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The Nontrivial Zeros of Period Polynomials of Modular Forms Lie on the Unit Circle

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We show that all but five of the zeros of the period polynomial associated to a Hecke cusp form are on the unit circle.

1 Introduction

Let $\mathcal{M}_k(\Gamma)$ be the space of holomorphic modular forms of weight k for the full modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. It is well known that $\mathcal{M}_k(\Gamma)$ has dimension $\frac{k}{12} + O(1)$ and a modular form $f \in \mathcal{M}_k$ has $\frac{k}{12} + O(1)$ inequivalent zeros in a fundamental domain $\Gamma \setminus \mathcal{H}$. The study of the natural question of the distribution of the zeros of modular forms dates back to the 1960s and has seen some renewed interest thanks to the recent progress on the quantum unique ergodicity (QUE) conjecture.

In the simplest case of Eisenstein series, it was conjectured by Rankin in 1968 and proved by Rankin and Swinnerton-Dyer [12] that all the zeros, in the standard fundamental domain, of the series

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k}$$

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lie on the geodesic arc $\{z \in \mathcal{H} : |z| = 1, 0 \leq \Re z \leq 1/2\}$ and as $k \to \infty$ they become uniformly distributed on this unit arc. A similar result for the cuspidal Poincare series was proved by Rankin [13]. For generalizations of these results to other Fuchsian groups and to weakly holomorphic modular functions, see [1, 3, 7], among many others.

In contrast to these cases, for the cuspidal Hecke eigenforms, it is a consequence of the recent proof of the holomorphic QUE by Holowinsky and Soundararajan [8] that the zeros are uniformly distributed. More precisely, we have the following theorem:

Theorem (Holowinsky and Soundararajan [8]). Let $\{f_k\}$ be a sequence of cuspidal Hecke eigenforms of weight k. Then as $k \to \infty$ the zeros of f_k become equidistributed with respect to the normalized hyperbolic measure $\frac{3}{\pi}\frac{dx\,dy}{y^2}$. □

For some recent work on the zeros of holomorphic Hecke cusp forms that lie on the geodesic segments of the standard fundamental domain, see [5].

In this note, we turn our attention from the zeros of modular forms to the zeros of their period polynomials.

It is well known that Γ is generated by the elliptic transformations $S = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $U = \pm\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with the defining relations $S^2 = U^3 = \pm I$.

Let P_{k-2} be the space of all complex polynomials of degree at most $k-2$. For $p(z) \in P_{k-2}$, $A \in \mathrm{PSL}(2, \mathbb{C})$ acts on $p(z)$ in the usual way via

$$(p|A)(z) := (cz+d)^{k-2}\, p\left(\frac{az+b}{cz+d}\right).$$

Let P^-_{k-2} be the space of odd polynomials of degree $k-2$ and

$$W^- = W^-_{k-2} = \{p \in P^-_{k-2};\ p|(I+S) = p|(I+U+U^2) = 0\}.$$

For $f(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z}$ a Hecke eigenform of even integral weight $k = w+2$ and level 1, let $L(f,s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ be its associated L-function. The odd period polynomial for f is defined by

$$r^-_f(X) := \sum_{\substack{n=1 \\ n\,\mathrm{odd}}}^{w-1} (-1)^{\frac{n-1}{2}} \binom{w}{n} n!\,(2\pi)^{-n-1} L(f, n+1) X^{w-n}. \tag{1.1}$$

The basic result of Eichler–Shimura theory is:

Theorem (Eichler–Shimura). Let $\mathcal{S}_k(\Gamma)$ be the space of cusp forms for Γ. Then the map

$$r^- : \mathcal{S}_k(\Gamma) \to W^-$$
$$f \to r_f^-(X)$$

is an isomorphism. □

In the light of the theorem of Eichler and Shimura, studying the zeros of period polynomials is as natural as studying the zeros of modular forms. In this paper, we prove the following:

Theorem 1.1. If f is a Hecke eigenform, then the odd period polynomial $r_f^-(X)$ has simple zeros at 0, ± 2, and $\pm 1/2$ and double zeros at ± 1. The rest of its zeros are complex numbers on the unit circle. □

Figure 1 illustrates Theorem 1.1 in the case of f a cusp form of weight $w = 34$. Note that in this example the spacing between zeros is quite regular. From the proof of Theorem 1.1 it will become clear that this is a general phenomenon. It is worth noting that for an arbitrary cusp form which is not a Hecke eigenform the zeros of $r_f^-(X)$ need not be on the unit circle. This can be thought as analogous to the fact that for a general modular form f which is not a Hecke eigenform, the zeros of f need not be uniformly distributed.

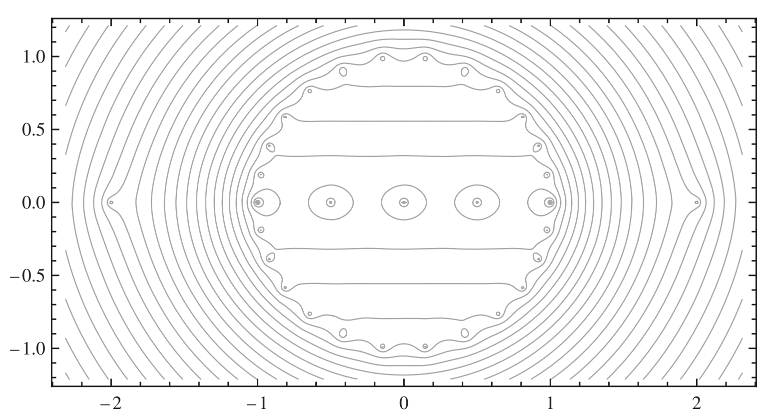

Fig. 1. A contour plot of $\log |r_f^-(z)|$ for f one of the cusp forms of weight $w = 34$, illustrating the zeros at ± 2, $\pm\frac{1}{2}$, and 0, with the remaining zeros on the unit circle.

In the case of zeros of modular forms, the uniform distribution result is a remarkable consequence of the deep QUE conjecture, which is now a theorem due to Holowinsky and Soundararajan for holomorphic eigenforms. The fact that the uniform distribution of the zeros of modular forms follows from the QUE conjecture was first observed by Nonnenmacher and Voros [11], Shiffman and Zelditch [15], and Rudnick [14].

In the case of the zeros of period polynomials, as we will show in the next section, Theorem 1.1 follows using simple function theory arguments together with the deep theorem of Deligne which is the Ramanujan–Petersson conjecture in the case of holomorphic cuspforms.

Finally, it is worth noting that the proof of our Theorem 1.1 can be applied without much difficulty to show that the zeros of some special period polynomials are also on the unit circle. More precisely these are the polynomials associated to the cusp forms $R_n(z)$, $0 \le n \le w = k-2$ characterized by the property

$$r_n(f) := n!(2\pi)^{n-1} L(f, n+1) = (f, R_n), \quad \forall f \in \mathcal{S}_k(\Gamma).$$

Here, (f, R_n) is the Petersson inner product of f and R_n. $R_n(z)$ has the following Poincare type series representation, due to Cohen [2]. For $0 < n < w$, $\tilde{n} = w - n$, and $c_{k,n} = i^{\tilde{n}+1} 2^{-w} \binom{w}{n} \pi$, we have

$$R_n(z) = c_{k,n}^{-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma} (az+b)^{-n-1} (cz+d)^{-\tilde{n}-1}.$$

A special case of [9, Theorem 1] gives that the odd period polynomial of R_n for n even and $0 < n < w$ is given by the Bernoulli type polynomial

$$(-1)^{k/2+n/2} 2^{-w} r_{R_n}^{-}(X) = \left[\frac{B_{\tilde{n}+1}^{0}(X)}{\tilde{n}+1} - \frac{B_{n+1}^{0}(X)}{n+1} \right] \mid (I - S), \tag{1.2}$$

where

$$B_{n+1}^{0}(X) = \sum_{\substack{i=0 \\ i \neq 1}}^{n+1} \binom{n}{i} B_i X^{n+1}.$$

The polynomials in (1.2) can be closely approximated by $\sin(2\pi x) + x^w \sin(2\pi/x)$ which then can be used to show that their nontrivial zeros are on the unit circle. The period polynomials of the cusp forms R_n can be seen as complementary to the Ramanujan polynomials which can be thought in terms of the period polynomials of the Eisenstein series (see [4, 9]). Recently, it was shown by Murty et al. [10] that the zeros of the Ramanujan polynomials also lie on the unit circle. In this context, see also [6].

2 Period Polynomial of Hecke Eigenforms

For $f(z)=\sum_{n=1}^{\infty} a(n)\,\mathrm{e}^{2\pi i n z}\in\mathcal{S}_k(\Gamma)$, a Hecke eigenform, we let $L(f,s)=\sum_{n=1}^{\infty} a(n)n^{-s}$ be its associated L-function and

$$r_f^-(X) := \sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{w-1} (-1)^{\frac{n-1}{2}} \binom{w}{n} n!(2\pi)^{-n-1} L(f,n+1) X^{w-n} \tag{2.1}$$

the odd part of its period polynomial.

The L-function satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L(f,s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)L(f,k-s).$$

It follows from the functional equation that $r_f^-(X)$ is self-reciprocal, that is,

$$r_f^-(X) = X^w r_f^-(1/X) \tag{2.2}$$

and it follows from the modularity of f (specifically that $f(\frac{z-1}{z}) = z^k f(z)$) that

$$r_f^-(X) + X^w r_f^-\left(1-\frac{1}{X}\right) + (X-1)^w r_f^-\left(\frac{-1}{X-1}\right) = 0. \tag{2.3}$$

By Eichler–Shimura theory the vector space of polynomials of degree $\le k-2$ spanned by the set of $r_f^-(X)$ as f runs through Hecke eigenforms of weight k is precisely the space of odd polynomials P of degree $\le k-3$ for which

$$P(x) + x^{k-2} P\left(\frac{-1}{x}\right) \equiv 0 \tag{2.4}$$

and

$$P(x) + x^{k-2} P\left(1-\frac{1}{x}\right) + (x-1)^{k-2} P\left(\frac{-1}{x-1}\right) \equiv 0. \tag{2.5}$$

Lemma 2.1. Any polynomial $P(X) \in W_{k-2}^{-}$ has "trivial zeros" at ± 2, $\pm\frac{1}{2}$, and 0. □

Proof. Since P is odd, we have $P(0)=0$ and we only have to verify that $P(1)=P'(1)=P(2)=P(1/2)=0$. We substitute $x=1$ into (2.5), noting that

$$\lim_{x\to 1}(x-1)^{k-2}P\left(\frac{-1}{x-1}\right)=0$$

since P has degree smaller than $k-2$. Thus, $P(1)=0=P(-1)$. Now we substitute $x=-1$ into (2.5) to obtain

$$P(2)+2^{k-2}P(1/2)=0,$$

while from $x=2$ in (2.4), we have

$$P(2)-2^{k-2}P(1/2)=0.$$

Thus, $P(1/2)=P(2)=0$. We differentiate (2.4) to obtain

$$P'(x)+(k-2)x^{k-3}P\left(\frac{-1}{x}\right)+x^{k-4}P'\left(\frac{-1}{x}\right)\equiv 0.$$

Substituting $x=1$ here gives

$$P'(1)+P'(-1)=0.$$

But P is odd, so P' is even which means that $P'(-1)=P'(1)$. Therefore, $P'(1)=P'(-1)=0$ and so we have verified that all of the trivial zeros are where we said they would be. ■

To understand the rest of the zeros of the odd period polynomial $r_f^-(x)$ attached to a Hecke eigenform $f \in \mathcal{S}_k(\Gamma)$, we look at

$$p_f^-(X):=\frac{2\pi r_f^-(X)}{(-1)^{w/2}(2\pi)^{-w}(w-1)!}=\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{w-1}(-1)^{\frac{n-1}{2}}\frac{(2\pi X)^n}{n!}L(f,w-n+1)$$
$$=\sum_{m=0}^{w/2-1}\frac{(-1)^m(2\pi X)^{2m+1}}{(2m+1)!}L(f,w-2m). \tag{2.6}$$

Since $L(f, w-2m)$ is close to 1 for small values of m, we see that the initial terms of the above series are close to the initial terms of the series

$$\sin(2\pi X) = \sum_{m=0}^{\infty} \frac{(-1)^m (2\pi X)^{2m+1}}{(2m+1)!}.$$

The idea now is to study the zeros of

$$\sin(2\pi x) + x^N \sin(2\pi/x).$$

It follows from (2.2) that $p_f^-(X)$ may be written as

$$p_f^-(X) = q_f(X) + X^w q_f(1/X), \tag{2.7}$$

where

$$q_f(X) = \sum_{m=0}^{[(w-6)/4]} \frac{(-1)^m (2\pi X)^{2m+1}}{(2m+1)!} L(f, w-2m) + \frac{L(f, \frac{w+2}{2})(2\pi X)^{\frac{w}{2}}}{2(\frac{w}{2})!}. \tag{2.8}$$

Note that when $k \equiv 2 \bmod 4$ the last term does not appear, since in this case the functional equation implies that $L(f, k/2) = 0$. Note also that q_f and r_f^- have real coefficients, since $L(f, s)$ is real on the real axis.

To prove that the nontrivial zeros of $r_f^-(X)$ are on the unit circle, we need several lemmas. First, we can replace $\sin 2\pi z$ above by an entire function $r(z)$. The crucial idea is the following lemma.

Lemma 2.2. Let $r(z)$ be an entire function and for an even $N \in \mathbb{N}$, let

$$F_N(z) := r(z) + z^N r(z^{-1}).$$

Let

$$R(\theta) := \Re r(e^{i\theta}) \quad \text{and} \quad I(\theta) := \Im r(e^{i\theta}).$$

For $j = 0, \ldots, 2M-1$ et $\mathcal{I}_j$ denote the interval $[\frac{\pi}{2M} + \frac{\pi}{M} j, \frac{\pi}{2M} + \frac{\pi}{M}(j+1)] \subset \mathbb{R}$. Then

(a) For $\theta_j = \frac{\pi}{2M} + \frac{\pi}{M} j$, if $I(\theta_j) = 0$, then $F_{2M}(e^{i\theta_j}) = 0$.

(b) If $I(\theta) \neq 0$ for $\theta \in \mathcal{I}_j$, then $F_{2M}((e^{i\theta}) = 0$ for some $\theta \in \mathcal{I}_j$. □

Proof. Let

$$f_N(z) = z^{-N/2} F_N(z) = z^{-M} r(z) + z^M r(z^{-1}),$$

where $2M = N$. Note that f_N and F_N have the same zeros on $|z| = 1$. Since f_N is real when $|z| = 1$, it suffices to look at the real-valued function

$$\Re f_N(e^{i\theta}) = 2\cos(M\theta) R(\theta) + 2\sin(M\theta) I(\theta). \tag{2.9}$$

Using (2.9), part (a) of the Lemma is clear.

To see part (b) note that if $I(\theta) \neq 0$, then $\Re f_N(e^{i\theta}) = 0$ will have a solution when

$$-\tan M\theta = \frac{R(\theta)}{I(\theta)}. \tag{2.10}$$

If $I(\theta) \neq 0$ for $\theta \in \mathcal{I}_j$, then in the interval $\mathcal{I}_j$ the function $R(\theta)/I(\theta)$ is bounded and continuous and hence (2.10) will have a solution. ■

Lemma 2.3. Let

$$S(z) = \sin(2\pi z) - \sin(2\pi/z).$$

Then $S(z)$ has precisely 10 zeros in the annulus $A := \{z : 4/5 \leq |z| \leq 5/4\}$. Moreover, on the boundary of the annulus, $|S(z)| > 1$. □

Proof. One can count the number of zeros in the annulus by numerically integrating the logarithmic derivative $S'(z)/S(z)$ on the boundary of the annulus. On the boundary, $S'(z)/S(z)$ is bounded by $2\pi(\frac{5}{4})^2 < 10$, so numerically integrating with a step size of 1/100 is more than adequate to determine the integral with an error $< \frac{1}{2}$. This is easily done on any modern computer algebra system. ■

The zeros of $S(z)$ are illustrated in the contour plot in Figure 2.

Lemma 2.4. Let f be a Hecke eigenform of weight k for the full modular group and let $L(f, s) = \sum a(n) n^{-s}$ be its associated L-function. Then for $\sigma \geq 3k/4$, we have

$$|L(f, \sigma) - 1| \leq 4 \times 2^{-k/4} \tag{2.11}$$

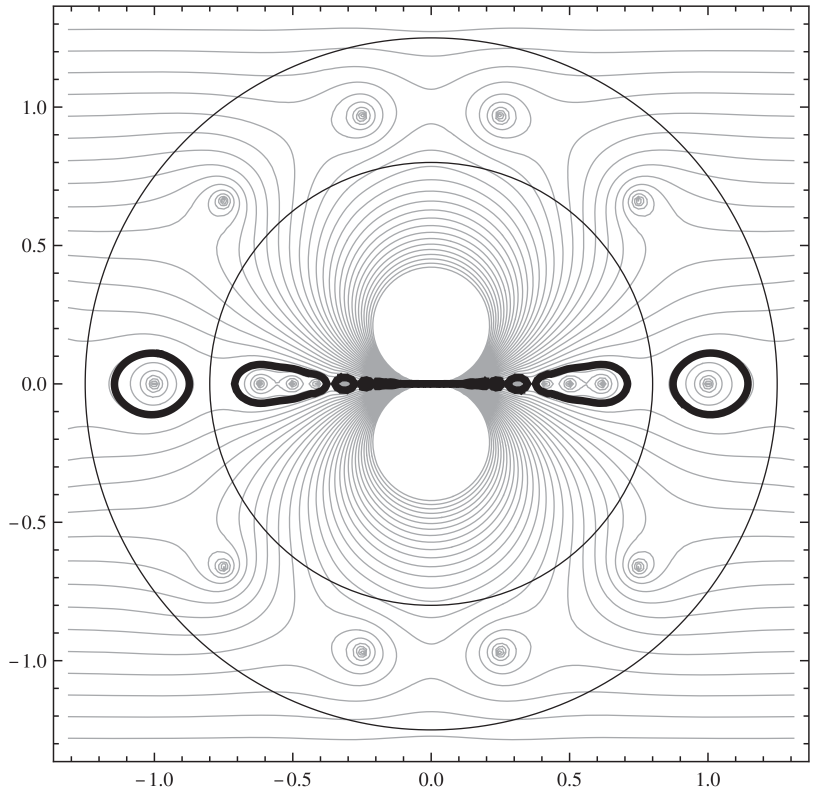

Fig. 2. A contour plot of $\log|S(z)|$. The darker contour is the set where $\log|S(z)| = \log(1.5)$.

and for σ an integer with $\sigma \geq k/2$, we have

$$L(f, \sigma) \leq 2k^{1/2} \log 2k + 1. \tag{2.12}$$

□

Proof. If $\sigma \geq 3k/4$, then we are in the region of absolute convergence. By Deligne's Theorem, we have

$$|L(f, \sigma) - 1| \leq \sum_{n=2}^{\infty} \frac{d(n)}{n^{\sigma-(k-1)/2}} \leq \sum_{n=2}^{\infty} \frac{d(n)}{n^{k/4}} = \zeta(k/4)^2 - 1 \leq 2(\zeta(k/4) - 1).$$

We have

$$\zeta(k/4) - 1 = 2^{-k/4} + \sum_{n=3}^{\infty} n^{-k/4} \leq 2^{-k/4} + \int_2^{\infty} u^{-k/4}\, \mathrm{d}u \leq 2 \times 2^{-k/4}$$

for $k \geq 12$. This proves (2.11).

Next, if $\sigma \geq k/2+1$, we estimate $L(f,\sigma)$ trivially by $L(f,\sigma) \leq \zeta(3/2)^2 < 7$. If $k/2 \leq \sigma \leq k/2+1$, then using standard methods, we have for $\sigma = k/2$,

$$\Gamma(k/2)L(f,k/2) = 2\sum_{n=1}^{\infty} \frac{a(n)}{n^{k/2}} \int_{2\pi n}^{\infty} \mathrm{e}^{-x} x^{k/2} \frac{\mathrm{d}x}{x}.$$

Thus,

$$|L(f,\sigma)| \leq 2\Gamma(k/2)^{-1} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/2}} \int_{2\pi n}^{\infty} \mathrm{e}^{-x} x^{k/2} \frac{\mathrm{d}x}{x}.$$

We split the sum over n at k. The terms with $n \leq k$ are

$$\leq \sum_{n\leq k} \frac{d(n)}{n^{1/2}}$$

as is seen by completing the integrals down to 0. Now

$$\sum_{n\leq k} \frac{d(n)}{n^{1/2}} = \sum_{mn\leq k} \frac{1}{(mn)^{1/2}} \leq \sum_{m\leq k} \frac{1}{m^{1/2}} \int_0^{k/m} u^{-1/2}\,\mathrm{d}u = 2k^{1/2} \sum_{m\leq k} \frac{1}{m} \leq 2k^{1/2} \log 2k$$

for $k \geq 5$. The tail of the series is

$$\begin{aligned} &= 2\Gamma(k/2)^{-1} \sum_{n=k+1}^{\infty} \frac{d(n)}{\sqrt{n}} \int_{2\pi n}^{\infty} \mathrm{e}^{-x} x^{k/2} \frac{\mathrm{d}x}{x} \\ &\leq 2\Gamma(k/2)^{-1} \sum_{n=k+1}^{\infty} \frac{d(n)}{\sqrt{n}} \mathrm{e}^{-\pi n} \int_{2\pi n}^{\infty} \mathrm{e}^{-x/2} x^{k/2} \frac{\mathrm{d}x}{x}. \end{aligned}$$

The integral is

$$= 2^{k/2} \int_{\pi n}^{\infty} \mathrm{e}^{-x} x^{k/2} \frac{\mathrm{d}x}{x} \leq 2^{k/2}\Gamma(k/2).$$

Using $d(n) \leq 2\sqrt{n}$, we have that the tail is

$$\leq 4 \times 2^{k/2} \sum_{n=k+1}^{\infty} \mathrm{e}^{-\pi n} \leq 4 \times 2^{k/2} \mathrm{e}^{-\pi k} < 1.$$

Note that $2k^{1/2} \log 2k + 1 > 7$ for $k \geq 3$. The proof is complete. ∎

Lemma 2.5. Let q_f be as in (2.8). Then, for $z \le 5/4$ and $k \ge 80$ we have

$$\sin 2\pi z - q_f(z)| \le \frac{1}{100}.$$

□

Proof. We have

$$\sin 2\pi z = \sum_{m=0}^{\infty} (-1)^m \frac{(2\pi z)^{2m+1}}{(2m+1)!}$$

and

$$q_f(z) = \sum_{m=0}^{[(w-6)/4]} \frac{(-1)^m (2\pi z)^{2m+1}}{(2m+1)!} L(f, w-2m) + \frac{L(f, \frac{w+2}{2})(2\pi z)^{\frac{w}{2}}}{2(\frac{w}{2})!}.$$

Thus,

$$\begin{aligned} |\sin 2\pi z - q_f(z)| &\le \sum_{m \le w/8} \frac{(5\pi/2)^{2m+1}}{(2m+1)!} |L(f, w-2m) - 1| \\ &\quad + \sum_{w/8 < m < w/4} \frac{(5\pi/2)^{2m+1}}{(2m+1)!} (|L(f, w-2m)| + 1) + \sum_{m > w/4} \frac{(5\pi/2)^{2m+1}}{(2m+1)!} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

say. Now by Lemma 2.4 we have

$$\Sigma_1 \le 4 \times 2^{-k/4} \sum_{m \le w/8} \frac{(5\pi/2)^{2m+1}}{(2m+1)!} \le 4 \times 2^{-k/4} \times e^{5\pi/2}.$$

We can combine estimates for Σ_2 and Σ_3. Again using Lemma 2.4 we have

$$\Sigma_2 + \Sigma_3 \le (2\sqrt{k} \log 2k + 2) \sum_{w/8 < m} \frac{(5\pi/2)^{2m+1}}{(2m+1)!}.$$

We can bound the sum using

$$\begin{aligned} \sum_{m=r+1}^{\infty} \frac{x^m}{m!} &= \frac{x^{r+1}}{(r+1)!} \left(1 + \frac{x}{r+2} + \frac{x^2}{(r+2)(r+3)} + \cdots\right) \\ &\le \frac{x^{r+1}}{(r+1)!} \frac{1}{(1 - \frac{x}{r+1})} = \frac{x^{r+1}}{r!(r+1-x)} < \frac{(ex)^{r+1}}{r^r(r+1-x)}, \end{aligned}$$

the last line uses $r! > (r/e)^r$. Using this above, we have

$$\Sigma_2 + \Sigma_3 \leq (2\sqrt{k}\log 2k + 2)\frac{(5\pi e/2)^{k/2+1}}{(k/2)^{k/2}(k/2 + 1 - 5\pi/2)}.$$

For $k \geq 80$, we have $\Sigma_1 + \Sigma_2 + \Sigma_3 < 0.01$. ■

Lemma 2.6. If $k \geq 80$, then the function

$$Q_f(z) := q_f(z) - q_f(1/z)$$

has at most 10 zeros in the annulus A. □

This follows from Rouché's theorem using Lemmas 2.3 and 2.5.

Corollary 2.7. If $k \geq 80$, then

$$\Im q_f(e^{i\theta})$$

has at most 10 zeros in $0 \leq \theta < 2\pi$. Moreover, $\Im q_f(e^{i\theta}) = 0$ at $\theta = 0$ and at $\theta = \pi$. □

Proof. If z is on the unit circle, then $\Im q_f(z) = -i Q_f(z)$ so any zero of $\Im q_f(z)$ has to be a zero of $Q_f(z)$. But $Q_f(z)$ has at most 10 zeros on the annulus A of which the unit circle is a subset. Since $r_f(\pm 1) = 0$, we see from (2.7) that $q_f(\pm 1) = 0$, so $\Im q_f(e^{i\theta}) = 0$ for $\theta = 0$ and $\theta = \pi$. ■

We now combine these lemmas to prove:

Theorem 2.8. Let f be a cusp form of weight $k \geq 80$ for $\mathrm{SL}(2, \mathbb{Z})$, $w = k - 2$ and $p_f^-(z) = q_f(z) + z^w q_f(1/z)$ be its odd period polynomial of degree $w - 1$. Then $p_f^-(z)$ has all but five of its zeros on the unit circle. The five trivial zeros of $p_f(z)$are at $z = 0, 2, -2, 1/2, -1/2$. It has double zeros at $z = 1, -1$. □

Proof. First recall that we have shown that each period polynomial has simple zeros at $z = 0, 2, -2, 1/2, -1/2$ and double zeros at $1, -1$. To prove that the rest of the zeros are on the unit circle we let $r(z) = q_f(z)$ and $N = w = k - 2$ in Lemma 2.2.

By the corollary, there are 10 zeros of $\Im(q_f(e^{i\theta}))$ in the interval $[0, 2\pi)$ and hence by part (b) of Lemma 2.2 for each of the $N - 10$ intervals among the N intervals $\mathcal{I}_j =$

$[\frac{\pi}{2M} + \frac{\pi}{M} j, \frac{\pi}{2M} + \frac{\pi}{M}(j+1)]$, $j = 0, N-1$ for which $\Im(q_f(e^{i\theta})) \neq 0$, $p_f(e^{i\theta}) = 0$ for some $\theta \in \mathcal{I}_j$ This gives at least $N - 10$ zeros on the unit circle. Among the 10 discarded intervals in which $\Im(q_f(e^{i\theta})$ vanish, we have also excluded the intervals that contain $\theta = 0$ and $\theta = \pi$ where $p_f(z)$ has double zeros. Hence, we have at least $N - 10 + 4 = w - 6$ zeros on the unit circle. Since the degree of $p_f(z)$ is $w - 1$ together with the five zeros at $z = 0, 2, -2, 1/2, -1/2$, this covers all the zeros and finishes the proof of the theorem. ■

Finally, Theorem 1.1 follows from Theorem 2.8 together with the fact that for the weights $k \leq 80$ the statement can be verified numerically, as we have done.

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