ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF SOME NONSELFADJOINT PROBLEMS

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1. *Introduction and statement of the results*

In [4] we studied the question of the location of the spectrum of the linear elliptic eigenvalue problem

$$\mathscr{L}u = \lambda m u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $m \in L^\infty(\Omega)$ is a real-valued (possibly indefinite) weight function, $m \not\equiv 0$. We proved in particular that if the top-order coefficients of $\mathscr{L}$ are real-valued and smooth, and if $m \geqslant 0$ in the bounded domain Ω, then (1) has a discrete spectrum and the eigenvalues condense along the positive axis: for arbitrary ε with $0 < \varepsilon < \pi/2$ all the eigenvalues λ, except possibly a finite number of them, lie in the sector $G_\varepsilon := \{\zeta \in \mathbb{C} : |\arg \zeta| < \varepsilon\}$. The generalized eigenfunctions are relatively complete, in a sense made precise there. Here we consider the question of the asymptotic distribution of the eigenvalues. For simplicity we assume $\mathscr{L}$ to be of second order with real-valued coefficient functions; it is however straightforward to extend the results to higher-order problems and/or complex-valued lower order coefficients.

Let $\Omega \subset \mathbb{R}^N$ $(N \geqq 1)$ be a bounded domain having smooth boundary $\partial\Omega$, and let $\mathscr{L}$:

$$\mathscr{L}u = -\sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{k=1}^{N} a_k \frac{\partial u}{\partial x_k} + a_0 u$$

be a strongly uniformly elliptic differential expression with real-valued coefficient functions $a_{jk} = a_{kj} \in C^1(\bar{\Omega})$, $a_k, a_0 \in L^\infty(\Omega)$, $a_0 \geqq 0$. Further, let $m \in L^\infty(\Omega)$ be a weight function with $m \geqq 0$, $m \not\equiv 0$. Setting $\Omega_+ := \{x \in \Omega : m(x) > 0\}$, we assume that $\operatorname{meas}(\Omega_+ \backslash \operatorname{int} \Omega_+) = 0$ (which holds, for example, if m is continuous). For $t \geqq 0$ let $n(t)$ denote the number of eigenvalues λ of problem (1) with $\operatorname{Re}\lambda \leqq t$.

THEOREM 1. *$n(t) \sim c t^{\frac{1}{2}N}$ as $t \to +\infty$, where $c = \int_\Omega m(x)^{\frac{1}{2}N} \mu_{\mathscr{L}_0}(x)\, dx$ and*

$$\mu_{\mathscr{L}_0}(x) := (2\pi)^{-N} \int_{\{\xi \in \mathbb{R}^N : \Sigma a_{jk}(x)\xi_j \xi_k < 1\}} d\xi.$$

This result is well-known in the standard situation $m = \mathbb{1}$ (e.g. Agmon [1]). In the present generality it is obtained by considering $\mathscr{L}$ as a lower order perturbation of the formally selfadjoint differential expression

$$\mathscr{L}_0 := -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{jk} \frac{\partial}{\partial x_k}\right),$$

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applying the results of e.g. Pleijel [**6**] or Fleckinger and Lapidus [**2**] on the asymptotic distribution of the eigenvalues of the variational problem

$$(1_0) \qquad \mathscr{L}_0 u = \lambda m u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and using an abstract perturbation result which should be of independent interest.

Let $\mathscr{H}$ be an (infinite-dimensional) separable Hilbert space over $\mathbb{C}$, and for $0 < p \leqq \infty$ let C_p denote the two-sided ideal in $\mathscr{L}(\mathscr{H})$ consisting of the (compact) operators A for which the eigenvalues of $(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity, form an l_p-sequence. We say that $\lambda \in \mathbb{C}$ is a characteristic value of A if there is an $x \neq 0$ such that $x = \lambda A x$; of course $\lambda \neq 0$, and x is an eigenfunction of A corresponding to the eigenvalue λ^{-1}. We denote the range and null space of A by $R(A)$ and $N(A)$ respectively.

THEOREM 2. *Let $A := H(I+S)$, where $H \in \mathscr{L}(\mathscr{H})$ is compact, selfadjoint, nonnegative (that is, $(Hx, x) \geqq 0$ $\forall x \in \mathscr{H}$) and belongs to the class C_p for some $p < \infty$, while $S \in \mathscr{L}(\mathscr{H})$ is compact and such that $(I+S)$ is invertible.*

Let $\mathscr{H}$ be orthogonally decomposed as $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where $\mathscr{H}_1 = \overline{R(H)}$ and $\mathscr{H}_2 = \mathscr{N}(\mathscr{H})$, and suppose that

$$(*) \qquad \mathscr{H}_1 \cap (I+S)^{-1} \mathscr{H}_2 = \{0\}.$$

If there exists a nondecreasing function ϕ on $\mathbb{R}^+$ with $\phi(t) \to +\infty$ as $t \to +\infty$, satisfying

$$(2) \quad \frac{\phi(s)}{\phi(t)} \leqq \left(\frac{s}{t}\right)^\gamma \text{ for all sufficiently large } t < s \text{ and some constant } \gamma \text{ with } 0 < \gamma < p,$$

and such that $\lim_{t \to +\infty} n(t, H)/\phi(t) = 1$, then

$$\lim_{t \to +\infty} n(t, A)/n(t, H) = 1.$$

Here $n(t, A)$ and $n(t, H)$ denote the distribution functions of the characteristic values of A and H, respectively (or since by [**4**, Theorem 1] almost all characteristic values of A lie in the sector G_ε, $n(t, A)$ is the number of characteristic values λ of A with $\operatorname{Re} \lambda \leqq t$). Theorem 2 extends an assertion of Keldyš [**5**] (compare [**3**, Theorem 11.1]) to the case where H may have nontrivial nullspace, and relies on the results of [**4**]. Condition (2) is trivially satisfied for $\phi(t) = ct^\gamma$ $(c > 0)$.

REMARKS. 1. It is not clear whether Theorems 1 and 2 can also be obtained by a limiting procedure, looking first at definite problems, since in the nonselfadjoint case there are no monotonicity arguments available (compare with [**2**, proof of Theorem 3.1]).

2. The results on the variational eigenvalue problem (1_0) in [**2**, **6**], as well as the perturbation result in [**4**], hold for weight functions $m \in L^\infty(\Omega)$ which may change sign. We have, however, not been able to prove that the spectrum of (1) and (1_0) still have the same asymptotic distribution in the case when m is indefinite rather than semidefinite.

3. There are various recent results on perturbations of (abstract) selfadjoint operators preserving the asymptotic properties of the spectrum (for example, Ramm [**7**]). These results do not apply here since the perturbed mappings are not normal.

2. *Proofs*

Once Theorem 2 is proved, the assertion of Theorem 1 follows immediately along the lines of Hess [**4**, proof of Theorem 2], using the results of [**2**, **6**]. We therefore do not reproduce it.

The proof of Theorem 2 parallels that of [**3**, Theorem 11.1]; we give only the details of those steps which differ. We start with the following auxiliary result which is a variant of the Lemma in [**4**]:

LEMMA 1. *Let* $H \in \mathscr{L}(\mathscr{H})$ *be compact, selfadjoint and nonnegative, and let* $T \in \mathscr{L}(\mathscr{H})$ *be compact. Decompose* $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, *where* $\mathscr{H}_1 = \overline{R(H)}$, $\mathscr{H}_2 = N(H)$. *For* $0 < \varepsilon < \pi$ *let* $F_\varepsilon := \{\zeta \in \mathbb{C} : \varepsilon \leqq \arg\zeta \leqq 2\pi - \varepsilon\}$. *Then*

$$\lim_{\substack{\lambda \in F_\varepsilon \\ |\lambda| \to \infty}} \| T(I-\lambda H)^{-1}|_{\mathscr{H}_1} \| = 0.$$

To $S \in \mathscr{L}(\mathscr{H})$ as given in Theorem 2, we associate the (compact) operator T such that $I - T = (I+S)^{-1}$. Fix ε with $0 < \varepsilon < \pi$. For $\lambda \in F_\varepsilon$ the factorization

$$I - \lambda A = (I - T(I-\lambda H)^{-1})(I - \lambda H)(I+S)$$

holds. The following assertions are proved in [**4**]:

(i) $I - \lambda A$ is injective for $\lambda \in F_{\varepsilon, R} := \{\zeta \in F_\varepsilon : |\zeta| \geqq R\}$, provided R is sufficiently large;
(ii) $(I - T(I-\lambda H)^{-1})^{-1}$ is bounded on $F_{\varepsilon, R}$;
(iii) $\mathscr{H}_1$ is invariant under H and A and equals the closed linear hull of all the generalized eigenvectors of A belonging to characteristic values λ $(\neq \infty)$.

For $\lambda \in F_{\varepsilon, R}$ let $A(\lambda) := \lambda^{-1}((I-\lambda A)^{-1} - I) = A(I-\lambda A)^{-1}$ denote the Fredholm resolvent. As in [**3**, p. 279] we have

$$A(\lambda) - H(\lambda) = A(\lambda)\, T(I-\lambda H)^{-1}$$

and thus

$$A(\lambda) = H(\lambda)(I + C(\lambda)),$$

where

$$C(\lambda) = (I - T(I-\lambda H)^{-1})^{-1} - I.$$

LEMMA 2. $\| C(\lambda)|_{\mathscr{H}_1} \| \to 0$ *as* $\lambda \in F_\varepsilon$, $|\lambda| \to \infty$.

Proof. Contrary to the assertion, we assume there exist sequences $(x_n) \subset \mathscr{H}_1$ with $\| x_n \| = 1$, $(\lambda_n) \subset F_\varepsilon$ with $|\lambda_n| \to \infty$, and $(f_n) \subset \mathscr{H}$ with $0 < c \leqq \|f_n\|$, such that $C(\lambda_n) x_n = f_n$. Since, by (ii), $(I - T(I-\lambda_n H)^{-1})^{-1}$ is bounded on $F_{\varepsilon, R}$, we infer that $c \leqq \|f_n\| \leqq C$ for some C. Now

$$x_n = (I - T(I-\lambda_n H)^{-1})(x_n + f_n)$$

and, decomposing $v \in \mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ as $v = v^1 + v^2$,

$$0 = f_n - T(I-\lambda_n H)^{-1} f_n^2 - T(I-\lambda_n H)^{-1}(x_n + f_n^1). \tag{3}$$

The last term in (3) tends to 0 by Lemma 1. Since $T(I-\lambda_n H)^{-1} f_n^2 = T f_n^2$, we thus have

$$f_n^1 + (I-T) f_n^2 \to 0 \quad (n \to \infty).$$

As T is compact, we conclude that (for a subsequence) $f_n^1 \to f^1$, $f_n^2 \to f^2$, with $f = f^1 + f^2 \neq 0$ and $f^1 + (I-T)f^2 = 0$. This contradicts hypothesis (*).

Now we note that the trace $\mathrm{sp}\,(A(\lambda)^p) = \mathrm{sp}\,(A(\lambda)^p\,|_{\mathscr{H}_1})$. Thus all the remaining steps of the proof of [3, Theorem 11.1, pp. 278–283] carry through unchanged. This proves Theorem 2.

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