A RECIPROCITY FOR SYMMETRIC ALGEBRAS

by MARKUS LINCKELMANN[†]

(*CNRS, Université Paris* 7*, UFR Mathématiques, 2 place Jussieu,* 75251 *Paris Cedex* 05*, France*)

and BERNARD STALDER

(*Université de Lausanne, Institut de Mathématiques,* 1015 *Lausanne, Switzerland*)

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Abstract

The aim of this note is to show that the reciprocity property of group algebras given by B. Stalder can be deduced from formal properties of symmetric algebras, as exposed by M. Broué, for instance.

Let $\mathcal{O}$ be a commutative ring. By an $\mathcal{O}$-algebra we always mean a unitary associative algebra over $\mathcal{O}$. Given an $\mathcal{O}$-algebra A, we denote by A^0 the opposite algebra of A. An A-module is a unitary left module, unless stated otherwise. A right A-module can be considered as a left A^0-module. If A, B are $\mathcal{O}$-algebras, we mean by an A-B-bimodule always a bimodule whose left and right $\mathcal{O}$-module structures coincide; in other words, any A-B-bimodule can be regarded as $A \otimes_{\mathcal{O}} B^0$-module. For an A-A-bimodule M we set $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. In particular, $A^A = Z(A)$, the centre of A. If A, B, C are $\mathcal{O}$-algebras, M is an A-B-bimodule and N is an A-C-bimodule, we consider the space $\mathrm{Hom}_A(M, N)$ of left A-module homomorphisms from M to N as B-C-bimodules via $(b.\varphi.c)(m) = \varphi(mb)c$. Similarly, if furthermore N' is a C-B-bimodule, we consider the space $\mathrm{Hom}_{B^0}(M, N')$ of right B-module homomorphisms from M to N' as C-A-bimodules via $(c.\psi.a)(m) = c\psi(am)$. In particular, the $\mathcal{O}$-dual $M^* = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ becomes a B-A-bimodule via $(b.\tau.a)(m) = \tau(amb)$. Here $a \in A$, $b \in B$, $c \in C$, $m \in M$, $\varphi \in \mathrm{Hom}_A(M, N)$, $\psi \in \mathrm{Hom}_{B^0}(M, N')$ and $\tau \in M^*$.

An $\mathcal{O}$-algebra A is called *symmetric* if A is finitely generated projective as $\mathcal{O}$-module and if A is isomorphic to its $\mathcal{O}$-dual $A^* = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as A-A-bimodule. The image $s \in A^*$ of 1_A under any A-A-isomorphism $\Phi : A \cong A^*$ satisfies $\Phi(a) = a.s = s.a$ for all $a \in A$; that is, s is symmetric and the map $a \mapsto a.s$ is a bimodule isomorphism $A \cong A^*$. Any such linear form is called a *symmetrizing form of* A. The choice of a symmetrizing form on A is thus equivalent to the choice of a bimodule isomorphism $A \cong A^*$.

THEOREM 1 *Let A, B be symmetric $\mathcal{O}$-algebras and let M, N be A-B-bimodules which are finitely generated projective as left and right modules. We have a bifunctorial $\mathcal{O}$-linear isomorphism*

$$(M^* \otimes_A N)^B \cong (N \otimes_B M^*)^A$$

which is canonically determined by the choice of symmetrizing forms of A and B.

[†] Corresponding author: linckelm@math.jussieu.fr

Proof. Let $s \in A^*$ and $t \in B^*$ be symmetrizing forms on A and B, respectively. It is well known (see [**1**], also [**3**, Appendix]) that there is an isomorphism of B-A-bimodules

$$\begin{cases} \mathrm{Hom}_A(M, A) & \cong M^* \\ f & \mapsto s \circ f \end{cases}$$

which is functorial in M. Moreover, since M and N are finitely generated projective as left and right modules, we have an isomorphism of B-B-bimodules

$$\begin{cases} \mathrm{Hom}_A(M, A) \otimes_A N & \cong \mathrm{Hom}_A(M, N) \\ f \otimes n & \mapsto (m \mapsto f(m)n) \end{cases}$$

which is functorial in both M and N. Taking B-fixpoints yields $(M^* \otimes_A N)^B \cong (\mathrm{Hom}_A(M, A) \otimes_A N)^B \cong (\mathrm{Hom}_A(M, N))^B = \mathrm{Hom}_{A\otimes B^0}(M, N)$. Similarly, there is an isomorphism of B-A-bimodules

$$\begin{cases} \mathrm{Hom}_{B^0}(M, B) & \cong M^* \\ g & \mapsto t \circ g \end{cases}$$

and we have an isomorphism of A-A-bimodules

$$\begin{cases} N \otimes_B \mathrm{Hom}_{B^0}(M, B) & \cong \mathrm{Hom}_{B^0}(M, N) \\ n \otimes g & \mapsto (m \mapsto ng(m)). \end{cases}$$

As before, taking A-fixpoints yields $(N \otimes_B M^*)^A \cong (N \otimes_B \mathrm{Hom}_{B^0}(M, B))^A \cong (\mathrm{Hom}_{B^0}(M, N))^A = \mathrm{Hom}_{A\otimes B^0}(M, N)$.

REMARK The proof of Theorem 1 shows that the two expressions in the statement of Theorem 1 are isomorphic to $\mathrm{Hom}_{A\otimes B^0}(M, N)$. In particular, for $M = N$, this induces algebra structures on $(M^* \otimes_A M)^B$ and $(M \otimes_B M^*)^A$.

Taking derived functors of the fixpoint functors in Theorem 1 yields the following consequence on Hochschild cohomology.

COROLLARY 1 *With the notation and assumptions of Theorem* 1*, we have an isomorphism of graded $\mathcal{O}$-modules* $HH^*(B, M^* \otimes_A N) \cong HH^*(A, N \otimes_B M^*)$.

Proof. Let P be a projective resolution of M as A-B-bimodule. Then $P^* = \mathrm{Hom}_{\mathcal{O}}(P, \mathcal{O})$ is an $\mathcal{O}$-injective resolution of M^*. Thus $N \otimes_B P^*$ and $P^* \otimes_A N$ are $\mathcal{O}$-injective resolutions of $N \otimes_B M^*$ and $M^* \otimes_A N$, respectively. Using Theorem 1, we have isomorphisms of cochain complexes $\mathrm{Hom}_{B\otimes_{\mathcal{O}} B^0}(B, P^* \otimes_A N) \cong (P^* \otimes_A N)^B \cong (N \otimes_B P^*)^A \cong \mathrm{Hom}_{A\otimes_{\mathcal{O}} A^0}(A, N \otimes_B P^*)$. Taking cohomology yields the statement.

Let A be an $\mathcal{O}$-algebra. Following the terminology in [**2**,**3**] (which generalizes [**4**]), an *interior* A-algebra is an $\mathcal{O}$-algebra B endowed with a unitary algebra homomorphism $\sigma : A \to B$. If A, B are $\mathcal{O}$-algebras, C is an interior B-algebra and M is an A-B-bimodule, we set $\mathrm{Ind}_M(C) = \mathrm{End}_{C^0}(M \otimes_B C)$, considered as an interior A-algebra via the homomorphism $A \to \mathrm{Ind}_M(C)$ sending a to the C^0-endomorphism given by left multiplication with a on $M \otimes_B C$.

THEOREM 2 *Let A, B be symmetric $\mathcal{O}$-algebras and let M be an A-B-bimodule which is finitely generated projective as left and right module. There is a canonical anti-isomorphism of $\mathcal{O}$-algebras*

$$(\mathrm{Ind}_M(B))^A \cong (\mathrm{Ind}_{M^*}(A))^B .$$

Proof. We have $\mathrm{Ind}_M(B) = \mathrm{End}_{B^0}(M)$ and $\mathrm{Ind}_{M^*}(A) = \mathrm{End}_{A^0}(M^*)$. Since taking $\mathcal{O}$-duality is a contravariant functor, this algebra is isomorphic to $\mathrm{End}_A(M)^0$. Taking fixpoints completes the proof.

The group algebra $\mathcal{O}G$ of a finite group G is a symmetric algebra. More precisely, $\mathcal{O}G$ has a canonical symmetrizing form, namely the form $s : \mathcal{O}G \to \mathcal{O}$ mapping a group element $g \in G$ to zero if $g \neq 1$ and to 1 if $g = 1$. Following the terminology of Puig [**4**], an interior G-algebra is an $\mathcal{O}$-algebra endowed with a group homomorphism $\sigma : G \to A^\times$. Such a group homomorphism extends uniquely to an $\mathcal{O}$-algebra homomorphism $\mathcal{O}G \to A$, and thus A becomes an interior $\mathcal{O}G$-algebra (and vice versa). If H is a subgroup of G and B an interior H-algebra, the *induced algebra* $\mathrm{Ind}_H^G(B)$ defined in [**4**] is the $\mathcal{O}$-module $\mathcal{O}G \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} \mathcal{O}G$ endowed with the multiplication $(x \otimes b \otimes y)(x' \otimes b' \otimes y) = (x \otimes byx'b' \otimes y')$ provided that $yx' \in H$, and 0 otherwise, where $x, y, x', y' \in G$ and $b, b' \in B$. The algebra $\mathrm{Ind}_H^G(B)$ is viewed as an interior G-algebra with the structural homomorphism mapping $x \in G$ to $\sum_{y\in[G/H]} xy \otimes 1_B \otimes y^{-1}$. For $B = \mathcal{O}H$, we have the obvious identification $\mathrm{Ind}_H^G(\mathcal{O}H) = \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G$, with multiplication given by $(x \otimes y)(x' \otimes y') = x \otimes yx'y'$ if $yx' \in H$ and 0 otherwise, where $x, y, x', y' \in G$. The previous notion of algebra induction is consistent with this concept.

LEMMA 1 *Let G be a finite group, H a subgroup of G and let B be an interior H-algebra. Set $M = \mathcal{O}G_H$. There is an isomorphism of $\mathcal{O}$-algebras*

$$\begin{cases} \mathrm{Ind}_H^G(B) & \cong \mathrm{Ind}_M(B) \\ (x \otimes b \otimes y) & \mapsto (z \otimes c \ \mapsto x \otimes byzc \textit{ if } yz \in H \textit{ and } 0 \textit{ otherwise}) , \end{cases}$$

where $x, y, z \in G$ and $b, c \in B$.

Proof. Straightforward verification.

THEOREM 3 (Stalder [**5**]) *Let G be a finite group, let H, K be subgroups of G. Consider $\mathcal{O}G$ as $\mathcal{O}H$-$\mathcal{O}K$-bimodule via multiplication in $\mathcal{O}G$. Then there is an isomorphism of $\mathcal{O}$-algebras*

$$\begin{cases} (\mathrm{Ind}_H^G(\mathcal{O}H))^K & \xrightarrow{\sim} (\mathrm{Ind}_K^G(\mathcal{O}K))^H \\ \displaystyle\sum_{k\in[K/K_{(x\otimes y)}]} kx \otimes yk & \longmapsto \displaystyle\sum_{h\in[H/H_{(x\otimes y)}]} hx \otimes yh, \end{cases}$$

where $K_{(x\otimes y)}$ is the stabilizer in K of $x \otimes y \in \mathrm{Ind}_H^G(\mathcal{O}H)$ under the action of K and $H_{(x\otimes y)}$ is the stabilizer in H of $x \otimes y \in \mathrm{Ind}_K^G(\mathcal{O}K)$ under the action of H.

There are (at least) three ways to go about the proof of Theorem 3: by explicit verification or by interpreting Theorem 3 as special case of either Theorem 1 or Theorem 2. We sketch the three different proofs.

Proof 1 *of Theorem* 3. The image of the set $G \times G$ in $\mathrm{Ind}_H^G(\mathcal{O}H) = \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G$ is an $\mathcal{O}$-basis which is permuted under the action of K by conjugation. Thus the subalgebra $(\mathrm{Ind}_H^G(\mathcal{O}H))^K$ of K-stable elements has as $\mathcal{O}$-basis the set of relative traces $\mathrm{Tr}_{K_{(x\otimes y)}}^K(x \otimes y)$, where $x, y \in G$. If $x, x', y, y' \in G$ and $k \in K$ such that

$$kx \otimes yk = x' \otimes y'$$

in $\mathrm{Ind}_H^G(\mathcal{O}H)$, there is a (necessarily unique) $h \in H$ such that $kx = x'h$ and $yk = hy'$, which in turn is equivalent to the equality

$$hx \otimes yh = (x') \otimes (y')$$

in $\mathrm{Ind}_K^G(\mathcal{O}K)$. Thus the map $x \otimes y \mapsto x \otimes y$ induces a bijection between the sets of K-orbits and of H-orbits of the images of $G \times G$ in $\mathrm{Ind}_H^G(\mathcal{O}H)$ and $\mathrm{Ind}_K^G(\mathcal{O}K)$, respectively. In particular, with the notation above, we have $k \in K_{(x\otimes y)}$ if and only if $h \in H_{(x\otimes y)}$, and the correspondence $k \mapsto h$ induces a group isomorphism $K_{(x\otimes y)} \cong H_{(x\otimes y)}$. From this it follows that the map given in Theorem 3 is an $\mathcal{O}$-linear isomorphism. It remains to verify that this is an algebra homomorphism. In $\mathrm{Ind}_H^G(\mathcal{O}H)$, multiplication is given by

$$(x \otimes y)(z \otimes t) = \begin{cases} x \otimes yzt & \text{if } yz \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where $x, y, z, t \in G$. If $yz \in H$, then in $\mathrm{Ind}_K^G(\mathcal{O}K)$, the elements $(yz)z^{-1} \otimes t^{-1}(yz)^{-1}$ and $z^{-1} \otimes t^{-1}$ are in the same H-orbit, and the multiplication in $\mathrm{Ind}_K^G(\mathcal{O}K)$ yields $(x^{-1} \otimes y^{-1})((yz)z^{-1} \otimes t^{-1}(yz)^{-1}) = x^{-1} \otimes t^{-1}z^{-1}y^{-1}$, and this corresponds precisely to the bijection between the sets of K-orbits and H-orbits of the images of the set $G \times G$ in $\mathrm{Ind}_K^G(\mathcal{O}K)$ and $\mathrm{Ind}_H^G(\mathcal{O}H)$, respectively.

Proof 2 *of Theorem* 3. We are going to apply Theorem 1 to the particular case where $A = \mathcal{O}H$, $B = \mathcal{O}K$, $M = N = \mathcal{O}G$ viewed as A-B-bimodule (through the inclusions $H \subseteq G$, $K \subseteq G$). This yields an $\mathcal{O}$-linear isomorphism

$$((\mathcal{O}G)^* \otimes_{\mathcal{O}H} \mathcal{O}G)^K \cong (\mathcal{O}G \underset{\mathcal{O}K}{\otimes} (\mathcal{O}G)^*)^H .$$

Composing this with the canonical isomorphism $(\mathcal{O}G)^* \cong \mathcal{O}G$ mapping $f \in (\mathcal{O}G)^*$ to $\sum_{x\in G} f(x^{-1})x$ yields the isomorphism in Theorem 3.

Proof 3 *of Theorem* 3. Applying Theorem 2 and the above lemma to $A = \mathcal{O}K$, $B = \mathcal{O}H$ and $M = \mathcal{O}G$ as A-B-bimodule yields an anti-isomorphism $(\mathrm{Ind}_H^G(\mathcal{O}H))^K \cong (\mathrm{Ind}_K^G(\mathcal{O}K))^H$. The map sending $x \otimes y$ to $y \otimes x$ is an anti-automorphism of $\mathrm{Ind}_H^G(\mathcal{O}H)$ which induces an anti-automorphism of $(\mathrm{Ind}_H^G(\mathcal{O}H))^K$. Composing both maps yields again the isomorphism in Theorem 3.

REMARK The proof 3 of Theorem 3 is essentially the proof given in [**5**, §11].

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