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Logrolling and the Paradox of Voting: Are They Really Logically Equivalent? A Comment

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In his paper "Vote-Trading and the Voting Paradox: A Proof of Logical Equivalence" David Koehler[1] claims to have proved the equivalence of logrolling and the paradox of voting. Unfortunately, the proofs given by Koehler do not have the general validity he intends them to have. Indeed, it is easy to show that logrolling does not necessarily imply the paradox of voting, nor does the paradox of voting imply the existence of logrolling situations.

I begin by first demonstrating with the help of a counter-example that certain logrolling situations do not imply the paradox of voting. This is not very surprising, since to prove that logrolling implies the paradox, it has been necessary to assume the absence of a certain kind of complementarity among issues.[2] In the counter-example a group of three voters, V_1, V_2 and V_3, has to decide two issues x and y. The two issues each contain two alternatives, namely, x_1, x_2 and y_1, y_2. In Matrix 1 the utility payoffs for voters V_1, V_2 and V_3 are given for the four possible outcomes (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_2).

Matrix 1

Moves available to all voters	x_1	x_2
y_1	8, 8, 8	2, 10, 2
y_2	6, 2, 10	10, 6, 6

With simple majority voting x_2 and y_2 will receive a majority of votes, so that (x_2, y_2) will be the outcome selected by the group. This result follows, because x_1 is dominated by x_2 for V_2, and y_1 by y_2 for V_3. With no knowledge about the behavior of the other voters V_2 will, therefore, vote for x_2 and V_3 for y_2. V_1 knows this situation and consequently casts his votes for x_2 and y_2, since by doing so his most preferred outcome (x_2, y_2) will get a majority.

But now voters V_2 and V_3 prefer (x_1, y_1) to (x_2, y_2) because of the higher payoffs $8>6$. And they can obtain this outcome by making a logrolling agreement. According to this agreement V_2 votes for x_1 as against x_2, which is against his immediate interest, to get in exchange V_3's vote for y_1 as against y_2, which is contrary to V_3's immediate advantage. This exchange of votes is, however, favorable to both, since y_1 strongly dominates y_2 for V_2, and x_1 strongly dominates x_2 for V_3. Consequently, a typical logrolling situation is present.

It is now quite obvious that in the case given above, logrolling does not imply the paradox of voting. For (x_2, y_1) as well as (x_1, y_2) are preferred to (x_1, y_1) by only one voter. (x_1, y_1) is a stable outcome. Note, moreover, that $(x_2, y_2)P_1(x_2, y_1)$, $(x_2, y_2)P_1(x_1, y_2)$ and $(x_1, y_1)P_1(x_2, y_1)$, $(x_1, y_1)P_1(x_1, y_2)$, where P_1 means "preferred to by V_1." As a consequence just the kind of complementarity is present, the absence of which had to be assumed to prove that logrolling situations imply the paradox of voting.[2]

I proceed to show that cyclical group preferences do not generally imply the presence of logrolling situations. First, it has to be mentioned that Koehler has not given a proof for cycles containing more than three alternatives. But it is obvious that cyclical group preference orderings with more than three alternatives A_1 can exist. In general, we have:

$$A_1PA_2PA_3P \cdots PA_{n-1}PA_nPA_1,$$

where P means "preferred to by a majority of the group." Koehler's proof is only concerned with the case, in which $n=3$, and no proof has been given for $n>3$.

Secondly, consider Koehler's case with $n=3$:

$$A_1PA_2PA_3PA_1.$$

Even in this case the paradox of voting need not imply logrolling situations. For assume that A_1, A_2 and A_3 belong all to one issue A. For example, the A_i may represent three different bridges, out of which one has to be selected by the group, since only one bridge can be built because of budgetary

[1] David H. Koehler, "Vote-Trading and the Voting Paradox: A Proof of Logical Equivalence," *American Political Science Review*, 69 (September, 1975), 954–960.

[2] Peter Bernholz, "Logrolling, Arrow-Paradox and Decision Rules: A Generalization," *Kyklos*, 22 (fasc. 1, 1974), 53, assumption A5.

restrictions. In this case substantive logrolling is obviously not possible, since there are no two issues concerning which votes can be exchanged by group members.

The above considerations prove not only that Koehler has failed to show the logical equivalence of logrolling and the paradox of voting but also that such a proof is impossible. What Koehler has demonstrated—but only for three group members and for two issues—is that *under certain conditions* logrolling and the paradox of voting are equivalent. But his belief that he had given a general proof prevented him from trying to find these conditions. As mentioned above, one of these conditions is the absence of a certain kind of complementarity among the relevant issues.

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