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A LOWER BOUND ON THE GROWTH OF WORD HYPERBOLIC GROUPS

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Abstract

We give a linear lower bound on the exponential growth rate of a non-elementary subgroup of a word hyperbolic group, with respect to the number of generators for the subgroup.

1. *Introduction*

Let G be a finitely generated group and A a finite set of generators for G. By $|x|_A$ we denote the *word length* of an element $x \in G$ in the generators A, that is, the length of the shortest word in the alphabet $A^{\pm 1}$ representing x. Let $B_A(n)$ denote the ball $\{g \in G \mid |g|_A \leqslant n\}$ of radius n in G with respect to A. The *exponential growth rate* of the pair (G, A) is the limit

$$\lambda(G, A) = \lim_{n\to\infty} \sqrt[n]{\#B_A(n)}.$$

This limit exists due to the submultiplicativity property of the function $\#B(n)$; see, for example, [**8**, VI.C, Proposition 56]. The *uniform exponential growth rate* $\lambda(G)$ of G is the infimum $\inf_A \lambda(G, A)$ over all finite generating sets A of G. Clearly, $\lambda(G, A) \geqslant 1$ and hence $\lambda(G) \geqslant 1$ for any G and A. A finitely generated group G is said to be of *exponential growth* if $\lambda(G, A) > 1$ for some (which, in fact, implies for any) finite generating set A.

In this paper, we restrict our attention to the behaviour of the exponential growth rate of a group with respect to the choice of its generating set. So far, this behaviour remains far from being well understood. In particular, in general it is very difficult to determine whether a given finitely generated group G realizes its uniform exponential growth rate, that is, it has a finite generating set A with $\lambda(G, A) = \lambda(G)$.

This question has a relative in differential geometry, where it is required to determine which Riemannian metrics (with appropriate normalization), if any, achieve the minimum value of the volume entropy on a given compact manifold; see, for example, [**3**].

Apart from several simple examples, no classes of groups are known for which the exponential growth rate achieves its infimum on some generating set. Namely, it is easy to see that a finitely generated free group has this realization property and $\lambda(F_k) = \lambda(F_k, A_k) = 2k - 1$ for a free group F_k of rank k freely generated by A_k, see [**8**, Proposition VII.13]. The infimum $\lambda(G)$ is also achieved for free products

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$G = \mathbb{Z}_p * F_k$ with prime p (see [**15**]). For a fundamental group S_g of an orientable surface of genus $g \geqslant 2$, only a lower bound for $\lambda(G)$ is known, namely, $\lambda(S_g) \geqslant 4g-3$ (see [**8**, Proposition VII.15]).

Sambusetti [**13**] has shown that $\lambda(G, A) > \lambda(G)$ for any finite A when G is a free product $G = G_1 * G_2$ with G_1 non-Hopfian and G_2 non-trivial. It is an open question whether the same holds for non-Hopfian groups G themselves, in particular, for Baumslag–Solitar groups.

By a recent result of Wilson [**16**] (see also [**2**]), there are finitely generated groups of exponential growth with $\lambda(G) = 1$ (these are said to have *non-uniform exponential growth*). This answers a known question due to Gromov [**6**]. On the other hand, there are known several classes of groups of uniform exponential growth, that is, with $\lambda(G) > 1$, (see [**8**, Ch. VIIB]). In particular, non-elementary word hyperbolic groups are of this class [**9**]. (Note that a word hyperbolic group has exponential growth if and only if it is non-elementary.)

As the class of word hyperbolic groups is, in a geometric sense, a generalization of free groups, it is natural to conjecture that word hyperbolic groups realize their uniform exponential growth rate. However, even for this case, the question seems to be highly non-trivial. In particular, as observed in [**1**], an affirmative answer to the question would imply that a non-elementary word hyperbolic group be Hopfian. Note that word hyperbolic groups that are *torsion free* are known to be Hopfian [**14**].

In this paper, we try to do a step towards the proof of the conjecture that for non-elementary word hyperbolic groups the infimum of the exponential growth rate is achieved on some of its generating sets. Namely, our main result is the following.

THEOREM 1. *Let G be a word hyperbolic group. Then there is a number $\alpha > 0$, effectively calculated from G, such that, for any finitely generated non-elementary subgroup H of G and a finite generating set C for H,*

$$\lambda(H, C) \geqslant \alpha \# C.$$

NOTE ON THE EFFECTIVENESS. There are several natural ways to present a word hyperbolic group as a finite object. All of them are known to be algorithmically equivalent. For example, G can be given by a finite presentation supplied with a constant of hyperbolicity. (Even just a finite presentation of G is enough, see [**11**].)

As an immediate consequence of Theorem 1, we get the following.

COROLLARY. Let G be a non-elementary word hyperbolic group. Then there is a number N, effectively calculated from G, with the following property: if the uniform exponential growth $\lambda(G)$ is achieved on some generating set A for G then $\# A < N$.

Another result, which, in fact, is also an easy consequence of Theorem 1, restricts generating sets A as in the Corollary in even a stronger way. To formulate it we treat an n-element generating set A for a group G as an epimorphism $\gamma : F_n \to G$ of a free group F_n with a fixed basis $(x_1, x_2, \dots, x_n)$. In other words, A is viewed as an *ordered list* of elements $(x_1^\gamma, x_2^\gamma, \dots, x_n^\gamma)$. Then $\operatorname{Aut} F_n$ acts in a natural way on the n-element generating sets of G by $A^\phi = (x_1^{\phi\gamma}, x_2^{\phi\gamma}, \dots, x_n^{\phi\gamma})$.

THEOREM 2. *Let G be a non-elementary word hyperbolic group. Then there exists a finite n-element generating set A of G, effectively constructed from G, satisfying the following property: the uniform exponential growth of G is achieved on some generating set of G if and only if the infimum of the exponential growth rates is achieved* inside *the orbit $A^{\mathrm{Aut}\,F_n}$, that is,*

$$\lambda(G) = \lambda(G, B_0) \text{ for some } B_0 \iff \inf\{\lambda(G,B) \mid B \in A^{\mathrm{Aut}\,F_n}\} = \lambda(G, B_0) \text{ for some } B_0 \in A^{\mathrm{Aut}\,F_n}.$$

This theorem raises an intriguing question of how the growth rate $\lambda(G, A)$ behaves inside an orbit under the action of $\mathrm{Aut}\,F_n$. For example, what happens with $\lambda(G, A)$ when applying an elementary Nielsen transformation to A?

2. *Word hyperbolic groups*

Let X be a metric space with length function $|x-y|$. *The Gromov inner product* $\langle x, y\rangle_z$ of two points $x, y \in X$ with respect to a point $z \in X$ is defined by

$$\langle x, y\rangle_z = \tfrac{1}{2}(|x-z| + |y-z| - |x-y|).$$

Recall that X is called *δ-hyperbolic* if, for any $x, y, z, u \in X$,

$$\langle x, y\rangle_z \geqslant \min\{\langle x, u\rangle_z, \langle y, u\rangle_z\} - \delta.$$

Given a group G and its generating set A, the Cayley graph $C(G, A)$ is naturally viewed as a metric space with the path metric with edges of length 1. A group G is *word hyperbolic* [**7**] if its Cayley graph $C(G, A)$ with respect to a finite generating set A is δ-hyperbolic for some $\delta \geqslant 0$.

For the rest of the paper, we fix a group G, a finite set A of generators for G and a constant δ of hyperbolicity. As δ-hyperbolicity is preserved under enlarging δ, we assume for convenience that $\delta > 0$.

We start with listing several well-known facts about a δ-hyperbolic geodesic metric space X. We use a notation $[x, y]$ for a (possibly non-unique, but fixed once chosen) geodesic segment joining two points $x, y \in X$.

FACT 1 (see, for example, [**5**, Ch. 2, Proposition 21]). Geodesic triangles in X are 4δ-thin. This means that for any $x, y, z \in X$, if $y' \in [x, y]$, $z' \in [x, z]$ and $|y' - x| = |z' - x| \leqslant \langle y, z\rangle_x$, then $|y' - z'| \leqslant 4\delta$.

We call two subsets $U, V \subseteq X$ *r-close* if U and V lie in the Hausdorf r-neighbourhood of each other; that is, for any $x \in U$ there is $y \in V$ with $|x - y| \leqslant r$ and *vice versa*.

An easy consequence of Fact 1 is the following.

FACT 2. If $|x - z| \leqslant r$ and $|y - z| \leqslant r$, then $[x, y]$ is $(r + 4\delta)$-close to z. If $|x - z| \leqslant r$ and $|y - w| \leqslant r$, then $[x, y]$ is $(r + 8\delta)$-close to $[z, w]$.

For technical convenience, we introduce a stronger 'directed' version of closeness for paths.

Definition 1. Let $p : I \to X$ and $q : J \to X$ be two paths in a metric space X where I and J are (closed or open) intervals in $\mathbb{R}$. We say that p and q are *strictly r-close* if $\operatorname{Im} p$ and $\operatorname{Im} q$ are r-close and the closeness is 'monotone' in the following sense. There is a binary relation $R \in I \times J$ with the following properties.

(i) If xRy, then $|p(x) - q(y)| \leqslant r$.

(ii) For any $x \in I$ there is at least one $y \in J$ with xRy and *vice versa*, for any $y \in J$ there is at least one $x \in I$ with xRy.

(iii) If xRx', yRy' and $x \leqslant x'$, then $y \leqslant y'$.

The following properties of this notion are obvious from the definition. If p, q are strictly r-close and q, t are strictly s-close, then p and t are strictly $(r+s)$-close. If $p : I \to X$ and q are r-close and $p' : I' \to X$, $I' \subseteq I$, is a subpath of p, then p' is r-close to a subpath q' of q.

We consider only paths that come in the natural way from broken geodesics of the form $S = \bigcup_{i=0}^{k-1} [x_i, x_{i+1}]$. We simply say in that case that two broken geodesics S and S' are strictly r-close.

The following fact is an immediate consequence of Fact 1.

Fact 3. For any $x, y, z \in X$, geodesic segments $[x, y]$ and $[x, z]$ have strictly 4δ-close initial subsegments S and T of length $\langle y, z\rangle_x$. Moreover, any subsegment of S is strictly 4δ-close to a subsegment of T of the same length.

Fact 4. For any $x, y, z \in X$, the segment $[x, z]$ and the broken geodesics $[x, y] \cup [y, z]$ are strictly r-close for $r = \langle x, z\rangle_y + 4\delta$.

To see this, we take the unique points $u \in [x, y]$, $v \in [y, z]$ and $w \in [x, z]$ with $|x - u| = |x - w| = \langle y, z\rangle_x$, $|y - u| = |y - v| = \langle x, z\rangle_y$ and $|z - v| = |z - w| = \langle x, y\rangle_z$. Then

$$[x, y] \cup [y, z] = [x, u] \cup ([u, y] \cup [y, v]) \cup [v, z]$$

where $[x, u]$ is strictly 4δ-close to $[x, w]$, $[v, z]$ is strictly 4δ-close to $[w, z]$ and $[u, y] \cup [y, v]$ is r-close to w.

The following is a variant of a known 'broken geodesics' lemma.

Lemma 1. *Let $x_1, x_2, \dots, x_k$ $(k \geqslant 3)$ be points in a δ-hyperbolic geodesic metric space X. Suppose that*

$$\langle x_{i-1}, x_{i+1}\rangle_{x_i} + \langle x_i, x_{i+2}\rangle_{x_{i+1}} < |x_i - x_{i+1}| - 3\delta, \quad i = 2, 3, \dots, k-2. \tag{1}$$

Then the following assertions are true:

(a) $|x_1 - x_k| \geqslant \sum_{i=1}^{k-1} |x_i - x_{i+1}| - 2\sum_{i=2}^{k-1} \langle x_{i-1}, x_{i+1}\rangle_{x_i} - 2(k-3)\delta$;

(b) $[x_1, x_k]$ *and* $\bigcup_{i=1}^{k-1} [x_i, x_{i+1}]$ *are strictly r-close for*

$$r = \max_i \langle x_{i-1}, x_{i+1}\rangle_{x_i} + 14\delta.$$

Proof. First we prove that

$$\langle x_1, x_k\rangle_{x_{k-1}} \leqslant \langle x_{k-2}, x_k\rangle_{x_{k-1}} + \delta. \tag{2}$$

Indeed, this is trivial when $k = 3$. For $k > 3$, using the inductive hypothesis and (1) we have

$$\begin{aligned}\langle x_1, x_{k-2}\rangle_{x_{k-1}} &= |x_{k-1} - x_{k-2}| - \langle x_1, x_{k-1}\rangle_{x_{k-2}} \\ &\geqslant |x_{k-1} - x_{k-2}| - \langle x_{k-3}, x_{k-1}\rangle_{x_{k-2}} - \delta \\ &> \langle x_{k-2}, x_k\rangle_{x_{k-1}} + 2\delta.\end{aligned}$$

Now from

$$\langle x_{k-2}, x_k\rangle_{x_{k-1}} \geqslant \min\{\langle x_1, x_{k-2}\rangle_{x_{k-1}}, \langle x_1, x_k\rangle_{x_{k-1}}\} - \delta$$

we get (2).

If $k = 3$, part (a) becomes an equality. To prove part (a) for $k > 3$, by (2) we have

$$\begin{aligned}|x_1 - x_k| &= |x_1 - x_{k-1}| + |x_{k-1} - x_k| - 2\langle x_1, x_k\rangle_{x_{k-1}} \\ &\geqslant |x_1 - x_{k-1}| + |x_{k-1} - x_k| - 2\langle x_{k-2}, x_k\rangle_{x_{k-1}} - 2\delta\end{aligned}$$

and one can use an easy induction.

We now prove part (b). First, for any point $y \in \bigcup_{i=1}^{k-1}[x_i, x_{i+1}]$ we find a point on $[x_1, x_k]$ that is $(r - 4\delta)$-close to y. Assume that $y \in [x_{i-1}, x_i]$ for $3 \leqslant i \leqslant k-1$ and y satisfies

$$\langle x_{i-1}, x_{i+1}\rangle_{x_i} + 2\delta \leqslant |y - x_i| \leqslant |x_{i-1} - x_i| - \langle x_{i-2}, x_i\rangle_{x_{i-1}} - \delta. \tag{3}$$

By (2),

$$\begin{aligned}\langle x_1, x_{i-1}\rangle_{x_i} &= |x_{i-1} - x_i| - \langle x_1, x_i\rangle_{x_{i-1}} \\ &\geqslant |x_{i-1} - x_i| - \langle x_{i-2}, x_i\rangle_{x_{i-1}} - \delta.\end{aligned} \tag{4}$$

Hence

$$|y - x_i| \leqslant \langle x_1, x_{i-1}\rangle_{x_i}.$$

By Fact 1, for $y' \in [x_1, x_i]$ with $|y' - x_i| = |y - x_i|$ we have $|y - y'| \leqslant 4\delta$.

By δ-hyperbolicity,

$$\langle x_{i-1}, x_k\rangle_{x_i} \geqslant \min\{\langle x_1, x_{i-1}\rangle_{x_i}, \langle x_1, x_k\rangle_{x_i}\} - \delta.$$

By (2) with $(x_1, \dots, x_k) := (x_k, x_{k-1}, \dots, x_{i-1})$,

$$\langle x_{i-1}, x_k\rangle_{x_i} \leqslant \langle x_{i-1}, x_{i+1}\rangle_{x_i} + \delta.$$

By (1), the right-hand side of (4) is greater than $\langle x_{i-1}, x_{i+1}\rangle_{x_i} + 2\delta$ and we conclude that

$$\langle x_1, x_k\rangle_{x_i} \leqslant \langle x_{i-1}, x_{i+1}\rangle_{x_i} + 2\delta \leqslant |y - x_i| = |y' - x_i|$$

and hence

$$|y' - x_1| \leqslant \langle x_i, x_k\rangle_{x_1}.$$

Then for $y'' \in [x_1, x_k]$ with $|y'' - x_1| = |y' - x_1|$ we get $|y' - y''| \leqslant 4\delta$. Thus, for any $y \in [x_{i-1}, x_i]$, $3 \leqslant i \leqslant k-1$, satisfying (3) we found a point $y'' \in [x_1, x_k]$ with $|y - y''| \leqslant 8\delta$. Any $y^* \in [x_{i-1}, x_i]$ is $(r - 12\delta)$-close to a point $y \in [x_{i-1}, x_i]$ satisfying (3). In fact, from the way we choose y'' we easily conclude that $[x_{i-1}, x_i]$ is strictly $(r - 4\delta)$-close to a subsegment $[u_{i-1}, v_i]$ of $[x_1, x_k]$.

Now consider the case when $y \in [x_1, x_2] \cup [x_{k-1}, x_k]$. Let, for example, $y \in [x_{k-1}, x_k]$. As we have seen above, if

$$|y - x_k| \leqslant |x_{k-1} - x_k| - \langle x_{k-2}, x_k \rangle_{x_{k-1}} - \delta,$$

then y is 4δ-close to a point $y' \in [x_1, x_k]$. Again, we find a subsegment $[u_{k-1}, v_k]$ such that $[x_{k-1}, x_k]$ is strictly $(r-4\delta)$-close to $[u_{k-1}, v_k]$. Similarly we find $[u_1, v_2]$.

As $|u_i - x_i| \leqslant r - 4\delta$ and $|v_i - x_i| \leqslant r - 4\delta$ any point between u_i and v_i is r-close to x_i by Fact 2. This easily implies that $[x_1, x_k]$ and $\bigcup_{i=1}^{k-1} [x_i, x_{i+1}]$ are strictly r-close. □

Lemma 2. *Let $r > 0$ and x, y, x', y' be points in a δ-hyperbolic geodesic metric space X. If*

$$|x - y| + |x' - y'| \geqslant |x - x'| + |y - y'| + 2r,$$

then geodesic segments $[x, y]$ and $[x', y']$ have strictly 8δ-close subsegments $[u, v]$ and $[u', v']$ with $|u - v|, |u' - v'| \geqslant r$.

Proof. By Fact 3, the geodesic segments $[x, y]$ and $[x, y']$ have strictly 4δ-close initial subsegments of length $\langle y, y' \rangle_x$, and the geodesic segments $[x, y']$ and $[x', y']$ have strictly 4δ-close terminal subsegments of length $\langle x, x' \rangle_{y'}$. Hence, there is a subsegment S of $[x, y']$ of length

$$\langle y, y' \rangle_x + \langle x, x' \rangle_{y'} - |x - y'| = \tfrac{1}{2}(|x - y| + |x' - y'| - |x - x'| - |y - y'|) \geqslant r,$$

which is strictly 4δ-close to subsegments of both $[x, y]$ and $[x', y']$. The corresponding subsegments of $[x, y]$ and $[x', y']$ have the same length as S, again by Fact 3. □

Now we turn to our word hyperbolic group G. We always identify elements of G with vertices of its Cayley graph $C(G, A)$. In particular, if $g, h \in G$ then $[g, h]$ denotes a geodesic segment in $C(G, A)$ joining the vertices g and h. We also use the notation $|g|$ for the word length of an element $g \in G$ in our fixed generating set A, and $\langle g, h \rangle$ for the Gromov product of g and h with respect to the identity element 1 of G, that is,

$$\langle g, h \rangle = \langle g, h \rangle_1 = \tfrac{1}{2}(|g| + |h| - |g^{-1}h|).$$

By $\|g\|$ we denote the length of a shortest element in the conjugacy class of an element $g \in G$. It is easy to see that

$$\|g\| = \min_{x \in C(G,A)} |gx - x| \tag{5}$$

where $x \in C(G, A)$ runs over all points in $C(G, A)$, not only the vertices. We say that g is *cyclically minimal* if $\|g\| = |g|$.

We apply Lemma 1 to the sequence $1, g, g^2, \ldots, g^n$. This immediately gives the following.

Corollary 1. *If $|g| - 2\langle g, g^{-1} \rangle > 3\delta$ and $n \geqslant 2$, then*

$$|g^n| \geqslant n|g| - 2(n-1)\langle g, g^{-1} \rangle - 2(n-2)\delta$$

and $[1, g^n]$ and $\bigcup_{i=0}^{n-1} [g^i, g^{i+1}]$ are strictly r-close for $r = \langle g, g^{-1} \rangle + 14\delta$.

LEMMA 3. *We have the following.*

(a) *If* $|g| - 2\langle g, g^{-1}\rangle > 3\delta$, *then*

$$\|g\| \geqslant |g| - 2\langle g, g^{-1}\rangle - 2\delta.$$

(b) *If* $\|g\| > 4\delta$, *then*

$$\|g\| \leqslant |g| - 2\langle g, g^{-1}\rangle + 4\delta.$$

In particular, if g is cyclically minimal, then $\langle g, g^{-1}\rangle \leqslant 2\delta$.

(c) *If* $\|g\| > 7\delta$, *then $h^{-1}gh$ is cyclically minimal for some h with*

$$|h| \leqslant \tfrac{1}{2}(|g| - \|g\|) + 5\delta.$$

Proof. (a) If $g = h^{-1}g_0h$ then $(1/n)|g^n| \leqslant |g_0| + (2/n)|h|$. This implies that $\lim_{n\to\infty}(1/n)|g^n| \leqslant \|g\|$ and the assertion follows from Corollary 1.

(b) Let $t = \min\{\frac{1}{2}|g|, \langle g, g^{-1}\rangle\}$. Choose $u \in [1, g]$ with $|u - 1| = t$. Then $\langle g, u\rangle = t$ and

$$\langle g^{-1}, g^{-1}u\rangle = |g^{-1}u - 1| = |u - g| = |g| - t \geqslant t.$$

Hence,

$$\langle u, g^{-1}u\rangle \geqslant \min\{\langle g, u\rangle, \langle g, g^{-1}\rangle, \langle g^{-1}, g^{-1}u\rangle\} - 2\delta \geqslant t - 2\delta,$$

which implies that

$$\|g\| \leqslant |gu - u| = |u - g^{-1}u| = |g| - 2\langle u, g^{-1}u\rangle \leqslant |g| - 2t + 4\delta.$$

Now from $\|g\| > 4\delta$ we get $t < \frac{1}{2}|g|$ and, hence, $t = \langle g, g^{-1}\rangle$.

(c) Let $h^{-1}gh$ be cyclically minimal and h have the minimal possible length with this property. For any $u \in [h, gh]$,

$$\begin{aligned}|u - gu| &\leqslant |u - gh| + |gh - gu| \\ &= |u - gh| + |h - u| \\ &= |h - gh| = \|g\|\end{aligned}$$

and hence, by (5), $|u - gu| = \|g\|$. This implies that $|u - 1| \geqslant |h|$, as if $|u - 1| < |h|$, then for some $h' \in [h, gh]$, $h' \in G$, we would get $|h'| < |h - 1| = |h|$ and $|h'^{-1}gh'| = |h' - gh'| = \|g\|$ contrary to the choice of h. Taking u with $|u - h| = \langle 1, gh\rangle_h$ we have $\langle gh, u\rangle_h = |u - h| = \langle 1, gh\rangle_h$ and, hence,

$$\langle 1, u\rangle_h \geqslant \min\{\langle 1, gh\rangle_h, \langle gh, u\rangle_h\} - \delta = \langle 1, gh\rangle_h - \delta.$$

On the other hand,

$$\langle 1, u\rangle_h = \tfrac{1}{2}(|h| + |u - h| - |u - 1|) \leqslant \tfrac{1}{2}\langle 1, gh\rangle_h.$$

This gives $\langle 1, gh\rangle_h \leqslant 2\delta$.

Similarly we get

$$\langle g, h\rangle_{gh} = \langle 1, g^{-1}h\rangle_h \leqslant 2\delta.$$

Now application of Lemma 1(a) to the points $1, h, gh, g$ yields

$$|g| \geqslant |h^{-1}gh| + 2|h| - 10\delta,$$

that is, $|h| \leqslant \frac{1}{2}(|g| - \|g\|) + 5\delta$ as required. □

Definition 2. Let $g \in G$ be an element of infinite order. We set

$$\begin{aligned} E(g) &= \{h \in G \mid h^{-1}g^t h = g^{\varepsilon t} \text{ for some integer } t \neq 0 \text{ and } \varepsilon = \pm 1\}, \\ E^+(g) &= \{h \in G \mid h^{-1}g^t h = g^t \text{ for some integer } t \neq 0\}, \\ E^*(g) &= \{h \in E^+(g) \mid h \text{ has finite order}\}, \end{aligned}$$

and

$$E^-(g) = E(g) \setminus E^+(g).$$

Lemma 4. *For any $g \in G$ of infinite order:*
(a) *$E(g)$ is the maximal elementary subgroup of G containing g;*
(b) *$E^*(g)$ is a finite normal subgroup of $E(g)$;*
(c) *$h^2 \in E^*(g)$ for any $h \in E^-(g)$.*

Proof. (a) Clearly, $E(g)$ stabilizes the two-point set $\{g^{-\infty}, g^{+\infty}\}$ in the hyperbolic boundary of G. Hence, $E(g)$ is an elementary subgroup of G. For every elementary subgroup H of G, if $g \in H$, then H contains $\langle g \rangle$ as a subgroup of finite index. Hence, $\langle g^t \rangle$ is normal in H for some $t > 0$ that precisely means that $H \subseteq E(g)$.

(b) By part (a), $\langle g \rangle$ has finite index in $E(g)$. For some $t > 0$, the subgroup $\langle g^t \rangle$ is normal and hence central in $E^+(g)$. By a theorem of Schur (see [**12**, Theorem 7.57]), in a group with the centre of finite index, the set of torsion elements is a normal subgroup. Hence, $E^*(g)$ is a normal subgroup of $E^+(g)$. It is finite as a torsion subgroup in a finite extension of an infinite cyclic group. As $|E(g) : E^+(g)| \leqslant 2$, for $h \in E^-(g)$ we have $h^{-1}E^*(g)h \subseteq E^+(g)$ and hence $h^{-1}E^*(g)h = E^*(g)$.

(c) By definition, for $h \in E^-(g)$ we have $h^{-1}g^t h = g^{-t}$ for some $t \neq 0$. As $\langle g \rangle$ has finite index in $E(g)$, for any $h \in E(g)$ of infinite order we have $g^m = h^n$ for some $m, n \neq 0$. This implies that $E^-(g)$ cannot contain elements of infinite order. $\square$

Lemma 5. *Let $g \in G$ be cyclically minimal and $|g| > 7\delta$. Then $|h| \leqslant 140\delta$ for any $h \in E^*(g)$.*

Proof. Let $S_n = \bigcup_{i=-n}^{n-1}[g^i, g^{i+1}]$. By Lemma 1(b) S_n is strictly 16δ-close to any geodesic segment $[g^{-n}, g^n]$. So any such geodesic segment $[g^{-n}, g^n]$ intersects $B(16\delta)$. Hence, there is an infinite geodesics $L = (g^{-\infty}, g^{+\infty})$ also intersecting $B(16\delta)$. Let $x \in L$, $|x - 1| \leqslant 16\delta$.

Now let $h \in E^*(g)$. Then any two of the infinite geodesics L, hL and h^2L are strictly 16δ-close (see, for example, [**5**, Ch. 7, Corollary 3(b)]. Let $y \in L$, $|y - hx| \leqslant 16\delta$. Without loss of generality we assume that y lies on the ray $[x, g^{+\infty})$ (the case $y \in [x, g^{-\infty})$ is symmetric). Then $|z - hy| \leqslant 16\delta$ for some $z \in [y, g^{+\infty})$. We have

$$|z - y| \geqslant |hy - hx| - 32\delta \geqslant |h^2x - hx| - 48\delta = |hx - x| - 48\delta$$

and

$$\begin{aligned} |h^2x - x| &\geqslant |z - x| - 32\delta \\ &= |z - y| + |y - x| - 32\delta \\ &\geqslant 2|hx - x| - 96\delta. \end{aligned}$$

Assume that $|h| > 140\delta$. Then $|hx - x| \geqslant |h| - 32\delta > 108\delta$ and, hence,

$$\langle x, h^2x\rangle_{hx} \leqslant 48\delta < \tfrac{1}{2}|hx - x| - 3\delta.$$

Applying Lemma 1(a) to the sequence $x, hx, \ldots, h^n x$ we see that $|h^n x - x| > 0$ for any $n > 0$. This means that h has infinite order. However, this is a contradiction with $h \in E^*(g)$. Hence, $|h| \leqslant 140\delta$. □

3. *Proofs*

Observe in advance that all the numerical parameters in the formulation of lemmas in this section are effectively computed from the given word hyperbolic group G. With this observation, 'the effectiveness' part in Theorems 1 and 2 will immediately follow from the proof of 'the existence' part. Below we are only concerned about 'the existence' part.

The main point in the proof of Theorem 1 is the following. Given a non-elementary subgroup H of G generated by a finite set C, we construct a finite set $Y \subseteq H$ with the following properties.

(i) Every $y \in Y$ is expressed as a product of at most D generators in $C^{\pm 1}$ where $D = D(G)$ depends only on G.

(ii) $\#Y \geqslant \beta\#C$ for some $\beta > 0$, $\beta = \beta(G)$.

(iii) Y freely generates a free subgroup of H.

It is easy to see that this is enough for proving Theorem 1. Indeed, properties (i)–(iii) imply that $B_C(t) \cap H$ contains at least $((3\beta/D)\#C)^{t-1}$ elements and hence

$$\lambda(H, C) \geqslant \frac{3\beta}{D}\#C$$

as required.

To construct Y we use a 'big powers' approach. This means that we choose a certain hyperbolic element $d \in H$ and use big powers of d as 'blocks' to construct a basis Y for a free subgroup in H. We perform the choice of d in two steps. The first step (Lemma 7) is simply to pick up a hyperbolic element $b \in H$ given by a product of at most two generators in C. However, taking $d = b$ is not enough for our purpose because the construction of Y is based on the fact that the length of any element of $E^+(d) \setminus E^*(d)$ is large enough compared with the lengths of elements in C in the generators A. To achieve this we use a big power of b (in fact, $d = b^R v$ for R large enough and some $v \in C$). This is done at the second step, Lemma 8.

The main technical tool for dealing with big powers is the following lemma. Again, it can be treated as a variant of a known fact on word hyperbolic groups; see, for example, [**10**, Lemma 2.1].

LEMMA 6. *Let $N = \#B(200\delta) + 10$. Let $g \in G$ be cyclically minimal and $|g| > 7\delta$. Let $x, y \in G$ and suppose that, for some $m, n > 0$ and $\varepsilon = \pm 1$, geodesic segments $[x, xg^n]$ and $[y, yg^{\varepsilon m}]$ have strictly 8δ-close subsegments S and T. If $|S| \geqslant N|g|$, then $x^{-1}y \in E^+(g)$ if $\varepsilon = 1$ and $x^{-1}y \in E^-(g)$ if $\varepsilon = -1$.*

Proof. Let

$$X = \bigcup_{i=0}^{n-1} [xg^i, xg^{i+1}] \quad \text{and} \quad Y = \bigcup_{i=0}^{m-1} [yg^{\varepsilon i}, yg^{\varepsilon(i+1)}].$$

By Lemma 3(b) and Corollary 1, X is strictly 16δ-close to $[x, xg^n]$ and Y is strictly 16δ-close to $[y, yg^{\varepsilon m}]$. Let S_0 and T_0 be the sections of X and Y that are strictly 16δ-close to S and T, respectively. We have $S_0 \overset{16\delta}{\sim} S \overset{8\delta}{\sim} T \overset{16\delta}{\sim} T_0$ where $\overset{r}{\sim}$ stands for strict r-closeness. For a point $z \in S_0 \cup T_0$ we denote by $\tau(z)$ a (non-unique but fixed once chosen) point $\tau(z) \in T$ determined by the strict closeness, with

$$|z - \tau(z)| \leqslant 24\delta \text{ or } 16\delta, \text{ respectively.}$$

The lengths of S_0 and T_0 are at least

$$|S| - 48\delta > (N-7)|g|.$$

Hence, for some i_0, j_0 and $k \geqslant N - 8$,

$$xg^{i_0}, xg^{i_0+1}, \ldots, xg^{i_0+k} \in S_0, \quad yg^{j_0}, yg^{j_0+\varepsilon}, \ldots, yg^{j_0+k\varepsilon} \in T_0.$$

We assume that xg^{i_0} and yg^{j_0} are 'the leftmost', that is, S_0 and T_0 start at some points in $(xg^{i_0-1}, xg^{i_0}]$ and $(yg^{j_0-\varepsilon}, yg^{j_0}]$, respectively.

Up to interchanging X and Y from the start of the proof, we may assume also that $\tau(xg^{i_0})$ is 'left to' $\tau(yg^{j_0})$ in T. This means that $\tau(xg^{i_0}) = \tau(v)$ for some $v \in \bigcup_{j \leqslant j_0} [yg^{j-\epsilon}, yg^j] \cap T_0$. As T_0 starts at a point on $[yg^{j_0-\epsilon}, yg^{j_0}]$ we have $v \in [yg^{j_0-\epsilon}, yg^{j_0}]$.

We denote $u = xg^{i_0}$. Observe that, for any $0 \leqslant t \leqslant k-1$, the point $xg^t x^{-1} u = xg^{i_0+t}$ belongs to S_0, and the point $yg^{\varepsilon t} y^{-1} v$ belongs to

$$yg^{\varepsilon t} y^{-1} [yg^{j_0-\varepsilon}, yg^{j_0}] = [yg^{j_0+(t-1)\varepsilon}, yg^{j_0+t\varepsilon}]$$

and, hence, to T_0.

Now we prove that

$$|xg^t x^{-1} u - yg^{\varepsilon t} y^{-1} v| < 200\delta, \quad t = 0, 1, \ldots, k-1. \tag{6}$$

Fix $t \leqslant k-1$. As both $\tau(yg^{\varepsilon t} y^{-1} v)$ and $\tau(yg^{j_0})$ lie on the geodesic segment $[\tau(u), \tau(yg^{j_0+\varepsilon t})] \subseteq T$, we have

$$\begin{aligned} &|\tau(u) - \tau(yg^{\varepsilon t} y^{-1} v)| + |\tau(yg^{\varepsilon t} y^{-1} v) - \tau(yg^{j_0+\varepsilon t})| \\ &\quad = |\tau(u) - \tau(yg^{j_0})| + |\tau(yg^{j_0}) - \tau(yg^{j_0+\varepsilon t})| = |\tau(u) - \tau(yg^{j_0+\varepsilon t})|. \end{aligned}$$

Using the shorthand '$\alpha \approx \beta$ up to γ' for '$|\alpha - \beta| \leqslant \gamma$' we observe that

$$\begin{gathered} |\tau(yg^{\varepsilon t} y^{-1} v) - \tau(yg^{j_0+\varepsilon t})| \approx |yg^{\varepsilon t} y^{-1} v - yg^{j_0+\varepsilon t}| = |v - yg^{j_0}| \quad \text{up to } 32\delta, \\ |\tau(u) - \tau(yg^{j_0})| \approx |v - yg^{j_0}| \quad \text{up to } 32\delta, \\ |\tau(yg^{j_0}) - \tau(yg^{j_0+\varepsilon t})| \approx |g^t| \quad \text{up to } 32\delta. \end{gathered}$$

Hence,

$$|\tau(u) - \tau(yg^{\varepsilon t} y^{-1} v)| \approx |g^t| \quad \text{up to } 96\delta.$$

With

$$|\tau(u) - \tau(xg^t x^{-1} u)| \approx |g^t| \quad \text{up to } 48\delta$$

this implies that

$$|\tau(xg^t x^{-1} u) - \tau(yg^{\varepsilon t} y^{-1} v)| \leqslant 144\delta.$$

Hence,

$$|xg^t x^{-1} u - yg^{\varepsilon t} y^{-1} v| \leqslant 184\delta$$

as required.

Now observe that we may assume v in (6) to be a graph vertex, that is, $v \in G$. As $N - 8 > \#B(200\delta)$ there are $0 \leqslant s < t \leqslant k-1$ such that

$$(xg^s x^{-1}u)^{-1} y g^{\varepsilon s} y^{-1} v = (xg^t x^{-1}u)^{-1} y g^{\varepsilon t} y^{-1} v.$$

This finally gives

$$g^{t-s} = x^{-1} y g^{\varepsilon(t-s)} y^{-1} x. \qquad \square$$

LEMMA 7. *Let H be a subgroup of G generated by a finite set $C \subset G$. Suppose that*

$$L = \min_{h \in G} \max_{x \in C} |h^{-1}xh| \geqslant 30\delta.$$

Then, after conjugation of H and C by an appropriate element of G, there is cyclically minimal $b \in H$, $|b|_C \leqslant 2$, such that

$$L - 24\delta \leqslant |b| \leqslant 2L.$$

In addition, $\max_{x \in C} |x| \leqslant L + 26\delta$ after the conjugation.

Proof. We apply an inner automorphism of G to H and C to obtain $L = \max_{x \in C} |x|$. Let $y \in C$ such that $|y| = L$.

Our first step is to find $b \in H$ such that $|b|_C \leqslant 2$, $|b| \geqslant L - 8\delta$ and $\|b\| \geqslant |b| - 16\delta$.

If $\|y\| \geqslant L - 16\delta$, then we take y as b. Suppose that $\|y\| < L - 16\delta$. Then, by Lemma 3(a),

$$\langle y, y^{-1} \rangle \geqslant \min\{\tfrac{1}{2}(|y| - 3\delta), \tfrac{1}{2}(|y| - \|y\| - 2\delta)\} > 7\delta.$$

We prove that for some $x \in C^{\pm 1}$, $x \neq y^{\pm 1}$, we have $|x| \geqslant L - 8\delta$ and $\langle x, y \rangle < 4\delta$. Assume the converse, that is, for any $x \in C^{\pm 1}$, $x \neq y^{\pm 1}$, either $|x| < L - 8\delta$ or $\langle x, y \rangle \geqslant 4\delta$. Let w be the initial segment of y with $|w| = 4\delta$. We reach a contradiction with the definition of L by proving that $|w^{-1}xw| < L$ for any $x \in C$.

For $x := y^{\pm 1}$, from $\langle (w^{-1}y)^{-1}, y^{-1} \rangle = |w^{-1}y| > 4\delta$, $\langle y, w \rangle = |w| = 4\delta$ and $\langle y, y^{-1} \rangle > 4\delta$ we deduce by δ-hyperbolicity

$$\langle (w^{-1}y)^{-1}, w \rangle = \tfrac{1}{2}(|w^{-1}y| + |w| - |w^{-1}yw|) \geqslant 2\delta,$$

which implies that $|w^{-1}yw| < |w^{-1}y| + |w| = |y| = L$.

If $|x| < L - 8\delta$, then $|w^{-1}xw| \leqslant |x| + 2|w| < L$.

Now suppose that $x \neq y^{\pm 1}$, $|x| \geqslant L - 8\delta$ and $\langle x, y \rangle \geqslant 4\delta$. As $\langle w, y \rangle = 4\delta$ we get $\langle w, x \rangle \geqslant 3\delta$. The same for $x := x^{-1}$ gives $\langle w, x^{-1} \rangle \geqslant 3\delta$. As $|x| > 7\delta$ we have $\langle xw, x \rangle > 3\delta$, which gives $\langle xw, w \rangle \geqslant 2\delta$. Hence,

$$\langle w, x^{-1} \rangle + \langle xw, w \rangle = |w| + \tfrac{1}{2}(|x| - |w^{-1}xw|) \geqslant 5\delta$$

and we conclude that $|w^{-1}xw| < |x| \leqslant L$.

We have proved that $|x_0| \geqslant L - 8\delta$ and $\langle x_0, y \rangle < 4\delta$ for some $x_0 \in C^{\pm 1}$, $x_0 \neq y^{\pm 1}$. Now if $\|x_0\| \geqslant |x_0| - 16\delta$ we take x_0 as b. Let $\|x_0\| < |x_0| - 16\delta$. Then, by Lemma 3(a),

$$\langle x_0, x_0^{-1} \rangle \geqslant \min\{\tfrac{1}{2}(|x_0| - 3\delta), \tfrac{1}{2}(|x_0| - \|x_0\| - 2\delta)\} > 7\delta.$$

From $\langle x_0^{-1}, y^{-1} \rangle \geqslant 6\delta$ using $\langle x_0, x_0^{-1} \rangle > 6\delta$ and $\langle y, y^{-1} \rangle > 6\delta$ we would get $\langle x_0, y \rangle \geqslant 4\delta$. Hence, $\langle x_0^{-1}, y^{-1} \rangle < 6\delta$. Applying Lemma 1(a) to the sequence

$$1, x_0^{-1}, x_0^{-1}y, x_0^{-1}yx_0^{-1}, (x_0^{-1}y)^2, \ldots, (x_0^{-1}y)^k$$

we see that

$$|(x_0^{-1}y)^k| \geqslant k(|x_0|+|y|) - 2k\langle x_0, y\rangle - 2(k-1)\langle x_0^{-1}, y^{-1}\rangle - 2(2k-2)\delta.$$

This implies that

$$\begin{aligned}\|x_0^{-1}y\| &\geqslant |x_0|+|y| - 2\langle x_0, y\rangle - 2\langle x_0^{-1}, y^{-1}\rangle - 4\delta\\ &= |x_0^{-1}y| - 2\langle x_0^{-1}, y^{-1}\rangle - 4\delta\\ &> |x_0^{-1}y| - 16\delta.\end{aligned}$$

As

$$|x_0^{-1}y| = |x_0|+|y| - 2\langle x_0, y\rangle > 2L - 16\delta > L - 8\delta$$

we can take $x_0^{-1}y$ for b in this case.

Thus, we have found $b \in H$ such that $|b|_C \leqslant 2$, $|b| \geqslant L - 8\delta$ and $\|b\| \geqslant |b| - 16\delta$. Obviously, $\|b\| \leqslant 2L$. By Lemma 3(c), $h^{-1}bh$ is cyclically minimal for some h with $|h| \leqslant 13\delta$. Conjugating by h we increase $\max_{x\in C}|x|$ for at most 26δ. □

Lemma 8. *Let H be a non-elementary subgroup of G generated by a finite set $C \subset G$. Suppose that*

$$L = \min_{h\in G}\max_{x\in C}|h^{-1}xh| \geqslant 40\delta.$$

Let $M > 0$ be any number and N as in Lemma 6. Then, after conjugation of H and C by an appropriate element of G, there is a cyclically minimal element $d \in H$ such that

$$|d|_C \leqslant 5M + 200N + 300, \quad |d| \geqslant ML$$

and every $h \in E^+(d)$ may be represented as $h = d^t u$, $t \in \mathbb{Z}$ where $u \in E^*(d)$ and $|u| < (4N+12)L$.

In addition, $\max_{x\in C}|x| < (4N+10)L$ after the conjugation.

Proof. We find b by Lemma 7, after an appropriate conjugation of H and C. As H is non-elementary, $v \notin E(b)$ for some $v \in C$. We take

$$R = 4M + 200N + 300$$

and set

$$d = b^R v.$$

By Lemma 3(b), $\langle b, b^{-1}\rangle \leqslant 2\delta$. As $|b| \geqslant L - 24\delta$, by Corollary 1,

$$|b^R| \geqslant R(|b| - 6\delta) + 8\delta > \tfrac{1}{4}RL. \tag{7}$$

If $|b^R v b^R| \leqslant 2|b^R| - |v| - 2N|b|$, applying Lemma 2 to the points 1, b^R, $b^R v b^R$, $b^R v$ and then Lemma 6, we would obtain $b^R v b^R \in E^-(b)$ contrary to the assumption $v \notin E(b)$. Therefore,

$$|b^R v b^R| > 2|b^R| - |v| - 2N|b|.$$

This implies that

$$\langle d, d^{-1}\rangle = \tfrac{1}{2}(2|b^R v| - |b^R v b^R v|) \leqslant |b^R| - \tfrac{1}{2}|b^R v b^R| + \tfrac{3}{2}|v| \leqslant N|b| + 2|v|. \tag{8}$$

Consider an infinite broken geodesics

$$S_0 = \bigcup_{i=-\infty}^{\infty} [d^i, d^{i+1}].$$

By Corollary 1, any geodesics $[d^{-n}, d^n]$ is r-close to $\bigcup_{i=-n}^{n-1} [d^i, d^{i+1}]$ for

$$r = N|b| + 2|v| + 14\delta.$$

Hence, there is an infinite geodesics $T = (d^{-\infty}, d^{\infty})$ that is r-close to S_0. By Fact 4, $[1, b^R] \cup [b^R, b^R v]$ is $(|v| + 4\delta)$-close to $[1, b^R v]$. Hence, the infinite broken geodesics

$$S = \bigcup_{i=-\infty}^{\infty} ([d^i, d^i b^R] \cup [d^i b^R, d^{i+1}])$$

is r_1-close to T with

$$r_1 = N|b| + 3|v| + 18\delta.$$

Now let $h \in E^+(d)$. Since T and hT are 16δ-close (see [**5**, Ch. 7, Corollary 3(b)]), S and hS are both r_2-close to T, for

$$r_2 = r_1 + 16\delta = N|b| + 3|v| + 34\delta.$$

For $z \in S \cup hS$, we denote by $\tau(z)$ a point on T with $|z - \tau(z)| \leqslant r_2$. For any two successive subsegments $[\tau(d^i), \tau(d^i b^R)]$ and $[\tau(d^{i+1}), \tau(d^{i+1} b^R)]$ of T, we have $|\tau(d^i b^R) - \tau(d^{i+1})| \leqslant 2r_2 + |v|$. Hence, $[\tau(h), \tau(hb^R)]$ has an overlapping subsegment U with some $[\tau(d^i), \tau(d^i b^R)]$ of length

$$|U| \geqslant \tfrac{1}{2}(|\tau(h) - \tau(hb^R)| - 2r_2 - |v|) \geqslant \tfrac{1}{2}|b^R| - 2r_2 - \tfrac{1}{2}|v|.$$

As $|h - \tau(h)|, |hb^R - \tau(hb^R)| \leqslant r_2$ every $z \in [\tau(h), \tau(hb^R)]$ is $(r_2 + 8\delta)$-close to a point in $[h, hb^R]$ by Fact 2, and the same is true for $[\tau(d^i), \tau(d^i b^R)]$ and $[d^i, d^i b^R]$. Thus, for some $[x, y] \subset [h, hb^R]$ and $[x', y'] \subset [d^i, d^i b^R]$ we have $|x - x'|, |y - y'| \leqslant 2r_2 + 16\delta$ and

$$|x - y|, |x' - y'| \geqslant |U| - 2r_2 - 16\delta \geqslant \tfrac{1}{2}|b^R| - 4r_2 - \tfrac{1}{2}|v| - 16\delta.$$

Now using $L \geqslant 40\delta$, $|b| \leqslant 2L$, $|v| \leqslant L + 26\delta$, $R > 200N + 300$ and (7) it is straightforward to check that the right-hand side of the last inequality is greater than $2r_2 + N|b| + 16\delta$. Then, by Lemma 2, $[x, y]$ and $[x', y']$ have strictly 8δ-close subsegments of length $\geqslant N|b|$. Application of Lemma 6 to $[h, hb^R]$ and $[d^i, d^i b^R]$ gives

$$h^{-1} d^i \in E^+(b).$$

Assume that $h^{-1}d^i$ has infinite order. Then $\langle h^{-1}d^i \rangle \cap \langle b \rangle \neq 1$ and as $h^{-1}d^i \in E^+(d)$, also $\langle h^{-1}d^i \rangle \cap \langle d \rangle \neq 1$. Hence, $\langle b \rangle \cap \langle d \rangle \neq 1$, which implies that $d \in E^+(b)$, that is, $v \in E^+(b)$. However this contradicts the choice of v. Hence, $h^{-1}d^i$ is of finite order, that is, $h^{-1}d^i \in E^*(b) \cap E^*(d)$.

We have proved that any $h \in E^+(d)$ may be represented as $h = d^t u$ where $u \in E^*(b) \cap E^*(d)$. By Lemma 5,

$$|u| \leqslant 140\delta < 4L.$$

Now, as in the final step of the proof of Lemma 7, we conjugate H and C to make d cyclically minimal. By (7), (8) and Lemma 3(a) and (c), $f^{-1}df$ is cyclically minimal for some f with

$$|f| \leqslant \langle d, d^{-1}\rangle + 6\delta \leqslant N|b| + 2|v| + 6\delta < (2N+4)L.$$

Then, after conjugation by f we get $|u| < (4N+12)L$ and $\max_{x\in C}|x| < (4N+10)L$. Using (7) we also get

$$|d| \geqslant |b^R| - |v| - 2(2N+4)L \geqslant \tfrac{1}{4}RL - 2L - 2(2N+4)L > ML. \qquad \square$$

Proof of Theorem 1. First observe that we can assume that $\#C \geqslant \#B(40\delta)$, as otherwise we can take $\alpha = (\#B(40\delta))^{-1}$ in the theorem. This guarantees that the condition $L = \min_{h\in G}\max_{x\in C}|h^{-1}xh| \geqslant 40\delta$ of Lemma 8 is satisfied. We fix a number $R > 0$ (its value is given below). After an appropriate conjugation of H and C, we choose d by Lemma 8 with

$$M := 2R + 4N + 12.$$

where N is as in Lemma 6.

We claim that $B(RL) \cap E(d)$ is contained in a finite subgroup F of G. By the statement of Lemma 8, every $h \in E^+(d)$ may be represented as $h = d^t u$, $t \in \mathbb{Z}$, where $u \in E^*(d)$ and $|u| < (4N+12)L$. Observe that $|d^t| > |d|$ for $|t| \geqslant 2$ by Corollary 1. Then for any $h \in E^+(d) \setminus E^*(d)$,

$$|h| > |d| - (4N+12)L \geqslant ML - (4N+12)L > RL.$$

Hence,

$$B(RL) \cap E^+(d) \subseteq E^*(d).$$

If $B(RL) \cap E^-(d) = \varnothing$ we can take $F = E^*(d)$ and there is nothing else to prove. Let $v \in B(RL) \cap E^-(d)$. As $v^2 \in E^*(d)$ and $v^{-1}E^*(d)v = E^*(d)$ by Lemma 4, the subgroup $F = \langle E^*(d), v\rangle$ is finite. As $|E(d) : E^+(d)| = 2$ every $g \in E^-(d)$ is represented as

$$g = d^t uv, \quad u \in E^*(d), \quad t \in \mathbb{Z}, \quad |u| < (4N+12)L.$$

If $|g| \leqslant RL$, this implies that

$$|d^t| \leqslant |g| + |u| + |v| < 2RL + (4N+12)L = ML \leqslant |d|,$$

that is, $t = 0$ and so $g \in F$, as required.

According to [**4**, Ch. III.Γ, Theorem 3.2], the order of any finite subgroup of G is bounded by a number K depending only on G (in fact, $K = \#B(4\delta + 2)$). We choose a largest subset

$$W = \{w_1, w_2, \ldots, w_n\} \subseteq C$$

representing distinct right cosets modulo F. We have

$$\#W \geqslant \frac{1}{K}\#C.$$

Now take $R = 2(4N+10)$, $m = 3N$ and set

$$Y = \{w_i d^m w_i^{-1}\}.$$

Then the assertion (i) at the beginning of the section holds by construction. To check the assertion (ii) we observe that

$$|w_i d^m w_i^{-1}|_C \leqslant m|d|_C + 2,$$

which by the statement of Lemma 8 is bounded by

$$D = 3N(5M + 200N + 300) + 2.$$

It remains to prove that the elements of Y freely generate a free subgroup of H. We need to show that any non-cancellable product

$$p_1 p_2 \dots p_\ell, \quad p_i = v_i d^{mt_i} v_i^{-1}, \quad v \in W, \quad t_i \neq 0 \tag{9}$$

does not equal 1. Consider two neighbouring factors in (9)

$$vd^{mt}v^{-1}wd^{ms}w^{-1}.$$

If

$$|d^{mt}v^{-1}wd^{ms}| \leqslant |d^{mt}| + |d^{ms}| - |v^{-1}w| - 2N|d|$$

using Lemmas 2 and 6 we obtain $v^{-1}w \in E(d)$. However, then $|v^{-1}w| \leqslant 2(4N+10)L$ and hence $v^{-1}w \in F$ contrary to the choice of W. Therefore,

$$|d^{mt}v^{-1}wd^{ms}| > |d^{mt}| + |d^{ms}| - |v^{-1}w| - 2N|d|,$$

which implies that

$$|vd^{mt}v^{-1}wd^{ms}w^{-1}| > |vd^{mt}v^{-1}| + |wd^{ms}w^{-1}| - 4(|v| + |w|) - 2N|d|$$

and may be rewritten as

$$\langle (vd^{mt}v^{-1})^{-1}, wd^{ms}w^{-1} \rangle \leqslant N|d| + 2(|v| + |w|).$$

Using Corollary 1 we also get

$$|vd^{mt}v^{-1}| \geqslant |d^{mt}| - 2|v| > m|t|(|d| - 6\delta) - 2|v|.$$

Switching to the notation (9) and using the bound $(4N+10)L$ for lengths of elements of C we obtain

$$|p_i| \geqslant m(|d| - 6\delta) - 2(4N + 10)L$$

and

$$\langle p_i^{-1}, p_{i+1} \rangle \leqslant N|d| + 4(4N + 10)L.$$

For $m = 3N$, a straightforward calculation shows that

$$\langle p_{i-1}^{-1}, p_i \rangle + \langle p_i^{-1}, p_{i+1} \rangle < |p_i| - 3\delta.$$

Then, by Lemma 1(a), $|p_1 p_2 \dots p_\ell| > 0$ that is, $p_1 p_2 \dots p_\ell \neq 1$. □

Proof of Theorem 2. We switch to viewing generating sets of G as lists $A = (a_1, a_2, \dots, a_m)$ keeping the notation $\#A$ for the number of distinct elements in A.

Let A_0 be any finite set of generators for G. We choose N with $\alpha N > \lambda(G, A_0)$ where α is given by Theorem 1. By the theorem, $\lambda(G, C) > \lambda(G, A_0)$ for any C with $\#C > N$. For the required A, we take the list $\bar{A}_0 = (A_0, 1, \dots, 1)$ of length $n = 2N$ obtained from A_0 by filling with ones. Observe that $\lambda(G, A_0) = \lambda(G, \bar{A}_0)$. Thus, it suffices to prove that, for any C with $\#C \leqslant N$, the number $\lambda(G, C)$ lies in the set $\{\lambda(G, B) \mid B \in \bar{A}_0^{\operatorname{Aut} F_n}\}$. To do this, we use a simple 'stable equivalence' argument.

Observe that two lists

$$(g_1, g_2, \dots, g_k, \dots, g_n)$$

and

$$(g_1, g_2, \ldots, g_{k-1}, g_k h, g_{k+1}, \ldots, g_n)$$

lie in the same orbit under the action of $\operatorname{Aut} F_n$ if h is in the subgroup generated by all g_i for $i \neq k$, as one is obtained from another by a chain of elementary Nielsen transformation. Now $\lambda(G, C) = \lambda(G, \bar{C})$ for $\bar{C} = (C, 1, \ldots, 1)$ of length $n = 2N$ and as both A_0 and C are of length at most N,

$$\begin{aligned}\bar{C} &= (C, 1, \ldots, 1) \\ &\sim (C, A_0, 1, \ldots, 1) \\ &\sim (1, \ldots, 1, A_0, 1, \ldots, 1) \\ &\sim (A_0, 1, \ldots, 1) = \bar{A}_0\end{aligned}$$

where '$\sim$' stands for the equivalence under the action of $\operatorname{Aut} F_n$. □

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