ON THE ZEEMAN COMPARISON THEOREM FOR THE HOMOLOGY OF QUASI-NILPOTENT FIBRATIONS

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0. Introduction

THIS note constitutes a generalization of the Zeeman comparison theorem for spectral sequences [**9**]. Zeeman's theorem was based on hypotheses valid for the homology spectral sequence of a fibration with simply-connected base space; however, his hypotheses were stated purely algebraically and there was no assumption that the spectral sequence was derived from a filtered chain complex—merely that it was a first quadrant sequence and the differentials had the usual bidegrees. Thus Zeeman's version was more general than Moore's earlier comparison theorem [**5**]; it was also more general in that isomorphism assumptions were only made up to certain dimensions (and so only deduced up to certain dimensions).

We generalize Zeeman's theorem in two directions. The most important direction is that we cover the situation of a *quasi-nilpotent* fibration; this is a fibration

$$F \to E \to B \tag{0.1}$$

in which all spaces are connected, and $\pi_1 B$ operates nilpotently on $H_i F$, $i \geq 0$. We say that (0.1) is *strongly* quasi-nilpotent if it is quasi-nilpotent and if, in addition, $\pi_1 B$ is nilpotent.

Among the quasi-nilpotent fibrations we find the *nilpotent* fibrations [**2**]; these are fibrations (0.1) in which all spaces are connected and $\pi_1 E$ operates nilpotently [**3**] on $\pi_i F$, $i \geq 1$. If E, B are nilpotent spaces and F is connected then F is nilpotent and (0.1) is a nilpotent fibration, and also strongly quasi-nilpotent. A special case, then, of a nilpotent fibration which is also strongly quasi-nilpotent is

$$\tilde{X} \to X \to K(\pi_1 X, 1) \tag{0.2}$$

where X is nilpotent and $\tilde{X}$ is the universal cover of X. We obtain a very general theorem of *Whitehead* type by applying our comparison theorem to the homology spectral sequence of (0.2) (see Corollary 3.4); we remark that this spectral sequence violates not only assumption (iii) of [**9**] but also the refinement of (iii) mentioned on p. 58 of [**9**].

We should mention at this point that our results strengthen those of Zeeman even when the base is simply-connected—that is, in the case

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Zeeman was considering. For whereas, in Theorem 2 of [**9**], it was assumed that, in the map of fibrations

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

g induces isomorphisms g_*: $H_iE \to H_iE'$, $i \leq N$, and h induces isomorphisms h_*: $H_iB \to H_iB'$, $i \leq P$, in order to deduce that f induces isomorphisms f_*: $H_iF \to H_iF'$, $i < Q = \min(N, P-1)$ with f_*: $H_QF \to H_QF'$ surjective, we obtain (Theorem 3.2) the same conclusion where we weaken the hypotheses by only requiring g_*: $H_NE \to H_NE'$ and h_*: $H_PB \to H_PB'$ to be surjective instead of isomorphic. This improvement is essential for the deduction of a Whitehead theorem. Similarly, in our versions of Theorems 1 and 3 of [**9**]—that is, in Theorems 3.1 and 3.5—we make hypotheses which only involve surjection in the top dimensions. These improvements are rendered possible by strengthening the two fundamental lemmas of [**9**]—these improvements appear in § 2. We remark, with regard to Lemma 2.1 (which improves Lemma 3 of [**9**], insofar as the latter requires that α: $K \to K'$ be an *isomorphism* of finitely-filtered abelian grous), that only the improvement in Lemma 2.1(b) is exploited in this note. That in Lemma 2.1(a) would come into play if we worked in cohomology rather than homology, as in [**6**].

The second direction in which we generalize Zeeman's theorem is that we work modulo an acyclic Serre class C of nilpotent groups [**4**]. (Of course, in the original context of Zeeman's theorem this would amount to working modulo a Serre class of *abelian* groups in the classical sense [**7**].) Thus our assumptions and conclusions are all to be understood modulo C. We recall here the axioms on a Serre class C of nilpotent groups. Given a non-empty family C of nilpotent groups, we say that C *satisfies* (S) if, for any central extension of nilpotent groups $N \rightarrowtail G \twoheadrightarrow Q$,

$$N, Q \in C \Leftrightarrow G \in C.$$

We say that C *satisfies* (I) if

$$A \in C \Rightarrow A \text{ is finitely-generated},$$

where A is abelian; and that C *satisfies* (II) if

$$A \in C \Rightarrow \oplus A \in C$$

where the direct sum is taken over any indexing set. We say that C is *acyclic* if

$$A \in C \Rightarrow H_nA \in C, \qquad n \geq 1.$$

Then C is an acyclic Serre class if it is acyclic, satisfies (S), and satisfies (I) or (II). Our theorems apply to maps between *any* strongly quasi-nilpotent fibrations (0.1) if C is an acyclic Serre class satisfying (II); and to maps between strongly quasi-nilpotent fibrations with all spaces having finitely-generated homology in all dimensions if C is an acyclic Serre class satisfying (I). It is not hard to show that if (0.1) is a strong quasi-nilpotent fibration in which F and B have finitely-generated homology in all dimensions, then E also has finitely-generated homology in all dimensions. Since all the necessary theory of Serre classes of nilpotent groups was established in [**4**], there is no difficulty in achieving the desired generalization of the comparison theorem. We will need certain results on homology with nilpotent local coefficients and these are obtained in § 1; the argument, thanks to the results in [**4**], is no more complicated in the mod C case than in the absolute case $(C=\{1\})$. Corollary 1.4(*ii*), in the absolute case, was first noted by Dror.

In § 4 we use Corollary 3.4 to obtain a mod C version of the Blakers–Massey triad theorem in the nilpotent category, and in an appendix we discuss the modifications needed in our results if we only assume our fibration (0.1) to be quasi-nilpotent, that is, we no longer assume $\pi_1 B$ nilpotent.

We adopt a notation based on that of [**9**] but avoiding (we hope!) the more idiosyncratic features of that notation. We regard the Serre class C as fixed. Then if $\alpha\colon G \rightarrow G'$ is a homomorphism of nilpotent groups (in particular, of abelian groups) we write ιG, μG, εG to indicate that α is *C-bijective, C-injective, C-surjective* respectively; and we cross out the symbol ι, μ, ε to deny the assertion it represents. This notation enormously shortens our formulations; it is also appropriate since many of our assertions are proved by contradiction.

Unlike the statement of hypotheses in [**9**], which is purely algebraic, we state our hypotheses in topological guise in §§ 1 and 3. However, it is quite obvious that these hypotheses can be rendered purely algebraic by anybody conscientious enough to undertake the task.

It is a pleasure to acknowledge the value of conversations with Guido Mislin. In particular, he drew our attention to Quillen's paper [**6**], in Lemma 3.8 of which he formulated 'the core of the Zeeman comparison theorem'. Quillen worked in cohomology and his hypotheses were inappropriate to our concern in this paper; nevertheless his formulation provided the clue as to how to generalize Zeeman's argument.

1. Nilpotent local coefficients

Let X, X' be connected spaces with nilpotent fundamental groups; let C be an acyclic Serre class of nilpotent groups and suppose either that C

satisfies (*II*) or that C satisfies (*I*) and X, X' have finitely-generated homology groups; let $f\colon X \to X'$ be a map inducing a C-bijection $\pi_1 f\colon \pi_1 X \to \pi_1 X'$; let A, A' be nilpotent $\pi_1 X$-, $\pi_1 X'$-modules, respectively, and let $\phi\colon A \to A'$ be a module-map compatible with $\pi_1 f$. We prove three propositions relating to this situation.

PROPOSITION 1.1. *If* ιA; $\iota H_i X$, $i<n$; *and* $\varepsilon H_n X$, *then* $\iota H_i(X;A)$, $i<n$, *and* $\varepsilon H_n(X;A)$.

Recall that ιA means that $\phi\colon A \to A'$ is C-bijective; similarly $\varepsilon H_n X$ means that $H_n f\colon H_n X \to H_n X'$ is C-surjective.

Proof. We argue by induction on $c = \max(\mathrm{nil}_\pi A, \mathrm{nil}_{\pi'} A')$, where $\pi = \pi_1 X$, $\pi' = \pi_1 X'$. If $c=1$, then the coefficients are trivial modules and the conclusion easily follows from the universal coefficient theorem in homology. For the inductive step, we write $\Gamma^i = \Gamma^i_\pi A$, $\Gamma'^i = \Gamma^i_{\pi'} A'$, and consider the commutative diagram, with each vertical arrow C-bijective (Corollary 4.3 of [**4**]),

$$\begin{tikzcd}
\Gamma^c \arrow[r, tail] \arrow[d, "\phi^c"] & A \arrow[r, two heads] \arrow[d, "\phi"] & A/\Gamma^c \arrow[d, "\phi_c"] \\
\Gamma'^c \arrow[r, tail] & A' \arrow[r, two heads] & A'/\Gamma'^c
\end{tikzcd}$$

and the induced map of homology sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(X;A/\Gamma^c) \arrow[r] \arrow[d, "\Phi_1"] & H_i(X;\Gamma^c) \arrow[r] \arrow[d, "\Phi_2"] & H_i(X;A) \arrow[r] \arrow[d, "\Phi_3"] & H_i(X;A/\Gamma^c) \arrow[r] \arrow[d, "\Phi_4"] & H_{i-1}(X;\Gamma^c) \arrow[r] \arrow[d, "\Phi_5"] & \cdots \\
\cdots \arrow[r] & H_{i+1}(X';A'/\Gamma'^c) \arrow[r] & H_i(X';\Gamma'^c) \arrow[r] & H_i(X';A') \arrow[r] & H_i(X';A'/\Gamma'^c) \arrow[r] & H_{i-1}(X';\Gamma'^c) \arrow[r] & \cdots
\end{tikzcd}$$

If $i<n$, then Φ_1 is C-surjective, Φ_2, Φ_4 and Φ_5 are C-bijective, all by the inductive hypothesis, so Φ_3 is C-bijective. If $i=n$, then Φ_2, Φ_4 are C-surjective, Φ_5 is C-bijective, all by the inductive hypothesis, so Φ_3 is C-surjective.

PROPOSITION 1.2. *If* $\iota H_0(X;A)$, $\varepsilon H_1(X;A)$, $\varepsilon H_2 X$, *then* ιA.

Proof. We prove by induction on i that $\iota(A/\Gamma^i)$; the result will then follow by taking i sufficiently large. Now $H_0(X;A) = A/\Gamma^2 = A_\pi$, so we have $\iota(A/\Gamma^2)$. If $\iota(A/\Gamma^i)$, $i \geq 2$, consider the evident diagram

$$\begin{tikzcd}
H_1(X;A) \arrow[r] \arrow[d, "\Phi_1"] & H_1(X;A/\Gamma^i) \arrow[r] \arrow[d, "\Phi_2"] & H_0(X;\Gamma^i) \arrow[r] \arrow[d, "\Phi_3"] & H_0(X;A) \arrow[r] \arrow[d, "\Phi_4"] & H_0(X;A/\Gamma^i) \arrow[d, "\Phi_5"] \\
H_1(X';A') \arrow[r] & H_1(X';A'/\Gamma'^i) \arrow[r] & H_0(X';\Gamma'^i) \arrow[r] & H_0(X';A') \arrow[r] & H_0(X';A'/\Gamma'^i)
\end{tikzcd}$$

Now Φ_1 is C-surjective and Φ_4 is C-bijective by hypothesis. Since $\iota\pi_1X$ we have ιH_1X (Theorem 3.4 of **[4]**) so that, given εH_2X and the inductive hypothesis $\iota(A/\Gamma^i)$ we infer from Proposition 1.1 that Φ_2 and Φ_5 are C-bijective. It follows that Φ_3 is C-bijective. But $H_0(X;\Gamma^i)=\Gamma^i/\Gamma^{i+1}$ so that $\iota(\Gamma^i/\Gamma^{i+1})$. We now conclude that $\iota(A/\Gamma^{i+1})$ and the inductive step is complete.

PROPOSITION 1.3. *If* $\varepsilon H_0(X;A)$ *then* εA.

Proof. We first factor ϕ_2: $A/\Gamma^2\rightarrow A'/\Gamma'^2$ as

$$A/\Gamma^2 \xrightarrow{\alpha} A'/\Gamma^2_\pi A' \xrightarrow{\beta} A'/\Gamma'^2.$$

Here we construct $\Gamma^2_\pi A'$ by regarding A' as a π-module via $\pi_1 f$. Since $\iota\pi_1X$ we have ιH_1X, so that β is a C-bijection by Proposition 1.1. Thus, since ϕ_2 is C-surjective, so is α, and we must deduce that ϕ is C-surjective. This argument allows us, in proving Proposition 1.3, to assume that $\pi=\pi'$ and that π_1f is the identity.

We now exploit the commutative diagram, with $I\pi$ the augmentation ideal of π,

$$\begin{tikzcd}((I\pi)^{k-1}/(I\pi)^k)\otimes A_\pi \arrow[r, two heads] \arrow[d, "1\otimes\phi_2"] & \Gamma^k/\Gamma^{k+1} \arrow[d, "\phi^k_{k+1}"] \\ ((I\pi)^{k-1}/(I\pi)^k)\otimes A'_\pi \arrow[r, two heads] & \Gamma'^k/\Gamma'^{k+1}\end{tikzcd}, \quad k\geq 2,$$

where the horizontal arrow is given by $[\xi]\otimes[a]\mapsto[\xi a]$, $\xi\in(I\pi)^{k-1}$, $a\in A$. We note that $(I\pi)^{k-1}/(I\pi)^k=(I\pi)^{k-1}\otimes_\pi \mathbf{Z}$ and that, if H_1X is finitely-generated, so is π_1X and hence so is $(I\pi)^{k-1}$ as π-module. Thus on either hypothesis on the class C, we may infer that, ϕ_2 being C-surjective, so is $1\otimes\phi_2$ and hence also ϕ^k_{k+1}, $k\geq 2$. Thus by the 5-lemma we may prove, by induction on $(j-k)$, that ϕ^k_j: $\Gamma^k/\Gamma^j\rightarrow\Gamma'^k/\Gamma'^j$ is C-surjective, $2\leq k<j$. Setting $k=2$ and taking j sufficiently large we infer that ϕ^2: $\Gamma^2\rightarrow\Gamma'^2$ is C-surjective. Recourse to the diagram

$$\begin{tikzcd}\Gamma^2 \arrow[r, tail] \arrow[d, "\phi^2"] & A \arrow[r, two heads] \arrow[d, "\phi"] & A/\Gamma^2 \arrow[d, "\phi_2"] \\ \Gamma'^2 \arrow[r, tail] & A' \arrow[r, two heads] & A'/\Gamma'^2\end{tikzcd}$$

finally shows that ϕ is C-surjective.

We close this section by giving the explicit forms of these three propositions when X, X' are Eilenberg-MacLane spaces.

COROLLARY 1.4. *If* γ: $\pi\rightarrow\pi'$ *is a C-bijection;* A, A' *nilpotent* π-, π'-*modules, respectively;* ϕ: $A\rightarrow A'$ *a module map compatible with* γ, *then*

(*i*) *if* ιA, *then* $\iota H_i(\pi; A)$, $i \geqslant 0$;
(*ii*) *if* $\iota H_0(\pi; A)$, $\varepsilon H_1(\pi; A)$, *then* ιA;
(*iii*) *if* $\varepsilon H_0(\pi; A)$, *then* εA.

Proof. It is only necessary to observe (Theorem 3.4 of [**4**]) that if $\gamma\colon \pi \to \pi'$ is C-bijective, so is $H_i\gamma\colon H_i\pi \to H_i\pi'$, $i \geqslant 0$.

2. The Zeeman lemmas

In this section we improve the crucial Lemma 3 and 4 of [**9**] in order to be able to prove a strengthened form of the comparison theorem. We emphasize that, in this section, we are *not* concerned with strengthening the comparison theorem in order to handle quasi-nilpotent fibrations, but in order to be able to weaken the assumptions of Theorem 2 and 3 of [**9**], even in the case of simply-connected bases. This weakening of the assumptions is already necessary in order to be able to obtain the Whitehead theorem as a consequence of the comparison theorem. We will also work modulo an acyclic Serre class C instead of working absolutely, but this presents no additional difficulty.

Let $\alpha\colon A \to A'$ be a homomorphism of abelian groups. Provided that it is evident that we are referring to α, we will, as in Section 1, write ιA, μA, εA to mean that α is C-bijective, C-injective, C-surjective. By crossing out the symbol ι, μ, ε we will deny the truth of the assertion conveyed by the symbol.

Now let $\alpha\colon K \to K'$ be a homomorphism of finitely-filtered abelian groups. Let us write F_k for the kth term of either filtration. Then the following lemma improves Lemma 3 of [**9**].

LEMMA 2.1. (*a*) *If* μK *and* $\not\mu F_k/F_{k-1}$, *then* $\not\varepsilon F_j/F_{j-1}$ *for some* $j<k$.
(*b*) *If* εK *and* $\not\varepsilon F_j/F_{j-1}$, *then* $\not\mu F_k/F_{k-1}$ *for some* $k>j$.

Proof. We will be content to prove (*b*). We may assume, without loss of generality, that $F_N K = K$, $F_N K' = K'$. Let k be the largest integer such that $\not\varepsilon F_k/F_{k-1}$. Then $k \geqslant j$. Also it is easy to see by downward induction on l and the 5-lemma that εF_l, $k \leqslant l \leqslant N$. Thus εF_k, whence $\varepsilon F_k/F_{k-1}$. It follows that $k \neq j$, so that $k>j$, and that $\not\mu F_k/F_{k-1}$.

To explain the second lemma, we need notation. Let us consider the first quadrant spectral sequence E^r_{pq}, $r \geqslant 2$, with deg $d^r = (-r, r-1)$ and let us set (see [**9**])

$$\langle p, q\rangle = \bigoplus_{\substack{p' \leqslant p\\ q' \leqslant q}} E^2_{p'q'},\ [n, p, s] = d^{p-s}E^{p-s}_{p,n-p} = B^{p-s}_{s,n-1-s},\ 2 \leqslant s+2 \leqslant p \leqslant n,$$

$$[n, p, \infty] = E^\infty_{p,n-p},\ 0 \leqslant p \leqslant n. \tag{2.1}$$

Then if we consider $E^2_{p,n-p}$, $2 \leq p \leq n-1$, we see that it is filtered by the subgroups, expressed by the customary abuse of language as

$$0 \subseteq \operatorname{im} d^2 \subseteq \operatorname{im} d^3 \subseteq \cdots \subseteq \operatorname{im} d^{n-p} \subseteq \operatorname{im} d^{n+1-p} \subseteq \ker d^p \subseteq \cdots \subseteq \ker d^3 \subseteq \ker d^2 \subseteq E^2_{p,n-p},$$

and the associated graded group is given by

$$GrE^2_{p,n-p} = [n+1, p+2, p] + [n+1, p+3, p] + \cdots + [n+1, n+1, p] + [n, p, \infty] + [n, p, 0] + \cdots + [n, p, p-2]. \quad (2.2)$$

If $p=0$, 1 or n there is a simpler expression for the associated graded object; namely, if $p=0$, 1, then we stop (2.2) at $[n, p, \infty]$ and, if $p=n$, we start (2.2) at $[n, p, \infty]$. (We could simply adopt the convention that $[n, p, s]=0$ if $p=s+1$ or if $p=0$, 1 and then (2.2) holds for $0 \leq p \leq n$.)

We write $\iota'\langle p, q\rangle$ to mean that $\iota E^2_{p'q'}$ if $p'<q$, $q' \leq q$ and $\varepsilon E^2_{pq'}$ if $q' \leq q$; and prove, improving Lemma 4 of [**9**],

LEMMA 2.2. *If* $\iota'\langle p, n-1-s\rangle$ *then* $\iota[n, p, s]$.

Proof. Suppose $\not\varepsilon[n, p, s]$. Since $\varepsilon E^2_{p,n-p}$, it follows from Lemma 2.1(b) and (2.2) that $\not\mu[n, p, s_1]$ for some $s < s_1 \leq p-2$. Since $\iota E^2_{s_1,n-1-s_1}$ it follows from Lemma 2.1(a) that $\not\varepsilon[n, p_1, s_1]$ for some $s_1+2 \leq p_1 < p$. Thus, iterating this argument, we find sequences

$$s < s_1 < s_2 < \ldots, p > p_1 > p_2 > \cdots$$

with $\not\varepsilon[n, p_i, s_i]$. But this is absurd since we require $s_i+2 \leq p_i$.

The hypothesis $\not\mu[n, p, s]$ leads similarly to a contradiction.

3. The comparison theorem for maps of fibre spaces

We consider a map of fibrations, with all spaces connected,

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \\ E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd} \quad (3.1)$$

where it is assumed that $\pi_1 B$, $\pi_1 B'$ are nilpotent and operate nilpotently on HF, HF' respectively. We take an arbitrary acyclic Serre class C and we assume either that C satisfies (*II*) or that C satisfies (*I*) and the homology groups of F, F', B, B' are finitely-generated. We prove first

THEOREM 3.1. *If* $\iota H_q F$, $q < Q$, $\varepsilon H_Q F$, *and* $\iota H_p B$, $p < P$, $\varepsilon H_P B$, $P \geq 2$, *then* $\iota H_n E$, $n < N = \min(P, Q)$, $\varepsilon H_N E$.

Proof. The hypotheses immediately imply (Theorem 3.4 of [**4**]) that $\iota\pi_1 B$. It follows from Proposition 1.1 that ιE^2_{pq} for $p+q<N$ and εE^2_{pq} for $p+q=N$. It is then plain that we may pass through the spectral sequence to obtain

$$\iota E^\infty_{pq} \quad \text{for} \quad p+q<N, \qquad \varepsilon E^\infty_{pq} \quad \text{for} \quad p+q=N$$

from which the theorem follows immediately.

We next prove

THEOREM 3.2. *If $\iota H_n E$, $n<N$, $\varepsilon H_N E$; and $\iota H_p B$, $p<P$, $\varepsilon H_P B$, $P\geq 2$, then $\iota H_q F$, $q<Q=\min(N, P-1)$, $\varepsilon H_Q F$.*

We prove this as a consequence of the following more technical proposition.

PROPOSITION 3.3. *If $\iota H_n E$, $n<N$, $\varepsilon H_N E$; and $\iota H_p B$, $p<P$, $\varepsilon H_P B$, $P\geq 2$; and if $\iota H_q F$, $q<M$, then*

$$\text{if } M<Q,\ \iota E^2_{0M} \quad \text{and} \quad \varepsilon E^2_{1M}$$
$$\text{if } M=Q,\ \varepsilon E^2_{0M},$$

where $Q=\min(N, P-1)$.

Deduction of Theorem 3.2 *from Proposition* 3.3.

Given the hypotheses of Theorem 3.2, then it follows that $\iota\pi_1 B$ and that the hypotheses of Proposition 3.3 are certainly valid if $M=1$. If $Q>1$ we infer from Proposition 3.3 that ιE^2_{01} and εE^2_{11}. But $\varepsilon H_2 B$ by hypothesis so that, by Proposition 1.2, $\iota H_1 F$. Similarly if $Q=1$ we infer from Proposition 3.3 that εE^2_{01} and hence, by Proposition 1.3, that $\varepsilon H_1 F$. If $Q>1$ we may then take $M=2$. If $Q>2$ we infer, as above, that $\iota H_2 F$ and if $Q=2$ we infer that $\varepsilon H_2 F$. We continue in this way, finally obtaining $\iota H_q F$, $q<Q$, and $\varepsilon H_Q F$.

Proof of Proposition 3.3. From Proposition 1.1 we infer that $\iota'\langle P, M-1\rangle$. We first suppose $M\leq Q$ and prove εE^2_{0M}. For, if $\not\varepsilon E^2_{0M}$, then, by (2.2), either (*i*) $\not\varepsilon[M,0,\infty]$ or (*ii*) $\not\varepsilon[M+1,p,0]$ for some $2\leq p\leq M+1$.

(i) If $\not\varepsilon[M,0,\infty]$ it follows from Lemma 2.1(*b*), since $\varepsilon H_M E$, that $\not\mu[M,p_1,\infty]$ for some $0<p_1\leq M$. But $p_1\neq M$, since $[M,M,\infty]$ is a subgroup of $H_M B$ and $\iota H_M B$. Thus $0<p_1<M$. Since $\iota E^2_{p_1,M-p_1}$, it follows from Lemma 2.1(*a*) that $\not\varepsilon[M+1,p_2,p_1]$ for some $p_1+2\leq p_2\leq M+1$. But $\iota'\langle p_2, M-p_1\rangle$, contradicting Lemma 2.2.

(ii) If $\not\varepsilon[M+1,p,0]$ for some $2\leq p\leq M+1$, then, since $\varepsilon E^2_{p,M+1-p}$, it follows from Lemma 2.1(*b*) that $\not\mu[M+1,p,s]$ for some $1\leq s\leq p-2$. But $\iota'\langle p, M-s\rangle$, contradicting Lemma 2.2.

We next prove that, if $M<Q$, then μE^2_{0M}. For, if $\not\mu E^2_{0M}$ then, by (2.2), either (*iii*) $\not\mu[M,0,\infty]$ or (*iv*) $\not\mu[M+1,p,0]$ for some $2\leq p\leq M+1$.

(iii) Since $[M, 0, \infty]$ is a subgroup of $H_M E$ and $\iota H_M E$, it follows that $\mu[M, 0, \infty]$.

(iv) If $\not\mu[M+1, p, 0]$ for some $2 \leqslant p \leqslant M+1$, then, since $\iota E^2_{p,M+1-p}$, it follows from Lemma 2.1(*a*) that $\not\varepsilon[M+1, p, \infty]$ or $\not\varepsilon[M+2, p_1, p]$ for some $p+2 \leqslant p_1 \leqslant M+2$. The second possibility contradicts $\iota'\langle p_1, M+1-p\rangle$. Thus $\not\varepsilon[M+1, p, \infty]$ whence, since $\varepsilon H_{M+1}E$, it follows from Lemma 2.1(*b*) that $\not\mu[M+1, p_1, \infty]$ for some $p \leqslant p_1 \leqslant M+1$. This possibility, however, is excluded exactly as in the argument (*i*).

Finally we prove that, if $M < Q$, then εE^2_{1M}. For, if $\not\varepsilon E^2_{1M}$ then, by (2.2), either (*v*) $\not\varepsilon[M+1, 1, \infty]$ or (*vi*) $\not\varepsilon[M+2, p, 1]$, for some $3 \leqslant p \leqslant M+2$.

(v) If $\not\varepsilon[M+1, 1, \infty]$, it follows from Lemma 2.1(*b*), since $\varepsilon H_{M+1}E$, that $\not\mu[M+1, p, \infty]$ for some $1 < p \leqslant M+1$. We now argue as in (*i*).

(vi) If $\not\varepsilon[M+2, p, 1]$ for some $3 \leqslant p \leqslant M+2$ then, since $\varepsilon E^2_{p,M+2-p}$, it follows from Lemma 2.1(*b*) that $\not\mu[M+2, p, s]$ for some $1 < s \leqslant p-2$. But $\iota'\langle p, M+1-s\rangle$, contradicting Lemma 2.2.

Corollary 3.4. *Let f: $X \to X'$ be a map of nilpotent spaces and let C be an acyclic Serre class. Assume either that C satisfies (II) or that C satisfies (I) and X, X' are of finite type. Then, if $n \geqslant 2$, the following statements are equivalent.*

(*i*) $\iota H_i X$, $i < n$, *and* $\varepsilon H_n X$;

(*ii*) $\iota \pi_i X$, $i < n$, *and* $\varepsilon \pi_n X$;

(*iii*) $\iota \pi_1 X$, $\iota H_i \tilde{X}$, $i < n$ *and* $\varepsilon H_n \tilde{X}$.

Proof. The equivalence of (*ii*) and (*iii*) is classical. To establish the equivalence of (*i*) and (*iii*) we first observe that $\iota H_1 X$, $\varepsilon H_2 X$ together imply $\iota \pi_1 X$. We now consider the nilpotent fibrations

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \tilde{X}' \arrow[d] \\
X \arrow[r] \arrow[d] & X' \arrow[d] \\
K(\pi_1 X, 1) \arrow[r] & K(\pi_1 X', 1)
\end{tikzcd} \tag{3.2}$$

We next note that if X, X' are of finite type then the homology groups of all spaces in (3.2) are finitely generated. Thus we may, on the hypothesis (*i*) or (*iii*), apply Theorems 3.1, 3.2 with $P = \infty$. Then the implication (*iii*) $\Rightarrow$ (*i*) follows from Theorem 3.1 and the implication (*i*) $\Rightarrow$ (*iii*) follows from Theorem 3.2.

Remark. The equivalence of (*i*) and (*ii*), in the absolute case $C = \{1\}$, is known to Dror and has also been disovered independently by Toomer [**8**].

To complete the comparison theorem we revert to (3.1) and prove

THEOREM 3.5. *If* $\iota H_n E$, $n<N$, $\varepsilon H_N E$; *and* $\iota H_q F$, $q<Q$, $\varepsilon H_Q F$; *and* $\iota \pi_1 B$, *then* $\iota H_p B$, $p<P=\min(N, Q+1)$, $\varepsilon H_P B$.

Proof. We assume that $\iota H_p B$, $p<M$, $\varepsilon H_M B$ and show that $\iota H_M B$, $\varepsilon H_{M+1} B$, provided that $M<P$. This will prove the theorem since certainly we may begin with $M=1$. From our assumption we infer that $\iota'\langle M, Q-1\rangle$. We first prove that $\mu H_M B$. If $\not\mu E^2_{M0}$, then either (*i*) $\not\mu[M, M, \infty]$ or (*ii*) $\not\mu[M, M, p]$ for some $0\leqslant p\leqslant M-2$.

(i) If $\not\mu[M, M, \infty]$ then, since $\iota H_M E$, $\not\varepsilon[M, p_1, \infty]$ for some $0\leqslant p_1<M$. But $p_1\neq 0$ since $[M, 0, \infty]$ is a quotient of $H_M F$ and $\varepsilon H_M F$. Thus $0<p_1<M$. Now $\iota E^2_{p_1, M-p_1}$ so $\not\mu[M, p_1, s]$ for some $0\leqslant s\leqslant p_1-2$. But this contradicts $\iota\langle p_1, M-1-s\rangle$.

(ii) If $\not\mu[M, M, p]$ then, since $\not\iota E^2_{p, M-1-p}$, it follows that $\not\varepsilon[M, p_2, p]$ for some $p+2\leqslant p_2<M$. But this contradicts $\iota\langle p_2, M-1-p\rangle$.

We next prove that $\varepsilon H_{M+1} B$. If $\not\varepsilon E^2_{M+1,0}$, then either (*iii*) $\not\varepsilon[M+1, M+1, \infty]$ or (*iv*) $\not\varepsilon[M+1, M+1, p]$ for some $0\leqslant p\leqslant M-1$.

(iii) Since $\varepsilon H_{M+1} E$ it follows that $\varepsilon[M+1, m+1, \infty]$.

(iv) If $\not\varepsilon[M+1, M+1, p]$ then, since $\iota E^2_{p, M-p}$ if $p>0$ and εE^2_{0M} (for E^2_{0M} is a quotient of $H_M F$ and $\varepsilon H_M F$), it follows that $\not\mu[M, p, \infty]$ or $\not\mu[M, p, s]$ for some $0\leqslant s\leqslant p-2$. The second possibility is excluded since $\iota\langle p, M-1-s\rangle$. The first possibility implies, since $\iota H_M E$, that $\not\varepsilon[M, p_1, \infty]$ for some $0\leqslant p_1<p$ and is excluded exactly as in (*i*).

4. A Blakers–Massey triad theorem in nilpotent *C*-theory

In the authors' earlier paper [**4**], use was made of the implication (*ii*) $\Rightarrow$ (*i*) in Corollary 3.4 to study torsion phenomena in nilpotent spaces. However, in classical homotopy theory, it is often the converse implication (i) $\Rightarrow$ (ii) which plays a crucial role. In this section, we use this more delicate half of our extended Whitehead theorem to similarly extend the Blakers–Massey triad theorem.

A convenient source for the mod C Blakers–Massey triad theorem in the classical setting is ([**1**]; Paper 7) and we shall be content to show how to modify the proof contained therein so as to obtain the following more general statement:

THEOREM 4.4. *Suppose that* $(X; A, B)$ *is a CW-triad, that each of the spaces* X, A, B, $C=A\cap B$ *is (connected) nilpotent and that the inclusion* $C\subseteq X$ *induces a surjection of fundamental groups. Suppose further that* $\pi_r(X, A)\in S$, $r<q$, $q\geqslant 3$ *and* $\pi_r(X, B)\in S$, $r<p$, $p\geqslant 3$, *where* S *is an acyclic Serre class; either* S *satisfies (II) or* S *satisfies (I) and* X, A, B, C *are of finite type. Then the triad homotopy groups* $\pi_r(X; A, B)\in S$, $r<$

$p+q-1$ and the generalized Whitehead product $\pi_p(A, C) \otimes \pi_q(B, C) \to \pi_{p+q-1}(X; A, B)$ is a S-bijection.

Proof. We shall, as indicated in the preamble, restrict ourselves to commenting on the various places in the proof given in [**1**] where modification is necessary.

In Step 1, ([**1**]; p. 109), the triad $(X; A, B)$ is replaced by a triad $(X_1; A_1, B_1)$; here $X_1 = (X, \text{basepoint})^{([0,1],0)}$ is the (contractible) path space, $A_1 = (X, A)^{([0,1],0)}$, $B_1 = (X, B)^{([0,1],0)}$ *and* $C_1 = (X, C)^{([0,1],0)} = A_1 \cap B_1$. Clearly, the relative and triad homotopy groups of $(X_1; A_1, B_1)$ are the 'same' as those of $(X; A, B)$, and $(X_1; A_1, B_1)$ is a CW-triad by the result of Milnor. It is only necessary to check that each of the spaces X_1, A_1, B_1, C_1 is (connected) nilpotent. For A_1, for example, we have the fibration (up to homotopy)

$$A_1 \to A \to X,$$

where $A \to X$ is the inclusion. By assumption, $\pi_1(C) \twoheadrightarrow \pi_1(X)$, hence also $\pi_1(A) \to \pi_1(X)$, so that A_1 is connected. It then follows from the fact that A is nilpotent, using [**2**] (or [**3**]) that A_1 is also nilpotent.

In Step 2, ([**1**]; pp. 109, 110), the triad (X_1, A_1, B_1) is replaced by a triad $(X_2; A_2, B_2)$; here $X_2 = X_1 \times X_1$, $A_2 = A_1 \times B_1$, $B_2 = \text{diagonal in } X_2$ and $C_2 = A_2 \cap B_2$ is homeomorphic to C_1. Again, the relative and triad homotopy groups of $(X_2; A_2, B_2)$ are the 'same' as those of $(X_1; A_1, B_1)$, but to make $(X_2; A_2, B_2)$ a CW-triad it may be necessary to retopologize. Plainly, the spaces X_2, A_2, B_2, C_2 are nilpotent since X_1, A_1, B_1, C_1 are nilpotent.

Finally, in Step 3, ([**1**]; p. 111, 112), we use Corollary 3.4 to infer from the assertion $H_r(A_2, C_2) \in S$, $r < p+q-2$, that also $\pi_r(A_2, C_2) \in S$, $r < p+q-2$. In fact, the hypotheses $p \geq 3$, $q \geq 3$, needed elsewhere in the proof, imply a fortiori that $p+q-2 > 2$, thus insuring fulfillment of the dimensionality criterion in Corollary 3.4. (The hypotheses $p \geq 3$, $q \geq 3$ are needed only to establish the last clause of the theorem, involving the generalized Whitehead product. The 'vanishing' assertion $\pi_r(X; A, B) \in S$, $r < p+q-1$, only requires $p+q \geq 5$.) Furthermore, it is evident that one deduces the finite type of X_2, A_2, B_2, C_2 from that of X, A, B, C so that Corollary 3.4 ($(i) \Rightarrow (ii)$) may indeed be applied.

The rest of the argument in [**1**] is unchanged.

5. Appendix

So far as the theorems of § 3 are concerned, we only needed the assumption that the bases of our fibrations have nilpotent fundamental groups, in order to be able to apply nilpotent C-theory [**4**]. If we are prepared to assume that, in the map of fibrations (3.1), the induced map $\pi_1 B \twoheadrightarrow \pi_1 B'$ is an *isomorphism*, then we may dispense with the condition that $\pi_1 B$, $\pi_1 B'$ be nilpotent. In particular, in the *absolute* case ($C = \{1\}$) we will have no need of this hypothesis.

The precise modifications needed in our statements in Sections 1 and 3 are as follows; note that we now only need talk of a Serre class of *abelian* groups.

Propositions 1.1, 1.2, 1.3. We no longer assume $\pi_1 X$, $\pi_1 X'$ nilpotent, but now assume $\pi_1 f$ to be an isomorphism: the propositions then read exactly as stated, except that we must assume $\pi_1 X$ finitely-generated if C satisfies (I).

Corollary 1.4. We modify the hypothesis by assuming that γ is an isomorphism with π, π' arbitrary groups; we must further assume that π and $H_i\pi$ are finitely-generated if C satisfies (I). The conclusion of the corollary now follows.

Theorems 3.1, 3.2, 3.5. We no longer assume $\pi_1 B$, $\pi_1 B'$ nilpotent, but assume $\pi_1 B \twoheadrightarrow \pi_1 B'$ to be an isomorphism; the theorems then read exactly as stated. Note, however, that, with regard to Theorems 3.1, 3.2, there is a substantial price to be paid for dropping the requirement that $\pi_1 B$, $\pi_1 B'$ be nilpotent. For, under that hypothesis, we could actually *infer*, from the rest of the data of the theorems, that $\pi_1 B \twoheadrightarrow \pi_1 B'$ was a C-bijection.

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