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# A Remark on different lattice approximations and continuum limits for $\phi_2^4$-fields[1]

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**Abstract**—Consider the lattice approximation of a $\phi_2^4$-quantum field model with different lattice cutoffs $a'$ and $a$ in the free and interacting parts, respectively. In [1] it was shown that the corresponding continuum limit measure exists if $\lim_{a\to 0} a'|\log a|^{5/4} < \infty$ and it coincides with the original $\phi_2^4$-field measure if $\lim_{a\to 0} a'|\log a|^2 < \infty$. In this paper, a result is given indicating that the new continuum limit measure might be different from the original one if $a'$ is too big compared with $a$.

*Keywords:* Lattice approximation, quantum fields, $\phi_2^4$-model, inequalities, continuum limits

## 1. INTRODUCTION

Let $G_a$ be the free lattice field measure of mass $m_0 > 0$ and lattice spacing $a > 0$ on $a\mathbf{Z}^2 = \{an; n \in \mathbf{Z}^2\}$, and let

$$C^{(a)}(x-y) = \langle \phi_x \phi_y \rangle_{G_a},$$

where $\langle \cdot \rangle_*$ denotes the expectation with respect to $*$. $G_a$ is thus the (lattice) Gaussian measure with covariance $C^{(a)}$. $(\phi_x)_{x\in a\mathbf{Z}^2}$ is the coordinate process (a Gaussian field, called Euclidean free lattice field). One has by definition (see [3])

$$C^{(a)}(x-y) = (2\pi)^{-2} \int_{[-\frac{\pi}{a},\frac{\pi}{a}]^2} e^{ik\cdot(x-y)} \mu_a(k)^{-2} dk,$$

where

$$\mu_a(k) := \left( m_0^2 + 2a^{-2} \sum_{j=1}^{2} (1-\cos(ak_j)) \right)^{1/2} \quad \text{for } k = (k_1, k_2).$$

Let $\mu_0$ be the (Nelson's or Euclidean) free field measure on $\mathbf{R}^2$ of mass $m_0$, *i.e.*, Gaussian measure on $\mathcal{S}'(\mathbf{R}^2)$ with the covariance $(-\Delta + m_0^2)^{-1}$. Let

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$$\mu(k) := (m_0^2 + |k|^2)^{1/2},$$

where $k = (k_1, k_2)$, $|k|^2 = k_1^2 + k_2^2$, and let $f_{a,x}(\cdot)$ be the function whose Fourier transform is

$$\mathcal{F}(f_{a,x})(k) = (2\pi)^{-1} e^{-ik\cdot x} \mu_a(k)^{-1} \mu(k) 1_{[-\frac{\pi}{a},\frac{\pi}{a}]}(k_1) 1_{[-\frac{\pi}{a},\frac{\pi}{a}]}(k_2).$$

Denote by $\phi$ the coordinate process associated with $\mu_0$ (called Nelson's or Euclidean free field): $\phi$ is first defined as an element of $\mathcal{S}'(\mathbf{R}^2)$, so that $\phi(g)$ is the dualization of $g \in \mathcal{S}(\mathbf{R}^2)$ with $\phi \in \mathcal{S}'(\mathbf{R}^2)$. $\phi$ is then extended by continuity in $L^2(d\mu_0)$ to a linear process $\phi(g)$, with $g$ belonging to a larger space than $\mathcal{S}(\mathbf{R}^2)$. In fact this space contains functions of the form $f_{a,x}$, and it is easy to check that

$$\langle \phi(f_{a,x}) \phi(f_{a,y}) \rangle_{\mu_0} = \langle \phi_x \phi_y \rangle_{G_a}.$$

(see, *e.g.*, [3] - [5]). In this sense, we can realize the above Gaussian field $\phi_x$ on $a\mathbf{Z}^2$ by $\phi(f_{a,x})$ defined on $\mathcal{S}'(\mathbf{R}^2)$.

Let $g \in C_0^+(\mathbf{R}^2)$ be a given function and let $a' = a'(a) \geq a$ satisfying $a'\mathbf{Z}^2 \subset a\mathbf{Z}^2$ and $\lim_{a\to 0+} a'(a) = 0$.

Let $: \phi(f_{a,x})^4 :$ be the fourth Wick order of $\phi(f_{a,x})$, *i.e.*,

$$: \phi(f_{a,x})^4 := \phi(f_{a,x})^4 - 6C^{(a)}(0)\phi(f_{a,x})^2 + 3C^{(a)}(0)^2.$$

For any $\lambda > 0$, let $\mu_{\lambda,a,a'}$ be the probability measure on $\mathcal{S}'(\mathbf{R}^2)$ defined by

$$\mu_{\lambda,a,a'}(d\phi) := \frac{e^{-\lambda \int_{a'\mathbf{Z}^2} dx g(x) :\phi(f_{a,x})^4:} \mu_0(d\phi)}{\int_{\mathcal{S}'(\mathbf{R}^2)} e^{-\lambda \int_{a'\mathbf{Z}^2} dx g(x) :\psi(f_{a,x})^4:} \mu_0(d\psi)},$$

where we used the notation $\int_{a'\mathbf{Z}^2} dx$ to denote the lattice sum on $a'\mathbf{Z}^2$ with weight $a'^2$. By [1], we have that for $\lambda > 0$ small enough, there exists a sequence $a_n \to 0$ ( $n \to \infty$) such that $\mu_{\lambda,a_n,a_n'}$ converges weakly to a probability measure on $\mathcal{S}'(\mathbf{R}^2)$ as $n \to \infty$, if

$$\lim_{a_n \to 0} a_n' |\log a_n|^{5/4} < \infty;$$

and moreover, if

$$\lim_{a_n \to 0} a_n' |\log a_n|^2 < \infty,$$

then the limit is equal to the origin $\phi_2^4$-field measure given by

$$\mu_\lambda(d\phi) = \left( \int e^{-\lambda \int_{\mathbf{R}^2} g(x) :\phi(x)^4: dx} \mu_0(d\phi) \right)^{-1} e^{-\lambda \int_{\mathbf{R}^2} g(x) :\phi(x)^4: dx} \mu_0(d\phi),$$

where $: \phi(x)^4 :$ is the fourth Wick power of Nelson's free field $\phi(x)$ with respect to $\mu_0$ (see [3] for the definition of it).

It is an interesting question to ask whether the above weak limit coincides with the original measure $\mu_\lambda$. This will be discussed in the following section.

## 2. THE RESULT

It has been conjectured in [1] that for

$$\lim_{a\to 0} a'|\log a|^2 = +\infty,$$

the weak limit of $\mu_{\lambda,a,a'}$ as $a \to 0$ is different from $\mu_\lambda$.

A first result in this direction is given in the following: Let $A > 0$ be a constant such that the support of $g$ is contained in the ball $B(0, A)$ centered at 0 and of radius $A$, and let $f = f_{4A,0}$. Without loss of generality, we may and do assume that $A = \frac{1}{4}$. Then we have the following:

**Theorem 2.1** *For any $m \in \mathbf{N}$, if*

$$\lim_{a\to 0} a'(a)|\log a|^{2-\frac{1}{2m}} = +\infty,$$

*then*

$$\lim_{a\to 0} \frac{d^{2m}}{d\lambda^{2m}} \left( \int_{\mathcal{S}'(\mathbf{R}^2)} \phi(f)^2 \mu_{\lambda,a,a'}(d\phi) \right)\Big|_{\lambda=0} = +\infty.$$

*Remark* 1: Intuitively, Theorem 2.1 means that the effect of the perturbation (coming from the interaction term depending on $a'$) in a fixed direction does not decay too fast when $a \to 0$. This indeed makes it reasonable to expect that the weak limit of $\mu_{\lambda,a,a'}$ as $a \to 0$ is different from $\mu_\lambda$. Also, we want to remind the reader that

$$\begin{aligned} &\frac{d^{2m}}{d\lambda^{2m}} \left( \int_{\mathcal{S}'(\mathbf{R}^2)} \phi(f)^2 \mu_{\lambda}(d\phi) \right)\Big|_{\lambda=0} \\ = &\int_{\mathcal{S}'(\mathbf{R}^2)} : \phi(f_{1,0})^2 : \left( \int_{a'\mathbf{Z}^2} dx g(x) : \phi_x^4 : \right)^{2m} \mu_0(d\phi) < \infty. \end{aligned}$$

It would be, on the other hand, not enough to argue (as shortly mentioned in [1], [2]) that having a Gaussian measure $\mu$ and making a change $\mu \to \mu_n = e^{a_n Z_n}\mu$, with $a_n \to \infty$ and the law of the random variables $Z_n$ under $\mu$ converging to $N(0,1)$, that then the limit of $\mu_n$ would be orthogonal to $\mu$. A counter-example to this can easily be provided: given an infinite dimensional product Gaussian measure $\mu = \otimes\nu_n$, $\nu_n$ being the standard centered Gaussian measure on $\mathbf{R}$, if we change $\mu$ to a measure $\mu_{(m)} = \otimes_{n\neq m}\nu_n \otimes \nu'_m$ with some $\nu'_m \neq \nu_m$, then whatever $\{\nu'_m\}_{m\in\mathbf{N}}$ is, the sequence $\mu_{(m)}$ weakly converges to the original measure $\mu$, as $m \to \infty$.

*Proof of Theorem* 2.1. We first remark that

$$\begin{aligned} &\frac{d^{2m}}{d\lambda^{2m}} \left( \int_{\mathcal{S}'(\mathbf{R}^2)} \phi(f_{1,0})^2 \mu_{\lambda,a,a'}(d\phi) \right)\Big|_{\lambda=0} \\ = &\int_{\mathcal{S}'(\mathbf{R}^2)} : \phi(f_{1,0})^2 : \left( \int_{a'\mathbf{Z}^2} dx g(x) : \phi(f_{a,x})^4 : \right)^{2m} \mu_0(d\phi) \\ = &\; a'^{4m} \sum_{l_1,\cdots,l_{2m}\in\mathbf{Z}^2} \left( \prod_{k=1}^{2m} g(a'l_k) \right) \langle : \phi(f_{1,0})^2 : \prod_{k=1}^{2m} : \phi(f_{a,a'l_k})^4 : \rangle_{\mu_0}. \end{aligned} \tag{2.1}$$

Now, we shall calculate the expression on the right hand side.

Note that for any numbers $\alpha_1, \cdots, \alpha_{2m+1} \in \mathbf{R}$ and functions $f_1, \cdots, f_{2m+1} \in H_2^1(\mathbf{R}^2)$, we have by the definition of Wick powers that

$$\prod_{i=1}^{2m+1} : e^{\alpha_i f_i} : = : e^{\sum_{i=1}^{2m+1} \alpha_i f_i} : \times e^{\sum_{1 \le i < j \le 2m+1} \alpha_i \alpha_j \langle f_i f_j \rangle},$$

so

$$\langle \prod_{i=1}^{2m+1} : e^{\alpha_i f_i} : \rangle = e^{\sum_{1 \le i < j \le 2m+1} \alpha_i \alpha_j \langle f_i f_j \rangle}.$$

Both sides are analytic in $\alpha_1, \cdots, \alpha_{2m+1}$. Taking a Taylor expansion of both sides of this equality, and comparing the coefficients of the term $\alpha_1^2 \alpha_2^4 \cdots \alpha_{2m+1}^4$, we get that

$$\begin{aligned} & \frac{1}{2!(4!)^{2m}} \langle : f_1^2 : \prod_{i=2}^{2m+1} : f_i^4 : \rangle \\ = \ & \frac{1}{(4m+1)!} \Big( \text{coefficient of } \alpha_1^2 \alpha_2^4 \cdots \alpha_{2m+1}^4 \text{ in} \\ & \qquad \text{the expansion of } \big( \sum_{1 \le i < j \le 2m+1} \alpha_i \alpha_j \langle f_i f_j \rangle \big)^{4m+1} \Big) \end{aligned}$$

So we need some estimates of $\langle \phi(f_{1,0}) \phi(f_{a,a'l}) \rangle_{\mu_0}$ and $\langle \phi(f_{a,a'l_1}) \phi(f_{a,a'l_2}) \rangle_{\mu_0}$.

Let $\mathcal{F}^{-1}(\cdot)$ denote the inverse Fourier transform. Then in general, we have that for any $a > 0$ and $x, z \in \mathbf{R}^2$:

$$\begin{aligned} f_{a,x}(z) &= \frac{1}{2\pi} \int_{[-\frac{\pi}{a}, \frac{\pi}{a}]^2} e^{ik \cdot z} e^{-ik \cdot x} \mu_a(k)^{-1} \mu(k) dk \\ &= \mathcal{F}^{-1}(\mu_a^{-1} \mu)(z - x) = \mathcal{F}^{-1}(\mu_a^{-1} \mu)(x - z), \end{aligned}$$

so by the definition of $\mu(k)$, we have

$$(-\Delta + m_0^2)^{-1} f_{a,x}(z) = \mathcal{F}^{-1}(\mu_a^{-1} \mu^{-1})(z - x).$$

Therefore, for any $x, y \in \mathbf{R}^2$ and $a \ge b > 0$,

$$\begin{aligned} & \langle \phi(f_{a,x}) \phi(f_{b,y}) \rangle_{\mu_0} \\ = \ & \int_{\mathbf{R}^2} f_{a,x}(z) (-\Delta + m_0^2)^{-1} f_{b,y}(z) dz \\ = \ & \int_{\mathbf{R}^2} \mathcal{F}^{-1}(\mu_a^{-1} \mu)(x - z) \mathcal{F}^{-1}(\mu_b^{-1} \mu^{-1})(z - y) dz \\ = \ & \left( \mathcal{F}^{-1}(\mu_a^{-1} \mu) \right) * \left( \mathcal{F}^{-1}(\mu_b^{-1} \mu^{-1}) \right) (y - x) \\ = \ & (2\pi)^{-1} \mathcal{F}^{-1}(\mu_a^{-1} \mu \mu_b^{-1} \mu^{-1})(y - x) \\ = \ & (2\pi)^{-1} \mathcal{F}^{-1}(\mu_a^{-1} \mu_b^{-1})(y - x) \\ = \ & (2\pi)^{-2} \int_{[-\frac{\pi}{a}, \frac{\pi}{a}]^2} e^{ik \cdot (y - x)} \mu_a(k)^{-1} \mu_b(k)^{-1} dk. \end{aligned}$$

Therefore, since

$$e^{iz} + e^{-iz} = 2 \cos z$$

for any $z \in \mathbf{R}$, we have that for any $a < 1$,

$$\begin{aligned}
&\langle \phi(f_{1,0})\phi(f_{a,y})\rangle_{\mu_0} \\
= \;& (2\pi)^{-2}\int_{[-\pi,\pi]^2} e^{ik\cdot y}\Big[m_0^2 + 2\sum_{j=1}^{2}(1-\cos k_j)\Big]^{-1/2} \\
& \times\Big[m_0^2 + 2a^{-2}\sum_{j=1}^{2}(1-\cos(ak_j))\Big]^{-1/2} dk \\
= \;& (2\pi)^{-2}\int_{[-\pi,\pi]\times[0,\pi]} \cos(k\cdot y)\Big[m_0^2 + 2\sum_{j=1}^{2}(1-\cos k_j)\Big]^{-1/2} \\
& \times\Big[m_0^2 + 2a^{-2}\sum_{j=1}^{2}(1-\cos(ak_j))\Big]^{-1/2} dk.
\end{aligned}$$

Therefore,

$$\langle \phi(f_{1,0})\phi(f_{a,y})\rangle_{\mu_0} \le 2(2\pi)^{-2}m_0^2(2\pi\times\pi).$$

Moreover, since

$$\cos(k\cdot y) > \cos\frac{\pi}{4}$$

and

$$\sum_{j=1}^{2} a^{-2}(1-\cos(ak_j)) \le \sum_{j=1}^{2} a^{-2}|ak_j|^2 = |k|^2 \le 2\pi^2$$

for any $k \in [-\pi,\pi]^2$ and $y \in B(0,\frac{1}{4})$, we have

$$\langle \phi(f_{1,0})\phi(f_{a,y})\rangle_{\mu_0} \ge 2\cos\frac{\pi}{4}\times(2\pi)^{-2}\frac{1}{m_0^2+2\pi^2}(2\pi\times\pi).$$

Also,

$$\langle \phi(f_{a,a'l_1})\phi(f_{a,a'l_2})\rangle > 0 \quad \text{for any } l_1, l_2 \in \mathbf{Z}^2,$$

and

$$\langle \phi(f_{a,a'l_1})^2)\rangle = C^{(a)}(0) \sim |\log a| \quad \text{as } a \to 0,$$

in the sense that the quotient of the left hand side by the right hand side is lower and upper bounded by positive constants, as $a \to 0$. So by (2.1) and (2.2), we have that there exist constants $C_m, C'_m, C_{m,g} > 0$ such that

$$\begin{aligned}
&\frac{d^{2m}}{d\lambda^{2m}}\Big(\int_{\mathcal{S}'(\mathbf{R}^2)} \phi(f_{1,0})^4 \mu_{\lambda,a,a'}(d\phi)\Big)\Big|_{\lambda=0} \\
\ge \;& C_m a'^{4m} \sum_{l_1,\cdots,l_m\in\mathbf{Z}^2}\Big(\prod_{k=1}^{m} g(a'l_k)^2\Big) \\
& \times\langle\phi(f_{1,0})\phi(f_{a,a'l_1})\rangle_{\mu_0}^2\langle\phi(f_{a,a'l_1})^2\rangle_{\mu_0}^3\Big(\prod_{k=2}^{m}\langle\phi(f_{a,a'l_k})^2\rangle_{\mu_0}^4\Big)
\end{aligned}$$

$$
\begin{aligned}
&\geq C'_m a'^{2m} |\log a|^{4m-1} a'^{2m} \sum_{l_1,\cdots,l_m \in \mathbf{Z}^2} \prod_{k=1}^{m} g(a' l_k)^2 \\
&\geq C_{m,g} a'^{2m} |\log a|^{4m-1}.
\end{aligned}
$$

This completes the proof.

**REFERENCES**

1. S. Albeverio, M. S. Bernabei, and X. Y. Zhou. Lattice approximations and continuum limits of $\phi_2^4$-quantum fields, *J. Math. Phys.* 45, 149–178 (2004).
2. S. Albeverio and X. Y. Zhou. A Central Limit Theorem for the Fourth Wick Power of the Free Lattice Field, *Commun. Math. Phys*, 181, 1–10 (1996).
3. B. Simon. *The $P(\phi)_2$ Euclidean (Quantum) Field Theory.* Princeton University Press (1974).
4. J. Glimm and A. Jaffe. *Quantum physics. A functional integral point of view. Second edition.* Springer-Verlag, New York (1987).
5. S. Albeverio, R. Høegh-Krohn, J. E. Fenstad and T. Lindstrøm. *Nonstandard methods in stochastic analysis and mathematical physics.* Pure and Applied Mathematics, 122. Academic Press, Inc.