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Dirac Structures and Dixmier–Douady Bundles

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Dedicated to Richard Melrose on the occasion of his 60th birthday.

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A Dirac structure on a vector bundle V is a maximal isotropic sub-bundle E of the direct sum $V \oplus V^*$. We show how to associate to any Dirac structure a Dixmier–Douady bundle $\mathcal{A}_E$, that is, a $\mathbb{Z}_2$-graded bundle of C^*-algebras with typical fiber the compact operators on a Hilbert space. The construction has good functorial properties, relative to Morita morphisms of Dixmier–Douady bundles. As applications, we show that the Dixmier–Douady bundle $\mathcal{A}_G^{\mathrm{Spin}} \to G$ over a compact, connected Lie group (as constructed by Atiyah–Segal) is multiplicative, and we obtain a canonical "twisted Spin_c-structure" on spaces with group valued moment maps.

1 Introduction

A *Dixmier–Douady bundle* is a bundle of C^*-algebras $\mathcal{A} \to M$, with typical fiber $\mathbb{K}(\mathcal{H})$ the compact operators on a Hilbert space. A *Morita morphism* $\mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ of two such bundles is given by a proper map $\Phi\colon M_1 \to M_2$ of the underlying spaces, along with a bundle of bimodules $\Phi^*\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1$ locally modeled on $\mathbb{K}(\mathcal{H}_2) \circlearrowright \mathbb{K}(\mathcal{H}_1, \mathcal{H}_2) \circlearrowleft \mathbb{K}(\mathcal{H}_1)$. A classical result of Dixmier and Douady [11] states that the Morita isomorphism classes of such bundles over M are classified by the degree three cohomology group $H^3(M, \mathbb{Z})$. Dixmier–Douady bundles $\mathcal{A} \to M$ may thus be regarded as higher analogues

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of line bundles, with Morita isomorphisms replacing line bundle isomorphisms. An important example of a Dixmier–Douady bundle is the Clifford algebra bundle of a Euclidean vector bundle of even rank; a Morita isomorphism $\mathbb{C}l(V) \dashrightarrow \mathbb{C}$ amounts to a Spin_c-structure on V.

In this paper, we will relate the Dixmier–Douady theory to Dirac geometry. A (linear) *Dirac structure* $(\mathbb{V}, E)$ over M is a vector bundle $V \to M$ together with a subbundle

$$E \subset \mathbb{V} := V \oplus V^*,$$

such that E is maximal isotropic relative to the natural symmetric bilinear form on $\mathbb{V}$. Our main result is the construction of a *Dirac–Dixmier–Douady functor*, associating to any Dirac structure $(\mathbb{V}, E)$ a Dixmier–Douady bundle $\mathcal{A}_E$, and to every "strong" morphism of Dirac structures $(\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ a Morita morphism $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E'}$. For the Dirac structure given by $E = V^*$, we find that $\mathcal{A}_{V^*}$ is canonically Morita trivial, while $\mathcal{A}_V$ (for V of even rank) is canonically Morita isomorphic to the Clifford bundle $\mathbb{C}l(V)$.

The tangent bundle of a compact Lie group G carries an interesting Dirac structure $(\mathbb{T}G, E)$ known as the *Cartan–Dirac structure* [7, 31]. The associated Dixmier–Douady bundle $\mathcal{A}_E =: \mathcal{A}_G^{\mathrm{spin}}$ over G is related to the spin representation of the loop group LG. This bundle (or equivalently the corresponding bundle of projective Hilbert spaces) was described by Atiyah–Segal [6, Section 5]. From Alekseev et al. [1], it is known that the Cartan–Dirac structures is multiplicative, in the sense that the group multiplication lifts to a Dirac morphism

$$(\mathbb{T}G, E) \times (\mathbb{T}G, E) \dashrightarrow (\mathbb{T}G, E)$$

(associative in a suitable sense). Our theory therefore produces a Morita morphism

$$\mathrm{pr}_1^* \mathcal{A}_G^{\mathrm{Spin}} \otimes \mathrm{pr}_2^* \mathcal{A}_G^{\mathrm{Spin}} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}},$$

proving that the Atiyah–Segal bundle $\mathcal{A}_G^{\mathrm{Spin}}$ is *multiplicative*.

Another application is to the theory of q-Hamiltonian G-spaces, that is, spaces with G-valued moment maps $\Phi\colon M \to G$ [2]. Typical examples of such spaces are products of conjugacy classes in G. As observed by Bursztyn–Crainic [7], the structure of a q-Hamiltonian space on M defines a strong Dirac morphism $(\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G, E)$ to the Cartan–Dirac structure. Therefore, our theory gives a Morita morphism $\mathcal{A}_{TM} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$. On the other hand, as remarked above $\mathcal{A}_{TM}$ is canonically Morita isomorphic to the

Clifford bundle $\mathbb{C}l(TM)$, provided $\dim M$ is even (this is automatic if G is connected). One may think of the resulting Morita morphism

$$\mathbb{C}l(TM) \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$$

with underlying map Φ as a "twisted Spin_c-structure" on M (following the terminology of Wang [33] and Douglas [12]). It is the q-Hamiltonian analog of the canonical Spin_c-structure $\mathbb{C}l(TM) \dashrightarrow \mathbb{C}$ of a symplectic manifold.

Given a Dixmier–Douady bundle $\mathcal{A} \to M$, one defines the twisted K-homology group $K_0(M, \mathcal{A})$, as the K-homology of the C^*-algebra of sections of $\mathcal{A}$ (see [29]). If M is an even-dimensional manifold, $K_0(M, \mathbb{C}l(TM))$ contains a distinguished *Kasparov fundamental class* $[M]$. A Spin_c structure, given as a Morita morphism $\mathbb{C}l(TM) \dashrightarrow \mathbb{C}$, defines a push-forward to $K_0(\mathrm{pt}) = \mathbb{Z}$, and the image of $[M]$ under this map is the index of the associated Spin_c-Dirac operator. In the paper [20] (see also [21] for the case $G = \mathrm{SU}(2)$), we define the quantization of q-Hamiltonian G-spaces as push-forwards in twisted equivariant K-homology. The canonical twisted Spin_c-structure defined here is a key ingredient in that construction.

The organization of this paper is as follows. In Section 2 we consider Dirac structures and morphisms on vector bundles, and some of their basic examples. We observe that any Dirac morphism defines a path of Dirac structures inside a larger bundle. We introduce the "tautological" Dirac structure over the orthogonal group and show that group multiplication lifts to a Dirac morphism. Section 3 contains a quick review of some Dixmier–Douady theory. In Section 4 we give a detailed construction of Dixmier–Douady bundles from families of skew-adjoint real Fredholm operators. In Section 5 we observe that any Dirac structure on a Euclidean vector bundle gives such a family, by defining a family of boundary conditions for the operator $\frac{\partial}{\partial t}$ on the interval $[0, 1]$. Furthermore, to any Dirac morphism we associate a Morita morphism of the Dixmier–Douady bundles. In Section 7, we describe the construction of twisted Spin_c structures for q-Hamiltonian G-spaces. In Section 8, we show that the associated Hamiltonian loop group space carries a distinguished "canonical line bundle", generalizing constructions from [13, 22].

2 Dirac Structures and Dirac Morphisms

We begin with a review of linear Dirac structures on vector spaces and on vector bundles [1, 8]. In this paper, we will not consider any notions of integrability.

2.1 Dirac structures

For any vector space V, the direct sum $\mathbb{V} = V \oplus V^*$ carries a nondegenerate symmetric bilinear form extending the pairing between V and V^*,

$$\langle x_1, x_2 \rangle = \mu_1(v_2) + \mu_2(v_1), \quad x_i = (v_i, \mu_i).$$

A *morphism* $(\Theta, \omega)\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ is a linear map $\Theta\colon V \to V'$ together with a 2-form $\omega \in \wedge^2 V^*$. The composition of two morphisms $(\Theta, \omega)\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ and $(\Theta', \omega')\colon \mathbb{V}' \dashrightarrow \mathbb{V}''$ is defined as follows:

$$(\Theta', \omega') \circ (\Theta, \omega) = (\Theta' \circ \Theta, \omega + \Theta^*\omega').$$

Any morphism $(\Theta, \omega)\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ defines a relation between elements of $\mathbb{V}, \mathbb{V}'$ as follows:

$$(v, \alpha) \sim_{(\Theta,\omega)} (v', \alpha') \;\Leftrightarrow\; v' = \Theta(v),\; \alpha = \iota_v\omega + \Theta^*\alpha'.$$

Given a subspace $E \subset \mathbb{V}$, we define its *forward image* to be the set of all $x' \in \mathbb{V}'$ such that $x \sim_{(\Theta,\omega)} x'$ for some $x \in E$. For instance, V^* has forward image equal to $(V')^*$. Similarly, the *backward image* of a subspace $E' \subset \mathbb{V}'$ is the set of all $x \in \mathbb{V}$ such that $x \sim_{(\Phi,\omega)} x'$ for some $x' \in E'$. The backward image of $\{0\} \subset \mathbb{V}'$ is denoted $\ker(\Theta, \omega)$, and the forward image of $\mathbb{V}$ is denoted $\mathrm{ran}(\Theta, \omega)$.

A subspace E is called *Lagrangian* if it is maximal isotropic, that is, $E^\perp = E$. Examples are $V, V^* \subset \mathbb{V}$. The forward image of a Lagrangian subspace $E \subset \mathbb{U}$ under a Dirac morphism (Θ, ω) is again Lagrangian. On the set of Lagrangian subspaces with $E \cap \ker(\Theta, \omega) = 0$, the forward image depends continuously on E. The choice of a Lagrangian subspace $E \subset \mathbb{V}$ defines a (linear) *Dirac structure*, denoted $(\mathbb{V}, E)$. We say that (Θ, ω) defines a *Dirac* morphism

$$(\Theta, \omega)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E') \tag{1}$$

if E' is the *forward image* of E, and a *strong* Dirac morphism if, furthermore, $E \cap \ker(\Theta, \omega) = 0$. The composition of strong Dirac morphisms is again a strong Dirac morphism.

Example 2.1.

(a) Every morphism $(\Theta, \omega)\colon \mathbb{V} \dashrightarrow \mathbb{V}'$ defines a strong Dirac morphism $(\mathbb{V}, V^*) \dashrightarrow (\mathbb{V}', (V')^*)$.

(b) The zero Dirac morphism $(0,0)\colon (\mathbb{V}, E) \dashrightarrow (0,0)$ is strong if and only if $E \cap V = 0$.

(c) Given vector spaces V and V', any 2-form $\omega \in \wedge^2 V^*$ defines a Dirac morphism $(0,\omega)\colon (\mathbb{V}, V) \dashrightarrow (\mathbb{V}', (V')^*)$. It is a *strong* Dirac morphism if and only if ω is nondegenerate. (This is true in particular if $V' = 0$.)

(d) If $E = V$, a Dirac morphism $(\Theta, \omega)\colon (\mathbb{V}, V) \dashrightarrow (\mathbb{V}', E')$ is strong if and only if $\ker(\omega) \cap \ker(\Theta) = 0$. □

2.2 Paths of Lagrangian subspaces

The following observation will be used later on. Suppose (1) is a strong Dirac morphism. Then there is a distinguished path connecting the subspaces

$$E_0 = E \oplus (V')^*, \quad E_1 = V^* \oplus E', \tag{2}$$

of $\mathbb{V} \oplus \mathbb{V}'$, as follows. Define a family of morphisms $(j_t, \omega_t)\colon \mathbb{V} \dashrightarrow \mathbb{V} \oplus \mathbb{V}'$ interpolating between $(\mathrm{id} \oplus 0, 0)$ and $(0 \oplus \Theta, \omega)$:

$$j_t(v) = ((1-t)v, t\Theta(v)), \quad \omega_t = t\omega.$$

Then

$$\ker(j_t, \omega_t) = \begin{cases} 0, & t \neq 1, \\ \ker(\Theta, \omega), & t = 0. \end{cases}$$

Since (Θ, ω) is a strong Dirac morphism, it follows that E is transverse to $\ker(j_t, \omega_t)$ for all t. Hence, the forward images $E_t \subset \mathbb{V} \oplus \mathbb{V}'$ under (j_t, ω_t) are a continuous path of Lagrangian subspaces, taking on the values (2) for $t = 0, 1$. We will refer to E_t as the *standard path* defined by the Dirac morphism (1).

Given another strong Dirac morphism $(\Theta', \omega')\colon (\mathbb{V}', E') \dashrightarrow (\mathbb{V}'', E'')$, define a 2-parameter family of morphisms $(j_{tt'}, \omega_{tt'})\colon \mathbb{V} \dashrightarrow \mathbb{V} \oplus \mathbb{V}' \oplus \mathbb{V}''$ by

$$j_{tt'}(v) = ((1-t-t')v, t\Theta(v), t'\Theta'(\Theta(v))), \quad \omega_{tt'} = t\omega + t'(\omega + \Theta^*\omega')$$

Then

$$\ker(j_{tt'}, \omega_{tt'}) = \begin{cases} 0, & t + t' \neq 1, \\ \ker(\Theta, \omega), & t + t' = 1, \ t \neq 0, \\ \ker((\Theta', \omega') \circ (\Theta, \omega)), & t = 0, \ t' = 1. \end{cases}$$

In all cases, $\ker(j_{tt'}, \omega_{tt'}) \cap E = 0$, hence we obtain a continuous 2-parameter family of Lagrangian subspaces $E_{tt'} \subset \mathbb{V} \oplus \mathbb{V}' \oplus \mathbb{V}''$ by taking the forward images of E. We have,

$$E_{00} = E \oplus (V')^* \oplus (V'')^*, \quad E_{10} = V^* \oplus E' \oplus (V'')^*, \quad E_{01} = V^* \oplus (V')^* \oplus E''.$$

Furthermore, the path E_{s0} (resp. E_{0s}, $E_{1-s,s}$) is the direct sum of $(V'')^*$ (resp. of $(V')^*$, V^*) with the standard path defined by (Θ, ω) (resp. by $(\Theta', \omega') \circ (\Theta, \omega)$, (Θ', ω')).

2.3 The parity of a Lagrangian subspace

Let $\mathrm{Lag}(\mathbb{V})$ be the *Lagrangian Grassmannian* of $\mathbb{V}$, that is, the set of Lagrangian subspaces $E \subset \mathbb{V}$. It is a submanifold of the Grassmannian of subspaces of dimension $\dim V$. $\mathrm{Lag}(\mathbb{V})$ has two connected components, which are distinguished by the $\bmod 2$ dimension of the intersection $E \cap V$. We will say that E has *even* or *odd parity*, depending on whether $\dim(E \cap V)$ is even or odd. The parity is preserved under strong Dirac morphisms.

Proposition 2.2. Let $(\Theta, \omega)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ be a strong Dirac morphism. Then the parity of E' coincides with that of E. □

Proof. Clearly, E has the same parity as $E_0 = E \oplus (V')^*$, while E' has the same parity as $E_1 = V^* \oplus E'$. But the Lagrangian subspaces $E_0, E_1 \subset \mathbb{V} \oplus \mathbb{V}'$ have the same parity since they are in the same path component of $\mathrm{Lag}(\mathbb{V} \oplus \mathbb{V}')$. ■

2.4 Orthogonal transformations

Suppose V is a Euclidean vector space, with inner product B. Then the Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{V})$ is isomorphic to the orthogonal group of V, by the map associating to $A \in \mathrm{O}(V)$ the Lagrangian subspace

$$E_A = \left\{ \left((I - A^{-1})v, (I + A^{-1})\frac{v}{2} \right) \Big| \ v \in V \right\}.$$

Here B is used to identify $V^* \cong V$, and the factor of $\frac{1}{2}$ in the second component is introduced to make our conventions consistent with [1]. For instance,

$$E_{-I} = V, \quad E_I = V^*, \quad E_{A^{-1}} = (E_A)^{\mathrm{op}},$$

where we denote $E^{\mathrm{op}} = \{(v, -\alpha)|\ (v, \alpha) \in E\}$. It is easy to see that the Lagrangian subspaces corresponding to A_1 and A_2 are transverse if and only if $A_1 - A_2$ is invertible; more generally one has $E_{A_1} \cap E_{A_2} \cong \ker(A_1 - A_2)$. As a special case, taking $A_1 = A$, $A_2 = -I$ it follows that the parity of a Lagrangian subspace $E = E_A$ is determined by $\det(A) = \pm 1$.

Remark 2.3. The definition of E_A may also be understood as follows. Let V^- denote V with the opposite bilinear form $-B$. Then $V \oplus V^-$ with a split bilinear form $B \oplus (-B)$ is isometric to $\mathbb{V} = V \oplus V^*$ by the map $(a, b) \mapsto (a - b,\ (a + b)/2)$. This defines an inclusion $\kappa\colon \mathrm{O}(V) \hookrightarrow \mathrm{O}(V \oplus V^-) \cong \mathrm{O}(\mathbb{V})$. The group $\mathrm{O}(\mathbb{V})$ acts on Lagrangian subspaces, and one has $E_A = \kappa(A) \cdot V^*$. □

2.5 Dirac structures on vector bundles

The theory developed above extends to (continuous) vector bundles $V \to M$ in a straightforward way. Thus, Dirac structures $(\mathbb{V}, E)$ are now given in terms of Lagrangian subbundles $E \subset \mathbb{V} = V \oplus V^*$. Given a Euclidean metric on V, the Lagrangian sub-bundles are identified with sections $A \in \Gamma(\mathrm{O}(V))$. A Dirac morphism $(\Theta, \omega)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ is a vector bundle map $\Theta\colon V \to V'$ together with a 2-form $\omega \in \Gamma(\wedge^2 V^*)$, such that the fiberwise maps and 2-forms define Dirac morphisms $(\Theta_m, \omega_m)\colon (\mathbb{V}_m, E_m) \dashrightarrow (\mathbb{V}'_{\Phi(m)}, E'_{\Phi(m)})$. Here Φ is the map on the base underlying the bundle map Θ.

Example 2.4. For any Dirac structure $(\mathbb{V}, E)$, let $U := \mathrm{ran}(E) \subset V$ be the projection of E along V^*. If U is a sub-bundle of V, then the inclusion $U \hookrightarrow V$ defines a strong Dirac morphism, $(\mathbb{U}, U) \dashrightarrow (\mathbb{V}, E)$. More generally, if $\Phi\colon N \to M$ is such that $U := \Phi^* \mathrm{ran}(E) \subset \Phi^* V$ is a sub-bundle, then Φ together with fiberwise inclusion defines a strong Dirac morphism $(\mathbb{U}, U) \dashrightarrow (\mathbb{V}, E)$. For instance, if $(\mathbb{V}, E)$ is invariant under the action of a Lie group, one may take Φ to be the inclusion of an orbit. □

2.6 The Dirac structure over the orthogonal group

Let X be a vector space, and put $\mathbb{X} = X \oplus X^*$. The trivial bundle $V_{\mathrm{Lag}(\mathbb{X})} = \mathrm{Lag}(\mathbb{X}) \times X$ carries a *tautological Dirac structure* $(\mathbb{V}_{\mathrm{Lag}(\mathbb{X})}, E_{\mathrm{Lag}(\mathbb{X})})$, with fiber $(E_{\mathrm{Lag}(\mathbb{X})})_m$ at $m \in \mathrm{Lag}(\mathbb{X})$ the

Lagrangian subspace labeled by m. Given a Euclidean metric B on X, we may identify $\mathrm{Lag}(\mathbb{X}) = \mathrm{O}(X)$; the tautological Dirac structure will be denoted by $(\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)})$. It is equivariant for the conjugation action on $\mathrm{O}(X)$. We will now show that the tautological Dirac structure over $\mathrm{O}(X)$ is multiplicative, in the sense that group multiplication lifts to a strong Dirac morphism. Let $\Sigma\colon V_{\mathrm{O}(X)} \times V_{\mathrm{O}(X)} \to V_{\mathrm{O}(X)}$ be the bundle map, given by the group multiplication on $V_{\mathrm{O}(X)}$ viewed as a semi-direct product $\mathrm{O}(X) \ltimes X$. That is,

$$\Sigma((A_1, \xi_1), (A_2, \xi_2)) = (A_1 A_2,\ A_2^{-1}\xi_1 + \xi_2). \tag{3}$$

Let σ be the 2-form on $V_{\mathrm{O}(X)} \times V_{\mathrm{O}(X)}$, given at $(A_1, A_2) \in \mathrm{O}(X) \times \mathrm{O}(X)$ as follows:

$$\sigma_{(A_1,A_2)}((\xi_1, \xi_2), (\zeta_1, \zeta_2)) = \frac{1}{2}(B(\xi_1,\ A_2\zeta_2) - B(A_2\xi_2, \zeta_1)). \tag{4}$$

Similar to [1, Section 3.4] we have the following.

Proposition 2.5. The map Σ and 2-form σ define a strong Dirac morphism

$$(\Sigma, \sigma)\colon (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}) \times (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}).$$

This morphism is associative in the sense that

$$(\Sigma, \sigma) \circ (\Sigma \times \mathrm{id}, \sigma \times 0) = (\Sigma, \sigma) \circ (\mathrm{id} \times \Sigma, 0 \times \sigma)$$

as morphisms $(\mathbb{V}, E) \times (\mathbb{V}, E) \times (\mathbb{V}, E) \dashrightarrow (\mathbb{V}, E)$. □

Outline of Proof. Given $A_1, A_2 \in \mathrm{O}(X)$ let $A = A_1 A_2$, and put

$$e(\xi) = \left((I - A^{-1})\xi,\ (I + A^{-1})\frac{\xi}{2}\right),\ \xi \in X. \tag{5}$$

Define $e_i(\xi_i)$ similarly for A_1 and A_2. One checks that

$$e_1(\xi_1) \times e_2(\xi_2) \sim_{(\Sigma,\sigma)} e(\xi)$$

if and only if $\xi_1 = \xi_2 = \xi$. The straightforward calculation is left to the reader. It follows that every element in $E_{\mathrm{O}(X)}|_A$ is related to a unique element in $E_{\mathrm{O}(X)}|_{A_1} \times E_{\mathrm{O}(X)}|_{A_2}$. ■

2.7 Cayley transform and exponential map

The trivial bundle $V_{\wedge^2 X} = \wedge^2 X \times X$ carries a Dirac structure $(\mathbb{V}_{\wedge^2 X},\ E_{\wedge^2 X})$, with fiber at $a \in \wedge^2 X$ the graph $\mathrm{Gr}_a = \{(\iota_\mu a, \mu) |\ \mu \in X^*\}$. It may be viewed as the restriction of the tautological Dirac structure under the inclusion $\wedge^2 X \hookrightarrow \mathrm{Lag}(\mathbb{X}),\ a \mapsto \mathrm{Gr}_a$. Use a Euclidean metric B on X to identify $\wedge^2 X = \mathfrak{o}(X)$, and write $(\mathbb{V}_{\mathfrak{o}(X)},\ E_{\mathfrak{o}(X)})$. The orthogonal transformation corresponding to the Lagrangian subspace Gr_a is given by the Cayley transform $\frac{I+a/2}{I-a/2}$. Hence, the bundle map

$$\Theta\colon V_{\mathfrak{o}(X)} \to V_{\mathrm{O}(X)}, \quad (a, \xi) \mapsto \left(\frac{I+a/2}{I-a/2}, \xi\right)$$

together with the zero 2-form define a strong Dirac morphism

$$(\Theta, 0)\colon (\mathbb{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)},\ E_{\mathrm{O}(X)}),$$

with underlying map the Cayley transform. On the other hand, we may also try to lift the exponential map $\exp\colon \mathfrak{o}(X) \to \mathrm{O}(X)$. Let

$$\Pi\colon V_{\mathfrak{o}(X)} \to V_{\mathrm{O}(X)}, \quad (a, \xi) \mapsto \left(\exp(a), \frac{I - \mathrm{e}^{-a}}{a}\xi\right), \tag{6}$$

the exponential map for the semi-direct product $\mathfrak{o}(X) \ltimes X \to \mathrm{O}(X) \ltimes X$. Define a 2-form ϖ on $V_{\mathfrak{o}(X)}$ by

$$\varpi_a(\xi_1, \xi_2) = -B\left(\frac{a - \sinh(a)}{a^2}\xi_1, \xi_2\right). \tag{7}$$

The following is parallel to [1, Section 3.5].

Proposition 2.6. The map Π and the 2-form ϖ define a Dirac morphism

$$(\Pi, -\varpi)\colon (\mathbb{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}).$$

It is a strong Dirac morphism over the open subset $\mathfrak{o}(V)_\natural$, where the exponential map has maximal rank. □

Outline of Proof. Let $a \in \mathfrak{o}(X)$ and $A = \exp(a)$ be given. Let $e(\xi)$ be as in (5), and define $e_0(\xi) = (a\xi, \xi)$. One checks by straightforward calculation that

$$e_0(\xi) \sim_{(\Pi, -\varpi)} e(\xi)$$

proving that $(\Pi, -\varpi) \colon (\mathbb{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)})$ is a Dirac morphism. Suppose now that the exponential map is regular at a. By the well-known formula for the differential of the exponential map, this is equivalent to invertibility of Π_a. An element of the form $(a\xi, \xi)$ lies in $\ker(\Theta, \omega)$ if and only if $\Pi_a(a\xi) = 0$ and $\xi = \iota_{a\xi}\varpi_a$. The first condition shows that $a\xi = 0$, and then the second condition gives $\xi = 0$. Hence $e_0(\xi) \sim_{(\Pi, -\varpi)} 0 \Rightarrow \xi = 0$. Conversely, if Π_a is not invertible, and $\xi \neq 0$ is an element in the kernel, then $(a\xi, \xi) \sim_{(\Pi, -\varpi)} 0$. ■

3 Dixmier–Douady Bundles and Morita Morphisms

We give a quick review of Dixmier–Douady bundles, geared towards applications in twisted K-theory. For more information, we refer to the articles [6, 11, 16–18, 29] and the monograph [27]. Dixmier–Douady bundles are also known as *Azumaya bundles*.

3.1 Dixmier–Douady bundles

A *Dixmier–Douady bundle* is a locally trivial bundle $\mathcal{A} \to M$ of $\mathbb{Z}_2$-graded C^*-algebras, with typical fiber $\mathbb{K}(\mathcal{H})$ the compact operators on a $\mathbb{Z}_2$-graded (separable) complex Hilbert space, and with structure group $\mathrm{Aut}(\mathbb{K}(\mathcal{H})) = \mathrm{PU}(\mathcal{H})$, using the strong operator topology. The tensor product of two such bundles $\mathcal{A}_1, \mathcal{A}_2 \to M$ modeled on $\mathbb{K}(\mathcal{H}_1), \mathbb{K}(\mathcal{H}_2)$ is a Dixmier–Douady bundle $\mathcal{A}_1 \otimes \mathcal{A}_2$ modeled on $\mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For any Dixmier–Douady bundle $\mathcal{A} \to M$ modeled on $\mathbb{K}(\mathcal{H})$, the bundle of opposite C^*-algebras $\mathcal{A}^{\mathrm{op}} \to M$ is a Dixmier–Douady bundle modeled on $\mathbb{K}(\mathcal{H}^{\mathrm{op}})$, where $\mathcal{H}^{\mathrm{op}}$ denotes the opposite (or conjugate) Hilbert space.

3.2 Morita isomorphisms

A *Morita isomorphism* $\mathcal{E} \colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ between two Dixmier–Douady bundles over M is a $\mathbb{Z}_2$-graded bundle $\mathcal{E} \to M$ of Banach spaces, with a fiberwise $\mathcal{A}_2 - \mathcal{A}_1$ bimodule structure

$$\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1$$

that is locally modeled on $\mathbb{K}(\mathcal{H}_2) \circlearrowright \mathbb{K}(\mathcal{H}_1,\mathcal{H}_2) \circlearrowleft \mathbb{K}(\mathcal{H}_1)$. Here $\mathbb{K}(\mathcal{H}_1,\mathcal{H}_2)$ denotes the $\mathbb{Z}_2$-graded Banach space of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. In terms of the associated principal bundles, a Morita isomorphism is given by a lift of the structure group $\mathrm{PU}(\mathcal{H}_2)\times \mathrm{PU}(\mathcal{H}_1^{\mathrm{op}})$ of $\mathcal{A}_2\otimes\mathcal{A}_1^{\mathrm{op}}$ to $\mathrm{PU}(\mathcal{H}_2\otimes\mathcal{H}_1^{\mathrm{op}})$. The composition of two Morita isomorphisms $\mathcal{E}\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ and $\mathcal{E}'\colon \mathcal{A}_2 \dashrightarrow \mathcal{A}_3$ is given by $\mathcal{E}'\circ\mathcal{E}=\mathcal{E}'\otimes_{\mathcal{A}_2}\mathcal{E}$, the fiberwise completion of the algebraic tensor product over $\mathcal{A}_2$. In local trivializations, it is given by the composition $\mathbb{K}(\mathcal{H}_2,\mathcal{H}_3)\times\mathbb{K}(\mathcal{H}_1,\mathcal{H}_2)\to\mathbb{K}(\mathcal{H}_1,\mathcal{H}_3)$.

Example 3.1.

(a) A Morita isomorphism $\mathcal{E}\colon \mathbb{C} \dashrightarrow \mathcal{A}$ is called a *Morita trivialization* of $\mathcal{A}$, and amounts to a Hilbert space bundle $\mathcal{E}$ with an isomorphism $\mathcal{A}=\mathbb{K}(\mathcal{E})$.

(b) Any $*$-bundle isomorphism $\phi\colon \mathcal{A}_1\to\mathcal{A}_2$ may be viewed as a Morita isomorphism $\mathcal{A}_1 \dashrightarrow \mathcal{A}_2$, by taking $\mathcal{E}=\mathcal{A}_2$ with the $\mathcal{A}_2-\mathcal{A}_1$-bimodule action $x_2\cdot y\cdot x_1 = x_2\, y\phi(x_1)$.

(c) For any Morita isomorphism $\mathcal{E}\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ there is an *opposite* Morita isomorphism $\mathcal{E}^{\mathrm{op}}\colon \mathcal{A}_2 \dashrightarrow \mathcal{A}_1$, where $\mathcal{E}^{\mathrm{op}}$ is equal to $\mathcal{E}$ as a real vector bundle, but with the opposite scalar multiplication. Denoting by $\chi:\mathcal{E}\to\mathcal{E}^{\mathrm{op}}$ the anti-linear map given by the identity map of the underlying real bundle, the $\mathcal{A}_1-\mathcal{A}_2$-bimodule action reads $x_1\cdot\chi(e)\cdot x_2=\chi(x_2^*\cdot e\cdot x_1^*)$. The Morita isomorphism $\mathcal{E}^{\mathrm{op}}$ is "inverse" to $\mathcal{E}$, in the sense that there are canonical bimodule isomorphisms

$$\mathcal{E}^{\mathrm{op}}\circ\mathcal{E}\cong\mathcal{A}_1,\quad \mathcal{E}\circ\mathcal{E}^{\mathrm{op}}\cong\mathcal{A}_2. \qquad\square$$

3.3 Dixmier–Douady theorem

The Dixmier–Douady theorem (in its $\mathbb{Z}_2$-graded version) states that the Morita isomorphism classes of Dixmier–Douady bundles $\mathcal{A}\to M$ are classified by elements

$$\mathrm{DD}(\mathcal{A})\in H^3(M,\mathbb{Z})\times H^1(M,\mathbb{Z}_2),$$

called the *Dixmier–Douady class* of $\mathcal{A}$. Write $\mathrm{DD}(\mathcal{A})=(x,y)$. Letting $\hat{\mathcal{A}}$ be the Dixmier–Douady bundle obtained from $\mathcal{A}$ by forgetting the $\mathbb{Z}_2$-grading, the element x is the obstruction to the existence of a (ungraded) Morita trivialization $\hat{\mathcal{E}}\colon \mathbb{C}\dashrightarrow\hat{\mathcal{A}}$. The class y corresponds to the obstruction of introducing a compatible $\mathbb{Z}_2$-grading on $\hat{\mathcal{E}}$. In more detail, given a loop $\gamma\colon S^1\to M$ representing a homology class $[\gamma]\in H_1(M,\mathbb{Z})$, choose a Morita trivialization $(\gamma, \hat{\mathcal{F}})\colon \mathbb{C} \dashrightarrow \hat{\mathcal{A}}$. Then $y([\gamma]) = \pm 1$, depending on whether or not $\hat{\mathcal{F}}$ admits a compatible $\mathbb{Z}_2$-grading.

(a) The opposite Dixmier–Douady bundle $\mathcal{A}^{\mathrm{op}}$ has class $\mathrm{DD}(\mathcal{A}^{\mathrm{op}}) = -\mathrm{DD}(\mathcal{A})$.

(b) If $\mathrm{DD}(\mathcal{A}_i) = (x_i, y_i),\ i = 1, 2$, are the classes corresponding to two Dixmier–Douady bundles $\mathcal{A}_1, \mathcal{A}_2$ over M, then [6, Proposition 2.3]

$$\mathrm{DD}(\mathcal{A}_1 \otimes \mathcal{A}_2) = (x_1 + x_2 + \tilde{\beta}(y_1 \cup y_2),\ y_1 + y_2),$$

where $y_1 \cup y_2 \in H^2(M, \mathbb{Z}_2)$ is the cup product, and $\tilde{\beta}\colon H^2(M, \mathbb{Z}_2) \to H^3(M, \mathbb{Z})$ is the Bockstein homomorphism.

Remark 3.2. Let $\mathcal{A} \to M$ be a Dixmier–Douady bundle. Choose an open cover $\{U_i, i \in I\}$ of M such that the restriction $\mathcal{A}|_{U_i}$ is Morita trivial, and pick a Morita trivialization $\mathcal{E}_i\colon \mathbb{C} \to \mathcal{A}|_{U_i}$. On overlaps $U_{ij} = U_i \cap U_j$, the Morita trivializations are related by $\mathbb{Z}_2$-graded line bundles $L_{ij} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}_i, \mathcal{E}_j)$ (cf. 3.4 below). By construction, these come with trivializations $\theta_{ijk}\colon L_{ij} \otimes L_{jk} \otimes L_{ki} \cong \mathbb{C}$ on triple overlaps, satisfying a coherence condition on quadruple overlaps. One thus obtains a ($\mathbb{Z}_2$-graded) "gerbe" in the description of Hitchin [15, Section 1.2]. □

3.4 2-isomorphisms

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be given Dixmier–Douady bundles over M.

Definition 3.3. A *2-isomorphism* between two Morita isomorphisms

$$\mathcal{E}, \mathcal{E}'\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$$

is a continuous bundle isomorphism $\mathcal{E} \to \mathcal{E}'$, intertwining the norms, the $\mathbb{Z}_2$-gradings and the $\mathcal{A}_2 - \mathcal{A}_1$-bimodule structures. □

Equivalently, a 2-isomorphism may be viewed as a trivialization of the $\mathbb{Z}_2$-graded Hermitian line bundle

$$L = \mathrm{Hom}_{\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}') \tag{8}$$

given by the fiberwise bimodule homomorphisms. Any two Morita bimodules are related by (8) as $\mathcal{E}' = \mathcal{E} \otimes L$. It follows that the set of 2-isomorphism classes of Morita isomorphisms $\mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ is either empty, or is a principal homogeneous space (torsor) for the group $H^2(M, \mathbb{Z}) \times H^0(M, \mathbb{Z}_2)$ of $\mathbb{Z}_2$-graded line bundles.

Example 3.4. Suppose the Morita isomorphisms $\mathcal{E}$ and $\mathcal{E}'$are connected by a continuous path $\mathcal{E}_s$ of Morita isomorphisms, with $\mathcal{E}_0 = \mathcal{E},\ \mathcal{E}_1 = \mathcal{E}'$. Then they are 2-isomorphic, in fact, $L_s = \mathrm{Hom}_{\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}_s)$ is a path connecting (8) to the trivial line bundle. □

Example 3.5. Suppose $\mathcal{A}_s,\ s \in [0, 1]$ is a continuous family of Dixmier–Douady-bundles over M, that is, their union defines a Dixmier–Douady bundle $\mathcal{A} \to [0, 1] \times M$. Then there exists a continuous family of isomorphisms $\phi_s \colon \mathcal{A}_0 \to \mathcal{A}_s$, that is, an isomorphism $\mathrm{pr}_2^* \mathcal{A}_0 \cong \mathcal{A}$ of bundles over $[0, 1] \times M$. (The existence of such an isomorphism is clear in terms of the associated principal $\mathrm{PU}(\mathcal{H})$-bundles.) By composing with ϕ_0^{-1} if necessary, we may assume $\phi_0 = \mathrm{id}$. Any other such family of isomorphisms $\phi'_s \colon A_0 \to \mathcal{A}_s,\ \phi'_0 = \mathrm{id}$ is related to ϕ_s by a family L_s of line bundles, with L_0 the trivial line bundle. We conclude that the homotopy of Dixmier–Douady bundles $\mathcal{A}_s$ gives a distinguished 2-isomorphism class of isomorphisms $\mathcal{A}_0 \to \mathcal{A}_1$. □

3.5 Clifford algebra bundles

Suppose that $V \to M$ is a Euclidean vector bundle of rank n. A Spin_c *structure on* V is given by an orientation on V together with a lift of the structure group of V from $\mathrm{SO}(n)$ to $\mathrm{Spin}_c(n)$, where $n = \mathrm{rk}(V)$. According to Connes [10] and Plymen [24], this is equivalent to Definition 3.6 in terms of Dixmier–Douady bundles.

Recall that if n is even, then the associated bundle of complex Clifford algebras $\mathbb{C}\mathrm{l}(V)$ is a Dixmier–Douady bundle, modeled on $\mathbb{C}\mathrm{l}(\mathbb{R}^n) = \mathrm{End}(\wedge \mathbb{C}^{n/2})$. In this case, a Spin_c structure may be defined to be a Morita trivialization $\mathcal{S} \colon \mathbb{C} \dashrightarrow \mathbb{C}\mathrm{l}(V)$, with $\mathcal{S}$ being the associated *spinor bundle*. To include the case of odd rank, it is convenient to introduce

$$\tilde{V} = V \oplus \mathbb{R}^n, \quad \widetilde{\mathbb{C}\mathrm{l}}(V) := \mathbb{C}\mathrm{l}(\tilde{V}).$$

Definition 3.6. A Spin_c *structure* on a Euclidean vector bundle V is a Morita trivialization

$$\tilde{\mathcal{S}} \colon \mathbb{C} \dashrightarrow \widetilde{\mathbb{C}\mathrm{l}}(V).$$

The bundle $\tilde{S}$ is called the corresponding *spinor bundle*. An isomorphism of two Spin_c structures is a 2-isomorphism of the defining Morita trivializations. □

If n is even, one recovers $\mathcal{S}$ by composing with the Morita isomorphism $\widetilde{\mathbb{C}\mathrm{l}}(V) \dashrightarrow \mathbb{C}\mathrm{l}(V)$. The Dixmier–Douady class (x, y) of $\widetilde{\mathbb{C}\mathrm{l}}(V)$ is the obstruction to the existence of a Spin_c structure: In fact, x is the third integral Stiefel–Whitney class $\tilde{\beta}(w_2(V)) \in H^3(M, \mathbb{Z})$, while y is the first Stiefel–Whitney class $w_1(V) \in H^1(M, \mathbb{Z}_2)$, that is, the obstruction to orientability of V.

Any two Spin_c-structures on V differ by a $\mathbb{Z}_2$-graded Hermitian line bundle, and an isomorphism of Spin_c structures amounts to a trivialization of this line bundle. Observe that there is a Morita trivialization

$$\wedge \tilde{V}^{\mathbb{C}} \colon \mathbb{C} \dashrightarrow \widetilde{\mathbb{C}\mathrm{l}}(V \oplus V) = \widetilde{\mathbb{C}\mathrm{l}}(V) \otimes \widetilde{\mathbb{C}\mathrm{l}}(V)$$

defined by the complex structure on $\tilde{V} \oplus \tilde{V} \cong \tilde{V} \otimes \mathbb{R}^2$. Hence, given a Spin_c structure, we can define the Hermitian line bundle

$$K_{\tilde{S}} = \mathrm{Hom}_{\widetilde{\mathbb{C}\mathrm{l}}(V \oplus V)}(\tilde{S} \otimes \tilde{S},\ \wedge \tilde{V}^{\mathbb{C}}). \tag{9}$$

(If n is even, one may omit the $\sim$'s.) This is the *canonical line bundle* of the Spin_c structure. If the Spin_c structure on V is defined by a complex structure J, then the canonical bundle coincides with $\det(V_-) = \wedge^{n/2} V_-$, where $V_- \subset V^{\mathbb{C}}$ is the $-\mathrm{i}$ eigenspace of J.

3.6 Morita morphisms

It is convenient to extend the notion of Morita isomorphisms of Dixmier–Douady bundles, allowing nontrivial maps on the base. A *Morita morphism*

$$(\Phi, \mathcal{E}) \colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2 \tag{10}$$

of bundles $\mathcal{A}_i \to M_i,\ i = 1, 2$ is a continuous map $\Phi \colon M_1 \to M_2$ together with a Morita isomorphism $\mathcal{E} \colon \mathcal{A}_1 \dashrightarrow \Phi^* \mathcal{A}_2$. A given map Φ lifts to such a Morita morphism if and only if $\mathrm{DD}(\mathcal{A}_1) = \Phi^* \mathrm{DD}(\mathcal{A}_2)$. Composition of Morita morphisms is defined as $(\Phi', \mathcal{E}') \circ (\Phi, \mathcal{E}) = (\Phi' \circ \Phi,\ \Phi^* \mathcal{E}' \circ \mathcal{E})$. If $\mathcal{E} \colon \mathbb{C} \dashrightarrow \mathcal{A}$ is a Morita trivialization, we can think of $\mathcal{E}^{\mathrm{op}} \colon \mathcal{A} \dashrightarrow \mathbb{C}$ as a Morita morphism covering the map $M \to \mathrm{pt}$. As mentioned in Section 1, a Morita morphism (10) such that Φ is *proper* induces a push-forward map in twisted K-homology.

3.7 Equivariance

The Dixmier–Douady theory generalizes to the G-equivariant setting, where G is a compact Lie group. G-equivariant Dixmier–Douady bundles over a G-space M are classified by $H^3_G(M,\mathbb{Z}) \times H^1_G(M,\mathbb{Z}_2)$. If M is a point, a G-equivariant Dixmier–Douady bundle $\mathcal{A} \to \mathrm{pt}$ is of the form $\mathcal{A} = \mathbb{K}(\mathcal{H})$, where $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space with an action of a central extension $\hat{G}$ of G by $\mathrm{U}(1)$. (It is a well-known fact that $H^3_G(\mathrm{pt},\mathbb{Z}) = H^3(BG,\mathbb{Z})$ classifies such central extensions.) The definition of Spin_c structures in terms of Morita morphisms extends to the G-equivariant in the obvious way.

4 Families of skew-adjoint real Fredholm operators

In this section, we will explain how a continuous family of skew-adjoint Fredholm operators on a bundle of real Hilbert spaces defines a Dixmier–Douady bundle. The construction is inspired by ideas in Atiyah-Segal [6], Carey–Mickelsson–Murray [9, 23], and Freed–Hopkins–Teleman [14, Section 3].

4.1 Infinite-dimensional Clifford algebras

We briefly review the spin representation for infinite-dimensional Clifford algebras. Excellent sources for this material are the book [25] by Plymen and Robinson and the article [5] by Araki.

Let $\mathcal{V}$ be an infinite-dimensional real Hilbert space, and $\mathcal{V}^{\mathbb{C}}$ its complexification. The Hermitian inner product on $\mathcal{V}^{\mathbb{C}}$ will be denoted $\langle\cdot,\cdot\rangle$, and the complex conjugation map by $v \mapsto v^*$. Just as in the finite-dimensional case, one defines the Clifford algebra $\mathbb{C}\mathrm{l}(\mathcal{V})$ as the $\mathbb{Z}_2$-graded unital complex algebra with odd generators $v \in \mathcal{V}$ and relations, $vv = \langle v, v\rangle$. The Clifford algebra carries a unique anti-linear anti-involution $x \mapsto x^*$ extending the complex conjugation on $\mathcal{V}^{\mathbb{C}}$, and a unique norm $||\cdot||$ satisfying the C^*-condition $||x^*x|| = ||x||^2$. Thus, $\mathbb{C}\mathrm{l}(\mathcal{V})$ is a $\mathbb{Z}_2$-graded pre-C^*-algebra.

A *(unitary) module over* $\mathbb{C}\mathrm{l}(\mathcal{V})$ is a complex $\mathbb{Z}_2$-graded Hilbert space $\mathcal{E}$ together with a $*$-homomorphism $\varrho\colon \mathbb{C}\mathrm{l}(\mathcal{V}) \to \mathcal{L}(\mathcal{E})$ preserving $\mathbb{Z}_2$-gradings. Here, $\mathcal{L}(\mathcal{E})$ is the $*$-algebra of bounded linear operators, and the condition on the grading means that $\varrho(v)$ acts as an odd operator for each $v \in \mathcal{V}^{\mathbb{C}}$.

We will view $\mathcal{L}(\mathcal{V})$ (the bounded $\mathbb{R}$-linear operators on $\mathcal{V}$) as an $\mathbb{R}$-linear subspace of $\mathcal{L}(\mathcal{V}^{\mathbb{C}})$. Operators in $\mathcal{L}(\mathcal{V})$ will be called *real*. A real skew-adjoint operator $J \in \mathcal{L}(\mathcal{V})$ is called an *orthogonal complex structure* on $\mathcal{V}$ if it satisfies $J^2 = -I$. Note $J^* = -J = J^{-1}$, so that $J \in \mathrm{O}(\mathcal{V})$.

The orthogonal complex structure defines a decomposition $\mathcal{V}^{\mathbb{C}} = \mathcal{V}_+ \oplus \mathcal{V}_-$ into maximal isotropic subspaces $\mathcal{V}_\pm = \ker(J \mp \mathrm{i}) \subset \mathcal{V}^{\mathbb{C}}$. Note $v \in \mathcal{V}_+ \Leftrightarrow v^* \in \mathcal{V}_-$. Define a Clifford action of $\mathbb{C}\mathrm{l}(\mathcal{V})$ on $\wedge\mathcal{V}_+$ by the formula

$$\rho(v) = \sqrt{2}(\epsilon(v_+) + \iota(v_-)),$$

writing $v = v_+ + v_-$ with $v_\pm \in \mathcal{V}_\pm$. Here ,$\epsilon(v_+)$ denotes exterior multiplication by v_+, while the contraction $\iota(v_-)$ is defined as the unique derivation such that $\iota(v_-)w = \langle v_-^*, w\rangle$ for $w \in \mathcal{V}^{\mathbb{C}} \subset \wedge\mathcal{V}^{\mathbb{C}}$. Passing to the Hilbert space completion one obtains a unitary $\mathbb{Z}_2$-graded Clifford module

$$\mathcal{S}_J = \overline{\wedge\mathcal{V}_+},$$

called the *spinor module* or *Fock representation* defined by J.

The equivalence problem for Fock representations was solved by Shale and Stinespring [32]. See also [25, Theorem 3.5.2]. Let $\mathcal{L}_{\mathrm{HS}}(\mathcal{V})$ be the space of real Hilbert–Schmidt operators on $\mathcal{V}$.

Theorem 4.1 (Shale–Stinespring). The $\mathbb{C}\mathrm{l}(\mathcal{V})$-modules $\mathcal{S}_1$ and $\mathcal{S}_2$ defined by orthogonal complex structures J_1 and J_2 are unitarily isomorphic (up to possible reversal of the $\mathbb{Z}_2$-grading) if and only if $J_1 - J_2 \in \mathcal{L}_{\mathrm{HS}}(\mathcal{V})$. In this case, the unitary operator implementing the isomorphism is unique up to a scalar $z \in \mathrm{U}(1)$. The implementer has even or odd parity, according to the parity of $\frac{1}{2}\dim\ker(J_1 + J_2) \in \mathbb{Z}$. □

Definition 4.2. [14, 30, p. 193] Two orthogonal complex structures J_1 and J_2 on a real Hilbert space $\mathcal{V}$ are called *equivalent* (written $J_1 \sim J_2$) if their difference is Hilbert–Schmidt. An equivalence class of complex structures on $\mathcal{V}$ (resp. on $\mathcal{V} \oplus \mathbb{R}$) is called an even (resp. odd) *polarization* of $\mathcal{V}$. □

By Theorem 4.1, the $\mathbb{Z}_2$-graded C^*-algebra $\mathbb{K}(\mathcal{S}_J)$ depends only on the equivalence class of J, in the sense that there exists a canonical identification $\mathbb{K}(\mathcal{S}_{J_1}) \equiv \mathbb{K}(\mathcal{S}_{J_2})$ whenever $J_1 \sim J_2$. That is, any polarization of $\mathcal{V}$ determines a Dixmier–Douady algebra.

4.2 Skew-adjoint Fredholm operators

Suppose D is a real skew-adjoint (possibly unbounded) Fredholm operator on $\mathcal{V}$, with dense domain $\mathrm{dom}(D) \subset \mathcal{V}$. In particular, D has a finite-dimensional kernel, and 0 is an isolated point of the spectrum. Let J_D denote the real skew-adjoint operator,

$$J_D = i\,\mathrm{sign}\left(\frac{1}{i}D\right)$$

(using functional calculus for the self-adjoint operator $\frac{1}{i}D$). Thus J_D is an orthogonal complex structure on $\ker(D)^\perp$, and vanishes on $\ker(D)$. If $\ker(D) = 0$, then we may also write $J_D = \frac{D}{|D|}$. The same definition of J_D also applies to complex skew-adjoint Fredholm operators. We have the following.

Proposition 4.3. Let D be a (real or complex) skew-adjoint Fredholm operator, and Q a skew-adjoint Hilbert–Schmidt operator. Then $J_{D+Q} - J_D$ is Hilbert–Schmidt. □

The following simple proof was shown to us by Gian-Michele Graf.

Proof. Choose $\epsilon > 0$ so that the spectrum of $D, D+Q$ intersects the set $|z| < 2\epsilon$ only in $\{0\}$. Replacing D with $D + i\epsilon$ if necessary, and noting that $J_{D+i\epsilon} - J_D$ has a finite rank, we may thus assume that 0 is not in the spectrum of D or of $D+Q$. One then has the following presentation of J_D as a Riemannian integral of the resolvent $R_z(D) = (D-z)^{-1}$,

$$J_D = -\frac{1}{\pi}\int_{-\infty}^{\infty} R_t(D)\,\mathrm{d}t,$$

convergent in the strong topology. Using a similar expression for J_{D+Q} and the second resolvent identity $R_t(D+Q) - R_t(D) = -R_t(D+Q)\,Q\,R_t(D)$,

we obtain

$$J_{D+Q} - J_D = \frac{1}{\pi}\int_{-\infty}^{\infty} R_t(D+Q)\,Q\,R_t(D)\,\mathrm{d}t.$$

Let $a > 0$ be such that the spectrum of $D,\ D+Q$ does not meet the disk $|z| \le a$. Then $||R_t(D)||,\ ||R_t(D+Q)|| \le (t^2 + a^2)^{-1/2}$ for all $t \in \mathbb{R}$. Hence

$$||R_t(D+Q)\,Q\,R_t(D)||_{\mathrm{HS}} \le \frac{1}{t^2 + a^2}||Q||_{\mathrm{HS}},$$

using $||AB||_{\mathrm{HS}} \le ||A||\,||B||_{\mathrm{HS}}$. Since $\int (t^2 + a^2)^{-1}\,\mathrm{d}t = \pi/a$, we obtain the estimate

$$||J_{D+Q} - J_D||_{\mathrm{HS}} \le \frac{1}{a}||Q||_{\mathrm{HS}}. \tag{11}$$

■

A real skew-adjoint Fredholm operator D on $\mathcal{V}$ will be called of *even* (resp. *odd*) *type* if $\ker(D)$ has even (resp. odd) dimension. As in [14, Section 3.1], we associate to any D of even type the even polarization defined by the orthogonal complex structures $J \in \mathrm{O}(\mathcal{V})$ such that $J - J_D$ is Hilbert–Schmidt. For D of odd type, we similarly obtain an odd polarization by viewing J_D as an operator on $\mathcal{V} \oplus \mathbb{R}$ (equal to 0 on $\mathbb{R}$).

Two skew-adjoint real Fredholm operators D_1 and D_2 on $\mathcal{V}$ will be called *equivalent* (written $D_1 \sim D_2$) if they define the same polarization of $\mathcal{V}$, and hence the same Dixmier–Douady algebra $\mathcal{A}$. Equivalently, D_i have the same parity and $J_{D_1} - J_{D_2}$ is Hilbert–Schmidt. In particular, $D \sim D + Q$ whenever Q is a skew-adjoint Hilbert–Schmidt operator. In the even case, we can always choose Q so that $D + Q$ is invertible, while in the odd case we can choose such a Q after passing to $\mathcal{V} \oplus \mathbb{R}$.

Remark 4.4. The estimate (11) show that for fixed D (such that D and $D + Q$ have trivial kernels), the difference $J_{D+Q} - J_D \in \mathcal{L}_{\mathrm{HS}}(\mathcal{X})$ depends continuously on Q in the Hilbert–Schmidt norm. On the other hand, it also depends continuously on D relative to the norm resolvent topology [28, p. 284]. This follows from the integral representation of $J_{D+Q} - J_D$, together with resolvent identities such as

$$R_t(D') - R_t(D) = R_t(D')R_1(D')^{-1}(R_1(D') - R_1(D))R_1(D)^{-1}R_t(D)$$

giving estimates $||R_t(D') - R_t(D)|| \leq (t^2 + a^2)^{-1}\, ||R_1(D') - R_1(D)||$ for $a > 0$ such that the spectrum of D and D' does not meet the disk of radius a. □

4.3 Polarizations of bundles of real Hilbert spaces

Let $\mathcal{V} \to M$ be a bundle of real Hilbert spaces, with typical fiber $\mathcal{X}$ and with structure group $\mathrm{O}(\mathcal{X})$ (using the norm topology). A polarization on $\mathcal{V}$ is a family of polarizations on $\mathcal{V}_m$, depending continuously on m. To make this precise, fix an orthogonal complex structure $J_0 \in \mathrm{O}(\mathcal{X})$, and let $\mathcal{L}_{\mathrm{res}}(\mathcal{X})$ be the Banach space of bounded linear operators S such that $[S, J_0]$ is Hilbert–Schmidt, with norm $\|S\| + \|[S, J_0]\|_{\mathrm{HS}}$. Define the *restricted orthogonal group* $\mathrm{O}_{\mathrm{res}}(\mathcal{X}) = \mathrm{O}(\mathcal{X}) \cap \mathcal{L}_{\mathrm{res}}(\mathcal{X})$, with the subspace topology. It is a Banach Lie group, with Lie algebra $\mathfrak{o}_{\mathrm{res}}(\mathcal{X}) = \mathfrak{o}(\mathcal{X}) \cap \mathcal{L}_{\mathrm{res}}(\mathcal{X})$. The unitary group $\mathrm{U}(\mathcal{X}) = \mathrm{U}(\mathcal{X}, J_0)$ relative to J_0, equipped with the norm topology is a Banach subgroup of $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$. For more details on the restricted orthogonal group, we refer to Araki [5] or Pressley–Segal[26].

Definition 4.5. An even *polarization* of the real Hilbert space bundle $\mathcal{V} \to M$ is a reduction of the structure group $\mathrm{O}(\mathcal{X})$ to the restricted orthogonal group $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$. An odd polarization of $\mathcal{V}$ is an even polarization of $\mathcal{V} \oplus \mathbb{R}$. □

Thus, a polarization is described by a system of local trivializations of $\mathcal{V}$ the transition functions of which are continuous maps into $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$. Any global complex structure on $\mathcal{V}$ defines a polarization, but not all polarizations arise in this way.

Proposition 4.6. Suppose $\mathcal{V} \to M$ comes equipped with a polarization. For $m \in M$, let $\mathcal{A}_m$ be the Dixmier–Douady algebra defined by the polarization on $\mathcal{V}_m$. Then $\mathcal{A} = \bigcup_{m \in M} \mathcal{A}_m$ is a Dixmier–Douady bundle. □

Proof. We consider the case of an even polarization (for the odd case, replace $\mathcal{V}$ with $\mathcal{V} \oplus \mathbb{R}$). By assumption, the bundle $\mathcal{V}$ has a system of local trivializations with transition functions in $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$. Let $\mathcal{S}_0$ be the spinor module over $\mathbb{C}\mathrm{l}(\mathcal{X})$ defined by J_0, and $\mathrm{PU}(\mathcal{S}_0)$ the projective unitary group with the strong operator topology. A version of the Shale–Stinespring theorem [25, Theorem 3.3.5] says that an orthogonal transformation is implemented as a unitary transformation of $\mathcal{S}_0$ if and only if it lies in $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$, and in this case the implementer is unique up to scalar. According to Araki [5, Theorem 6.10(7)], the resulting group homomorphism $\mathrm{O}_{\mathrm{res}}(\mathcal{X}) \to \mathrm{PU}(\mathcal{S}_0)$ is continuous. That is, $\mathcal{A}$ admits the structure group $\mathrm{PU}(\mathcal{S}_0)$ with the strong topology. ■

In terms of the principal $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$-bundle $\mathcal{P} \to M$ defined by the polarization of $\mathcal{V}$, the Dixmier–Douady bundle is an associated bundle

$$\mathcal{A} = \mathcal{P} \times_{\mathrm{O}_{\mathrm{res}}(\mathcal{X})} \mathbb{K}(\mathcal{S}_0).$$

4.4 Families of skew-adjoint Fredholm operators

Suppose now that $D = \{D_m\}$ is a family of (possibly unbounded) real skew-adjoint Fredholm operators on $\mathcal{V}_m$, depending continuously on $m \in M$ in the norm resolvent sense [28, p. 284]. That is, the bounded operators $(D_m - I)^{-1} \in \mathcal{L}(\mathcal{V}_m)$ define a continuous section of the bundle $\mathcal{L}(\mathcal{V})$ with the norm topology. The map $m \mapsto \dim\ker(D_m)$ is locally constant mod 2. The family D will be called of even (resp. odd) type if all $\dim\ker(D_m)$ are even (resp. odd). Each D_m defines an even (resp. odd) polarization of $\mathcal{V}_m$, given by the complex structures on $\mathcal{V}_m$ or $\mathcal{V}_m \oplus \mathbb{R}$ the difference with J_{D_m} of which is Hilbert–Schmidt.

Proposition 4.7. Let $D = \{D_m\}$ be a family of (possibly unbounded) real skew-adjoint Fredholm operators on $\mathcal{V}_m$, depending continuously on $m \in M$ in the norm resolvent sense. Then the corresponding family of polarizations on $\mathcal{V}_m$ depends continuously on m in the sense of Definition 4.5. That is, D determines a polarization of $\mathcal{V}$. □

Proof. We assume that the family D is of even type. (The odd case is dealt with by adding a copy of $\mathbb{R}$.) We will show the existence of a system of local trivializations

$$\phi_\alpha : \mathcal{V}|_{U_\alpha} = U_\alpha \times \mathcal{X}$$

and skew-adjoint Hilbert–Schmidt perturbations $Q_\alpha \in \Gamma(\mathcal{L}_{\mathrm{HS}}(\mathcal{V}|_{U_\alpha}))$ of $D|_{U_\alpha}$, continuous in the Hilbert–Schmidt norm (The sub-bundle $\mathcal{L}_{\mathrm{HS}}(\mathcal{V}) \subset \mathcal{L}(\mathcal{V})$ carries a topology, where a sections is continuous at $m \in M$ if its expression in a local trivialization of $\mathcal{V}$ near m is continuous. (This is independent of the choice of trivialization.), so that

(i) $\ker(D_m + Q_\alpha|_m) = 0$ for all $m \in U_\alpha$, and
(ii) $\phi_\alpha \circ J_{D+Q_\alpha} \circ \phi_\alpha^{-1} = J_0$.

The transition functions $\chi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \to \mathrm{O}(\mathcal{X})$ will then take values in $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$: Indeed, by Proposition 4.3 the difference $J_{D+Q_\beta} - J_{D+Q_\alpha}$ is Hilbert–Schmidt, and (using (11) and Remark 4.4) it is a continuous section of $\mathcal{L}_{\mathrm{HS}}(\mathcal{V})$ over $U_\alpha \cap U_\beta$. Conjugating by ϕ_α, and using (ii) it follows that

$$\chi_{\alpha\beta}^{-1} \circ J_0 \circ \chi_{\alpha\beta} - J_0 \ : \ U_\alpha \cap U_\beta \to \mathcal{L}(\mathcal{X}) \tag{12}$$

takes values in Hilbert–Schmidt operators, and is continuous in the Hilbert–Schmidt norm. Hence the $\chi_{\alpha\beta}$ are continuous functions into $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$.

It remains to construct the desired system of local trivializations. It suffices to construct such a trivialization near any given $m_0 \in M$. Pick a continuous family of skew-adjoint Hilbert–Schmidt operators Q so that $\ker(D_{m_0} + Q_{m_0}) = 0$. (We may even take Q of finite rank.) Hence $J_{D_{m_0}+Q_{m_0}}$ is a complex structure. Choose an isomorphism $\phi_{m_0} : \mathcal{V}_{m_0} \to \mathcal{X}$ intertwining $J_{D_{m_0}+Q_{m_0}}$ with J_0, and extend to a local trivialization $\phi : \mathcal{V}|_U \to U \times \mathcal{X}$ over a neighborhood U of m_0. We may assume that $\ker(D_m + Q_m) = 0$ for $m \in U$, defining complex structures $J_m = \phi_m \circ J_{D_m+Q_m} \circ \phi_m^{-1}$. By construction $J_{m_0} = J_0$, and hence $||J_m - J_0|| < 2$ after U is replaced by a smaller neighborhood if necessary.

By [25, Theorem 3.2.4], Condition (ii) guarantees that

$$g_m = (I - J_m J_0)\, |I - J_m J_0|^{-1}$$

gives a well-defined continuous map $g\colon U \to \mathrm{O}(\mathcal{X})$ with $J_m = g_m J_0 g_m^{-1}$. Hence, replacing ϕ with $g \circ \phi$ we obtain a local trivialization satisfying (i) and (ii). ∎

To summarize, a continuous family $D = \{D_m\}$ of skew-adjoint real Fredholm operators on $\mathcal{V}$ determines a polarization of $\mathcal{V}$. The fibers $\mathcal{P}_m$ of the associated principal $\mathrm{O}_{\mathrm{res}}(\mathcal{X})$-bundle $\mathcal{P} \to M$ defining the polarization are given as the set of isomorphisms of real Hilbert spaces $\phi_m\colon \mathcal{V}_m \to \mathcal{X}$ such that $J_0 - \phi_m J_{D_m} \phi_m^{-1}$ is Hilbert–Schmidt. In turn, the polarization determines a Dixmier–Douady bundle $\mathcal{A} \to M$.

We list some elementary properties of this construction:

(a) Suppose $\mathcal{V}$ has a finite rank. Then $\mathcal{A} = \mathbb{C}\mathrm{l}(\mathcal{V})$ if the rank is even, and $\mathcal{A} = \mathbb{C}\mathrm{l}(\mathcal{V} \oplus \mathbb{R})$ if the rank is odd. In both cases, $\mathcal{A}$ is canonically Morita isomorphic to $\widetilde{\mathbb{C}\mathrm{l}}(V)$.

(b) If $\ker(D) = 0$ everywhere, the complex structure $J = D|D|^{-1}$ gives a global a spinor module $\mathcal{S}$, defining a Morita trivialization

$$\mathcal{S}\colon \mathbb{C} \dashrightarrow \mathcal{A}.$$

(c) If $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ and $D = D' \oplus D''$, the corresponding Dixmier–Douady algebras satisfy $\mathcal{A} \cong \mathcal{A}' \otimes \mathcal{A}''$, provided the kernels of D' or D'' are even-dimensional. If both D' and D'' have odd-dimensional kernels, we obtain $\mathcal{A} \otimes \mathbb{C}\mathrm{l}(\mathbb{R}^2) \cong \mathcal{A}' \otimes \mathcal{A}''$. In any case, $\mathcal{A}$ is canonically Morita isomorphic to $\mathcal{A}' \otimes \mathcal{A}''$.

(d) Combining the three items above, it follows that if $\mathcal{V}' = \ker(D)$ is a subbundle of $\mathcal{V}$, then there is a canonical Morita isomorphism

$$\widetilde{\mathbb{C}\mathrm{l}}(\mathcal{V}') \dashrightarrow \mathcal{A}.$$

(e) Given a G-equivariant family of skew-adjoint Fredholm operators (with G a compact Lie group) one obtains a G-Dixmier–Douady bundle.

Suppose D_1 and D_2 are two families of skew-adjoint Fredholm operators as in Proposition 4.7. We will call these families equivalent and write $D_1 \sim D_2$ if they define the same polarization of $\mathcal{V}$, and therefore the same Dixmier–Douady bundle $\mathcal{A} \to M$. We stress that different polarizations can induce *isomorphic* Dixmier–Douady bundles, however, the isomorphism is usually not canonical.

5 From Dirac structures to Dixmier–Douady bundles

We will now use the constructions from the last section to associate to every Dirac structure $(\mathbb{V}, E)$ over M a Dixmier–Douady bundle $\mathcal{A}_E \to M$, and to every strong Dirac morphism $(\Theta, \omega)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')$ a Morita morphism. The construction is functorial "up to 2-isomorphisms".

5.1 The Dixmier–Douady algebra associated to a Dirac structure

Let $(\mathbb{V}, E)$ be a Dirac structure over M. Pick a Euclidean metric on V, and let $\mathcal{V} \to M$ be the bundle of real Hilbert spaces with fibers

$$\mathcal{V}_m = L^2([0,1], V_m).$$

Let $A \in \Gamma(\mathrm{O}(V))$ be the orthogonal section corresponding to E, as in Section 2.4. Define a family $D_E = \{(D_E)_m,\ m \in M\}$ of operators on $\mathcal{V}$, where $(D_E)_m = \frac{\partial}{\partial t}$ with domain

$$\mathrm{dom}((D_E)_m) = \{f \in \mathcal{V}_m \mid \dot{f} \in \mathcal{V}_m,\ \ f(1) = -A_m f(0)\}. \tag{13}$$

The condition that the distributional derivative $\dot{f}$ lies in $L^2 \subset L^1$ implies that f is absolutely continuous; hence the boundary condition $f(1) = -A_m f(0)$ makes sense. The unbounded operators $(D_E)_m$ are closed and skew-adjoint (see, e.g. [28, Chapter VIII]). By Proposition A.4 in the Appendix, the family D_E is continuous in the norm resolvent sense, hence it defines a Dixmier–Douady bundle $\mathcal{A}_E$ by Proposition 4.7.

The kernel of the operator $(D_E)_m$ is the intersection of $V_m \subset \mathcal{V}_m$ (embedded as constant functions) with the domain (13). That is,

$$\ker((D_E)_m) = \ker(A_m + I) = V_m \cap E_m.$$

Proposition 5.1. Suppose $E \cap V$ is a sub-bundle of V. Then there is a canonical Morita isomorphism

$$\widetilde{\mathbb{C}\mathrm{l}}(E \cap V) \dashrightarrow \mathcal{A}_E.$$

In particular, there are canonical Morita isomorphisms

$$\mathbb{C} \dashrightarrow \mathcal{A}_{V^*}, \quad \widetilde{\mathbb{C}l}(V) \dashrightarrow \mathcal{A}_V. \qquad \square$$

Proof. Since $\ker(D_E) \cong E \cap V$ is then a sub-bundle of $\mathcal{V}$, the assertion follows from item (d) in Section 4.4. ■

Remark 5.2. The definition of $\mathcal{A}_E$ depends on the choice of a Euclidean metric on V. However, since the space of Euclidean metrics is contractible, the bundles corresponding to two choices are related by a canonical 2-isomorphism class of isomorphisms. See Example 3.5. □

Remark 5.3. The Dixmier–Douady class $\mathrm{DD}(\mathcal{A}_E) = (x, y)$ is an invariant of the Dirac structure $(\mathbb{V}, E)$. It may be constructed more directly as follows: Choose V' such that $V \oplus V' \cong X \times \mathbb{R}^N$ is trivial. Then $E \oplus (V')^*$ corresponds to a section of the orthogonal bundle, or equivalently to a map $f\colon X \to \mathrm{O}(N)$. The class $\mathrm{DD}(\mathcal{A}_E)$ is the pull-back under f of the class over $\mathrm{O}(N)$ the restriction to each component of which is a generator of $H^3(\cdot, \mathbb{Z})$, respectively, $H^1(\cdot, \mathbb{Z}_2)$. (See Proposition 6.2.) However, not all classes in $H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}_2)$ are realized as such pull-backs. □

The following proposition shows that the polarization defined by D_E depends very much on the choice of E, while it is not affected by perturbations of D_E by skew-adjoint multiplication operators M_μ. Let $L^\infty([0, 1], \mathfrak{o}(V))$ denote the Banach bundle with fibers $L^\infty([0, 1], \mathfrak{o}(V_m))$. Its continuous sections μ are given in local trivialization of V by continuous maps to $L^\infty([0, 1], \mathfrak{o}(X))$. Fiberwise multiplication by μ defines a continuous homomorphism

$$L^\infty([0, 1], \mathfrak{o}(V)) \to \mathcal{L}(\mathcal{V}), \quad \mu \mapsto M_\mu.$$

Proposition 5.4.

(a) Let E and E' be two Lagrangian sub-bundles of V. Then $D_E \sim D_{E'}$ if and only if $E = E'$.

(b) Let $\mu \in \Gamma(L^\infty([0, 1], \mathfrak{o}(V)))$, defining a continuous family of skew-adjoint multiplication operators $M_\mu \in \Gamma(\mathcal{L}(\mathcal{V}))$. For any Lagrangian sub-bundle $E \subset \mathbb{V}$ one has

$$D_E + M_\mu \sim D_E. \qquad \square$$

The proof is given in the Appendix, see Propositions A.2 and A.3.

5.2 Paths of Lagrangian sub-bundles

Suppose E_s, $s \in [0,1]$ is a path of Lagrangian sub-bundles of V, and $A_s \in \Gamma(\mathrm{O}(V))$ the resulting path of orthogonal transformations. In Example 3.5, we remarked that there is a path of isomorphisms $\phi_s \colon \mathcal{A}_{E_0} \to \mathcal{A}_{E_s}$ with $\phi_0 = \mathrm{id}$, and the 2-isomorphism class of the resulting isomorphism $\phi_1 \colon \mathcal{A}_{E_0} \to \mathcal{A}_{E_1}$ does not depend on the choice of ϕ_s. It is also clear from the discussion in Example 3.5 that the isomorphism defined by a concatenation of two paths is 2-isomorphic to the composition of the isomorphisms defined by the two paths.

If the family E_s is differentiable, there is a distinguished choice of the isomorphism $\mathcal{A}_{E_0} \to \mathcal{A}_{E_1}$, as follows.

Proposition 5.5. Suppose that $\mu_s := -\frac{\partial A_s}{\partial s} A_s^{-1}$ defines a continuous section of $L^\infty([0,1], \mathfrak{o}(V))$. Let $M_\gamma \in \Gamma(\mathrm{O}(\mathcal{V}))$ be the orthogonal transformation given fiberwise by pointwise multiplication by $\gamma_t = A_t A_0^{-1}$. Then

$$M_\gamma \circ D_{E_0} \circ M_\gamma^{-1} = D_{E_1} + M_\mu \sim D_{E_1}.$$

Thus, M_γ induces an isomorphism $\mathcal{A}_{E_0} \to \mathcal{A}_{E_1}$. □

Proof. We have

$$f(1) = -A_0 f(0) \;\Leftrightarrow\; (M_\gamma f)(1) = -A_1 (M_\gamma f)(0),$$

which shows $M_\gamma(\mathrm{dom}(D_{E_0})) = \mathrm{dom}(D_{E_1})$, and

$$A_t A_0^{-1} \frac{\partial}{\partial t}(A_0 A_t^{-1} f) = \frac{\partial f}{\partial t} + \mu_t f. \qquad ■$$

Example 5.6.

(a) Suppose E corresponds to $A = \exp(a)$ with $a \in \Gamma(\mathfrak{o}(V))$. Then $A_s = \exp(sa)$ defines a path from $A_0 = I$ and $A_1 = A$. Hence, we obtain an isomorphism $\mathcal{A}_{V^*} \to \mathcal{A}_E$. (The 2-isomorphism class of this isomorphism may depend on the choice of a.)

(b) Any 2-form $\omega \in \Gamma(\wedge^2 V^*)$ defines an orthogonal transformation of $\mathbb{V}$, given by $(v, \alpha) \mapsto (v, \alpha - \iota_v \omega)$. Let E^ω be the image of the Lagrangian sub-bundle $E \subset \mathbb{V}$

under this transformation. The corresponding orthogonal transformations A and A^ω are related by

$$A^\omega = (A - \omega(A - I))(I - \omega(A - I))^{-1},$$

where we identified the 2-form ω with the corresponding skew-adjoint map $\omega \in \Gamma(\mathfrak{o}(V))$. Replacing ω with $s\omega$, one obtains a path E_s from $E_0 = E$ to $E_1 = E^\omega$, defining an isomorphism $\mathcal{A}_E \to \mathcal{A}_{E^\omega}$. □

5.3 The Dirac–Dixmier–Douady functor

Having assigned a Dixmier–Douady bundle to every Dirac structure on a Euclidean vector bundle V,

$$(\mathbb{V}, E) \rightsquigarrow \mathcal{A}_E \tag{14}$$

we will now associate a Morita morphism to every strong Dirac morphism:

$$((\Theta, \omega) \colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}', E')) \ \rightsquigarrow\ ((\Phi, \mathcal{E}) \colon \mathcal{A}_E \dashrightarrow \mathcal{A}_{E'}). \tag{15}$$

Here, $\Phi \colon M \to M'$ is underlying the map on the base. Theorem 5.7 states that (15) is compatible with compositions "up to 2-isomorphism". Thus, if we take the morphisms for the category of Dixmier–Douady bundles to be the 2-isomorphism classes of Morita morphisms, and if we include the Euclidean metric on V as part of a Dirac structure, the construction (14) and (15) defines a functor. We will call this the *Dirac–Dixmier–Douady functor*.

The Morita isomorphism $\mathcal{E} \colon \mathcal{A}_E \dashrightarrow \Phi^*\mathcal{A}_{E'} = \mathcal{A}_{\Phi^*E'}$ in (15) is defined as a composition

$$\mathcal{A}_E \dashrightarrow \mathcal{A}_{E \oplus \Phi^*(V')^*} \cong \mathcal{A}_{V^* \oplus \Phi^* E'} \dashrightarrow \mathcal{A}_{\Phi^* E'}, \tag{16}$$

where the middle map is induced by the path E_s from $E_0 = E \oplus \Phi^*(V')^*$ to $E_1 = V^* \oplus \Phi^* E'$, constructed as in Section 2.2. By composing with the Morita isomorphisms $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E \oplus \Phi^*(V')^*}$ and $\mathcal{A}_{V^* \oplus \Phi^* E'} \dashrightarrow \mathcal{A}_{E'}$ this gives the desired Morita morphism $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E'}$.

Theorem 5.7. (i) The composition of the Morita morphisms $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E'}$ and $\mathcal{A}_{E'} \dashrightarrow \mathcal{A}_{E''}$ defined by two strong Dirac morphisms (Θ, ω) and (Θ', ω') is 2-isomorphic to the Morita morphism $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E''}$ defined by $(\Theta', \omega') \circ (\Theta, \omega)$. (ii) The Morita morphism

$\mathcal{A}_E \dashrightarrow \mathcal{A}_E$ defined by the Dirac morphism $(\mathrm{id}_V, 0)\colon (\mathbb{V}, E) \dashrightarrow (\mathbb{V}, E)$ is 2-isomorphic to the identity. □

Proof. (i) By pulling everything back to M, we may assume that $M = M' = M''$ and that Θ and Θ' induce the identity map on the base. As in Section 2.2, consider the three Lagrangian sub-bundles

$$E_{00} = E \oplus (V')^* \oplus W^*, \quad E_{10} = V^* \oplus E' \oplus W^*, \quad E_{01} = V^* \oplus (V')^* \oplus E''$$

of $\mathbb{V} \oplus \mathbb{V}' \oplus \mathbb{V}''$. We have canonical Morita isomorphisms

$$\mathcal{A}_E \dashrightarrow \mathcal{A}_{E_{00}}, \quad \mathcal{A}_{E'} \dashrightarrow \mathcal{A}_{E_{10}}, \quad \mathcal{A}_{E''} \dashrightarrow \mathcal{A}_{E_{01}}.$$

The morphism (16) may be equivalently described as a composition

$$\mathcal{A}_E \dashrightarrow \mathcal{A}_{E_{00}} \cong \mathcal{A}_{E_{10}} \dashrightarrow \mathcal{A}_{E'},$$

since the path from E_{00} to E_{10} (constructed as in Section 2.2) is just the direct sum of W^* with the standard path from $E \oplus (V')^*$ to $V^* \oplus E'$. Similarly, one describes the morphism $\mathcal{A}_{E'} \dashrightarrow \mathcal{A}_{E''}$ as

$$\mathcal{A}_{E'} \dashrightarrow \mathcal{A}_{E_{10}} \cong \mathcal{A}_{E_{01}} \dashrightarrow \mathcal{A}_{E''}.$$

The composition of the Morita morphisms $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E'} \dashrightarrow \mathcal{A}_{E''}$ defined by (Θ, ω), (Θ', ω') is hence given by

$$\mathcal{A}_E \dashrightarrow \mathcal{A}_{E_{10}} \cong \mathcal{A}_{E_{01}} \cong \mathcal{A}_{E_{01}} \dashrightarrow \mathcal{A}_{E''}.$$

The composition $\mathcal{A}_{E_{10}} \cong \mathcal{A}_{E_{01}} \cong \mathcal{A}_{E_{01}}$ is 2-isomorphic to the isomorphism defined by the concatenation of standard paths from E_{00} to E_{10} to E_{01}. As observed in Section 2.2 this concatenation is homotopic to the standard path from E_{00} to E_{01}, which defines the morphism $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E''}$ corresponding to $(\Theta', \omega') \circ (\Theta, \omega)$.

(ii) We will show that the Morita morphism $\mathcal{A}_E \dashrightarrow \mathcal{A}_{E_0} \cong \mathcal{A}_{E_1} \dashrightarrow \mathcal{A}_E$ defined by $(\mathrm{id}_V, 0)$ is homotopic to the identity. Here, $E_0 = E \oplus V^*$, $E_1 = V^* \oplus E$, and the isomorphism $\mathcal{A}_{E_0} \cong \mathcal{A}_{E_1}$ is defined by the standard path E_t connecting E_0 and E_1. By definition,

E_t is the forward image of E under the morphism $(j_t, 0)\colon \mathbb{V} \dashrightarrow \mathbb{V} \oplus \mathbb{V}$ where

$$j_t\colon V \to V \oplus V, \quad y \mapsto ((1-t)y, ty).$$

It is convenient to replace j_t by the isometry,

$$\tilde{j}_t = (t^2 + (1-t)^2)^{-1/2} j_t.$$

This is homotopic to j_t (e.g., by linear interpolation), hence the resulting path $\tilde{E}_t$ defines the same 2-isomorphism class of isomorphisms $\mathcal{A}_{E_0} \to \mathcal{A}_{E_1}$.

The splitting of $V \oplus V$ into $V_t := \mathrm{ran}(\tilde{j}_t)$ and $V_t^\perp$ defines a corresponding orthogonal splitting of $\mathbb{V} \oplus \mathbb{V}$. The subspace $\tilde{E}_t$ is the direct sum of the intersections

$$\tilde{E}_t \cap \mathbb{V}_t^\perp = \mathrm{ann}(V_t) = (V_t^\perp)^*, \quad \tilde{E}_t \cap \mathbb{V}_t =: \tilde{E}_t'.$$

This defines a Morita isomorphism

$$\mathcal{A}_{\tilde{E}_t} \dashrightarrow \mathcal{A}_{\tilde{E}_t'}.$$

On the other hand, the isometric isomorphism $V \to V_t$ given by $\tilde{j}_t$ extends to an isomorphism $\mathbb{V} \to \mathbb{V}_t$, taking E to $\tilde{E}_t'$. Hence $\mathcal{A}_{\tilde{E}_t'} \cong \mathcal{A}_E$ canonically. In summary, we obtain a family of Morita isomorphisms

$$\mathcal{A}_E \dashrightarrow \mathcal{A}_{E_0} \cong \mathcal{A}_{\tilde{E}_t} \dashrightarrow \mathcal{A}_{\tilde{E}_t'} \cong \mathcal{A}_E.$$

For $t=1$ this is the Morita isomorphism defined by $(\mathrm{id}_V, 0)$, while for $t=0$ it is the identity map $\mathcal{A}_E \to \mathcal{A}_E$. ■

5.4 Symplectic vector bundles

Suppose that $V \to M$ is a vector bundle, equipped with a fiberwise symplectic form $\omega \in \Gamma(\wedge^2 V^*)$. Given a Euclidean metric B on V, the 2-form ω is identified with a skew-adjoint operator R_ω, defining a complex structure $J_\omega = R_\omega/|R_\omega|$ and a resulting spinor module $\mathcal{S}_\omega\colon \mathbb{C} \dashrightarrow \mathbb{C}\mathrm{l}(V)$. (We may work with $\mathbb{C}\mathrm{l}(V)$ rather than $\widetilde{\mathbb{C}\mathrm{l}}(V)$, since V has an even rank.)

Proposition 5.8. The Morita isomorphism

$$S_\omega^{\mathrm{op}}: \mathbb{C}\mathrm{l}(V) \dashrightarrow \mathbb{C}$$

defined by the Spin_c-structure S_ω is 2-isomorphic to the Morita isomorphism $\mathbb{C}\mathrm{l}(V) \dashrightarrow \mathcal{A}_V$, followed by the Morita isomorphism $\mathcal{A}_V \dashrightarrow \mathbb{C}$ defined by the strong Dirac morphism $(0, \omega)\colon (\mathbb{V}, V) \dashrightarrow (0, 0)$ (cf. Example 2.1(c)). □

Proof. Consider the standard path for the Dirac morphism $(0, \omega)\colon (\mathbb{V}, E) \dashrightarrow (0, 0)$,

$$E_t = \{((1-t)v, \alpha)|\ t\iota_v\omega + (1-t)\alpha = 0\} \subset \mathbb{V},$$

defining $\mathcal{A}_V = \mathcal{A}_{E_0} \cong \mathcal{A}_{E_1} = \mathcal{A}_{V^*} \dashrightarrow \mathbb{C}$. The path of orthogonal transformations defined by E_t is

$$A_t = \frac{tR_\omega - \frac{1}{2}(1-t)^2}{tR_\omega + \frac{1}{2}(1-t)^2}.$$

We will replace A_t with a more convenient path $\tilde{A}_t$,

$$\tilde{A}_t = -\exp(t\pi J_\omega).$$

We claim that this is homotopic to A_t with the same endpoints. Clearly, $A_0 = -I = -\tilde{A}_0$ and $A_1 = I = \tilde{A}_1$. By considering the action on any eigenspace of R_ω, one checks that the spectrum of both $J_\omega A_t$ and $J_\omega \tilde{A}_t$ is contained in the half space $\mathrm{Re}(z) \geq 0$, for all $t \in [0, 1]$. Hence

$$J_\omega A_t + I, \quad J_\omega \tilde{A}_t + I \tag{17}$$

are invertible for all $t \in [0, 1]$. The Cayley transform $C \mapsto (C - I)/(C + I)$ gives a diffeomorphism from the set of all $C \in \mathrm{O}(V)$ such that $C + I$ is invertible onto the vector space $\mathfrak{o}(V)$. By using the linear interpolation of the Cayley transforms one obtains a homotopy between $J_\omega A_t$ and $J_\omega \tilde{A}_t$, and hence of A_t and $\tilde{A}_t$.

By Proposition 5.5, the path $\tilde{A}_t$ defines an orthogonal transformation $M_\gamma \in \mathrm{O}(\mathcal{V})$, taking the complex structure J_0 for $E_0 = V^*$ to a complex structure $J_1 = M_\gamma \circ J_0 \circ M_\gamma^{-1}$ in the equivalence class defined by D_{E_1}. Consider the orthogonal decomposition $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ with $\mathcal{V}' = \ker(D_V) \cong V$. Let J'' be the complex structure on $\mathcal{V}''$ defined by D_V, and put

$J' = J_\omega$. Since

$$M_\gamma \circ D_{V^*} \circ M_\gamma^{-1} = D_V + \pi J_\omega,$$

we see that $J_1 = J' \oplus J''$, hence $\mathcal{S}_1 = \mathcal{S}' \otimes \mathcal{S}'' = \mathcal{S}_\omega \otimes \mathcal{S}''$. The Morita isomorphism $\mathbb{C}\mathrm{l}(V) \dashrightarrow \mathcal{A}_V$ is given by the bimodule $\mathcal{E} = \mathcal{S}'' \otimes \mathbb{C}\mathrm{l}(V)$. Since $\mathbb{C}\mathrm{l}(\mathcal{V}) = \mathcal{S}_\omega \otimes \mathcal{S}_\omega^{\mathrm{op}}$, it follows that $\mathcal{E} = \mathcal{S}'' \otimes \mathbb{C}\mathrm{l}(V) = \mathcal{S}_1 \otimes \mathcal{S}_\omega^{\mathrm{op}}$, and

$$\mathcal{S}_1^{\mathrm{op}} \otimes_{\mathcal{A}_V} \mathcal{E} = \mathcal{S}_\omega^{\mathrm{op}}.$$

■

6 The Dixmier–Douady Bundle Over the Orthogonal Group

6.1 The bundle $\mathcal{A}_{\mathrm{O}(X)}$

As a special case of our construction, let us consider the tautological Dirac structure $(\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)})$ for a Euclidean vector space X. Let $\mathcal{A}_{\mathrm{O}(X)}$ be the corresponding Dixmier–Douady bundle; its restriction to $\mathrm{SO}(X)$ will be denoted $\mathcal{A}_{\mathrm{SO}(X)}$. The Dirac morphism $(\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}) \times (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)})$ gives rise to a Morita morphism

$$\mathrm{pr}_1^* \mathcal{A}_{\mathrm{O}(X)} \otimes \mathrm{pr}_2^* \mathcal{A}_{\mathrm{O}(X)} \dashrightarrow \mathcal{A}_{\mathrm{O}(X)},$$

which is associative up to 2-isomorphisms.

Proposition 6.1.

(a) There is a canonical Morita morphism $\mathbb{C} \dashrightarrow \mathcal{A}_{\mathrm{O}(X)}$ with underlying map the inclusion of the group unit, $\{I\} \hookrightarrow \mathrm{O}(X)$.

(b) For any orthogonal decomposition $X = X' \oplus X''$, there is a canonical Morita morphism

$$\mathrm{pr}_1^* \mathcal{A}_{\mathrm{O}(X')} \otimes \mathrm{pr}_2^* \mathcal{A}_{\mathrm{O}(X'')} \dashrightarrow \mathcal{A}_{\mathrm{O}(X)}$$

with underlying map the inclusion $\mathrm{O}(X') \times \mathrm{O}(X'') \hookrightarrow \mathrm{O}(X)$. □

Proof. The proposition follows since the restriction of $E_{\mathrm{O}(X)}$ to I is X^*, while the restriction to $\mathrm{O}(X') \times \mathrm{O}(X'')$ is $E_{\mathrm{O}(X')} \times E_{\mathrm{O}(X'')}$. ■

The action of $\mathrm{O}(X)$ by conjugation lifts to an action on the bundle $\mathbb{V}_{\mathrm{O}(X)}$, preserving the Dirac structure $E_{\mathrm{O}(X)}$. Hence, $\mathcal{A}_{\mathrm{O}(X)}$ is an $\mathrm{O}(X)$-equivariant Dixmier–Douady bundle.

The construction of $\mathcal{A}_{\mathrm{O}(X)}$, using the family of boundary conditions given by orthogonal transformations, is closely related to a construction given by Atiyah–Segal in [6], who also identify the resulting Dixmier–Douady class. The result is most nicely stated for the restriction to $\mathrm{SO}(X)$; for the general case, use an inclusion $\mathrm{O}(X) \hookrightarrow \mathrm{SO}(X \oplus \mathbb{R})$.

Proposition 6.2. [6, Proposition 5.4] Let $(x, y) = \mathrm{DD}(\mathcal{A}_{\mathrm{SO}(X)})$ be the Dixmier–Douady class.

(a) For $\dim X \geq 3$ and $\dim X \neq 4$, the class x generates $H^3(\mathrm{SO}(X), \mathbb{Z}) = \mathbb{Z}$.

(b) For $\dim X \geq 2$, the class y generates $H^1(\mathrm{SO}(X), \mathbb{Z}_2) = \mathbb{Z}_2$. □

Atiyah–Segal's proof uses an alternative construction $\mathcal{A}_{\mathrm{SO}(X)}$ in terms of loop groups (see below). Another argument is sketched in Appendix B.

6.2 Pull-back under exponential map

Let $(\mathbb{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)})$ be as in Section 2.7, and let $\mathcal{A}_{\mathfrak{o}(X)}$ be the resulting $\mathrm{O}(X)$-equivariant Dixmier–Douady bundle. Since $E_{\mathfrak{o}}(X)|_a = \mathrm{Gr}_a$, its intersection with $X \subset \mathbb{X}$ is trivial, and so $\mathcal{A}_{\mathfrak{o}(X)}$ is Morita trivial. Recall the Dirac morphism $(\Pi, -\varpi)\colon (\mathbb{V}_{\mathfrak{o}(X)}, E_{\mathfrak{o}(X)}) \dashrightarrow (\mathbb{V}_{\mathrm{O}(X)}, E_{\mathrm{O}(X)})$, with underlying map $\exp\colon \mathfrak{o}(X) \to \mathrm{O}(X)$. We had shown that it is a strong Dirac morphism over the subset $\mathfrak{o}(X)_\natural$ where the exponential map has maximal rank, or equivalently where $\Pi_a = (I - e^{-a})/a$ is invertible. One hence obtains a Morita morphism

$$\mathcal{A}_{\mathfrak{o}(X)}|_{\mathfrak{o}(X)_\natural} \dashrightarrow \mathcal{A}_{\mathrm{O}(X)}.$$

Together with the Morita trivialization $\mathbb{C} \dashrightarrow \mathcal{A}_{\mathfrak{o}(X)}$, this gives a Morita trivialization of $\exp^* \mathcal{A}_{\mathrm{O}(X)}$ over $\mathfrak{o}(X)_\natural$.

On the other hand, $\exp^* E_{\mathrm{O}(X)}$ is the Lagrangian sub-bundle of $\mathfrak{o}(X) \times \mathbb{X}$ defined by the map $a \mapsto \exp(a) \in \mathrm{O}(X)$. Replacing $\exp(a)$ with $\exp(sa)$, one obtains a homotopy E_s between $E_1 = \exp^* E_{\mathrm{O}(X)}$ and $E_0 = X^*$, hence another Morita trivialization of $\exp^* \mathcal{A}_{\mathrm{O}(X)}$ (defined over all of $\mathfrak{o}(X)$). Let $L \to \mathfrak{o}(X)_\natural$ be the $\mathrm{O}(X)$-equivariant line bundle relating these two Morita trivializations.

Proposition 6.3. Over the component containing 0, the line bundle $L \to \mathfrak{o}(X)_\natural$ is $\mathrm{O}(X)$-equivariantly trivial. In other words, the two Morita trivializations of $\exp^* \mathcal{A}_{\mathrm{O}(X)}|_\natural$ are 2-isomorphic over the component of $\mathfrak{o}(X)_\natural$ containing 0. □

Proof. The linear retraction of $\mathfrak{o}(X)$ onto the origin preserves the component of $\mathfrak{o}(X)_\natural$ containing 0. Hence it suffices to show that the $\mathrm{O}(X)$-action on the fiber of L at 0 is trivial. But this is immediate since both Morita trivializations of $\exp^* \mathcal{A}_{\mathrm{O}(X)}$ at $0 \in \mathfrak{o}(X)_\natural$ coincide with the obvious Morita trivialization of $\mathcal{A}_{\mathrm{O}(X)}|_e$. ■

6.3 Construction via loop groups

The bundle $\mathcal{A}_{\mathrm{SO}(X)}$ has the following description in terms of loop groups (cf. [6]). Fix a Sobolev level $s > 1/2$, and let $\mathcal{P}\,\mathrm{SO}(X)$ denote the Banach manifold of paths $\gamma : \mathbb{R} \to \mathrm{SO}(X)$ of Sobolev class $s + 1/2$ such that $\pi(\gamma) := \gamma(t+1)\gamma(t)^{-1}$ is constant. (Recall that for manifolds Q and P, the maps $Q \to P$ of Sobolev class greater than $k + \dim Q/2$ are of class C^k.) The map

$$\pi : \mathcal{P}\,\mathrm{SO}(X) \to \mathrm{SO}(X), \;\; \gamma \mapsto \pi(\gamma)$$

is an $\mathrm{SO}(X)$-equivariant principal bundle, with structure group the loop group $L\,\mathrm{SO}(X) = \pi^{-1}(e)$. Here, elements of $\mathrm{SO}(X)$ acts by multiplication from the left, while loops $\lambda \in L\,\mathrm{SO}(X)$ acts by $\gamma \mapsto \gamma\lambda^{-1}$. Let $\mathcal{X} = L^2([0,1], X)$ carry the complex structure J_0 defined by $\frac{\partial}{\partial t}$ with anti-periodic boundary conditions, and let $\mathcal{S}_0$ be the resulting spinor module. The action of the group $L\,\mathrm{SO}(X)$ on $\mathcal{X}$ preserves the polarization defined by J_0, and defines a continuous map $L\,\mathrm{SO}(X) \to \mathrm{O}_{\mathrm{res}}(\mathcal{X})$. Using its composition with the map $\mathrm{O}_{\mathrm{res}}(\mathcal{X}) \to \mathrm{PU}(\mathcal{S}_0)$, we have the following.

Proposition 6.4. The Dixmier–Douady bundle $\mathcal{A}_{\mathrm{SO}(X)}$ is an associated bundle $\mathcal{P}\,\mathrm{SO}(X) \times_{L\,\mathrm{SO}(X)} \mathbb{K}(\mathcal{S}_0)$. □

Proof. Given $\gamma \in \mathcal{P}\,\mathrm{SO}(X)$, consider the operator M_γ on $\mathcal{X} = L^2([0,1], X)$ of pointwise multiplication by γ. As in Proposition 5.5, we see that M_γ takes the boundary conditions $f(1) = -f(0)$ to $(M_\gamma f)(1) = -\pi(\gamma)(M_\gamma f)(0)$, and induces an isomorphism $\mathbb{K}(\mathcal{S}_0) = \mathcal{A}_I \to \mathcal{A}_{\pi(\gamma)}$. This defines a map

$$\mathcal{P}\,\mathrm{SO}(X) \times \mathbb{K}(\mathcal{S}_0) \to \mathcal{A}_{\mathrm{SO}(X)}$$

with underlying map $\pi : \mathcal{P}\,\mathrm{SO}(X) \to \mathrm{SO}(X)$. This map is equivariant relative to the action of $L\,\mathrm{SO}(X)$, and descends to the desired bundle isomorphism. ■

In particular, $\pi^*\mathcal{A}_{\mathrm{SO}(X)} = \mathcal{P}\,\mathrm{SO}(X) \times \mathbb{K}(\mathcal{S}_0)$ has a Morita trivialization defined by the trivial bundle $\mathcal{E}_0 = \mathcal{P}\,\mathrm{SO}(X) \times \mathcal{S}_0$. The Morita trivialization is $\widehat{L\,\mathrm{SO}}(X) \times \mathrm{SO}(X)$-equivariant, using the central extension of the loop group obtained by pull-back of the central extension $\mathrm{U}(\mathcal{S}_0) \to \mathrm{PU}(\mathcal{S}_0)$.

7 q-Hamiltonian G-spaces

In this section, we will apply the correspondence between Dirac structures and Dixmier–Douady bundles to the theory of group-valued moment maps [2]. Most results will be immediate consequences of the functoriality properties of this correspondence. Throughout this section, G denotes a Lie group, with Lie algebra $\mathfrak{g}$. We denote by $\xi^{\mathrm{L}}, \xi^{\mathrm{R}} \in \mathfrak{X}(G)$ the left and the right invariant vector fields defined by the Lie algebra element $\xi \in \mathfrak{g}$, and by $\theta^{\mathrm{L}}, \theta^{\mathrm{R}} \in \Omega^1(G, \mathfrak{g})$ the Maurer–Cartan forms, defined by $\iota(\xi^{\mathrm{L}})\theta^{\mathrm{L}} = \iota(\xi^{\mathrm{R}})\theta^{\mathrm{R}} = \xi$. For sake of comparison, we begin with a quick review of ordinary Hamiltonian G-spaces from the Dirac geometry perspective.

7.1 Hamiltonian G-spaces

A Hamiltonian G-space is a triple (M, ω_0, Φ_0) consisting of a G-manifold M, an invariant 2-form ω_0 and an equivariant *moment map* $\Phi_0 \colon M \to \mathfrak{g}^*$ such that

(i) $\mathrm{d}\omega_0 = 0$,

(ii) $\iota(\xi_M)\omega_0 = -\mathrm{d}\langle \Phi_0, \xi\rangle,\ \ \xi \in \mathfrak{g}$,

(iii) $\ker(\omega_0) = 0$.

Conditions (ii) and (iii) may be rephrased in terms of Dirac morphisms. Let $E_{\mathfrak{g}^*} \subset \mathbb{T}\mathfrak{g}^*$ be the Dirac structure spanned by the sections

$$e_0(\xi) = (\xi^\sharp,\ \langle \mathrm{d}\mu, \xi\rangle), \qquad \xi \in \mathfrak{g}.$$

Here $\xi^\sharp \in \mathfrak{X}(\mathfrak{g}^*)$ is the vector field generating the co-adjoint action (i.e., $\xi^\sharp|_\mu = (\mathrm{ad}_\xi)^*\mu$), and $\langle \mathrm{d}\mu, \xi\rangle \in \Omega^1(\mathfrak{g}^*)$ denotes the 1-form defined by ξ. Then Conditions (ii) and (iii) hold if and only if

$$(\mathrm{d}\Phi_0, \omega_0) \colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}\mathfrak{g}^*, E_{\mathfrak{g}^*})$$

is a strong Dirac morphism. Using the Morita isomorphism $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}_{TM}$, and putting $\mathcal{A}_{\mathfrak{g}^*}^{\mathrm{Spin}} := \mathcal{A}_{E_{\mathfrak{g}^*}}$ we obtain a G-equivariant Morita morphism

$$(\Phi_0, \mathcal{E}_0)\colon \widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}_{\mathfrak{g}^*}^{\mathrm{Spin}}.$$

Since $E_{\mathfrak{g}^*} \cap T\mathfrak{g}^* = 0$, the zero Dirac morphism $(T\mathfrak{g}^*, E_{\mathfrak{g}^*}) \dashrightarrow (0, 0)$ is strong, hence it defines a Morita trivialization $\mathcal{A}_{\mathfrak{g}^*}^{\mathrm{Spin}} \dashrightarrow \mathbb{C}$. From Proposition 5.8, we see that the resulting equivariant Spin_c structure $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathbb{C}$ is 2-isomorphic to the Spin_c structure defined by the symplectic form ω_0. (Since symplectic manifolds are even-dimensional, we may work with $\mathbb{C}\mathrm{l}(TM)$ in place of $\widetilde{\mathbb{C}\mathrm{l}}(TM)$.)

7.2 q-Hamiltonian G-spaces

An $\mathrm{Ad}(G)$-invariant inner product B on $\mathfrak{g}$ defines a closed bi-invariant 3-form

$$\eta = \tfrac{1}{12} B(\theta^{\mathrm{L}}, [\theta^{\mathrm{L}}, \theta^{\mathrm{L}}]) \in \Omega^3(G).$$

A q-Hamiltonian G-manifold [2] is a G-manifold M, together with an invariant 2-form ω, and an equivariant *moment map* $\Phi\colon M \to G$ such that

(i) $\mathrm{d}\omega = -\Phi^*\eta$,
(ii) $\iota(\xi_M)\omega = -\tfrac{1}{2}\Phi^* B((\theta^{\mathrm{L}} + \theta^{\mathrm{R}}), \xi)$,
(iii) $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$ everywhere.

The simplest examples of q-Hamiltonian G-spaces are the conjugacy classes in G, with moment map the inclusion $\Phi\colon \mathcal{C} \hookrightarrow G$. Again, the definition can be re-phrased in terms of Dirac structures. Let $E_G \subset \mathbb{T}G$ be the Lagrangian sub-bundle spanned by the sections

$$e(\xi) = (\xi^\sharp,\ \tfrac{1}{2} B(\theta^{\mathrm{L}} + \theta^{\mathrm{R}}, \xi)), \quad \xi \in \mathfrak{g}.$$

Here $\xi^\sharp = \xi^{\mathrm{L}} - \xi^{\mathrm{R}} \in \mathfrak{X}(G)$ is the vector field generating the conjugation action. E_G is the *Cartan–Dirac structure* introduced by Alekseev, Ševera and Strobl [7, 31]. As shown by Bursztyn–Crainic [7], Conditions (ii) and (iii) above hold if and only if

$$(\mathrm{d}\Phi, \omega)\colon (\mathbb{T}M, TM) \dashrightarrow (\mathbb{T}G, E_G)$$

is a strong Dirac morphism. Let

$$\mathcal{A}_G^{\mathrm{Spin}} := \mathcal{A}_{E_G}$$

be the G-equivariant Dixmier–Douady bundle over G defined by the Cartan–Dirac structure. The strong Dirac morphism $(\mathrm{d}\Phi, \omega)$ determines a Morita morphism $\mathcal{A}_{TM} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$. Since $\mathcal{A}_{TM}$ is naturally Morita isomorphic to $\widetilde{\mathbb{C}\mathrm{l}}(TM)$ we obtain a distinguished 2-isomorphism class of G-equivariant Morita morphisms

$$(\Phi, \mathcal{E})\colon \widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}. \tag{18}$$

Definition 7.1. The Morita morphism (18) is called the *canonical twisted* Spin_c *structure* for the q-Hamiltonian G-space (M, ω, Φ). □

Remark 7.2.

(a) Equation (18) generalizes the usual Spin_c structure for a symplectic manifold. Indeed, if $G = \{e\}$ we have $\mathcal{A}_G^{\mathrm{Spin}} = \mathbb{C}$, and a q-Hamiltonian G-space is just a symplectic manifold. Proposition 5.8 shows that the composition $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}_{TM} \dashrightarrow \mathbb{C}$ in that case is 2-isomorphic to the Morita trivialization defined by an ω-compatible almost complex structure.

(b) The tensor product $\widetilde{\mathbb{C}\mathrm{l}}(TM) \otimes \widetilde{\mathbb{C}\mathrm{l}}(TM) = \widetilde{\mathbb{C}\mathrm{l}}(TM \oplus TM)$ is canonically Morita trivial (see Section 3.5). Hence, the twisted Spin_c structure on a q-Hamiltonian G-space defines a G-equivariant Morita trivialization

$$\mathbb{C} \dashrightarrow \Phi^*(\mathcal{A}_G^{\mathrm{Spin}})^{\otimes 2}. \tag{19}$$

One may think of (19) as the counterpart to the canonical line bundle. Indeed, for $G = \{e\}$, (19) is a Morita isomorphism from the trivial bundle over M to itself. It is thus given by a Hermitian line bundle, and from (a) above one sees that this is the *canonical line bundle* associated to the Spin_c structure of (M, ω). □

Remark 7.3. In terms of the trivialization $TG = G \times \mathfrak{g}$ given by the left-invariant vector fields ξ^{L}, the Cartan–Dirac structure $(\mathbb{T}G, E_G)$ is just the pull-back of the tautological Dirac structure $(\mathbb{V}_{\mathrm{O}(\mathfrak{g})}, E_{\mathrm{O}(\mathfrak{g})})$ under the adjoint action $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$. Similarly, $\mathcal{A}_G^{\mathrm{Spin}}$ is simply the pull-back of $\mathcal{A}_{\mathrm{O}(\mathfrak{g})} \to \mathrm{O}(\mathfrak{g})$ under the map $\mathrm{Ad}\colon G \to \mathrm{O}(\mathfrak{g})$. □

In many cases q-Hamiltonian G-spaces have even dimension, so that we may use the usual Clifford algebra bundle $\mathbb{C}l(TM)$ in (18).

Proposition 7.4. Let (M, ω, Φ) be a connected q-Hamiltonian G-manifold. Then $\dim M$ is even if and only if $\mathrm{Ad}_{\Phi(m)} \in \mathrm{SO}(\mathfrak{g})$ for all $m \in M$. In particular, this is the case if G is connected. □

Proof. This is proved in [4], but follows much more easily from the following Dirac-geometric argument. The parity of the Lagrangian sub-bundle $TM \subset \mathbb{T}M$ is given by $(-1)^{\dim M} = \pm 1$. By Proposition 2.2, the parity is preserved under strong Dirac morphisms. Hence it coincides with the parity of E_G over $\Phi(M)$, and by Remark 7.3 this is the same as the parity of the tautological Dirac structure $E_{\mathrm{O}(\mathfrak{g})}$ over $\mathrm{Ad}(\Phi(M)) \subset \mathrm{O}(\mathfrak{g})$. The latter is given by $\det(\mathrm{Ad}_\Phi) = \pm 1$. This shows $\det(\mathrm{Ad}_\Phi) = (-1)^{\dim M}$. ■

As a noteworthy special case, we have the following.

Corollary 7.5. A conjugacy class $\mathcal{C} = \mathrm{Ad}(G)g \subset G$ of a compact Lie group G is even dimensional if and only if $\det(\mathrm{Ad}_g) = 1$. □

7.3 Stiefel–Whitney classes

The existence of a Spin_c structure on a symplectic manifold implies the vanishing of the third integral Stiefel–Whitney class $W^3(M) = \tilde{\beta}(w_2(M))$, while of course $w_1(M) = 0$ by orientability. For q-Hamiltonian spaces we have the following statement.

Corollary 7.6. For any q-Hamiltonian G-space,

$$W^3(M) \equiv \tilde{\beta}(w_2(M)) = \Phi^* x, \quad w_1(M) = \Phi^* y.$$

where $(x, y) \in H^3(G, \mathbb{Z}) \times H^1(G, \mathbb{Z}_2)$ is the Dixmier–Douady class of $\mathcal{A}_G^{\mathrm{Spin}}$. A similar statement holds for the G-equivariant Stiefel–Whitney classes. □

Remark 7.7.

(a) The result gives in particular a description of $w_1(\mathcal{C})$ and $\tilde{\beta}(w_2(\mathcal{C}))$ for all conjugacy classes $\mathcal{C} \subset G$ of a compact Lie group.

(b) If G is simply connected, so that $H^1(G,\mathbb{Z}_2)=0$, it follows that $w_1(M)=0$. Hence q-Hamiltonian spaces for simply connected groups are orientable. In fact, there is a canonical orientation [4].

(c) Suppose G is simple and simply connected. Then x is $\mathsf{h}^\vee$ times the generator of $H^3(G,\mathbb{Z})=\mathbb{Z}$, where $\mathsf{h}^\vee$ is the dual Coxeter number of G. This follows from Remark 7.3, since

$$\mathrm{Ad}^*\colon H^3(\mathrm{SO}(\mathfrak{g}),\mathbb{Z})=\mathbb{Z}\to H^3(G,\mathbb{Z})=\mathbb{Z}$$

is multiplication by $\mathsf{h}^\vee$. We see that a conjugacy class $\mathcal{C}$ of G admits a Spin_c structure if and only if the pull-back of the generator of $H^3(G,\mathbb{Z})$ is $\mathsf{h}^\vee$-torsion. Examples of conjugacy classes not admitting a Spin_c structure may be found in [19]. □

7.4 Fusion

Let mult: $G\times G\to G$ be the group multiplication, and denote by $\sigma\in\Omega^2(G\times G)$ the 2-form

$$\sigma=-\tfrac{1}{2}B(\mathrm{pr}_1^*\theta^{\mathrm{L}},\ \mathrm{pr}_2^*\theta^{\mathrm{R}}), \tag{20}$$

where $\mathrm{pr}_j\colon G\times G\to G$ are the two projections. By [1, Theorem 3.9] the pair $(\mathrm{d}\,\mathrm{mult},\sigma)$ define a strong G-equivariant Dirac morphism

$$(\mathrm{d}\,\mathrm{mult},\sigma)\colon(\mathbb{T}G,E_G)\times(\mathbb{T}G,E_G)\dashrightarrow(\mathbb{T}G,E_G).$$

This can also be seen using Remark 7.3 and Proposition 2.5, since left trivialization of TG intertwines $\mathrm{d}\,\mathrm{mult}$ with the map Σ from (3), taking (20) to the 2-form σ on $V_{\mathrm{O}(\mathfrak{g})}\times V_{\mathrm{O}(\mathfrak{g})}$. It induces a Morita morphism

$$(\mathrm{mult},\mathcal{E})\colon\ \mathrm{pr}_1^*\mathcal{A}_G^{\mathrm{Spin}}\otimes\mathrm{pr}_2^*\mathcal{A}_G^{\mathrm{Spin}}\dashrightarrow\mathcal{A}_G^{\mathrm{Spin}}. \tag{21}$$

If (M,ω,Φ) is a q-Hamiltonian $G\times G$-space, then M with diagonal G-action, 2-form $\omega_{\mathrm{fus}}=\omega+\Phi^*\sigma$, and moment map $\Phi_{\mathrm{fus}}=\mathrm{mult}\circ\Phi\colon M\to G$ defines a q-Hamiltonian G-space

$$(M,\omega_{\mathrm{fus}},\Phi_{\mathrm{fus}}). \tag{22}$$

The space (22) is called the *fusion* of (M, ω, Φ). Conditions (ii) and (iii) hold since

$$(\mathrm{d}\Phi_{\mathrm{fus}}, \omega_{\mathrm{fus}}) = (\mathrm{d}\,\mathrm{mult}, \sigma) \circ (\mathrm{d}\Phi, \omega) \tag{23}$$

is a composition of strong Dirac morphisms, while (i) follows from $\mathrm{d}\sigma = \mathrm{mult}^* \eta - \mathrm{pr}_1^* \eta - \mathrm{pr}_2^* \eta$. The Dirac–Dixmier–Douady functor (Theorem 5.7) shows that the twisted Spin_c structures are compatible with fusion, in the following sense.

Proposition 7.8. The Morita morphism $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$ for the q-Hamiltonian G-space $(M, \omega_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$ is equivariantly 2-isomorphic to the composition of Morita morphisms

$$\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathrm{pr}_1^* \mathcal{A}_G^{\mathrm{Spin}} \otimes \mathrm{pr}_2^* \mathcal{A}_G^{\mathrm{Spin}} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$$

defined by the twisted Spin_c-structure for (M, ω, Φ), followed by (21). □

7.5 Exponentials

Let $\exp\colon \mathfrak{g} \to G$ be the exponential map. The pull-back $\exp^* \eta$ is equivariantly exact, and admits a canonical primitive $\varpi \in \Omega^2(\mathfrak{g})$ defined by the homotopy operator for the linear retraction onto the origin.

Remark 7.9. Explicit calculation shows [3] that ϖ is the pull-back of the 2-form (denoted by the same letter) $\varpi \in \Gamma(\wedge^2 V^*_{\mathfrak{o}(\mathfrak{g})}) \cong C^\infty(\mathfrak{o}(\mathfrak{g}), \wedge^2 \mathfrak{g}^*)$ from Section 2.7 under the adjoint map, $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$. Using the inner product to identify $\mathfrak{g}^* \cong \mathfrak{g}$, the Dirac structure $E_{\mathfrak{g}^*} \equiv E_{\mathfrak{g}}$ is the pull-back of the Dirac structure $E_{\mathfrak{o}(\mathfrak{g})}$ by the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$. □

The differential of the exponential map together with the 2-form ϖ define a Dirac morphism

$$(\mathrm{d}\exp, -\varpi)\colon (\mathbb{T}\mathfrak{g},\ E_{\mathfrak{g}}) \dashrightarrow (\mathbb{T}G,\ E_G)$$

which is a strong Dirac morphism over the open subset $\mathfrak{g}_\natural$ where exp has maximal rank. See [1, Proposition 3.12], or Proposition 2.6.

Let (M, Φ_0, ω_0) be a Hamiltonian G-space with $\Phi_0(M) \subset \mathfrak{g}_\natural$, and $\Phi = \exp \Phi_0$, $\omega = \omega_0 - \Phi_0^* \varpi$. Then $(\mathrm{d}\Phi, \omega) = (\mathrm{d}\exp, -\varpi) \circ (\mathrm{d}\Phi_0, \omega_0)$ is a strong Dirac morphism, hence (M, ω, Φ) is a q-Hamiltonian G-space. It is called the *exponential* of the Hamiltonian G-space (M, ω_0, Φ_0).

The canonical twisted Spin_c structure for (M, ω, Φ) can be composed with the Morita trivialization $\Phi^*\mathcal{A}_G^{\mathrm{Spin}} = \Phi_0^* \exp^* \mathcal{A}_G^{\mathrm{Spin}} \dashrightarrow \mathbb{C}$ defined by the Morita trivialization of $\exp^* \mathcal{A}_G^{\mathrm{Spin}}$, to produce an ordinary equivariant Spin_c structure. On the other hand, we have the equivariant Spin_c structure defined by the symplectic form ω_0.

Proposition 7.10. Suppose (M, ω_0, Φ_0) is a Hamiltonian G-space, such that Φ_0 takes values in the zero component of $\mathfrak{g}_\natural \subset \mathfrak{g}$. Let (M, ω, Φ) be its exponential. Then the composition

$$\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \Phi^*\mathcal{A}_G^{\mathrm{Spin}} \dashrightarrow \mathbb{C}$$

is 2-isomorphic to the Morita morphism $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathbb{C}$ given by the canonical Spin_c-structure for ω_0. (We could also write $\mathbb{C}\mathrm{l}(TM)$ in place of $\widetilde{\mathbb{C}\mathrm{l}}(TM)$ since $\dim M$ is even.)
□

Proof. Proposition 6.3 shows that over the zero component of $\mathfrak{g}_\natural$, the Morita trivialization of $\exp^* \mathcal{A}_G^{\mathrm{Spin}}$ is 2-isomorphic to the composition of the Morita isomorphism $\mathcal{A}_{\mathfrak{g}}^{\mathrm{Spin}} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$ induced by $(\mathrm{d}\exp, -\varpi)$, with the Morita trivialization of $\mathcal{A}_{\mathfrak{g}}^{\mathrm{Spin}}$ (induced by the Dirac morphism $(T\mathfrak{g}^*, E_{\mathfrak{g}}) \dashrightarrow (0, 0)$). The result now follows from Theorem 5.7. ■

7.6 Reduction

In this section, we will show that the canonical twisted Spin_c structure is well-behaved under reduction. Let (M, ω, Φ) be a q-Hamiltonian $K \times G$-space. Thus, Φ has two components Φ_K and Φ_G, taking values in K and G, respectively. Suppose $e \in G$ a regular value of Φ_G, so that $Z = \Phi_G^{-1}(e)$ is a smooth $K \times G$-invariant submanifold. Let $\iota\colon Z \to M$ be the inclusion. The moment map condition shows that the G-action is locally free on Z, and that $\iota^*\omega$ is G-basic. Let us assume for simplicity that the G-action on Z is actually free. Then

$$M_{\mathrm{red}} = Z/G$$

is a smooth K-manifold, the G-basic 2-form $\iota^*\omega$ descends to a 2-form ω_{red} on M_{red}, and the restriction $\Phi|_Z$ descends to a smooth K-equivariant map $\Phi_{\mathrm{red}}\colon M_{\mathrm{red}} \to K$.

Proposition 7.11. [2] The triple $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, \Phi_{\mathrm{red}})$ is a q-Hamiltonian K-space. In particular, if $K = \{e\}$ it is a symplectic manifold. □

We wish to relate the canonical twisted Spin_c-structures for M_{red} to that for M. We need the following lemma.

Lemma 7.12. There is a $G \times K$-equivariant Morita morphism

$$\widetilde{\mathbb{C}\mathrm{l}}(TM)|_Z \dashrightarrow \widetilde{\mathbb{C}\mathrm{l}}(TM_{\mathrm{red}}), \tag{24}$$

with underlying map the quotient map $\pi : Z \to M_{\mathrm{red}}$. □

Proof. Consider the exact sequences of vector bundles over Z,

$$0 \to Z \times \mathfrak{g} \to TZ \to \pi^* TM_{\mathrm{red}} \to 0, \tag{25}$$

where the first map is inclusion of the generating vector fields, and

$$0 \to TZ \to TM|_Z \to Z \times \mathfrak{g}^* \to 0, \tag{26}$$

where the map $TM|_Z \to \mathfrak{g}^* \cong \mathfrak{g} = T_e G$ is the restriction $(\mathrm{d}\Phi)|_Z$. (We are writing $\mathfrak{g}^*$ in (26) to avoid confusion with the copy of $\mathfrak{g}$ in (25).) The Euclidean metric on TM gives orthogonal splittings of both exact sequences, hence it gives a $K \times G$-equivariant direct sum decomposition

$$TM|_Z = \pi^* TM_{\mathrm{red}} \oplus Z \times (\mathfrak{g} \oplus \mathfrak{g}^*). \tag{27}$$

The standard symplectic structure

$$\omega_{\mathfrak{g} \oplus \mathfrak{g}^*}((v_1, \mu_1), (v_2, \mu_2)) = \mu_1(v_2) - \mu_2(v_1) \tag{28}$$

defines a $K \times G$-equivariant Spin_c structure on $Z \times (\mathfrak{g} \oplus \mathfrak{g}^*)$, and gives the desired equivariant Morita isomorphism. ■

Note that the restriction of the Morita morphism $\widetilde{\mathbb{C}\mathrm{l}}(TM) \dashrightarrow \mathcal{A}^{\mathrm{Spin}}_{K \times G}$ to $Z \subset M$ takes values in $\mathcal{A}^{\mathrm{Spin}}_{K \times G}|_{K \times \{e\}}$. Let

$$\mathcal{A}^{\mathrm{Spin}}_{K \times G}|_{K \times \{e\}} \dashrightarrow \mathcal{A}^{\mathrm{Spin}}_{K} \tag{29}$$

be the Morita isomorphism defined by the Morita trivialization of $\mathcal{A}_G^{\mathrm{Spin}}|_{\{e\}}$. The twisted Spin_c structure for (M, ω, Φ) descends to the twisted Spin_c structure for the G-reduced space $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, \Phi_{\mathrm{red}})$, in the following sense.

Theorem 7.13 (Reduction). Suppose (M, ω, Φ) is a q-Hamiltonian $K \times G$-manifold, such that e is a regular value of Φ_G and such that G acts freely on $\Phi_G^{-1}(e)$. The diagram of $K \times G$-equivariant Morita morphisms

$$\begin{array}{ccc} \widetilde{\mathbb{C}\mathrm{l}}(TM)|_Z & \dashrightarrow & \mathcal{A}^{\mathrm{Spin}}_{K\times G}|_{K\times\{e\}} \\ \Big\downarrow & & \Big\downarrow \\ \widetilde{\mathbb{C}\mathrm{l}}(TM_{\mathrm{red}}) & \dashrightarrow & \mathcal{A}^{\mathrm{Spin}}_{K} \end{array}$$

commutes up to equivariant 2-isomorphism. Here the vertical maps are given by (24) and (29). □

The proof uses the following normal form result for $TM|_Z$.

Lemma 7.14. For a suitable choice of invariant Euclidean metric on TM. the decomposition $TM|_Z = \pi^* TM_{\mathrm{red}} \oplus Z \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ from (27) is compatible with the 2-forms. That is,

$$\omega|_Z = \pi^* \omega_{\mathrm{red}} \oplus \omega_{\mathfrak{g}\oplus\mathfrak{g}^*}.$$ □

Proof. We will construct $K \times G$-equivariant splittings of the exact sequences (25) and (26) so that (27) is compatible with the 2-forms. (One may then take an invariant Euclidean metric on $TM|_Z$ for which these splittings are orthogonal, and extend to TM.) Begin with an arbitrary $K \times G$-invariant splitting

$$TM|_Z = TZ \oplus F.$$

Since $F \cap \ker(\omega) = 0$, the sub-bundle $F^\omega \subset TM|_Z$ (the set of vectors ω-orthogonal to all vectors in F) has codimension $\mathrm{codim}(F^\omega) = \dim F = \dim \mathfrak{g}$. The moment map condition shows that ω is nondegenerate on $F \oplus Z \times \mathfrak{g}$. Hence $(Z \times \mathfrak{g}) \cap F^\omega = 0$, and therefore

$$TM|_Z = (Z \times \mathfrak{g}) \oplus F^\omega.$$

Let $\phi\colon TM|_Z \to Z \times \mathfrak{g}$ be the projection along F^ω. The subspace

$$F' = \{v - \tfrac{1}{2}\phi(v) |\ v \in F\}$$

is again an invariant complement to TZ in $TM|_Z$, and it is *isotropic* for ω. Indeed, if $v_1, v_2 \in F$,

$$\omega(v_1 - \tfrac{1}{2}\phi(v_1),\ v_2 - \tfrac{1}{2}\phi(v_2)) = \tfrac{1}{2}\omega(v_1, v_2 - \phi(v_2)) + \tfrac{1}{2}\omega(v_1 - \phi(v_1), v_2)$$

vanishes since $v_i - \phi(v_i) \in F^\omega$. The restriction of $(\mathrm{d}\Phi_G)|_Z\colon TM|_Z \to \mathfrak{g}^*$ to F' identifies $F' = Z \times \mathfrak{g}^*$. We have hence shown the existence of an invariant decomposition $TM|_Z = TZ \oplus Z \times \mathfrak{g}^*$ where the second summand is embedded as an ω-isotropic subspace, and such that $(\mathrm{d}\Phi_G)|_Z$ is projection to the second summand. From the G-moment map condition

$$\iota(\xi_M)\omega|_Z = -\tfrac{1}{2}\Phi_G^* B((\theta^{\mathrm{L}} + \theta^{\mathrm{R}})|_Z, \xi) = -B((\mathrm{d}\Phi_G)|_Z, \xi), \quad \xi \in \mathfrak{g},$$

we see that the induced 2-form on the sub-bundle $Z \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ is just the standard one, $\omega_{\mathfrak{g}\oplus\mathfrak{g}^*}$. The ω-orthogonal space $Z \times (\mathfrak{g} \oplus \mathfrak{g}^*)^\omega$ defines a complement to $Z \times \mathfrak{g} \subset TZ$, and is hence identified with $\pi^* TM_{\mathrm{red}}$. ■

Proof of Theorem 7.13. Let $\Theta\colon TM|_Z \dashrightarrow TM_{\mathrm{red}}$ be the bundle morphism given by projection to the first summand in (27), followed by the quotient map. Then

$$(\Theta, \omega_{\mathfrak{g}\oplus\mathfrak{g}^*})\colon (\mathbb{T}M|_Z, TM|_Z) \dashrightarrow (\mathbb{T}M_{\mathrm{red}}, TM_{\mathrm{red}}),$$

is a strong Dirac morphism, and the resulting Morita morphism $\mathcal{A}_{TM}|_Z \dashrightarrow \mathcal{A}_{TM_{\mathrm{red}}}$ fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{C}\mathrm{l}}(TM)|_Z & \dashrightarrow & \mathcal{A}_{TM}|_Z \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{C}\mathrm{l}}(TM_{\mathrm{red}}) & \dashrightarrow & \mathcal{A}_{TM_{\mathrm{red}}} \end{array} \tag{30}$$

On the other hand, letting $\mathrm{pr}_1 : T(K \times G)|_{K\times\{e\}} \to TK$ be projection to the first summand, we have

$$(\mathrm{pr}_1, 0) \circ (\mathrm{d}\Phi|_Z, \omega|_Z) = (\mathrm{d}\Phi_{\mathrm{red}}, \omega_{\mathrm{red}}) \circ (\Theta, \omega_{\mathfrak{g}\oplus\mathfrak{g}^*}),$$

so that the resulting diagram of Morita morphisms

$$\begin{array}{ccc} \mathcal{A}_{TM}|_Z & \dashrightarrow & \mathcal{A}^{\mathrm{Spin}}_{K\times G}|_{K\times\{e\}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{TM_{\mathrm{red}}} & \dashrightarrow & \mathcal{A}^{\mathrm{Spin}}_K \end{array} \tag{31}$$

commutes up to 2-isomorphism. Placing (30) next to (31), the Theorem follows. ■

Remark 7.15. If e is a regular value of Φ_G, but the action of G on Z is not free, then the reduced space M_{red} is usually an orbifold. The theorem extends to this situation with obvious modifications. □

Remark 7.16. Reduction at more general values $g \in G$ may be expressed in terms of reduction at e, using the *shifting trick*: Let $G_g \subset G$ be the centralizer of g, and $\mathrm{Ad}(G)g^{-1} \cong G/G_{g^{-1}}$ the conjugacy class of g^{-1}. Then

$$M/\!\!/_g G := \Phi_G^{-1}(g)/G_g = (M \times \mathrm{Ad}(G)g^{-1})/\!\!/ G,$$

where $M \times \mathrm{Ad}(G).g^{-1}$ is the fusion product. Again, one finds that g is a regular value of Φ_G if and only if the G_g-action on $\Phi^{-1}(g)$ is locally free, and if the action is free then $M/\!\!/_g G$ is a q-Hamiltonian K-space. □

8 Hamiltonian LG-spaces

In his 1988 paper, Freed [13] argued that for a compact, simple and simply connected Lie group G, the canonical line bundle over the Kähler manifold LG/G (and over the other coadjoint orbits of the loop group) is a $\widehat{LG}$-equivariant Hermitian line bundle $K \to LG/G$, where the central circle of $\widehat{LG}$ acts with a weight $-2\mathsf{h}^\vee$, where $\mathsf{h}^\vee$ is the dual Coxeter number. In [22], this was extended to more general Hamiltonian LG-spaces.

In this section, we will use the correspondence between Hamiltonian LG-spaces and q-Hamiltonian G-spaces to give a new construction of the canonical line bundle, in which it is no longer necessary to assume G simply connected. We begin by recalling the definition of a Hamiltonian LG-space. Let G be a compact Lie group, with a given invariant inner product B on its Lie algebra. We fix $s > 1/2$, and take the loop group LG to be the Banach Lie group of maps $S^1 \to G$ of Sobolev class $s + 1/2$. Its Lie algebra $L\mathfrak{g}$ consists of maps $S^1 \to \mathfrak{g}$ of Sobolev class $s + 1/2$. We denote by $L\mathfrak{g}^*$ the $\mathfrak{g}$-valued 1-forms on S^1 of Sobolev class $s - 1/2$, with the gauge action $g \cdot \mu = \mathrm{Ad}_g(\mu) - g^*\theta^{\mathrm{R}}$. A *Hamiltonian LG-manifold* is a Banach manifold N with an action of LG, an invariant (weakly) symplectic 2-form $\sigma \in \Omega^2(N)$, and a smooth LG-equivariant map $\Psi\colon N \to L\mathfrak{g}^*$ satisfying the moment map condition

$$\iota(\xi^\sharp)\sigma = -\mathrm{d}\langle \Psi, \xi\rangle, \quad \xi \in L\mathfrak{g}.$$

Here the pairing between elements of $L\mathfrak{g}^*$ and of $L\mathfrak{g}$ is given by the inner product B followed by integration over S^1.

Suppose now that G is connected, and let $\mathcal{P}G$ be the space of paths $\gamma\colon \mathbb{R} \to G$ of Sobolev class $s + 1/2$ such that $\pi(\gamma) = \gamma(t+1)\gamma(t)^{-1}$ is constant. The map $\pi\colon \mathcal{P}G \to G$ taking γ to this constant is a G-equivariant principal LG-bundle, where $a \in G$ acts by $\gamma \mapsto a\gamma$ and $\lambda \in LG$ acts by $\gamma \mapsto \gamma\lambda^{-1}$. One has $\mathcal{P}G/G \cong L\mathfrak{g}^*$ with quotient map $\gamma \mapsto \gamma^{-1}\dot{\gamma}\mathrm{d}t$. Let $\tilde{N} \to N$ be the principal G-bundle obtained by pull-back of the bundle $\mathcal{P}G \to L\mathfrak{g}^*$, and $\tilde{\Psi}\colon \tilde{N} \to \mathcal{P}G$ the lifted moment map. Then $\tilde{\Psi}$ is $LG \times G$-equivariant. Since the LG-action on $\mathcal{P}G$ is a principal action, the same is true for the action on $\tilde{N}$. Assuming that Ψ (hence $\tilde{\Psi}$) is *proper*, one obtains a smooth *compact* manifold $M = \tilde{N}/LG$ with an induced G-map $\Phi\colon M \to G = \mathcal{P}G/LG$.

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{\Psi}} & \mathcal{P}G \\ {\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G} \\ M & \xrightarrow[\Phi]{} & G \end{array}$$

In [2], it was shown how to obtain an invariant 2-form ω on M, making (M, ω, Φ) into a q-Hamiltonian G-spaces. This construction sets up a 1–1 correspondence between Hamiltonian LG-spaces with proper moment maps and q-Hamiltonian spaces.

As noted in Remark 7.2, the canonical twisted Spin_c structure for (M, ω, Φ) defines a G-equivariant Morita trivialization of the bundle $\mathcal{E}\colon \mathbb{C} \dashrightarrow \Phi^*\mathcal{A}_G^{\mathrm{Spin}^{\otimes 2}}$ over M. On the other hand, let $\widehat{LG}^{\mathrm{Spin}}$ be the pull-back of the basic central extension $L\,\widehat{\mathrm{SO}(\mathfrak{g})}$ under the adjoint action. By the discussion in Section 6.3, the pull-back bundle $\mathcal{A}_G^{\mathrm{Spin}}$ to $\mathcal{P}G$ has a canonical $\widehat{LG}^{\mathrm{Spin}} \times G$-equivariant Morita trivialization,

$$\mathcal{S}_0\colon \mathbb{C} \dashrightarrow \pi_G^*\mathcal{A}_G^{\mathrm{Spin}},$$

where the central circle of $\widehat{LG}^{\mathrm{Spin}}$ acts with weight 1. Tensoring $\mathcal{S}_0$ with itself, and pulling everything back to $\hat{N}$ we obtain two Morita trivializations $\pi_M^*\mathcal{E}$ and $\tilde{\Psi}^*(\mathcal{S}_0 \otimes \mathcal{S}_0)$ of the Dixmier–Douady bundle $\mathcal{C}$ over $\tilde{N}$, given by the pull-back of $\mathcal{A}_G^{\mathrm{Spin}^{\otimes 2}}$ under $\Phi \circ \pi_M = \pi_G \circ \tilde{\Psi}$. Let

$$\tilde{K} := \mathrm{Hom}_{\mathcal{C}}(\tilde{\Psi}^*(\mathcal{S}_0 \otimes \mathcal{S}_0), \pi_M^*\mathcal{E}).$$

Then $\tilde{K}$ is a $\widehat{LG}^{\mathrm{Spin}} \times G$-equivariant Hermitian line bundle, where the central circle in $\widehat{LG}^{\mathrm{Spin}}$ acts with weight -2. Its quotient $K = \tilde{K}/G$ is the desired *canonical bundle* for the Hamiltonian LG-manifold N.

Remark 8.1. For G simple and simply connected, the central extension $\widehat{LG}^{\mathrm{Spin}}$ is the $\mathsf{h}^\vee$-th power of the "basic central" extension $\widehat{LG}$. We may thus also think of K_N as a $\widehat{LG}$-equivariant line bundle where the central circle acts with weight $-2\mathsf{h}^\vee$. □

The canonical line bundle is well-behaved under symplectic reduction. That is, if e is a regular value of Φ then $0 \in L\mathfrak{g}^*$ is also a regular value of Ψ, and $\Phi^{-1}(e) \cong \Psi^{-1}(0)$ as G-spaces. Assume that G acts freely on these level sets, so that $M/\!\!/G = N/\!\!/G$ is a symplectic manifold. The canonical line bundle for $M/\!\!/G$ is simply $K_{M/G} = K_N|_{\Psi^{-1}(0)}/G$. As in [22], one can sometimes use this fact to compute the canonical line bundle over moduli spaces of flat G-bundles over surfaces.

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Appendix A. Boundary conditions

In this section, we will prove several facts about the operator $\frac{\partial}{\partial t}$ on the complex Hilbert space $L^2([0,1],\mathbb{C}^n)$, with boundary conditions defined by $A\in \mathrm{U}(n)$,

$$\mathrm{dom}(D_A)=\{f\in L^2([0,1],\mathbb{C}^n)|\ \dot f\in L^2([0,1],\mathbb{C}^n),\ \ f(1)=-Af(0)\}.$$

Let $e^{2\pi i\lambda^{(1)}},\ldots,e^{2\pi i\lambda^{(n)}}$ be the eigenvalues of A, with corresponding normalized eigenvectors $v^{(1)},\ldots,v^{(n)}\in\mathbb{C}^n$. Then the spectrum of D_A is given by the eigenvalues $2\pi i(\lambda^{(r)}+k-\frac{1}{2})$, $k\in\mathbb{Z}$, $r=1,\ldots,n$ with eigenfunctions

$$\phi_k^{(r)}(t)=\exp\left(2\pi i\left(\lambda^{(r)}+k-\frac{1}{2}\right)t\right)v^{(r)}.$$

We define $J_A=i\,\mathrm{sign}(-iD_A)$; this coincides with $J_A=D_A/|D_A|$ if D_A has trivial kernel.

Proposition A.1. Let $A,A'\in\mathrm{U}(n)$. Then $J_{A'}-J_A$ is Hilbert–Schmidt if and only if $A'=A$.
□

Proof. Suppose $A'\neq A$. Let Π and Π' be the orthogonal projection operators onto $\ker(J_A-i)$, $\ker(J_{A'}-i)$. It suffices to show that $\Pi'-\Pi$ is not Hilbert–Schmidt, that is, that $(\Pi'-\Pi)^2$ is not trace class. Since

$$(\Pi-\Pi')^2=\Pi(I-\Pi')\Pi+(I-\Pi)\Pi'(I-\Pi).$$

is a sum of two positive operators, it suffices to show that $\Pi(I-\Pi')\Pi$ is not trace class. Let $\phi_l^{'(s)}$ be the eigenfunctions of $D_{A'}$, defined similar to those for D_A, with eigenvalues $2\pi i(\lambda^{'(s)}+l-\frac{1}{2})$. Indicating the eigenvalues and eigenfunctions for A' by a prime $'$, we have

$$\mathrm{tr}(\Pi(I-\Pi')\Pi)=\sum|\langle\phi_k^{(r)},\phi_l^{'(s)}\rangle|^2,$$

where the sum is over all k, r, l, s satisfying $\lambda^{(r)} + k - \frac{1}{2} > 0$ and $\lambda'^{(s)} + l - \frac{1}{2} \leq 0$. But

$$|\langle \phi_k^{(r)}, \phi_l'^{(s)} \rangle|^2 = \left| \frac{\langle v^{(r)}, v'^{(s)} \rangle (\mathrm{e}^{2\pi \mathrm{i}(\lambda'^{(s)} - \lambda^{(r)})} - 1)}{2\pi(\lambda'^{(s)} - \lambda^{(r)} + l - k)} \right|^2.$$

Since $A' \neq A$, we can choose r and s such that

$$\mathrm{e}^{2\pi \mathrm{i}\lambda^{(r)}} \neq \mathrm{e}^{2\pi \mathrm{i}\lambda'^{(s)}} \quad \text{and} \quad \langle v^{(r)}, v'^{(s)} \rangle \neq 0.$$

For such r and s, the enumerator is a nonzero constant, and the sum over k and l is divergent. ■

Proposition A.2. Given $A, A' \in \mathrm{U}(n)$, let

$$\gamma : [0, 1] \to \mathrm{Mat}_n(\mathbb{C})$$

be a continuous map with

$$A'\gamma(0) = \gamma(1)A,$$

and such that $\dot{\gamma} \in L^\infty([0, 1], \mathrm{Mat}_n(\mathbb{C}))$. Let M_γ be the bounded operator on $L^2([0, 1], \mathbb{C}^n)$ given as multiplication by γ. Then

$$M_\gamma J_A - J_{A'} M_\gamma$$

is Hilbert–Schmidt. □

Proof. This is a mild extension of Proposition(6.3.1) in Pressley–Segal [26, p. 82], and we will follow their line of argument. Using the notation from the proof of Proposition A.1, it suffices to show that $M_\gamma \Pi - \Pi' M_\gamma$ is Hilbert–Schmidt, or equivalently that both $(I - \Pi')M_\gamma \Pi$ and $\Pi' M_\gamma (I - \Pi)$ are Hilbert–Schmidt. We will give the argument for $\Pi' M_\gamma (I - \Pi)$, the discussion for $(I - \Pi')M_\gamma \Pi$ is similar. We must prove that

$$\begin{aligned} \mathrm{tr}((\Pi' M_\gamma (I - \Pi))(\Pi' M_\gamma (I - \Pi))^*) &= \mathrm{tr}(\Pi' M_\gamma (I - \Pi) M_\gamma^*) \\ &= \sum |\langle \phi_k'^{(r)} | M_\gamma | \phi_l^{(s)} \rangle|^2 < \infty, \end{aligned}$$

where the sum is over all k and r with $\lambda^{'(r)}+k-\frac{1}{2}>0$ and over all l and s with $\lambda^{(s)}+l-\frac{1}{2}\leq 0$. Changing the sum by only finitely many terms, we may replace this with a summation over all k, r, l and s such that $k>0$ and $l\leq 0$. Since $\langle\phi_k^{'(r)}|M_\gamma|\phi_l^{(s)}\rangle=\langle\phi_{k+n}^{'(r)}|M_\gamma|\phi_{l+n}^{(s)}\rangle$ for all $n\in\mathbb{Z}$, and since there are m terms with fixed $k-l=m$, the assertion is equivalent to

$$\sum_{r,s}\sum_{m>0} m\,|\langle\phi_0^{'(r)}|M_\gamma|\phi_m^{(s)}\rangle|^2<\infty. \tag{A.1}$$

To obtain this estimate, we use $\dot\gamma\in L^\infty([0,1],\mathrm{Mat}_n(\mathbb{C}))$. We have

$$\sum_{r,s}\sum_{m\in\mathbb{Z}}|\langle\phi_0^{'(r)}|M_{\dot\gamma}|\phi_m^{(s)}\rangle|^2=\sum_r||M_{\dot\gamma}^*\phi_0^{'(r)}||^2<\infty.$$

An integration by parts shows

$$\begin{aligned}\langle\phi_0^{'(r)}|M_{\dot\gamma}|\phi_m^{(s)}\rangle&=-2\pi\mathrm{i}(\lambda^{(s)}-\lambda^{'(r)}+m)\langle\phi_0^{'(r)}|M_\gamma|\phi_m^{(s)}\rangle\\&\quad+\langle\phi_0^{'(r)}(1)|\gamma(1)|\phi_m^{(s)}(1)\rangle-\langle\phi_0^{'(r)}(0)|\gamma(0)|\phi_m^{(s)}(0)\rangle.\end{aligned}$$

The boundary terms cancel since $A'\gamma(0)=\gamma(1)A$, and

$$\phi_0^{'(r)}(1)=-A'\phi_0^{'(r)}(0),\quad \phi_m^{(s)}(1)=-A\phi_m^{(s)}(0).$$

Hence we obtain

$$\sum_{r,s}\sum_{m\in\mathbb{Z}}(\lambda^{(s)}-\lambda^{'(r)}+m)^2\,|\langle\phi_0^{'(r)}|M_\gamma|\phi_m^{(s)}\rangle|^2<\infty$$

which implies (A.1). ■

Proposition A.3. Let $A\in\mathrm{U}(n)$, and let $\mu\in L^\infty([0,1],\mathfrak{u}(n))$. Consider $D_{A,\mu}=D_A+M_\mu$ with domain equal to that of D_A, and define $J_{A,\mu}$ similar to J_A. Then $J_{A,\mu}-J_A$ is Hilbert–Schmidt. □

Proof. Let $\gamma\in C([0,1],\mathrm{U}(n))$ be the solution of the initial value problem $\dot\gamma\gamma^{-1}=-\mu$ with $\gamma(0)=I$. Let $A=\gamma(1)A'$. The operator M_γ of multiplication by γ takes $\mathrm{dom}(D_{A'})$

to dom(D_A), and

$$M_\gamma D_{A'} M_\gamma^{-1} = D_A - \dot{\gamma}\gamma^{-1} = D_{A,\mu}.$$

Hence $M_\gamma J_{A'} M_{\gamma^{-1}} = J_{A,\mu}$. By Proposition A.2, $M_\gamma J_{A'} M_{\gamma^{-1}}$ differs from J_A by a Hilbert–Schmidt operator. ■

Let us finally consider the continuity properties of the family of operators D_A, $A \in \mathrm{U}(n)$. Recall [28, Chapter VIII] that the *norm resolvent topology* on the set of unbounded skewadjoint operators on a Hilbert space is defined by declaring that a net D_i converges to D if and only if $R_1(D_i) = (D_i - I)^{-1} \to R_1(D) = (D - I)^{-1}$ in norm. This then implies that $R_z(D_i) \to R_z(D)$ in norm, for any z with nonzero real part, and in fact $f(D_i) \to f(D)$ in norm for any bounded continuous function f. For bounded operators, convergence in the norm resolvent topology is equivalent to convergence in the norm topology.

Proposition A.4. The map $A \mapsto D_A$ is continuous in the norm resolvent topology. □

Proof. We will use that $||R_1(D)|| = ||(D - I)^{-1}|| < 1$ for any skew-adjoint operator D. Let us check continuity at any given $A \in \mathrm{U}(n)$. Given $a \in \mathfrak{u}(n)$, let us write $D_a = D_{\exp(a)A}$. We will prove continuity at A by showing that

$$||R_1(D_a) - R_1(D_0)|| \leq 3||a||.$$

Let $U_a \in \mathrm{U}(\mathcal{V})$ be the operator of pointwise multiplication by $\exp(ta) \in \mathrm{U}(V)$. Then

$$||U_a - U_0|| = \sup_{t \in [0,1]} ||\exp(ta) - I|| \leq ||a||.$$

The operator U_a takes the domain of D_0 to that of D_a, since $f(1) = -Af(0)$ implies $(U_a f)(1) = \exp(a) f(1) = -\exp(a) A f(0)$. Furthermore,

$$D_a = U_a (D_0 + M_a) U_a^{-1}$$

Hence

$$R_1(D_a) = U_a R_1(D_0 + M_a) U_a^{-1}.$$

The second resolvent identity $R_1(D_0+M_a)-R_1(D_0)=R_1(D_0+M_a)M_aR_1(D_0)$ shows

$$||R_1(D_0+M_a)-R_1(D_0)||\leq ||M_a||=||a||.$$

Hence

$$\begin{aligned}
&||R_1(D_a)-R_1(D_0)||\\
&\quad=||U_aR_1(D_0+M_a)U_a^{-1}-U_0R_1(D_0)U_0^{-1}||\\
&\quad\leq ||(U_a-U_0)R_1(D_0+M_a)U_a^{-1}||+||U_0R_1(D_0+M_a)(U_a^{-1}-U_0^{-1})||\\
&\qquad+||U_0(R_1(D_0+M_a)-R_1(D_0))U_0^{-1}||\\
&\quad\leq 2||a||\;||R_1(D_0+M_a)||+||R_1(D_0+M_a)-R_1(D_0)||<3||a||.
\end{aligned}$$

■

Appendix B. The Dixmier–Douady bundle over S^1

Let $S^1=\mathbb{R}/\mathbb{Z}$ carry the *trivial* action of S^1. The Morita isomorphism classes of S^1-equivariant Dixmier–Douady bundles $\mathcal{A}\to S^1$ are labeled by their class

$$\mathrm{DD}_{S^1}(\mathcal{A})\in H^3_{S^1}(S^1,\mathbb{Z})\times H^1(S^1,\mathbb{Z}_2).$$

The bundle corresponding to $x\in H^3_{S^1}(S^1,\mathbb{Z})=H^2_{S^1}(\mathrm{pt})=\mathbb{Z}$ and $y\in H^1(S^1,\mathbb{Z}_2)=H^0(\mathrm{pt},\mathbb{Z}_2)=\mathbb{Z}_2$ may be described as follows. Let $L_{(x,y)}\cong\mathbb{C}$ be the $\mathbb{Z}_2$-graded S^1-representation, of parity given by the parity of y, and with S^1-weight given by x. Choose a $\mathbb{Z}_2$-graded S^1-equivariant Hilbert space $\mathcal{H}$ with an equivariant isomorphism $\tau\colon\mathcal{H}\to\mathcal{H}\otimes L$ preserving $\mathbb{Z}_2$-gradings. Then τ induces an S^1-equivariant $*$-homomorphism

$$\bar{\tau}\colon\mathbb{K}(\mathcal{H})\to\mathbb{K}(\mathcal{H}\otimes L)=\mathbb{K}(\mathcal{H}),$$

preserving $\mathbb{Z}_2$-gradings. The bundle $\mathcal{A}\to S^1$ with Dixmier–Douady class (x,y) is obtained from the trivial bundle $[0,1]\times\mathbb{K}(\mathcal{H})$, using $\bar{\tau}$ to glue $\{0\}\times\mathbb{K}(\mathcal{H})$ and $\{1\}\times\mathbb{K}(\mathcal{H})$. Given another choice $\mathcal{H}',\tau'$, one obtains a Morita isomorphism $\mathcal{E}\colon\mathcal{A}\to\mathcal{A}'$, where $\mathcal{E}$ is obtained from a similar boundary identification for $[0,1]\times\mathbb{K}(\mathcal{H}',\mathcal{H})$.

A convenient choice of H, τ defining the bundle with $x=1$ and $y=1$ is as follows. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis of the form s_K, indexed by the subsets $K=\{k_1, k_2, \ldots\} \subset \mathbb{Z}$ such that $k_1 > k_2 > \cdots$ and $k_l = k_{l+1}+1$ for l sufficiently large. Let

$$m_K = \#\{k \in K | k > 0\} - \#\{k \in \mathbb{Z} - K | \, k \leq 0\}.$$

Let $\mathcal{H}$ carry the S^1-action such that s_K is a weight vector of weight m_K, and a $\mathbb{Z}_2$-grading, defined by the weight spaces of even/odd weight. Let $\tau(K) = \{k+1 | \, k \in K\}$. Then $m_{\tau(K)} = m_K + 1$, hence the automorphism $\tau : \mathcal{H} \to \mathcal{H}$ taking s_K to $s_{\tau(K)}$ has the desired properties.

The Hilbert space $\mathcal{H}$ can also be viewed as a spinor module. Let $\mathcal{V}$ be a real Hilbert space, with complexification $\mathcal{V}^{\mathbb{C}}$, and let $f_k, \; k \in \mathbb{Z}$ be vectors such that f_k together with f_k^* are an orthonormal basis. The elements s_K for $K = \{k_1, k_2, \ldots\}$ with $k_1 > k_2 > \cdots$ are written as formal infinite wedge products

$$s_K = f_{k_1} \wedge f_{k_2} \wedge \cdots$$

suggesting the action of the Clifford algebra: $\varrho(f_k)$ acts by exterior multiplication, while $\varrho(f_{k^*})$ acts by contraction. The automorphism $\tau \in \mathrm{U}(\mathcal{H})$ is an implementer of the orthogonal transformation $T \in \mathrm{O}(V)$,

$$T f_k = f_{k+1}, \quad T f_k^* = f_{k+1}^*. \tag{B.1}$$

Let us denote the resulting Dixmier–Douady bundle by $\mathcal{A}_{(1,1)}$.

Proposition B.1. The Dixmier–Douady bundle $\mathcal{A}_{(1,1)} \to S^1$ is equivariantly isomorphic to the Dixmier–Douady bundle $\mathcal{A} \to \mathrm{SO}(2) \cong S^1$, constructed as in Section 6. □

Proof. For $s \in \mathbb{R}$, let $A_s \in \mathrm{SO}(2)$ be the matrix of rotation by $2\pi s$, and let D_s be the skew-adjoint operator $\frac{\partial}{\partial t}$ on $L^2([0,1], \mathbb{R}^2)$ with boundary conditions $f(1) = -A_s f(0)$. The operator D_0 has an orthonormal system of eigenvectors $f_k, f_k^*, \; k \in \mathbb{Z}$ given by

$$f_k(t) = \mathrm{e}^{2\pi \mathrm{i}(k-\frac{1}{2})t} u,$$

with $u = \frac{1}{\sqrt{2}}(1, \mathrm{i})$. The eigenvalues for f_k, f_k^* are $\pm 2\pi\mathrm{i}(k - \frac{1}{2})$. We see that the $+\mathrm{i}$ eigenspace of $J = D_0/|D_0|$ is given by

$$\mathcal{V}_+ = \mathrm{span}\{\cdots, f_3, f_2, f_1, f_0^*, f_{-1}^*, \cdots\}.$$

There is a unique isomorphism of $\mathbb{C}\mathrm{l}(\mathcal{V})$-modules $\mathcal{S}_J \to \mathcal{H}$ taking the "vacuum vector" $1 \in \mathcal{S}_J = \overline{\wedge \mathcal{V}_+}$ to the "vacuum vector" $f_0 \wedge f_{-1} \wedge \cdots$.

For $s \in \mathbb{R}$, define orthogonal transformations $U_s \in \mathrm{O}(\mathcal{V})$, where U_s is pointwise multiplication by $t \mapsto A_{st}$. On f_k the operator U_s acts as multiplication by $\mathrm{e}^{2\pi\mathrm{i}st}$, and on f_k^* as multiplication by $\mathrm{e}^{-2\pi\mathrm{i}st}$. Hence

$$f_k^{(s)} = U_s f_k, \quad (f_k^{(s)})^* = U_s f_k^*$$

are the eigenvectors of D_s, with shifted eigenvalues $\pm 2\pi\mathrm{i}(k - \frac{1}{2} + s)$. The complex structure

$$J_s = U_s J U_s^{-1}$$

differs from $J_{D_s} = \mathrm{i}\,\mathrm{sign}(-\mathrm{i}D_s)$ by a finite rank operator. Hence, letting $\mathcal{S}_s$ denote the $\mathbb{C}\mathrm{l}(\mathcal{V})$-module defined by J_s, the fiber of $\mathcal{A} \to \mathrm{SO}(2)$ at $A(s)$ may be described as $\mathbb{K}(\mathcal{S}_s)$. The orthogonal transformation U_s extends to an orthogonal transformation of $\overline{\wedge \mathcal{V}}$, taking $\mathcal{S} = \overline{\wedge \mathcal{V}_+}$ to $\mathcal{S}_s = \overline{\wedge \mathcal{V}_{+,s}}$, where $\mathcal{V}_{\pm,s} = U_s \mathcal{V}_\pm$. Hence each $\mathcal{S}_s$ is identified with $\mathcal{S} \cong \mathcal{H}$ as a Hilbert space (not as a $\mathbb{C}\mathrm{l}(\mathcal{V})$-module). The identification $\mathbb{K}(\mathcal{S}_0) \cong \mathbb{K}(\mathcal{S}_1)$ is given by the choice of any isomorphism of $\mathbb{C}\mathrm{l}(\mathcal{V})$-modules $\mathcal{S}_0 \to \mathcal{S}_1$. In terms of the identifications with $\mathcal{H}$, such an isomorphism is given by an implementer of the orthogonal transformation U_1. The proof is completed by the observation that $U_1 = T$ (cf. (B.1)), which is implemented by τ. ∎

We are now in a position to outline an alternative argument for the computation of the Dixmier–Douady class of $\mathcal{A}_{\mathrm{SO}(n)}$, Proposition 6.2. Note that $\mathcal{A}_{\mathrm{SO}(n)}$ is equivariant under the conjugation action of $\mathrm{SO}(n)$. One has $H^3_{\mathrm{SO}(n)}(\mathrm{SO}(n), \mathbb{Z}) = \mathbb{Z}$ for $n \geq 2$, $n \neq 4$, and the natural maps to ordinary cohomology are isomorphisms for $n \geq 3$ and $n \neq 4$. Similarly $H^1_{\mathrm{SO}(n)}(\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2$ for $n \geq 2$, and the natural map to $H^1(\mathrm{SO}(n), \mathbb{Z}_2)$ is an isomorphism. On the other hand, the map $H^3_{\mathrm{SO}(n)}(\mathrm{SO}(n), \mathbb{Z}) \to H^3_{\mathrm{SO}(2)}(\mathrm{SO}(2), \mathbb{Z})$ (defined by the inclusion $\mathrm{SO}(2) \hookrightarrow \mathrm{SO}(n)$ as the upper left corner) is an isomorphism for $n \geq 2$ and $n \neq 4$, and likewise for $H^1(\cdot, \mathbb{Z}_2)$. It hence suffices to check that the bundle over $\mathrm{SO}(2)$ has *equivariant*

Dixmier–Douady class $(1, 1) \in \mathbb{Z} \times \mathbb{Z}_2$. But this is clear from our very explicit description of $\mathcal{A}_{\mathrm{SO}(2)}$.

References

[1] Alekseev, A., H. Bursztyn, and E. Meinrenken. "Pure spinors on Lie groups." *Astérisque* 327 (2009): 131–99.

[2] Alekseev, A., A. Malkin, and E. Meinrenken. "Lie group valued moment maps." *Journal of Differential Geometry* 48, no. 3 (1998): 445–95.

[3] Alekseev, A. and E. Meinrenken. "The non-commutative Weil algebra." *Inventiones Mathematicae* 139, no. 1 (2000): 135–72.

[4] Alekseev, A., E. Meinrenken, and C. Woodward. "Duistermaat–Heckman measures and moduli spaces of flat bundles over surfaces." *Geometry and Functional Analysis* 12, no. 1 (2002): 1–31.

[5] Araki, H. "Bogoliubov automorphisms and Fock representations of canonical anticommutation relations." *American Mathematical Society Contemporary Mathematics* 62 (1987): 23–141.

[6] Atiyah, M. and G. Segal. "Twisted K-theory." *Ukrainian Mathematical Bulletin* 1, no. 3 (2004): 291–334.

[7] Bursztyn, H. and M. Crainic. "Dirac Structures, Momentum Maps, and Quasi-Poisson Manifolds". *The Breadth of Symplectic and Poisson Geometry*, 1–40. Progress in Mathematics 232. Boston, MA: Birkhäuser, 2005.

[8] Bursztyn, H. and O. Radko. "Gauge equivalence of dirac structures and symplectic groupoids." *Annales de l' Institut Fourier (Grenoble)* 53, no. 1 (2003): 309–37.

[9] Carey, A., J. Mickelsson, and M. Murray. "Index theory, gerbes, and Hamiltonian quantization." *Communications in Mathematical Physics* 183, no. 3 (1997): 707–22.

[10] Connes, A. and G. Skandalis. "The longitudinal index theorem for foliations." *Kyoto University. Research Institute for Mathematical Sciences. Publications* 20, no. 6 (1984): 1139–183.

[11] Dixmier, J. and A. Douady. "Champs continus d'espaces hilbertiens et de C^*-algèbres." *Bulletin de la Societe Mathematique de France* 91 (1963): 227–84.

[12] Douglas, C. L. "On the twisted K-homology of simple Lie groups." *Topology* 45, no. 6 (2006): 955–88.

[13] Freed, D. S. "The geometry of loop groups." *Journal of Differential Geometry* 28, no. 2 (1988): 223–76.

[14] Freed, D., M. Hopkins, and C. Teleman. "Loop groups and twisted K-theory II". Preprint: arXiv:math.AT/0511232.

[15] Hitchin, N. "Lectures on special Lagrangian submanifolds." *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, 151–82. Cambridge, MA, 1999: American Mathematical Society, Providence, RI, 2001.

[16] Mathai, V., R. Melrose, and I. Singer. "The index of projective families of elliptic operators." *Geometry and Topology* 9 (2005): 341–73 (electronic).

[17] Mathai, V., R. Melrose, and I. Singer. "Fractional analytic index." *Journal of Differential Geometry* 74, no. 2 (2006): 265–92.

[18] Mathai, V., R. Melrose, and I. Singer. "The index of projective families of elliptic operators: The decomposable case." *Astérisque* 328 (2009): 255–96.

[19] Meinrenken, E. "On the quantization of conjugacy classes." *L' Enseignement Mathematique* 55, no. 1–2 (2009): 33–75.

[20] Meinrenken, E. "Twisted k-homology and group-valued moment maps" (2010): Preprint arXiv:1008.1261.

[21] Meinrenken, E. "Quantization of q-Hamiltonian SU(2)-spaces." *Geometric Aspects of Analysis and Mechanics, in Honor of Hans Duistermaat*. Basel: Birkhäuser, in press.

[22] Meinrenken, E. and C. Woodward,"Canonical bundles for Hamiltonian loop group manifolds." *Pacific Journal of Mathematics* 198, no. 2 (2001): 477–87.

[23] Mickelsson, J. *Gerbes and Quantum Field Theory*. Encyclopedia of Mathematical Physics, edited by T. Francoise and G. Naber. Amsterdam: Elsevier, 2006.

[24] Plymen, R. J. "Strong Morita equivalence, spinors and symplectic spinors." *Journal of Operator Theory* 16 (1986): no. 2, 305–24.

[25] Plymen, R. J. and P. L. Robinson. *Spinors in Hilbert Space*. Cambridge: Cambridge University Press, 1994.

[26] Pressley, A. and G. Segal. *Loop Groups*. Oxford: Oxford University Press, 1988.

[27] Raeburn, I. and D. Williams. *Morita equivalence and continuous-trace C^*-algebras.* Mathematical Surveys and Monographs 60. Providence, RI: American Mathematical Society, 1998.

[28] Reed, M. and B. Simon. *Functional Analysis*. New York: Academic Press, 1972.

[29] Rosenberg, J. "Continuous-trace algebras from the bundle theoretic point of view." *Journal of Australian Mathematical Society Series A* 47, no. 3 (1989): 368–81.

[30] Segal, G. "Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others)." *Astérisque* (1988) 161–162, Exp. No. 695, 4 (1989): 187–201 Séminaire Bourbaki, Vol. 1987/88.

[31] Ševera, P. and A. Weinstein. "Poisson geometry with a 3-form background." *Progress in Theoretical Physics Suppl.* no. 144 (2001): 145–54, Noncommutative geometry and string theory (Yokohama, 2001).

[32] Shale, D. and W. F. Stinespring. "Spinor representations of infinite orthogonal groups." *Journal of Mathematics and Mechanics* 14 (1965): 315–22.

[33] Wang, B.-L. "Geometric cycles, index theory and twisted k-homology." *Journal of Noncommutative Geometry* 2, no. 4 (2008): 497–552.