With $\sin A + \sin B + \sin C = \Delta / Rr$ (circumradius R, inradius r), some routine coordinate geometry leads eventually to the expected perpendicular distance r from the stationary point to any side. Furthermore, we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} > 0 \text{ and } \frac{\partial^2 f}{\partial x \, \partial y} = 0$$

at I, so the Hessian, $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y} \right)^2$, is positive here and I is indeed a minimum point.

Finally, note two sharp inequalities. For circumcentre and orthocentre we have

$$\frac{1}{\sin B \sin C} + \frac{1}{\sin C \sin A} + \frac{1}{\sin A \sin B} \geqslant 4,$$

and

$$\frac{\cos^2 A}{\sin B \sin C} + \frac{\cos^2 B}{\sin C \sin A} + \frac{\cos^2 C}{\sin A \sin B} \geqslant 1,$$

respectively.

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98.06 The relative size of the squares inscribed in a triangle

In this note, we consider a triangle ABC with area Δ and the square $\mathcal{S}_c$ with two consecutive vertices on the line of c, a third vertex on a, and the last on b (Figure 1). The square erected outwardly on c is transformed into square $\mathcal{S}_c$ by the homothety of similarity ratio $\frac{h_c}{h_c + c}$ centered at C, where h_c denotes the altitude perpendicular to c [1]. The side of $\mathcal{S}_c$ is thus

$$s_c = \frac{h_c}{h_c + c} c = \frac{2\Delta}{c + h_c}.$$

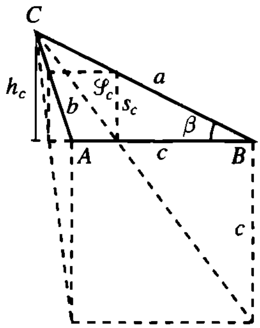

FIGURE 1

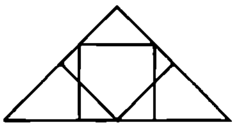

FIGURE 2

We analyse the relative size of the squares $\mathcal{S}_a$, $\mathcal{S}_b$, and $\mathcal{S}_c$. We prove in particular the result of [2] in a simpler, shorter, and more informative way: when the triangle is non-obtuse, the side lengths of the three squares differ by less than six percent from the longest square side.

One has for any triangle

$$\frac{s_c}{s_b} = \frac{b + h_b}{c + h_c} = \frac{b + c\sin\alpha}{c + b\sin\alpha} = f(b,c,\alpha), \qquad b > 0, c > 0,\ 0 < \alpha < \pi.$$

Since $\dfrac{\partial f}{\partial b} = \dfrac{c\cos^2\alpha}{(c + b\sin\alpha)^2}$, f is a strictly increasing or constant function of b when c and α are fixed, with range $\left(\sin\alpha, \frac{1}{\sin\alpha}\right)$ for $\alpha \neq \frac{\pi}{2}$ and range $\{1\}$ for $\alpha = \frac{\pi}{2}$. One has in particular $s_b = s_c$ if, and only if, b and c are the legs of an isosceles or right-angled triangle, with $\mathcal{S}_b = \mathcal{S}_c$ in the latter case (in the right-angled corner).

We now consider a non-obtuse triangle, that is, $c\cos\alpha \leqslant b \leqslant \frac{c}{\cos\alpha}$ or $\alpha = \frac{\pi}{2}$. We set

$$f(c\cos\alpha, c, \alpha) = \frac{\cos\alpha + \sin\alpha}{1 + \cos\alpha\sin\alpha} = g(\alpha),$$

which is invariant under $\alpha \to \frac{\pi}{2} - \alpha$. As f is an increasing function of b when c and α are fixed and as $f\left(\frac{c^2}{b}, c, \alpha\right) = 1/f(b, c, \alpha)$, the range of $f(b, c, \alpha) = s_c/s_b$ for a non-obtuse triangle with fixed c and α is thus $[g(\alpha), 1/g(\alpha)]$. Since $g'(\alpha) = \dfrac{\cos\alpha\sin\alpha(\sin\alpha - \cos\alpha)}{(1 + \cos\alpha\sin\alpha)^2}$, g is strictly decreasing from 1 to $\frac{2\sqrt{2}}{3}$ on $\left[0, \frac{\pi}{4}\right]$ before increasing symmetrically on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Thus the ratio s/s^* of the sides $s \leqslant s^*$ of two squares inscribed in the same non-obtuse triangle is in the range from $\frac{2\sqrt{2}}{3} \approx 0.9428$ to 1. The minimal value $\frac{2\sqrt{2}}{3}$ (for squares with areas in the ratio 8 : 9) corresponds to an isosceles right-angled triangle: the square of side s occupies then the middle third of the hypotenuse, whereas the side s^*, half as long as the legs, belongs to the square in the right-angled corner (Figure 2).

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98.07 The order of convergence of Newton's Method in special cases

It is well known that in the general case the Newton-Raphson method for solving non-linear equations is of convergence order 2. (Actually Simpson in 1740 was the first to write the method in its modern form using a derivative, so perhaps it should be called the N-R-S method). Note that 'convergence order p' means that if an iteration method is of the form

$$x_{i+1} = \phi(x_i)$$

and is converging to a root ζ, then the order is defined as p if

$$\lim_{i \to \infty} \frac{|x_{i+1} - \zeta|}{|x_i - \zeta|^p} = \frac{\phi^{(p)}(\zeta)}{p!} \tag{1}$$

(or equivalently if $|e_{i+1}| \approx k|e_i|^p$ where $e_i = x_i - \zeta$).

It follows that if

$$\phi'(\zeta) = \phi''(\zeta) = \qquad = \phi^{(p-1)}(\zeta) = 0$$

whereas

$$\phi^{(p)}(\zeta) \neq 0$$

then the order is indeed p. (For proof see e.g. [1].)

Now for Newton's method

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

and hence

$$\phi'(\zeta) = 0$$

(unless $f'(\zeta) = 0$), while

$$\phi''(\zeta) = \frac{f''(\zeta)}{f'(\zeta)} \neq 0 \text{ (usually).}$$