

Even powers of divisors and elliptic zeta values

Dedicated to Igor Frenkel on the occasion of his 50th birthday

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Abstract. We introduce and study *elliptic zeta values*, a two-parameter deformation of the values of Riemann's zeta function at positive integers. They are essentially Taylor coefficients of the logarithm of the elliptic gamma function, and inherit the functional equations of this function. Elliptic zeta values at even integers are related to Eisenstein series and thus to sums of odd powers of divisors. The elliptic zeta values at odd integers can be expressed in terms of generating series of sums of even powers of divisors.

1. Introduction

Let k be a positive integer. The generating function of the sum of k -1st powers of divisors of positive integers

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

converges in $|q| < 1$. If $k - 1$ is odd, it has interesting transformation properties under the modular group $\mathrm{SL}(2, \mathbb{Z})$. Indeed, it is closely related to the Eisenstein series

$$G_k(\tau) = \frac{1}{2} \sum_{(m,n) \neq (0,0)} (m\tau + n)^{-k}.$$

Let

$$D_k(q) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

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Then

$$G_k(\tau) = \zeta(k) + D_k(q), \quad q = e^{2\pi i\tau}, \quad k = 2, 4, 6, \dots,$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. For $k \geq 4$, these functions are modular forms of weight k :

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k G_k(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

See [3] for proofs of these facts, but notice that there G_k denotes in our notation $G_{2k}/2$. As $G_k(-1/\tau) = \tau^k G_k(\tau)$, it follows that, if k is even,

$$\lim_{\tau \rightarrow 0} \tau^k D_k(e^{2\pi i\tau}) = \zeta(k).$$

In other words, we may regard $\tau^k D_k(e^{2\pi i\tau})$ as a one-parameter deformation of the zeta value $\zeta(k)$, and this is also true if k is odd. For odd k , however, $D_k(q)$ does not have obvious modular properties. The purpose of this note is to show that $D_k(q)$ can be embedded into a two-parameter deformation of $\zeta(k)$, the *elliptic zeta value* at k , which obeys identities of modular type. These identities are essentially equivalent to modularity in the even case but are of a different nature in the odd case. They are related to (and a consequence of) the three term relations of the elliptic gamma function [1].

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2. Differences of modular forms

Let H be the upper half-plane $\mathrm{Im} \tau > 0$.

Proposition 2.1. *Let k be an even positive integer. Suppose $Z(\tau, \sigma)$ is a holomorphic function on $H \times H$ admitting an expansion*

$$Z(\tau, \sigma) = \sum_{n,m=0}^{\infty} a_{n,m} q^n r^m, \quad q = e^{2\pi i\tau}, \quad r = e^{2\pi i\sigma},$$

and obeying $Z(\tau, \sigma) = -Z(\sigma, \tau)$. Then the following statements are equivalent:

- (i) $Z(\tau, \sigma) = G(\tau) - G(\sigma)$ for some modular form G of weight k .
- (ii) Z obeys the three-term relations

$$\begin{aligned} Z(\tau, \sigma) &= Z(\tau, \tau + \sigma) + Z(\tau + \sigma, \sigma), \\ Z(\tau, \sigma) &= \tau^{-k} Z\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + \sigma^{-k} Z\left(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right), \end{aligned}$$

for all $\sigma, \tau \in H$ such that $\sigma/\tau \in H$.

Proof. It is easy to check that if G is a modular form then $G(\tau) - G(\sigma)$ obeys the three-term relations. Conversely, let us extend $a_{n,m}$ to all integers n, m by setting $a_{n,m} = 0$ if n or m is negative. The first three-term relation implies that $a_{n-m,m} + a_{n,m-n} = a_{n,m}$. It follows that $a_{n,m} = a_{n-m,m}$ if $n > m$ and $a_{n,m} = a_{n,m-n}$ if $m > n$. By the Euclidean algorithm we see that for $n, m > 0$, $a_{n,m} = a_{N,N}$, where $N = (n, m)$ is the greatest common divisor. But $a_{N,N} = 0$ since Z is odd under interchange of q and r . Thus only $a_{n,0} = a_{0,m}$ may be non-zero and $Z(\tau, \sigma) = g(\tau) - g(\sigma)$ with $g(\tau) = \sum_{n=1}^{\infty} a_{n,0} q^n$. In particular, $g(\tau + 1) = g(\tau)$. The second three-term relation can then be written as $h(\tau) = h(\sigma) - h(\sigma/\tau)\tau^{-k}$, where $h(\tau) = g(\tau) - \tau^{-k}g(-1/\tau)$ is holomorphic on the upper half plane. If we take the second partial derivative of this identity with respect to τ and σ , we obtain

$$zh''(z) + (k+1)h'(z) = 0,$$

with $z = \sigma/\tau$. Thus h' is homogeneous of degree $-k-1$ and $h(z) = a\tau^{-k} + b$ for some a, b . Inserting back in the identity for h shows that $b = -a$. Then $G(\tau) = g(\tau) + a$ obeys $G(\tau + 1) = G(\tau)$ and $G(-1/\tau) = \tau^k G(\tau)$ and is thus a modular form of weight k . \square

3. Elliptic zeta values

We now define two-parameter deformations of the values of the zeta function at positive integers, which we call *elliptic zeta values*:

$$(1) \quad Z_k(\tau, \sigma) = -\frac{(2\pi i)^k}{(k-1)!} \sum_{j=1}^{\infty} j^{k-1} \frac{q^j - (-1)^k r^j}{(1-q^j)(1-r^j)},$$

$$q = e^{2\pi i \tau}, \quad r = e^{2\pi i \sigma}, \quad k \in \mathbb{Z}_{\geq 1}.$$

Clearly, $Z_k(\tau, \sigma) = -(-1)^k Z_k(\sigma, \tau)$. The relation to the functions D_k is obtained by expanding the elliptic zeta values in a power series in q and r . The result is the following.

If k is even, then

$$Z_k(\tau, \sigma) = D_k(r) - D_k(q).$$

If k is odd, then

$$Z_k(\tau, \sigma) = D_k(q) + D_k(r) + 2 \sum_{(a,b)=1} D_k(q^a r^b).$$

The sum is over pairs of relatively prime pairs of positive integers a, b .

Theorem 3.1. (i) Let $k \geq 4$. Then Z_k obeys the three-term relations

$$Z_k(\tau, \sigma) = Z_k(\tau, \tau + \sigma) + Z_k(\tau + \sigma, \sigma),$$

$$Z_k(\tau, \sigma) = \tau^{-k} Z_k\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + (-\sigma)^{-k} Z_k\left(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right),$$

for all $\sigma, \tau \in H$ such that $\sigma/\tau \in H$.

(ii) $\lim_{\sigma \rightarrow 0} \lim_{\tau \rightarrow i\infty} \sigma^k Z_k(\tau, \sigma) = \zeta(k)$ if $k \geq 2$.

(iii) For $k = 1$, $\lim_{\sigma \rightarrow 0} \lim_{\tau \rightarrow i\infty} (\sigma Z_1(\tau, \sigma) + \ln(-2\pi i \sigma))$ is the Euler constant $\gamma = 0.577 \dots$ (the logarithm is real if σ is imaginary).

In particular, by taking the limit of Z_k as $\tau \rightarrow i\infty$ we obtain

$$\lim_{\sigma \rightarrow 0} (\sigma D_1(e^{2\pi i \sigma}) + \ln(-2\pi i \sigma)) = \gamma,$$

$$\lim_{\sigma \rightarrow 0} \sigma^k D_k(e^{2\pi i \sigma}) = \zeta(k), \quad k = 2, 3 \dots$$

We prove Theorem 3.1 in the next section.

Remark on $\mathrm{SL}(3, \mathbb{Z})$. The three-term relations in Theorem 3.1 have an interpretation as $\mathrm{SL}(3, \mathbb{Z})$ 1-cocycle properties parallel to the ones discussed in [1] for the elliptic gamma function. We only sketch the construction here as the details are the same as in the case of the elliptic gamma function [1]. First of all, one notices that the three-term relations, the symmetry $\tilde{Z}_k(\tau, \sigma) = -(-1)^k Z_k(\sigma, \tau)$ and the periodicity $Z_k(\tau + 1, \sigma) = Z_k(\tau, \sigma)$ continue to hold if we extend the domain of Z_k to $\mathrm{Im} \tau \neq 0, \mathrm{Im} \sigma \neq 0$, by setting

$$Z_k(-\tau, \sigma) = Z_k(\tau, -\sigma) = (-1)^k Z_k(\tau, \sigma).$$

We pass to homogeneous coordinates and set $\tilde{Z}_k(x_1, x_2, x_3) = x_3^{-k} Z_k(x_1/x_3, x_2/x_3)$. This function is homogeneous of degree $-k$ and holomorphic on the dense open set of \mathbb{C}^3 defined by the conditions $\mathrm{Im} x_i/x_3 \neq 0$ ($i = 1, 2$). Our identities relate values of \tilde{Z}_k at points related by the $\mathrm{SL}(3, \mathbb{Z})$ -action on \mathbb{C}^3 : if $k \geq 4$,

$$\tilde{Z}_k(x_1, x_2, x_3) = \tilde{Z}_k(x_1 + x_3, x_2, x_3) = \tilde{Z}_k(x_1, x_2 + x_3, x_3) = -\tilde{Z}_k(x_2, x_1, -x_3),$$

$$\tilde{Z}_k(x_1, x_2, x_3) = \tilde{Z}_k(x_1, x_1 + x_2, x_3) + \tilde{Z}_k(x_1 + x_2, x_2, x_3),$$

$$\tilde{Z}_k(x_1, x_2, x_3) = \tilde{Z}_k(-x_3, x_2, x_1) + \tilde{Z}_k(x_1, x_3, -x_2).$$

Let e_{ij} ($i \neq j$) be the elementary matrices in $\mathrm{SL}(3, \mathbb{Z})$, with 1 in the diagonal and at the position (i, j) , and zero elsewhere. These matrices generate $\mathrm{SL}(3, \mathbb{Z})$. A consequence of the identities is that if we set

$$\phi_{e_{12}}(x) = \tilde{Z}_k(x_1 - x_2, x_1, x_3), \quad \phi_{e_{32}}(x) = \tilde{Z}_k(x_2 - x_3, x_3, x_1), \quad \phi_{e_{ij}}(x) = 1, \quad j \neq 2,$$

($k \geq 4$) then ϕ extends (uniquely) to a 1-cocycle $(\phi_g)_{g \in \mathrm{SL}(3, \mathbb{Z})}$ of $\mathrm{SL}(3, \mathbb{Z})$ with values in the space of holomorphic functions with open dense domain in \mathbb{C}^3 , i.e., one has $\phi_{gh}(x) = \phi_g(x) + \phi_h(g^{-1}x)$ for all $g, h \in \mathrm{SL}(3, \mathbb{Z})$ and x in a dense open set.

4. The elliptic gamma function

Theorem 3.1 follows from the results of [1] where the properties of modular type for Ruijsenaars's elliptic gamma function [3] were discovered. In the normalization of [1], the elliptic gamma function is defined by the double infinite product

$$\Gamma(z, \tau, \sigma) = \prod_{j, \ell=0}^{\infty} \frac{1 - q^{j+1} r^{\ell+1} e^{-2\pi i z}}{1 - q^j r^{\ell} e^{2\pi i z}}, \quad z \in \mathbb{C}, \tau, \sigma \in H.$$

It obeys the functional relation $\Gamma(z + \sigma, \tau, \sigma) = \theta_0(z, \tau) \Gamma(z, \tau, \sigma)$, which is an elliptic version of Euler's functional equation for the gamma function. Here θ_0 denotes the modified theta function

$$\theta_0(z, \tau) = \prod_{j=0}^{\infty} (1 - q^{j+1} e^{-2\pi i z})(1 - q^j e^{2\pi i z}), \quad z \in \mathbb{C}, \tau \in H.$$

The Euler gamma function $\Gamma(z)$ is recovered in the limit

$$(2) \quad \Gamma(z) = \lim_{\sigma \rightarrow 0} \lim_{\tau \rightarrow i\infty} \theta_0(\sigma, \tau)^{1-z} \frac{\Gamma(\sigma z, \tau, \sigma)}{\Gamma(\sigma, \tau, \sigma)}.$$

Theorem 4.1 ([1], p. 54). *Suppose $\tau, \sigma, \sigma/\tau \in H$. Then*

$$\begin{aligned} \Gamma(z, \tau, \sigma) &= \Gamma(z + \tau, \tau, \sigma + \tau) \Gamma(z, \tau + \sigma, \sigma), \\ \Gamma(z/\tau, -1/\tau, \sigma/\tau) &= e^{i\pi Q(z; \tau, \sigma)} \Gamma((z - \tau)/\sigma, -\tau/\sigma, -1/\sigma) \Gamma(z, \tau, \sigma), \end{aligned}$$

for some polynomial $Q(z; \tau, \sigma)$ of degree three in z with coefficients in $\mathbb{Q}(\tau, \sigma)$.

It follows from the Weierstrass product representation

$$\Gamma(z + 1) = e^{-\gamma z} \prod_{j=1}^{\infty} (1 + z/j)^{-1} e^{jz}$$

that the values of ζ at positive integers are essentially Taylor coefficients at 1 of the logarithm of the Euler gamma function:

$$(3) \quad \ln \Gamma(z + 1) = -\gamma z + \sum_{j=2}^{\infty} \frac{\zeta(j)}{j} (-z)^j.$$

Here $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{j=1}^{n-1} 1/j - \ln n \right)$ is the Euler constant. The elliptic analog of this formula involves the elliptic zeta values:

$$(4) \quad \ln \frac{\Gamma(z + \sigma, \tau, \sigma)}{\Gamma(\sigma, \tau, \sigma)} = \sum_{j=1}^{\infty} \frac{Z_j(\tau, \sigma)}{j} (-z)^j.$$

This formula, with Z_j defined by eq. (1), is an easy consequence of the summation formula for $\ln \Gamma(z, \tau, \sigma)$ ([1], p. 51):

$$\ln \Gamma(z, \tau, \sigma) = -\frac{i}{2} \sum_{j=1}^{\infty} \frac{\sin(\pi j(2z - \tau - \sigma))}{j \sin(\pi j \tau) \sin(\pi j \sigma)}.$$

Proof of Theorem 3.1. The first claim of the theorem is proved by taking the logarithm of the identities of Theorem 4.1 and expanding them in powers of $z - \sigma$.

Taking the limit of (4), by using (2) to compare it with (3) implies (ii).

In the same way we can deduce (iii). However, if $n = 1$, we have to take into account the factor θ_0^{1-z} in (2): we obtain $\gamma = \lim_{\sigma \rightarrow 0} \lim_{\tau \rightarrow \infty} (\sigma Z_1(\tau, \sigma) + \ln \theta_0(\sigma, \tau))$. Since $\ln \theta_0(\sigma, \tau) = \ln(-2\pi i \sigma)$ plus terms that vanish in the limit, the proof is complete. \square

By using the explicit formula for the polynomial Q (see [1]), we also obtain the exceptional relations for $k = 1, 2, 3$:

Theorem 4.2. *Let $1 \leq k \leq 3$ and suppose $\tau, \sigma, \sigma/\tau \in H$. Then*

$$\begin{aligned} Z_k(\tau, \sigma) &= Z_k(\tau, \tau + \sigma) + Z_k(\tau + \sigma, \sigma), \\ Z_k(\tau, \sigma) &= \tau^{-k} Z_k\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + (-\sigma)^{-k} Z_k\left(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right) + i\pi a_k, \end{aligned}$$

where

$$a_1 = -\frac{1}{2} + \frac{1}{2\tau} - \frac{1}{2\sigma} + \frac{\sigma}{6\tau} + \frac{\tau}{6\sigma} + \frac{1}{6\tau\sigma}, \quad a_2 = -\frac{1}{\tau} + \frac{1}{\sigma} - \frac{1}{\tau\sigma}, \quad a_3 = \frac{1}{\tau\sigma}.$$

5. A direct proof of the three-term relations of Theorem 3.1

We thank Don Zagier for providing the following alternative direct proof of the three term relations of Theorem 3.1, which does not use the elliptic gamma function. As the case where k is even follows easily from the modular properties of the Eisenstein series, it is sufficient to consider the case where k is odd. The proof is based on an alternative formula for $Z_k(\tau, \sigma)$ for $k \geq 5$ odd, which could be seen as a more natural definition of the elliptic odd zeta values. Let $\varepsilon(n)$ be the sign of n ($\varepsilon(n) = 1, 0, -1$ if $n > 0, n = 0, n < 0$ respectively) and set $\varepsilon(a, b) = \frac{1}{2}(\varepsilon(a) + \varepsilon(b))$. Then

$$(5) \quad Z_k(\tau, \sigma) = \sum'_{a,b,c} \frac{\varepsilon(a, b)}{(a\tau + b\sigma + c)^k} \quad (k \text{ odd } \geq 5),$$

where \sum' means a sum over all $(a, b, c) \neq (0, 0, 0)$. To deduce this formula, rewrite (1) as

$$Z_k(\tau, \sigma) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell, m=0}^{\infty} \sum_{j=1}^{\infty} j^{k-1} (q^j + r^j) q^{j\ell} r^{jm}, \quad q = e^{2\pi i \tau}, \quad r = e^{2\pi i \sigma}, \quad k \text{ odd},$$

then use the Lipschitz formula $\sum_{n \in \mathbb{Z}} (\rho + n)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{j=1}^{\infty} j^{k-1} e^{2\pi i \rho j}$ ($\rho \in H, k \geq 2$).

By using (5) it is easy to prove Theorem 3.1 (i): the three-term relations follow from the obvious identities $\varepsilon(a-b, b) + \varepsilon(a, b-a) = \varepsilon(a, b)$, $\varepsilon(a, b) = \varepsilon(-c, b) + \varepsilon(a, c)$.

It should be possible to prove also the identities in the exceptional cases where $k = 1, 3$ using this method, but suitable summation procedures should be applied to make sense of the series (5), which are not absolutely convergent.

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