1}^{(n)} \cdot \left(\widehat{f}_{I-i+1}^{(n)} \cdot \ldots \cdot \widehat{f}_{j-1}^{(n)} - f_{I-i+1}^{(n)} \cdot \ldots \cdot f_{j-1}^{(n)}\right) \\
&\qquad \cdot C_{i,I-i+1}^{(m)} \cdot \left(\widehat{f}_{I-i+1}^{(m)} \cdot \ldots \cdot \widehat{f}_{j-1}^{(m)} - f_{I-i+1}^{(m)} \cdot \ldots \cdot f_{j-1}^{(m)}\right).
\end{aligned} \qquad (4.13)$$

As described above, we are not able to calculate the right-hand side of (4.13) explicitly, because the true chain ladder factors $f_l^{(n)}$ are not known. Therefore, we study the volatility of the estimators $\widehat{f}_l^{(n)}$ around the true values $f_l^{(n)}$ to get a reasonable range for the sizes on the right-hand side of (4.13). We therefore apply the conditional approach technique presented in Buchwalder *et al.* (2006). This technique proposes to sample new observations $\widetilde{f}_l^{(n)}$ for $\widehat{f}_l^{(n)}$, given the observations $S_l^{(n)}$ (conditional resampling). This means that the development period l in the time series is resampled at a time, based on the known volume $S_l^{(n)}$, for $l = 2, \ldots, J$.

Note that the observed chain ladder factor estimators satisfy (see (T1) and (3.2)):

$$\widehat{f}_{j-1}^{(n)} = f_{j-1}^{(n)} + \frac{\sigma_{j-1}^{(n)}}{S_{j-1}^{(n)}} \cdot \sum_{i=1}^{I-j+1} \sqrt{C_{i,j-1}^{(n)}} \cdot \varepsilon_{i,j}^{(n)}. \tag{4.14}$$

Now we generate a set of 'new' observations for $\widehat{f}_{j-1}^{(n)}$, as follows. Set:

$$\widetilde{f}_{j-1}^{(n)} = f_{j-1}^{(n)} + \frac{\sigma_{j-1}^{(n)}}{S_{j-1}^{(n)}} \cdot \sum_{i=1}^{I-j+1} \sqrt{C_{i,j-1}^{(n)}} \cdot \widetilde{\varepsilon}_{i,j}^{(n)} \qquad (1 \le n \le N, 2 \le j \le J) \tag{4.15}$$

with:

$$\left(\widetilde{\varepsilon}_{i,j}^{(n)}\right)_{n=1,\ldots,N, i=1,\ldots,I, j=2,\ldots,J} \text{independent copy of } \left(\varepsilon_{i,j}^{(n)}\right)_{n=1,\ldots,N, i=1,\ldots,I, j=2,\ldots,J}. \tag{4.16}$$

Observe that we have the following distributional equalities for the conditional distributions ($j = 1, \ldots, J-1$ and $n = 1, \ldots, N$):

$$\begin{aligned}\widetilde{f}_j^{(n)}|_{\mathcal{D}_I} &\stackrel{(d)}{=} \widetilde{f}_j^{(n)} | \left\{C_{i,l}^{(m)}, 1 \le m \le N, 1 \le i \le I, 1 \le l \le j\right\} \\ &\stackrel{(d)}{=} \widehat{f}_j^{(n)} | \left\{C_{i,l}^{(m)}, 1 \le m \le N, 1 \le i \le I, 1 \le l \le j\right\}.\end{aligned} \tag{4.17}$$

This means that, given $\mathcal{D}_I$, the random variable $\widetilde{f}_j^{(n)}$ has the same distribution as the observed estimator $\widehat{f}_j^{(n)}$, conditioned on all observations $C_{i,l}^{(m)}$ before time $j+1$. Therefore, we use $\widetilde{f}_j^{(n)}$ for deriving an estimate for the right-hand side of (4.13).

Given $\mathcal{D}_I$, the random variable $\widetilde{f}_j^{(n)}$ satisfies:

(1) the random variables $\widetilde{f}_0^{(n)}, \ldots, \widetilde{f}_{J-1}^{(n)}$ are conditionally independent given $\mathcal{D}_I$;

(2) $\widetilde{f}_j^{(n)}$ and $\widetilde{f}_l^{(m)}$ $(n \neq m)$ are conditionally independent given $\mathcal{D}_I$ if $j \neq l$;

(3) $\mathrm{E}\left[\widetilde{f}_{j-1}^{(n)} \,\middle|\, \mathcal{D}_I\right] = f_{j-1}^{(n)}$ for $1 \le j \le J$;

(4) $\mathrm{E}\left[\left(\widetilde{f}_{j-1}^{(n)}\right)^2 \,\middle|\, \mathcal{D}_I\right] = \left(f_{j-1}^{(n)}\right)^2 + \dfrac{\left(\sigma_{j-1}^{(n)}\right)^2}{S_{j-1}^{(n)}}$ for $1 \le j \le J$;

(5) $\mathrm{E}\left[\widetilde{f}_{j-1}^{(n)} \cdot \widetilde{f}_{l-1}^{(m)} \,\middle|\, \mathcal{D}_I\right] = f_{j-1}^{(n)} \cdot f_{l-1}^{(m)}$ for $1 \le j \le J$ if $j \neq l$; and

(6) $\mathrm{E}\left[\widetilde{f}_{j-1}^{(n)} \cdot \widetilde{f}_{j-1}^{(m)} \,\middle|\, \mathcal{D}_I\right] = f_{j-1}^{(n)} \cdot f_{j-1}^{(m)} + \frac{\sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)}}{S_{j-1}^{(n)} \cdot S_{j-1}^{(m)}} \cdot \sum_{i=1}^{-j+1} \sqrt{C_{i,j-1}^{(n)} C_{i,j-1}^{(m)}} \rho_{j-1}^{(n,m)}$ for $1 \le j \le J$.

In fact (1) and (2) are the crucial steps, which differ from the MSEP derivation of Braun (2004) (Braun uses only the conditional uncorrelatedness of the $\widehat{f}_j^{(n)}$). Using this conditional resampling technique we obtain conditional independence (see (1) and (2)), which leads to a product structure of the conditional estimation error. This product structure is the same as the one derived by Murphy (1994), however it is based on different arguments. (For a discussion we refer to Buchwalder *et al.* (2006); Mack *et al.* (2006); Gisler (2006); and Venter (2006).)

Properties (1) and (2) imply that:

$$\mathrm{E}\left[\widetilde{f}_{I-i+1}^{(n)} \cdot \widetilde{f}_{I-i+1}^{(m)} \cdot \ldots \cdot \widetilde{f}_{j-1}^{(n)} \cdot \widetilde{f}_{j-1}^{(m)} \,\middle|\, \mathcal{D}_I\right] = \prod_{l=I-i+1}^{j-1} \mathrm{E}\left[\widetilde{f}_l^{(n)} \cdot \widetilde{f}_l^{(m)} \,\middle|\, \mathcal{D}_I\right]. \tag{4.18}$$

Using (1) - (3) we have:

$$\begin{aligned} &\mathrm{E}\left[\left(\widetilde{f}_{I-i+1}^{(n)} \cdot \ldots \cdot \widetilde{f}_{j-1}^{(n)} - f_{I-i+1}^{(n)} \cdot \ldots \cdot f_{j-1}^{(n)}\right) \cdot \left(\widetilde{f}_{I-i+1}^{(m)} \cdot \ldots \cdot \widetilde{f}_{j-1}^{(m)} - f_{I-i+1}^{(m)} \cdot \ldots \cdot f_{j-1}^{(m)}\right) \middle| \mathcal{D}_I\right] \\ &\quad = \prod_{l=I-i+1}^{j-1} \mathrm{E}\left[\widetilde{f}_l^{(n)} \cdot \widetilde{f}_l^{(m)} \,\middle|\, \mathcal{D}_I\right] - \prod_{l=I-i+1}^{j-1} f_l^{(n)} \cdot f_l^{(m)}. \end{aligned} \tag{4.19}$$

Notice that (4.19) can be calculated for the random variables $\widetilde{f}_l^{(n)}$ explicitly, i.e. there is no approximation involved here. Using (6) we can calculate the covariance term in (4.19). This leads to the following estimator for the right-hand side of (4.13). We replace the unknown parameters $\sigma_l^{(n)}$, $f_l^{(n)}$ and $\rho_l^{(n,m)}$ by their estimators, and we obtain the following estimators for the estimation error terms in (4.12):

$$\widehat{\mathrm{E}}\left[\left(\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[C_{i,j}^{(n)} \,\middle|\, \mathcal{D}_I\right]\right) \cdot \left(\widehat{C}_{i,j}^{(m)} - \mathrm{E}\left[C_{i,j}^{(m)} \,\middle|\, \mathcal{D}_I\right]\right) \,\middle|\, \mathcal{D}_I\right] = C_{i,I-i+1}^{(n)} \cdot C_{i,I-i+1}^{(m)} \cdot \Delta_{i,j}^{(n,m)} \tag{4.20}$$

with $\Delta_{i,j}^{(n,m)}$ given by (3.12). $\Delta_{i,j}^{(n,m)}$ can be rewritten in a recursive form:

$$\begin{aligned} \Delta_{i,j}^{(n,m)} = &\widehat{f}_{j-1}^{(n)} \cdot \widehat{f}_{j-1}^{(m)} \cdot \Delta_{i,j-1}^{(n,m)} + \frac{\widehat{\sigma}_{j-1}^{(n)}}{S_{j-1}^{(n)}} \cdot \frac{\widehat{\sigma}_{j-1}^{(m)}}{S_{j-1}^{(m)}} \cdot \sum_{i=1}^{I-j+1} \sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}} \cdot \widehat{\rho}_{j-1}^{(n,m)} \\ &\cdot \prod_{l=I-i+1}^{j-2} \left(\widehat{f}_l^{(n)} \cdot \widehat{f}_l^{(m)} + \frac{\widehat{\sigma}_l^{(n)}}{S_l^{(n)}} \cdot \frac{\widehat{\sigma}_l^{(m)}}{S_l^{(m)}} \cdot \sum_{i=1}^{I-l} \sqrt{C_{i,l}^{(n)} \cdot C_{i,l}^{(m)}} \cdot \widehat{\rho}_l^{(n,m)}\right) \end{aligned} \tag{4.21}$$

with $\Delta_{i,I-i+1}^{(n,m)} = 0$.

Note, that for $n = m$ this is exactly the term given in Buchwalder *et al.* (2006, formula (4.22)), which corresponds to the estimation error term in Murphy (1994), but is different from the estimation error term in Mack (1993).

Hence, from (4.12) and (4.21) we obtain the following recursion for the estimator of the conditional estimation error of N correlated triangles:

$$\widehat{\mathrm{E}}\left[\left(\sum_{n=1}^{N}\widehat{C}_{i,j}^{(n)} - \mathrm{E}\left[\sum_{n=1}^{N} C_{i,j}^{(n)}\,\middle|\,\mathcal{D}_I\right]\right)^2\,\middle|\,\mathcal{D}_I\right]$$

$$= \sum_{n=1}^{N}\left(C_{i,I-i+1}^{(n)}\right)^2 \cdot \Delta_{i,j}^{(n,n)} + 2\cdot \sum_{1\le n<m\le N} C_{i,I-i+1}^{(n)} \cdot C_{i,I-i+1}^{(m)} \cdot \Delta_{i,j}^{(n,m)}. \qquad (4.22)$$

From (4.7) and (4.22) we obtain a recursive estimator for the MSEP for a single accident year of N correlated run-off triangles.

Result 4.2. For N correlated run off triangles, we have the following estimator for the conditional MSEP for a single accident year:

$$\widehat{\mathrm{MSEP}}\left(\sum_{n=1}^{N}\widehat{C}_{i,j}^{(n)}\right) = \sum_{n=1}^{N}\widehat{\mathrm{MSEP}}\left(\widehat{C}_{i,j}^{(n)}\right)$$

$$+ 2\cdot \sum_{1\le n<m\le N}\left(\Gamma_{i,j}^{(n,m)} + C_{i,I-i+1}^{(n)} \cdot C_{i,I-i+1}^{(m)} \cdot \Delta_{i,j}^{(n,m)}\right) \qquad (4.23)$$

where:

$$\widehat{\mathrm{MSEP}}\left(\widehat{C}_{i,j}^{(n)}\right) = \Gamma_{i,j}^{(n,n)} + \left(C_{i,I-i+1}^{(n)}\right)^2 \cdot \Delta_{i,j}^{(n,n)}. \qquad (4.24)$$

4.2 *Conditional MSEP for Aggregated Accident Years*

Consider two different accident years $i < k$. From our assumptions, we know that the ultimate losses $C_{i,J}^{(n)}$ and $C_{k,J}^{(m)}$ are independent for all $n, m \in \{1, \ldots, N\}$. Nevertheless, we have to be careful if we aggregate $\widehat{C}_{i,J}^{(n)}$ and $\widehat{C}_{k,J}^{(m)}$. If $n = m$, the estimators are no longer independent, since they use the same observations for estimating the age-to-age factors $f_j^{(n)}$. If $n \neq m$, the estimators are also no longer independent, since they use the dependent estimators $\widehat{f}_j^{(n)}$ and $\widehat{f}_j^{(m)}$ for estimating the age-to-age factors $f_j^{(n)}$ and $f_j^{(m)}$, respectively.

Firstly, we consider the following conditional MSEP:

$$\mathrm{MSEP}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}+\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}\right)=\mathrm{Var}\left[\sum_{n=1}^{N}\left(C_{i,J}^{(n)}+C_{k,J}^{(n)}\right)\middle|\mathcal{D}_I\right]$$
$$+\left(\sum_{n=1}^{N}\left(\widehat{C}_{i,J}^{(n)}+\widehat{C}_{k,J}^{(n)}\right)-\mathrm{E}\left[\sum_{n=1}^{N}\left(C_{i,J}^{(n)}+C_{k,J}^{(n)}\right)\middle|\mathcal{D}_I\right]\right)^2. \quad (4.25)$$

Using the independence of the different accident years, the first term on the right-hand side of (4.25) can easily be decoupled:

$$\mathrm{Var}\left[\sum_{n=1}^{N}\left(C_{i,J}^{(n)}+C_{k,J}^{(n)}\right)\middle|\mathcal{D}_I\right]=\mathrm{Var}\left[\sum_{n=1}^{N}C_{i,J}^{(n)}\middle|\mathcal{D}_I\right]+\mathrm{Var}\left[\sum_{n=1}^{N}C_{k,J}^{(n)}\middle|\mathcal{D}_I\right]. \quad (4.26)$$

However, for the second term (the conditional estimation error in (4.25) we have to be more careful. We have the following decomposition:

$$\left(\sum_{n=1}^{N}\left(\widehat{C}_{i,J}^{(n)}+\widehat{C}_{k,J}^{(n)}\right)-\mathrm{E}\left[\sum_{n=1}^{N}\left(C_{i,J}^{(n)}+C_{k,J}^{(n)}\right)\middle|\mathcal{D}_I\right]\right)^2$$
$$=\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{i,J}^{(n)}\middle|\mathcal{D}_I\right]\right)^2+\left(\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{k,J}^{(n)}\middle|\mathcal{D}_I\right]\right)^2$$
$$+2\cdot\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{i,J}^{(n)}\middle|\mathcal{D}_I\right]\right)\cdot\left(\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{k,J}^{(n)}\middle|\mathcal{D}_I\right]\right). \quad (4.27)$$

For the cross-product in formula (4.27) we obtain:

$$\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{i,J}^{(n)}\middle|\mathcal{D}_I\right]\right)\cdot\left(\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}-\mathrm{E}\left[\sum_{n=1}^{N}C_{k,J}^{(n)}\middle|\mathcal{D}_I\right]\right)$$
$$=\sum_{1\le n,m\le N}\left(\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[C_{i,J}^{(n)}\middle|\mathcal{D}_I\right]\right)\cdot\left(\widehat{C}_{k,J}^{(m)}-\mathrm{E}\left[C_{k,J}^{(m)}\middle|\mathcal{D}_I\right]\right). \quad (4.28)$$

Hence, we have the following decomposition for the conditional MSEP of the aggregated ultimate loss of two run-off triangles:

$$
\begin{aligned}
&\mathrm{MSEP}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}+\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}\right)\\
&\quad=\mathrm{MSEP}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)+\mathrm{MSEP}\left(\sum_{n=1}^{N}\widehat{C}_{k,J}^{(n)}\right)\\
&\qquad+2\cdot\sum_{n=1}^{N}\left(\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[C_{i,J}^{(n)}\,\middle|\,\mathcal{D}_I\right]\right)\cdot\left(\widehat{C}_{k,J}^{(n)}-\mathrm{E}\left[C_{k,J}^{(n)}\,\middle|\,\mathcal{D}_I\right]\right)\\
&\qquad+4\cdot\sum_{1\le n<m\le N}\left(\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[C_{i,J}^{(n)}\,\middle|\,\mathcal{D}_I\right]\right)\cdot\left(\widehat{C}_{k,J}^{(m)}-\mathrm{E}\left[C_{k,J}^{(m)}\,\middle|\,\mathcal{D}_I\right]\right).
\end{aligned}
\tag{4.29}
$$

This means that, in addition to the conditional MSEP of single accident years, we need to find estimates for the last two terms in (4.29). For this we proceed as in Section 4.1.2. We obtain:

$$
\begin{aligned}
&\left(\widehat{C}_{i,J}^{(n)}-\mathrm{E}\left[C_{i,J}^{(n)}\,\middle|\,\mathcal{D}_I\right]\right)\cdot\left(\widehat{C}_{k,J}^{(m)}-\mathrm{E}\left[C_{k,J}^{(m)}\,\middle|\,\mathcal{D}_I\right]\right)\\
&\quad=C_{i,I-i+1}^{(n)}\cdot\left(\widehat{f}_{I-i+1}^{(n)}\cdot\ldots\cdot\widehat{f}_{J-1}^{(n)}-f_{I-i+1}^{(n)}\cdot\ldots\cdot f_{J-1}^{(n)}\right)\\
&\qquad\cdot C_{k,I-k+1}^{(m)}\cdot\left(\widehat{f}_{I-k+1}^{(m)}\cdot\ldots\cdot\widehat{f}_{J-1}^{(m)}-f_{I-k+1}^{(m)}\cdot\ldots\cdot f_{J-1}^{(m)}\right).
\end{aligned}
\tag{4.30}
$$

Completely analogously, as in Section 4.1.2, the right-hand side of (4.30) is now estimated by the $P(\cdot\,|\,\mathcal{D}_I)$ average when replacing $\widehat{f}_j^{(n)}$ by the resampled values $\widetilde{f}_j^{(n)}$. Hence, (4.30) is estimated by ($i<k$):

$$
\begin{aligned}
&\widehat{\mathrm{E}}\left[\left(\widehat{C}_{i,J}^{(n)}-\mathrm{E}[C_{i,J}^{(n)}\,|\,\mathcal{D}_I]\right)\cdot\left(\widehat{C}_{k,J}^{(m)}-\mathrm{E}[C_{k,J}^{(n)}\,|\,\mathcal{D}_I]\right)\,\middle|\,\mathcal{D}_I\right]\\
&\quad=C_{i,I-i+1}^{(n)}\cdot C_{k,I-k+1}^{(m)}\cdot f_{I-k+1}^{(m)}\cdot\ldots\cdot f_{I-i}^{(m)}\\
&\qquad\cdot\left(\prod_{l=I-i+1}^{J-1}\mathrm{E}[\widetilde{f}_l^{(n)}\cdot\widetilde{f}_l^{(m)}\,|\,\mathcal{D}_I]-\prod_{l=I-i+1}^{J-1}f_l^{(n)}\cdot f_l^{(m)}\right).
\end{aligned}
\tag{4.31}
$$

If we plug in the estimators similarly to (4.20), we obtain Result 4.3.

Result 4.3. For N correlated run-off triangles, we have the following estimator for the conditional MSEP of the ultimate loss of aggregated accident years:

$$\widehat{\mathrm{MSEP}}\left(\sum_{i=I+2-J}^{I}\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)=\sum_{i=I+2-J}^{I}\widehat{\mathrm{MSEP}}\left(\sum_{n=1}^{N}\widehat{C}_{i,J}^{(n)}\right)$$
$$+2\cdot\sum_{I+2-J\le i<k\le I}\;\sum_{1\le n,m\le N}C_{i,I-i+1}^{(n)}\cdot\widehat{C}_{k,I-i+1}^{(m)}\cdot\Delta_{i,J}^{(n,m)}. \quad (4.32)$$

5. Comparison with the Braun Formula

If we compare Braun's (2004) bivariate formulae for the MSEP with our results (4.23) and (4.32) for the MSEP in the special case $N=2$, we see the following differences:

(1) There is a small distinction between our result (4.4) of the process variance and Braun's (2004) formula. It comes from the fact that $\rho_{j-1}^{(n,m),B}$, considered in Braun (2004), is not a correlation (see (3.15)).

(2) There is a second difference between our formulae and Braun's result; namely, we observe a difference in the estimation of the estimation error. The difference is of the same nature as the one in the univariate case $N=1$ (compare Buchwalder *et al.*, 2006; and Mack, 1993).

The main difference comes from the fact that in Braun (2004), the recursive formulae for $\Delta_{i,j}^{(n,m)}$ uses a linear approximation which involves the following terms:

$$\widehat{\Delta}_{i,j}^{(n,m)}=\widehat{f}_{j-1}^{(n)}\cdot\widehat{f}_{j-1}^{(m)}\cdot\widehat{\Delta}_{i,j-1}^{(n,m)}$$
$$+\frac{\widehat{\sigma}_{j-1}^{(n)}}{S_{j-1}^{(n)}}\cdot\frac{\widehat{\sigma}_{j-1}^{(m)}}{S_{j-1}^{(m)}}\cdot\sum_{i=1}^{I-j+1}\sqrt{C_{i,j-1}^{(n)}\cdot C_{i,j-1}^{(m)}}\cdot\widehat{\rho}_{j-1}^{(n,m)}\cdot\prod_{l=I-i+1}^{j-2}\widehat{f}_{l}^{(n)}\cdot\widehat{f}_{l}^{(m)}. \quad (5.1)$$

Our conditional resampling approach leads to the terms (4.21):

$$\Delta_{i,j}^{(n,m)}=\widehat{f}_{j-1}^{(n)}\cdot\widehat{f}_{j-1}^{(m)}\cdot\Delta_{i,j-1}^{(n,m)}+\frac{\widehat{\sigma}_{j-1}^{(n)}}{S_{j-1}^{(n)}}\cdot\frac{\widehat{\sigma}_{j-1}^{(m)}}{S_{j-1}^{(m)}}\cdot\sum_{i=1}^{I-j+1}\sqrt{C_{i,j-1}^{(n)}\cdot C_{i,j-1}^{(m)}}\cdot\widehat{\rho}_{j-1}^{(n,m)}$$
$$\cdot\prod_{l=I-i+1}^{j-2}\left(\widehat{f}_{l}^{(n)}\cdot\widehat{f}_{l}^{(m)}+\frac{\widehat{\sigma}_{l}^{(n)}}{S_{l}^{(n)}}\cdot\frac{\widehat{\sigma}_{l}^{(m)}}{S_{l}^{(m)}}\cdot\sum_{i=1}^{I-l}\sqrt{C_{i,l}^{(n)}\cdot C_{i,l}^{(m)}}\cdot\widehat{\rho}_{l}^{(n,m)}\right). \quad (5.2)$$

Hence, our approach gives a slightly larger estimate compared to Braun's estimate. In fact, Braun's result is a linear approximation from below to our result (see also Buchwalder *et al.*, 2006, for a similar fact in the univariate case).

Recall that we have used the conditional independence assumptions (1) and (2) in Section 4.1.2. These conditional independence assumptions have allowed for an explicit calculation of the estimator for the conditional estimation error. Braun (2004) does not have this independence assumption, and, therefore, cannot calculate an estimator for the estimation error explicitly. Indeed, at some point in Braun (2004) one replaces:

$$E\left[\left(\widehat{f}_l^{(n)}\right)^2 \cdot \left(\widehat{f}_r^{(n)}\right)^2 \middle| \mathcal{D}_I\right] \quad \text{by} \quad \left(f_l^{(n)} \cdot f_r^{(n)}\right)^2 \tag{5.3}$$

whereas, in our model we calculate this term explicitly:

$$E\left[\left(\widetilde{f}_l^{(n)}\right)^2 \cdot \left(\widetilde{f}_r^{(n)}\right)^2 \middle| \mathcal{D}_I\right] = \left(\left(f_l^{(n)}\right)^2 + \frac{\left(\sigma_l^{(n)}\right)^2}{S_l^{(n)}}\right) \cdot \left(\left(f_r^{(n)}\right)^2 + \frac{\left(\sigma_r^{(n)}\right)^2}{S_r^{(n)}}\right) \tag{5.4}$$

see (4.18) ff. This is exactly the difference between our estimator and the estimator in Braun (2004). In the practical examples at which we have looked, the numerical differences between the two formulae are rather small (or even negligible).

6. Example

We apply our methods to the data considered by Braun (2004) for $C_{i,j}^{(n)}$ (c.f. Tables 1 and 2 in Braun, 2004). The index $n = 1$ denotes motor third party liability (MTPL) insurance data and $n = 2$ is general liability (GL) insurance data. The data for MTPL are given in Table 2 and the data for GL are given in Table 3.

Now we calculate the estimated claims reserves $\widehat{C}_{i,J}^{(n)}$, the MSEP formulae of Result 3.5 and the corresponding parameter estimators. Note that all these calculations can be done easily and explicitly without any simulation behind them. We have done our calculations in a spreadsheet.

The estimators $\widehat{f}_j^{(n)}$ for the chain ladder factors $f_j^{(n)}$, $j \leq 13$, are calculated with formula (3.2). The estimators $\widehat{\sigma}_j^{(n)}$ and $\widehat{\rho}_j^{(n,m)}$, for $j \leq 12$, are calculated with the help of formulae (3.17) and (3.20), respectively. Note that the estimation of the parameters $\sigma_{13}^{(n)}$ $(n = 1, 2)$ and $\rho_{13}^{(1,2)}$ cannot be done with the help of (3.17) and (3.20), respectively. This is due to the lack of data in the tails. We therefore simply apply the formula proposed by Mack (1993):

$$\widehat{\sigma}_{13}^{(n)} \stackrel{def.}{=} \min\left\{\left(\widehat{\sigma}_{12}^{(n)}\right)^2 / \widehat{\sigma}_{11}^{(n)}, \widehat{\sigma}_{11}^{(n)}, \widehat{\sigma}_{12}^{(n)}\right\} \tag{6.1}$$

and

Table 2. MTPL run-off triangle (cumulative payments $C_{i,j}^{(1)}$)

Motor third party liability run-off triangle

AY/DY	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	114,423	247,961	312,982	344,340	371,479	371,102	380,991	385,468	385,152	392,260	391,225	391,328	391,537	391,428
2	152,296	305,175	376,613	418,299	440,308	465,623	473,584	478,427	478,314	479,907	480,755	485,138	483,974	
3	144,325	307,244	413,609	464,041	519,265	527,216	535,450	536,859	538,920	539,589	539,765	540,742		
4	145,904	307,636	387,094	433,736	463,120	478,931	482,529	488,056	485,572	486,034	485,016			
5	170,333	341,501	434,102	470,329	482,201	500,961	504,141	507,679	508,627	507,752				
6	189,643	361,123	446,857	508,083	526,562	540,118	547,641	549,605	549,693					
7	179,022	396,224	497,304	553,487	581,849	611,640	622,884	635,452						
8	205,908	416,047	520,444	565,721	600,609	630,802	648,365							
9	210,951	426,429	525,047	587,893	640,328	663,152								
10	213,426	509,222	649,433	731,692	790,901									
11	249,508	580,010	722,136	844,159										
12	258,425	686,012	915,109											
13	368,762	909,066												
14	394,997													

Source: Braun (2004)

Table 3. GL run-off triangle (cumulative payments $C^{(2)}_{i,j}$)

General liability run-off triangle

AY/DY	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	59,966	163,152	254,512	349,524	433,265	475,778	513,660	520,309	527,978	539,039	537,301	540,873	547,696	549,589
2	49,685	153,344	272,936	383,349	458,791	503,358	532,615	551,437	555,792	556,671	560,844	563,571	562,795	
3	51,914	170,048	319,204	425,029	503,999	544,769	559,475	577,425	588,342	590,985	601,296	602,710		
4	84,937	273,183	407,318	547,288	621,738	687,139	736,304	757,440	758,036	782,084	784,632			
5	98,921	278,329	448,530	561,691	641,332	721,696	742,110	752,434	768,638	768,373				
6	71,708	245,587	416,882	560,958	654,652	726,813	768,358	793,603	811,100					
7	92,350	285,507	466,214	620,030	741,226	827,979	873,526	896,728						
8	95,731	313,144	553,702	755,978	857,859	962,825	1,022,241							
9	97,518	343,218	575,441	769,017	934,103	1,019,303								
10	173,686	459,416	722,336	955,335	1,141,750									
11	139,821	436,958	809,926	1,174,196										
12	154,965	528,080	1,032,684											
13	196,124	772,971												
14	204,325													

Source: Braun (2004)

$$\widehat{\rho}_{13}^{(1,2)} \stackrel{def.}{=} \frac{\min\left\{\frac{\left(\widehat{\rho}_{12}^{(1,2)}\cdot\widehat{\sigma}_{12}^{(1)}\cdot\widehat{\sigma}_{12}^{(2)}\right)^2}{\left|\widehat{\rho}_{11}^{(1,2)}\cdot\widehat{\sigma}_{11}^{(1)}\cdot\widehat{\sigma}_{11}^{(2)}\right|}, \left|\widehat{\rho}_{12}^{(1,2)}\cdot\widehat{\sigma}_{12}^{(1)}\cdot\widehat{\sigma}_{12}^{(2)}\right|, \left|\widehat{\rho}_{11}^{(1,2)}\cdot\widehat{\sigma}_{11}^{(1)}\cdot\widehat{\sigma}_{11}^{(2)}\right|\right\}}{\widehat{\sigma}_{13}^{(1)}\cdot\widehat{\sigma}_{13}^{(2)}}. \tag{6.2}$$

For our purposes, i.e. to demonstrate the methods, these estimators are sufficient. However, in practice it is a serious issue choosing appropriate tail parameters, which needs a deep discussion which we do not want to enter here. Using our parameter estimators, we obtain the numerical values shown in Table 4.

Using the estimators $\widehat{f}_j^{(n)}$, $\widehat{\sigma}_j^{(n)}$ and $\widehat{\rho}_j^{(1,2)}$ $(n = 1, 2)$, we find the predictors for the ultimate claims $\widehat{C}_{i,J}^{(n)}$ (see Lemma 3.3) and the corresponding claims reserves defined by:

$$\widehat{R} = \sum_{i=2}^{14} \widehat{R}_i = \sum_{i=2}^{14} \left[\left(\widehat{C}_{i,14}^{(1)} - C_{i,14-i+1}^{(1)}\right) + \left(\widehat{C}_{i,14}^{(2)} - C_{i,14-i+1}^{(2)}\right)\right] \tag{6.3}$$

as well as the estimators for the conditional MSEP given in Result 3.5 (see Table 5).

Table 5 shows the estimated conditional process standard deviation (conditional estimation error)$^{1/2}$, the estimator for the conditional MSEP and the conditional standard error of prediction for the aggregated ultimate loss over all the different accident years.

We see that the results from Braun's (2004) formula and our formula presented in Result 3.5 are nearly the same. However, the fact that Braun's formula uses a linear approximation for the estimation error (see (5.3)), and that our method considers higher order terms according to (5.4), leads to slightly lower results in Braun's method. This is also confirmed by the findings in Buchwalder *et al.* (2006).

In the two last columns of Table 5, the results are given for: (1) the independent aggregation of the estimates for the two subportfolios; and (2) their aggregation, assuming perfect correlation between the two subportfolios. We see that these (rough) calculations lead to a prediction standard error which is about 52,000 lower and 81,000 higher, respectively, than the one taking the estimated correlation between the two subportfolios into account. Furthermore, note that using (4.10) leads to an estimate of 397,065 for the upper bound on the process standard deviation (c.f. Remarks 4.1, second remark). This result is only slightly higher than the estimate of 397,054 in Table 5.

Table 4. Parameter estimates for the two run-off triangles

	Development years												
	1	2	3	4	5	6	7	8	9	10	11	12	13
$\widehat{f}_j^{(1)}$	3.235	1.720	1.354	1.179	1.106	1.055	1.026	1.014	1.012	1.006	1.005	1.005	1.003
$\widehat{f}_j^{(2)}$	2.226	1.269	1.120	1.067	1.035	1.017	1.010	1.000	1.004	0.999	1.004	0.999	1.000
$\left(\widehat{\sigma}_j^{(1)}\right)^2$	17,642.53	7,027.84	1,432.51	685.21	144.32	209.99	50.81	52.03	136.96	43.45	2.66	54.03	2.66
$\left(\widehat{\sigma}_j^{(2)}\right)^2$	11,104.38	607.07	321.80	363.48	156.37	30.81	20.41	4.52	26.45	1.95	10.31	1.86	0.34
$\widehat{\rho}_j^{(1,2)}$	0.245	0.495	0.682	0.446	0.487	0.451	−0.172	0.802	0.337	0.687	−0.004	1.001	0.021

Table 5. Results for the whole portfolio consisting of the MTPL and GL subportfolios for aggregated accident years

	Braun (2004)	Result 3.5	Result 3.5 — Braun (2004)	Correlation = 0	Correlation = 1
Estimated reserves $\widehat{R}$	8,218,874	8,218,874	0	8,218,874	8,218,874
Process standard deviation	397,054	397,054	0	356,872	465,161
$\sqrt{\text{Estimation error}}$	318,807	318,841	34	285,946	362,477
Conditional $\widehat{\text{MSEP}}$	259,289,671,264	259,311,486,653	21,815,389	209,122,999,132	348,318,938,709
Prediction standard error	509,205	509,226	21	457,300	590,186

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APPENDIX

In this appendix we prove the unbiasedness of $\widetilde{\rho}_{j-1}^{(n,m)}$ defined in formula (3.18).

Proof of the unbiasedness of $\widetilde{\rho}_{j-1}^{(n,m)}$.
We define $\mathcal{B}_j^{(n)} = \left\{C_{1,j}^{(n)}, \ldots, C_{I-j+1,j}^{(n)}\right\}$. Note that, for $i \leq I-j+1$:

$$f_{j-1}^{(n)} = \mathrm{E}\left[\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} \,\middle|\, \mathcal{B}_{j-1}^{(n)}\right] = \mathrm{E}\left[\widehat{f}_{j-1}^{(n)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}\right]. \tag{A.1}$$

This implies that:

$$\begin{aligned} &\mathrm{E}\left[\widetilde{\rho}_{j-1}^{(n,m)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right] \\ &\quad = c_j \cdot \sum_{i=1}^{I-j+1} \frac{\sqrt{C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}}}{\sigma_{j-1}^{(n)} \cdot \sigma_{j-1}^{(m)}} \cdot \mathrm{Cov}\left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} - \widehat{f}_{j-1}^{(n)}, \frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} - \widehat{f}_{j-1}^{(m)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right). \end{aligned} \tag{A.2}$$

For the covariance we have:

$$\begin{aligned} &\mathrm{Cov}\left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}} - \widehat{f}_{j-1}^{(n)}, \frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} - \widehat{f}_{j-1}^{(m)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right) \\ &\quad = \mathrm{Cov}\left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}}, \frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right) - \mathrm{Cov}\left(\widehat{f}_{j-1}^{(n)}, \frac{C_{i,j}^{(m)}}{C_{i,j-1}^{(m)}} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right) \\ &\qquad - \mathrm{Cov}\left(\frac{C_{i,j}^{(n)}}{C_{i,j-1}^{(n)}}, \widehat{f}_{j-1}^{(m)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right) + \mathrm{Cov}\left(\widehat{f}_{j-1}^{(n)}, \widehat{f}_{j-1}^{(m)} \,\middle|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\right) \\ &\quad = \rho_{j-1}^{(n,m)} \sigma_{j-1}^{(n)} \sigma_{j-1}^{(m)} \left(C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}\right)^{-1/2} \left(1 - \frac{C_{i,j-1}^{(n)}}{S_{j-1}^{(n)}} - \frac{C_{i,j-1}^{(m)}}{S_{j-1}^{(m)}}\right) \\ &\qquad + \rho_{j-1}^{(n,m)} \sigma_{j-1}^{(n)} \sigma_{j-1}^{(m)} \frac{\sum_{i=1}^{I-j+1} \left(C_{i,j-1}^{(n)} (m)_{i,j-1}\right)^{1/2}}{S_{j-1}^{(n)} \cdot S_{j-1}^{(m)}}. \end{aligned} \tag{A.3}$$

Henceforth:

$$\begin{aligned}
&\mathrm{E}\big[\widetilde{\rho}_{j-1}^{(n,m)} \,\big|\, \mathcal{B}_{j-1}^{(n)}, \mathcal{B}_{j-1}^{(m)}\big] \\
&\quad = c_j \cdot \sum_{i=1}^{I-j+1} \rho_{j-1}^{(n,m)} \left(1 - \frac{C_{i,j-1}^{(n)}}{S_{j-1}^{(n)}} - \frac{C_{i,j-1}^{(m)}}{S_{j-1}^{(m)}}\right) + c_j \cdot \rho_{j-1}^{(n,m)} \frac{\left(\sum_{i=1}^{I-j+1} \left(C_{i,j-1}^{(n)} \cdot C_{i,j-1}^{(m)}\right)^{1/2}\right)^2}{S_{j-1}^{(n)} \cdot S_{j-1}^{(m)}} \\
&\quad = c_j \cdot \rho_{j-1}^{(n,m)} \cdot \left(I - j + 1 - 2 + w_j^2\right) = \rho_{j-1}^{(n,m)}. \qquad \text{(A.4)}
\end{aligned}$$

This completes the proof.