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Non-Scale Models of R&D-based Growth: The Market Solution

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Abstract

The market solution of a general R&D-based endogenous growth model is developed. The model is general in two respects: First, general formulations are used and restrictions are only introduced provided that these become necessary. Second, each factor of production (labour, capital and technological knowledge) is allowed to be productive in each sector (final output, capital goods and R&D). Since the resulting R&D-based growth model encompasses a large number of specific models, it can be viewed as a summary of this strand of the literature. The complete dynamic system as well as the balanced growth rates are derived. By employing numeric techniques, the gap between the decentralised and the centralised balanced growth path as well as the transitional dynamics implications are investigated.

KEYWORDS: R&D-based growth models, non-scale growth, market solution, dynamic systems

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1. Introduction

In an important paper, Eicher and Turnovsky (1999) formulate a general non-scale model of endogenous growth. The expression non-scale growth refers to the fact that the underlying model does not bear the scale-effect implication according to which the growth rate increases with the size (scale) of the economy. The elimination of the scale effect is clearly favourable since this implication appears to be in conflict with the empirical evidence (Jones, 1995a, 1995b). The authors further derive conditions under which positive and balanced growth is obtained. On this occasion, Eicher and Turnovsky focus on the centralised solution. This procedure has the advantage that the general framework encompasses investment-based as well as R&D-based models of growth. However, the authors themselves point to the fact that "*[t]his level of generality comes at the expense of having to abstract from issues related to specific microfoundations. We make these abstractions, not because we feel that such issues are unimportant. But to facilitate the identification of the characteristics common to alternative approaches.*" Eicher and Turnovsky (1999, p. 397).

The paper at hand picks up this important point and complements their original contribution. The market solution of a general non-scale R&D-based model of endogenous growth is developed. The model is general in two respects: First, general functions are used and restrictions on functional forms or parameters are only introduced provided that these become necessary. Second, each factor of production (labour, capital and technological knowledge) is allowed to be productive in each sector (final output, capital goods and R&D). The general model developed in this paper describes the class of R&D-based endogenous growth models of the increasing-variety type. It encompasses the first-generation of R&D-based models like the original Romer (1990a) model, the semi-endogenous growth model of Jones (1995a) and the hybrid non-scale model of R&D-based growth of Eicher and Turnovsky (1999, Section 3.1). Further examples are models with CES-CES technologies as used in Romer (1990b), models with complementarities among intermediate goods (Benhabib and Xie, 1994) as well as the lab-equipment approach (Rivera-Batiz and Romer, 1991).[1]

The benefits of this kind of work are manifold: First, by developing the decentralised solution, this paper demonstrates that the general non-scale R&D-based growth model of Eicher and Turnovsky (1999) indeed possesses a sound microeconomic interpretation. This conclusion is far from being trivial since general market equilibrium requires a consistent interplay between

[1] For a comprehensive overview of the literature see Jones (2005, especially Section 5). Moreover, the focus here is on R&D-based growth models of the increasing-variety type. See Arnold (2004) who, in the context of economic integration and growth, sets up a fairly general growth model including quality-upgrading models.

production technology, market structure and profit maximisation. The propositions derived in Eicher and Turnovsky turn out to be valid for the market economy as well. As a consequence, the range of applicability of their general results can be extended. This extension is of major importance from the perspective of positive economic theory since real-world economic dynamics most probably represent market outcomes instead of socially optimal solutions. Second, the discussion of the general model deepens our understanding of the underlying microeconomic structure. More specifically, the formulation of the decentralised model in non-parametric form forces us to be explicit on the conditions which are necessary for the existence of a market equilibrium. Third, the complete dynamic system governing the evolution of the decentralised solution is derived. Due to the generality of the set-up, the model is fairly flexible and accordingly requires less parameter restrictions than existing models (e.g. Rivera-Batiz and Romer, 1991) when calibrated and applied to empirical data. The resulting dynamic system is of independent interest since it serves as the basis for a number of important investigations including the analysis of the level of the balanced growth path as well as the investigation of transitional dynamics. Finally, the general R&D-based endogenous growth model developed in this paper encompasses a large number of specific models. Hence, it can be viewed as a summary of this strand of the literature.

One of the main implications of non-scale endogenous growth models lies in the fact that the long-run growth rates of the market economy and centralised economy coincide. This result bears the important implication that public policies targeting the balanced growth rate are simply not indicated. Nonetheless, the level of the balanced growth path resulting from the decentralised solution probably diverges from the level of the balanced growth path resulting from the centralised solution. This divergence may imply a dramatic loss in welfare. It would, therefore, be clearly desirable to get an idea about the magnitude of this gap. In order to answer this important question, the dynamic system governing the evolution of the market economy is applied and numerical techniques are employed. In addition, the dynamic system is employed to investigate the transitional dynamics implications.

The present paper is structured as follows: In Section 2, the market solution of a general non-scale R&D-based growth model is developed and the complete dynamic system is derived. In Section, 3 the balanced growth rates are determined. Section 4 discusses some specific benchmark models, which represent special cases. In Section 5, a general non-scale R&D-based growth model is applied to analyse the gap between the socially optimal and the decentralised balanced growth path and to determine the eigenvalues of the underlying dynamic system numerically. Finally, Section 6 offers a summary and conclusion.

2. A general non-scale R&D-based growth model

2.1. A non-formal sketch of the basic structure

The market equilibrium for the class of R&D-based endogenous growth models of the increasing-variety type is developed and the dynamic system governing the evolution of the decentralised solution is derived. The model is general in two respects: First, general formulations are used as far as possible and restrictions on the formal structure of the model are only introduced provided that these become necessary. Second, each input factor (labour, capital and technological knowledge) is allowed to be productive in each sector (final output, capital goods and R&D).

It is helpful to sketch at first the structure of the economy under consideration. On the production side there are three sectors. First, the final output (FO) sector produces a homogenous good that can be used for consumption or investment purposes. Second, the capital goods (CG) sector produces differentiated capital goods that serve as an input in the production of FO as well as R&D. Third, the R&D sector searches for new ideas, which are the (technical or legal) prerequisite for the production of CG. Households choose their level of consumption and inelastically supply one unit of labour each period of time.

The state variables are the stock of physical capital (K) and the number of designs (A). The model comprises three choice variables, namely the level of consumption (C), the share of labour (θ) and the share of CG (ϕ) devoted to the production of FO. Finally, since we have three distinct goods, there are three prices. FO serves as the numeraire, its price is set equal to unity. The price of CG is denoted as p and the price of designs as v, respectively.

2.2. Firms

Final-output sector

The FO sector comprises a large number of firms ordered on the interval $[0,1]$. Firms produce a homogenous good, which is sold in competitive markets and can be used for consumption or investment. More specifically, FO serves as an input in the production of differentiated CG. The original production function may be expressed as

$$Y = \bar{F}\left(\theta L,\, B_Y\right), \tag{1}$$

where Y denotes FO, θ ($0 \le \theta \le 1$) an allocation variable which gives the share of labour allocated to FO and L is the stock of labour which evolves according to $\dot{L} = nL$ (a “dot” above a variable denotes its derivative with respect to time) and $n \geq 0$. The production function (1) is assumed to satisfy $\partial \bar{F} / \partial \theta L > 0$, $\partial^2 \bar{F} / \partial \theta L^2 < 0$ and $\partial \bar{F} / \partial B_Y > 0$.

The index B_Y can be considered as an intermediate input, which is composed of an array of differentiated CG according to

$$B_Y = \left\{ \int_0^A \left[\phi(i) x(i) \right]^{\frac{\sigma_K}{\mu}} di \right\}^{\mu} \qquad \text{with } 0 < \sigma_K < 1 \text{ and } \mu \geq 1, \tag{2}$$

where $x(i)$ with i real valued and $i \in [0, A]$ denotes the number of CG of type i and $\phi(i)$ $[0 \leq \phi(i) \leq 1]$ gives the share of CG allocated to FO production.[2] The parameter A indicates the number of CG available at each point in time. A characteristic feature of this class of models is that this number is an endogenous variable; the law of motion of A is described below. The parameter μ captures the degree of complementarity between the differentiated CG. If $\mu = 1$ there is no complementarity implying that the $x(i)$ enter B_Y additively separable, while for $\mu > 1$ there is complementarity (Benhabib and Xie, 1994). The constant elasticity of substitution among CG is $\varepsilon_{FO} = (1 - \sigma_K / \mu)^{-1}$. Since $0 < \sigma_K < 1$ and $\mu \geq 1$ it follows that $1 < \varepsilon_{FO} < \infty$. The CG substitute imperfectly for each other (the elasticity of substitution is finite). This crucial assumption gives rise to monopolistic competition in the CG sector.

An important simplifying assumption is that the $x(i)$ enter the model completely symmetric. This assumption comprises two aspects: First, the $x(i)$ enter the index B_Y symmetric (as evident from the expression for B_Y above) and, second, all $x(i)$ are produced employing the same technology implying that the costs identical also. Accordingly, $x(i) = x$ for all i and B_Y can be simplified to read $B_Y = A(\phi x)^{\sigma_K}$. Defining aggregate capital by $K := qAx$ (which follows from the production technology for CG described below) and substituting into B_Y yields $B_Y = A^{1-\sigma_K} \left(\phi K \right)^{\sigma_K} q^{-\sigma_K}$. This form shows the Smith-Ethier effect according to which the number of CG (or services) A raises total factor productivity (Ethier, 1982).

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[2] A formulation of the kind $Y = F(\phi K)$ and $X = G[(1-\phi) K]$ is admissible and appropriate also for a decentralised set-up. One should recognise, however, that this formulation does not only describe the production technologies but in addition postulates that the factor inputs in both sectors add up to total supply. An equilibrium is realised by the fact that the allocation variable ϕ is derived such that factor inputs are indeed considered as optimal.

By substituting $B_Y = A^{1-\sigma_K} \left(\phi K\right)^{\sigma_K} q^{-\sigma_K}$ into the original production function (1) we get the transformed production function

$$Y = F\left(\theta L, \phi K, A\right). \tag{3}$$

The reason for the distinction between the original production function (1) and the transformed production function (3) lies in the fact that the former underlies basic relations, which describe market equilibrium (e.g. the demand function for CG). The latter formulation is used to describe the dynamics of the aggregate capital stock given by $\dot{K} = F\left(\theta L, \phi K, A\right) - \delta K - C$, where $\delta \geq 0$ denotes the constant rate of capital depreciation and C total consumption.

Finally, the production function (1) is required to exhibit constant returns to scale in the private inputs (L and x) in order to enable a competitive equilibrium in the FO sector.

Capital-goods sector

There is a large number of firms ordered on the interval $[0, A]$ manufacturing CG denoted as x. Every producer must at first invest in blueprints (designs) as the technical or legal prerequisite of production. The owner of a blueprint is the only producer of the respective CG. The market is structured monopolistically competitive. The representative CG producer can convert $q > 0$ units of FO into one CG, i.e. the production technology is proportional to the FO technology. Provided that there is no price differentiation, operating profits are given by

$$\pi(x) = \left[p(x) - q\,r\right]x, \tag{4}$$

where r denotes the (gross) interest rate and $p(x)$ the rental price of CG.[3]

The typical CG producer faces two demand schedules. One stems from the FO sector, while the other originates from R&D firms. Since there is a large number of firms in both sectors, the elasticities of substitution equal the respective price elasticities of demand denoted by ε_{FO} (FO) and $\varepsilon_{R\&D}$ (R&D). When

[3] The interest rate should be interpreted as the rental price of one unit of "raw capital" (output not consumed) per period of time (Romer, 1990b, p. 348).

allowing for $\varepsilon_{FO} \neq \varepsilon_{R\&D}$, the representative CG producer has incentives to differentiate prices across the two groups of demanders.[4]

With constant marginal costs ($q\,r$) and price elasticities given by ε_{FO} and $\varepsilon_{R\&D}$, the solution to the underlying monopoly pricing problem implies supply prices of $p^S_{FO} = \frac{\varepsilon_{FO}}{\varepsilon_{FO}-1} q\,r$ (FO) and $p^S_{R\&D} = \frac{\varepsilon_{R\&D}}{\varepsilon_{R\&D}-1} q\,r$ (R&D).[5]

Considering the demand side, both the FO sector and the CG sector are competitive and therefore the typical producer is willing to pay the marginal product for his/ her inputs. To simplify, we assume $\mu = 1$ (i.e. additive separability of CG in FO production applies). Hence, the inverse demand function originating from the FO sector may be expressed as

$$p^D_{FO} = \bar{F}_B\left(\theta L,\, B_Y\right)\sigma_K(\phi x)^{\sigma_K - 1}, \tag{5}$$

where p^D_{FO} denotes the demand price the FO firms are willing to pay. Similarly, the R&D technology (described below) implies that the inverse demand function from the R&D sector is (again it is assumed that CG enter the R&D technology additive separable)

$$p^D_{R\&D} = \bar{\bar{J}}_B\left[(1-\theta)\,L, (1-\theta)L_a, B_J\right]\eta^p_K\left[(1-\phi)x\right]^{\eta_K - 1} \tag{6}$$

with $\eta^p_K > 0$ and $0 < \eta_K < 1$,

where $p^D_{R\&D}$ denotes the demand price the R&D firms are willing to pay.

With price differentiation, the profit of the typical CG producer is $\pi(x) = (p_{FO} - q\,r)\phi x + (p_{R\&D} - q\,r)(1-\phi)x$ with p_{FO} and $p_{R\&D}$ denoting the equilibrium prices in the CG markets. By taking the equilibrium conditions $p_{FO} = p^D_{FO} = p^S_{FO}$ and $p_{R\&D} = p^D_{R\&D} = p^S_{R\&D}$ into account and noting $p^S_{FO} = \frac{\varepsilon_{FO}}{\varepsilon_{FO}-1} q\,r$ and $p^S_{R\&D} = \frac{\varepsilon_{R\&D}}{\varepsilon_{R\&D}-1} q\,r$, the profit function may be expressed as

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[4] Since $\varepsilon_{FO} = (1-\sigma_K)^{-1}$ and $\varepsilon_{R\&D} = (1-\eta^p_K)^{-1}$, $\varepsilon_{FO} = \varepsilon_{R\&D}$ would imply $\sigma_K = \eta^p_K$ and hence the model would be equivalent to the lab-equipment approach of Rivera-Batiz and Romer (1991).

[5] It is assumed that CG can be transformed costlessly back into raw capital. As a result, the typical CG producer solves the underlying monopoly pricing problem at each point in time.

$\pi(x)=p_{FO}^{D}\varepsilon_{FO}^{-1}\phi x+p_{R\&D}^{D}\varepsilon_{R\&D}^{-1}(1-\phi)x$. Moreover, using the demand functions (5) and (6) we get

$$\pi(x)=\overline{F}_B(.)\sigma_K(\phi x)^{\sigma_K}\,\varepsilon_{FO}^{-1}+\overline{\overline{J}}_B(.)\eta_K^p\left[(1-\phi)x\right]^{\eta_K}\,\varepsilon_{R\&D}^{-1}. \tag{7}$$

From equilibrium in the markets for CG ($p_{FO}^{D}=p_{FO}^{S}$ and $p_{R\&D}^{D}=p_{R\&D}^{S}$) as well as equilibrium in the market for raw capital (i.e. there is a unique rental rate of raw capital), we may express the equilibrium interest rate as follows

$$r=\frac{\varepsilon_{FO}-1}{\varepsilon_{FO}}\frac{\overline{F}_B(.)\sigma_K(\phi x)^{\sigma_K-1}}{q}=\frac{\varepsilon_{R\&D}-1}{\varepsilon_{R\&D}}\frac{\overline{\overline{J}}_B(.)\eta_K^p\left[(1-\phi)x\right]^{\eta_K-1}}{q}. \tag{8}$$

R&D sector

There is a large number of R&D firms ordered on the interval $[0,1]$ searching for new ideas (blueprints). The R&D technology is of the following shape

$$\dot{A}=\overline{\overline{J}}\left[A,(1-\theta)L,(1-\theta)L_a,B_J\right] \tag{9}$$

$$B_J=\left\{\int_0^A\left[(1-\phi(i))x(i)\right]^{\frac{\eta_K^p}{\beta}}\left[(1-\phi(i))x_a(i)\right]^{\frac{\eta_K^e}{\beta}}di\right\}^{\beta} \tag{10}$$

with $\eta_K^p>0$, $\eta_K^e<0$, $0<\eta_K^p+\eta_K^e<1$ and $\beta\geq1$.

It is further assumed that $\partial\overline{\overline{J}}/\partial A>0$, $\partial\overline{\overline{J}}/\partial(1-\theta)L>0$, $\partial^2\overline{\overline{J}}/\partial(1-\theta)L^2<0$, $\partial\overline{\overline{J}}/\partial(1-\theta)L_a<0$ and $\partial\overline{\overline{J}}/\partial B_J>0$. This general formulation deserves a thorough explanation: First, the preceding production function generalises the usual R&D technology in that CG (x) are considered to be productive in R&D as well. Second, $\partial\overline{\overline{J}}/\partial A>0$ captures the net effect of (intertemporal) knowledge spill-overs and “fishing out” effects (Jones and Williams, 2000). Third, following Jones (1995a) and Jones and Williams (2000), we allow for negative externalities associated with the sector-wide averages of private resources. This is present at two points: (i) $\partial\overline{\overline{J}}/\partial(1-\theta)L_a<0$ with L_a denoting the average stock of labour and (ii) $\partial B_J/\partial(1-\phi(i))x_a<0$ due to $\eta_K^e<0$. Both components capture (intratemporal) duplication externalities, which may be either accidental or intentional (as in the case of R&D races). Finally, it is assumed that there are constant returns to scale in the private inputs (L and x) at the level of the individual firm.[6]

Once more, using $x = K/(q\,A)$ allows us to express B_J as $B_J = A^{1-\eta_K} q^{-\eta_K} [(1-\phi)K]^{\eta_K^p} [(1-\phi)K_a]^{\eta_K^e}$ and, therefore, the transformed production function (in terms of aggregate capital) may be written as $\dot{A} = \bar{J}\left[A, (1-\theta)\,L, (1-\theta)L_a, (1-\phi)K, (1-\phi)K_a\right]$. Since in equilibrium $(1-\theta)\,L = (1-\theta)L_a$ and $(1-\phi)K = (1-\phi)K_a$, we can state this function in the form

$$\dot{A} = J\left[A, (1-\theta)\,L, (1-\phi)K\right]. \tag{11}$$

R&D technology: a specific example

Consider the following specific R&D technology

$$\dot{A} = \bar{\bar{J}}(.) = A^{\eta_A^{SO}} \left[(1-\theta)\,L\right]^{\eta_L} \int_0^A \left[(1-\phi(i))x(i)\right]^{\eta_K} di \tag{12}$$

with $\eta_L := \eta_L^p + \eta_L^e$, $\eta_K := \eta_K^p + \eta_K^e$, $\eta_A^{SO}, \eta_L^p, \eta_K^p > 0$,

$\eta_L^e, \eta_K^e < 0$, $0 < \eta_L, \eta_K < 1$ and $\eta_L^p + \eta_K^p = 1$.

There are constant returns to scale in private inputs at the level of the individual firm, i.e. $\eta_L^p + \eta_K^p = 1$. Moreover, there are negative externalities associated with the sector-wide averages of the private resources as indicated by $\eta_L^e, \eta_K^e < 0$.[7] Using $x = K/(qA)$ leads to

$$\dot{A} = J(.) = A^{\eta_A^{SO}+1-\eta_K} \left[(1-\theta)\,L\right]^{\eta_L} \left[(1-\phi)K\right]^{\eta_K} q^{-\eta_K} . \tag{13}$$

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[6] Constant returns to scale with respect to private inputs are necessary to get a formulation consistent with general equilibrium (zero profits and a finite demand for inputs). On the other hand, technological knowledge plays an important role in R&D. If we abstract from negative externalities associated with the sector-wide averages of private resources, this would imply heavily increasing returns in all three factor of production, which might be empirically implausible (e.g. Jones and Williams, 1998). In addition, the assumption of negative duplication externalities is theoretically plausible.

[7] In equilibrium, sector-wide averages of input factors are equal to the amount employed by the representative firm. This fact is used above to simplify the production technology.

The preceding formulation illustrates that there are two effects associated with the level of technology A: (i) The term $A^{\eta_A^{SO}}$ captures the net effect of (intertemporal) knowledge spill-overs and "fishing out" effects and (ii) since CG are productive in R&D, there is a Smith-Ethier effect as indicated by $A^{1-\eta_K}$.

The price of blueprints

The typical R&D firm sets the price of blueprints (designs) to extract the present value of the infinite profit stream, which accrues at first to the CG producer. This price (which equals the value of the typical CG firm) is given by $v(t) = \int_t^{\infty} \pi(\tau)\, e^{-R(t)}\, d\tau$ with $R(t) := \int_t^{\tau} [r(u) - \delta]\, du$. Differentiating the preceding integral equation with respect to time gives $\dot{v} = (r-\delta)\, v - \pi$. This equation can be interpreted as no-arbitrage condition for the two financial assets existing in this model. The reward of a consumption loan of size v amounts to $(r-\delta)v$, while the reward of an equity (issued by CG producers) of equal size is given by $\dot{v} + \pi$. Inserting the expressions for π and r derived above, one obtains the differential equation in v as

$$\dot{v} = \left[\frac{\varepsilon_{FO} - 1}{\varepsilon_{FO}} \frac{\overline{F}_B(.)\sigma_K (\phi x)^{\sigma_K - 1}}{q} - \delta \right] v \\ - \left\{ \overline{F}_B(.)\sigma_K (\phi x)^{\sigma_K} \varepsilon_{FO}^{-1} + \overline{\overline{J}}_B(.)\eta_K^p \left[(1-\phi)x\right]^{\eta_K} \varepsilon_{R\&D}^{-1} \right\} \tag{14}$$

Factor allocation conditions

Let us now turn to the factor allocation conditions. Profit-maximising firms reward the factors of production according to their (private) marginal product. Moreover, in equilibrium the wage rates are equalised across sectors so that (notice that original production functions are differentiated with respect to private labour input)

$$w = \overline{F}_{\theta L}(.) = v\, \overline{\overline{J}}_{(1-\theta)L}(.) \tag{15}$$

As for the CG, equation (8) gives a second condition which must hold for the intersectoral allocation of resources to be efficient. For any set $\{L, x, v, A\}$, equations (8) and (15) implicitly determine the optimal values of ϕ and θ. Using $x = K/(qA)$, these conditions may be expressed in general terms as $\phi = \phi(A, K, L, v)$ and $\theta = \theta(A, K, L, v)$.

Finally, it should be noted that (by Euler's theorem) factor payments according to marginal products together with constant returns to scale in private inputs implies zero profits in the R&D sector being compatible with free entry to this sector.

2.3. Households

The representative household is assumed to inelastically supply one unit of labour during each period of time and to maximise his/ her intertemporal utility. The instantaneous utility function is of the constant-intertemporal-elasticity-of-substitution type (CIES); a specific formulation is used to reduce notational effort. The dynamic problem reads as follows

$$\max_{\{C/L\}} \int_0^\infty \frac{(C/L)^{1-\gamma}-1}{1-\gamma} e^{-\rho t}\, dt$$

$$\text{s.t. } \dot{K} = (r-\delta)K + wL + A\pi - v\dot{A} - C;\ K(0)>0 \text{ and } A(0)>0, \tag{16}$$

where $\rho > 0$, $\gamma > 0$ and w denote the constant time preference rate, a constant preference parameter and the wage rate, respectively. From the necessary first-order conditions we get the Keynes-Ramsey rule describing the optimal consumption profile[8]

$$\dot{C} = \frac{C}{\gamma}\left[r-\delta-\rho-(1-\gamma)n\right]. \tag{17}$$

2.4. The dynamic system

The preceding discussion can be summarised by the following set of equations. The dynamic system shown below (together with appropriate endpoint conditions) governs the dynamics of the market solution for a broad class of R&D-based endogenous growth models of the increasing-variety type. Notice that the price of CG (p) has been eliminated.

$$\dot{K} = F\left(A, \theta L, \phi K\right) - \delta K - C \tag{18}$$

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[8] We assume that the sufficiency conditions are equally satisfied and that the transversality conditions for the stock of capital and technology are met. Moreover, convergence of the utility integral is assumed to apply.

$$\dot{A} = J\left[A, (1-\theta)L, (1-\phi)K\right] \tag{19}$$

$$\dot{C} = \frac{C}{\gamma}\left[\frac{\varepsilon_{FO}-1}{\varepsilon_{FO}} F_B(.) A^{1-\sigma_K} q^{-\sigma_K} \sigma_K (\phi K)^{\sigma_K - 1} - \delta - \rho - (1-\gamma)n\right] \tag{20}$$

$$\dot{v} = \left[\frac{\varepsilon_{FO}-1}{\varepsilon_{FO}} F_B(.) A^{1-\sigma_K} q^{-\sigma_K} \sigma_K (\phi K)^{\sigma_K - 1} - \delta\right] v \tag{21}$$

$$-\left\{F_B(.)\sigma_K A^{-\sigma_K} q^{-\sigma_K} (\phi K)^{\sigma_K} \varepsilon_{FO}^{-1} + J_B(.)\eta_K^p A^{-\eta_K} q^{-\eta_K} \left[(1-\phi)K\right]^{\eta_K} \varepsilon_{R\&D}^{-1}\right\}$$

$$\theta = \theta(A, K, L, v) \tag{22}$$

$$\phi = \phi(A, K, L, v) \tag{23}$$

For the readers convenience, the notation is summarised in Table 1 (some of the variables shown in Table 1 become relevant in course of the paper). In addition, it should be noticed that we have expressed the dynamic system in terms of aggregate capital K rather than in terms of differentiated CG x.

Table 1: Definition of variables

Y : output of final-output sector (y : scale-adjusted output)
J : output of R&D sector (j : scale-adjusted output)
L : population (supply of labour)
A : number of ideas (a : scale-adjusted number of ideas)
K : aggregate capital stock (k : scale-adjusted capital)
C : aggregate consumption (c : scale-adjusted consumption)
θ : share of labour allocated to final output ($0 \le \theta \le 1$)
ϕ : share of capital allocated to final output ($0 \le \phi \le 1$)
σ_z : elasticity of factor Z in final output production
η_z : elasticity of factor Z in R&D
v : price of one idea (v_a : scale-adjusted price)
ε : price elasticity of demand (capital goods)
δ : depreciation rate of capital
γ : elasticity of marginal utility w.r.t. consumption
ρ : time preference rate
n : growth rate of population

The dynamic system displayed in (18) to (23) is of independent interest. This is due to the fact that this system can be used to investigate a number of important issues. For instance, the distortions inherent in the market economy cause the level of the balanced growth path to diverge from the socially optimal level of the balanced growth path. The system displayed above (together with the corresponding system resulting from the social solution derived in Eicher and Turnovsky, 1999) can be used to investigate the magnitude of this gap. Moreover, the above displayed system serves as the basis for the analysis of transitional dynamics. The general stability properties can be analysed and the speed of convergence can be determined.

3. Balanced growth rates

As usual, a balanced growth path is defined by constant (possibly different) growth rates of the endogenous variables. This definition implies that the allocation variables (θ and ϕ) must be constant along the balanced growth path. In accordance with the stylised facts, we employ the auxiliary assumption stating that $\hat{Y} = \hat{K}$ along the balanced growth path (Romer, 1989). From $\hat{K} = Y/K - \delta - C/K$ it then follows that balanced growth further requires $\hat{K} = \hat{C}$. The balanced growth rates of K and A can be derived from $d[F(.)/K]/dt = 0$ and $d[J(.)/A]/dt = 0$ by noting that the allocation variables are constant. Carrying out the preceding instructions yields

$$(1-\sigma_K)\hat{K} - \sigma_A \hat{A} = \sigma_L \hat{L} \tag{24}$$

$$(1-\eta_A)\hat{A} - \eta_K \hat{K} = \eta_L \hat{L}, \tag{25}$$

where a “hat” above a variable denotes its growth rate, i.e. $\hat{K} := \dot{K}/K$ etc. The elasticities of production σ_z and η_z are defined by $\sigma_z := F_z(.)z/F(.)$ and $\eta_z := J_z(.)z/J(.)$ for $z = A, L, K$. These are exogenous constants in the Cobb-Douglas case and a function of the input vector in the more general CES case. Provided that $\hat{L} = n > 0$ equations (24) and (25) uniquely determine $\hat{K}$ and $\hat{A}$ given as follows

$$\hat{K} = \beta_K n \quad \text{with} \quad \beta_K := \frac{\sigma_L(1-\eta_A) + \eta_L \sigma_A}{(1-\eta_A)(1-\sigma_K) - \eta_K \sigma_A} \tag{26}$$

$$\hat{A} = \beta_A\, n \quad \text{with} \quad \beta_A := \frac{\eta_L\,(1-\sigma_K) + \eta_K\,\sigma_L}{(1-\eta_A)(1-\sigma_K) - \eta_K\,\sigma_A} \tag{27}$$

Eicher and Turnovsky (1999, Section 2.1) derive the conditions for positive and balanced growth applying to the social solution of a general R&D-based growth model. The results can equally be applied here as well since the underlying production functions and the resulting balanced growth rates are structurally identical for the decentralised and the centralised solution. Accordingly, $(1-\eta_A)(1-\sigma_K) - \eta_K\,\sigma_A > 0$ and $\sigma_K < 1$ is necessary and sufficient for positive growth. In addition, there are three conditions (each of which guarantees balanced growth) according to which the production functions in both sectors must be either: (i) constant returns to scale; (ii) of the Cobb-Douglas type or (iii) homogenously separable in the exogenously and endogenously growing factors. The model shows even growth ($\hat{K} = \hat{A}$) in the first and the third case and uneven growth ($\hat{K} \neq \hat{A}$) in the second case.

Provided that one of the preceding conditions for balanced growth applies, the balanced growth rates read as follows $\hat{Y} = \hat{K} = \hat{C} = \beta_K\, n$ and $\hat{A} = \beta_A\, n$.[9]

Several points appear worth being noted at this stage. First, the balanced growth rates are crucially determined by the structural characteristics of the technology side of the model given by the elasticities of the factors in the production of FO and R&D. Second, growth is characterised by the absence of a scale effect, i.e. growth is independent of the size of the economy. In addition, the balanced growth rates are proportional to the growth rate of the exogenously growing factor (labour). Third, one of the generalisations of the underlying model concerns the fact that capital goods are considered to be productive in R&D as well. Inspection of equation (26) shows that $\hat{Y} = \hat{K} = \hat{C} = \beta_K\, n$ increases with η_K. On the other hand, the effect on the growth rate of A shown in (27) is ambiguous.

Let us return to the comparison between the decentralised and the centralised solution. For the class of R&D-based models under consideration the (aggregate) production functions displayed in (18) and (19) are identical for the decentralised and the social solution. As a result, if $\hat{L} = n > 0$, then the balanced growth rates of the market economy and the balanced growth rates resulting from

[9] Since the balanced growth rates $\hat{K} = \beta_K\, n$ and $\hat{A} = \beta_A\, n$ were derived under the assumption $n > 0$, it cannot be concluded that these are zero provided that $n = 0$. In this case, the system of equations $(1-\sigma_K)\,\hat{K} - \sigma_A\,\hat{A} = 0$ and $(1-\eta_A)\,\hat{A} - \eta_K\,\hat{K} = 0$ is homogenous and determines merely the ratio of $\hat{K}$ and $\hat{A}$. Preferences (via the Keynes-Ramsey rule) become necessary to pin down the balanced growth rates; this remark applies to the Romer (1990a) model.

the centralised solution coincide. A well known example for the preceding proposition is the Jones (1995a) model. Although Jones stresses policy-ineffectiveness, his model also implies that policy measures targeting the balanced growth rate are simply not indicated because both growth rates coincide.

4. Some benchmark models

The general framework set up above includes a number of specific models as special cases. Examples comprise the original Romer (1990a) model, the semi-endogenous growth model of Jones (1995a) and the hybrid non-scale growth model of Eicher and Turnovsky (1999, Section 3.1). Further examples are the CES-CES technology used in Romer (1990b), models with complementarities among intermediate goods (Benhabib and Xie, 1994) as well as the lab-equipment approach of Rivera-Batiz and Romer (1991).

To illustrate, let us consider three specific examples. First, the well-known Romer (1990a) model results from the general framework set up above provided that $Y = \bar{F}(.) = (\theta L)^{\sigma_L} B_Y$, $B_Y = \int_0^A x(i)^{\sigma_K} di$ [i.e. $\mu = 1$ and $\phi(i) = 1$ due to $\eta_K^p = 0$]. The FO technology in terms of aggregate capital is $Y = F(.) = A^{1-\sigma_K} (\theta L)^{\sigma_L} K^{\sigma_K} q^{-\sigma_K}$. Moreover, $\dot{A} = J(.) = A(1-\theta) L$; notice that $B_J = 1$ due to $\eta_K^p = 0$.[10] It must be stressed that the balanced growth rates derived in Section 3 do not apply to this (scale) model since $n = 0$; for $n > 0$ growth would permanently accelerate. Therefore, equations (24) and (25) form a system of homogenous equations and a third equation (the Keynes-Ramsey rule) is needed to determine the balanced growth rate.

Next, let us consider the well-known Jones (1995a) model. This requires that $Y = \bar{F}(.) = (\theta L)^{\sigma_L} B_Y$, $B_Y = \int_0^A x(i)^{\sigma_K} di$ [i.e. $\mu = 1$ and $\phi(i) = 1$ from $\eta_K^p = 0$]. As before, the FO technology in terms of aggregate capital is $Y = F(.) = A^{1-\sigma_K} (\theta L)^{\sigma_L} K^{\sigma_K} q^{-\sigma_K}$. The R&D technology is $\dot{A} = J(.) = A^{\eta_A} (1-\theta) L^{\eta_L}$ with $\eta_L := \eta_L^p + \eta_L^e$, $\eta_L^p = 1$, $\eta_L^e < 0$ and $0 < \eta_A, \eta_L < 1$; notice that $B_J = 1$ since $\eta_K^p = 0$. The balanced growth rates are given by $\hat{K} - n = \hat{A} = \eta_L n / (1-\eta_A)$, which represent a special case of the general balanced growth rates (26) and (27).

Finally, we consider the decentralised solution of the hybrid non-scale model of R&D-based growth introduced by Eicher and Turnovsky (1999, Section 3.1), who provided the social solution for this model. This model is characterised by the fact that each input factor is productive in each sector and that the FO

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[10] The distinction between $\bar{\bar{J}}(.)$, $\bar{J}(.)$ and $J(.)$ vanishes since (i) CG are not productive in R&D and (ii) there are no externalities associated with the private input factor (labour) in R&D.

technology exhibits constant returns to scale in physical capital and labour. This model follows from $Y = \bar{F}(.) = (\theta L)^{\sigma_L} B_Y$ and $B_Y = \int_0^A \left[\phi(i)x(i)\right]^{\sigma_K} di$ (i.e. $\mu = 1$). The FO technology in terms of aggregate capital is then given by $Y = F(.) = A^{1-\sigma_K} (\theta L)^{\sigma_L} (\phi K)^{\sigma_K} q^{-\sigma_K}$ with $\sigma_L + \sigma_K = 1$. The R&D technology is $\dot{A} = \bar{\bar{J}}(.) = A^{\eta_A^{SO}} \left[(1-\theta) L\right]^{\eta_L} B_J$ with $\eta_L := \eta_L^p + \eta_L^e$, $0 < \eta_L^p < 1$, $\eta_L^e < 0$ and $0 < \eta_A^{SO}, \eta_L < 1$. The index B_J is given by $B_J = \int_0^A \left[(1-\phi(i))x(i)\right]^{\eta_K} di$ with $\eta_K := \eta_K^p + \eta_K^e$, $0 < \eta_K^p < 1$ and $\eta_K^e < 0$. Constant returns to scale in private inputs requires $\eta_L^p + \eta_K^p = 1$. In terms of aggregate capital, the R&D technology reads $\dot{A} = J(.) = A^{\eta_A^{SO}+1-\eta_K} \left[(1-\theta) L\right]^{\eta_L} \left[(1-\phi)K\right]^{\eta_K} q^{-\eta_K}$. In this case, the balanced growth rates are given by $\hat{K} - n = \hat{A} = (\eta_L + \eta_K)n/(1-\eta_A-\eta_K)$, where $\eta_A := \eta_A^{SO} + 1 - \eta_K$.[11]

5. An illustrative application

The hybrid non-scale model of R&D-based growth discussed above is employed to investigate two important issues. First, it has been argued above that the level of the balanced growth path probably differs between the market economy and the centralised solution. This difference may imply a dramatic loss in welfare. It would, therefore, be desirable to get an idea about the magnitude of this gap. Second, the importance of transitional dynamics vis-à-vis balanced growth dynamics is assessed by determining the (local) rate of convergence. This is important since the positive implications (scale effects) as well as the normative implications (due to divergence or coincidence of the growth rate) differ substantially along the transition path compared to the balanced growth path. As a consequence of the complexity of the model under study, the questions raised above must be analysed numerically.

The production side of the model can be expressed as follows (the production functions are stated in terms of aggregate capital):

$$Y = F(.) = A^{\sigma_A} (\theta L)^{\sigma_L} (\phi K)^{\sigma_K} q^{-\sigma_K} \tag{28}$$

with $\sigma_A, \sigma_L, \sigma_K > 0$; $\sigma_L + \sigma_K = 1$

$$\dot{A} = J(.) = A^{\eta_A} [(1-\theta) L]^{\eta_L} [(1-\phi) K]^{\eta_K} q^{-\eta_K} \tag{29}$$

[11] The fact that the balanced growth rates are exclusively determined by structural characteristics of the R&D technology (together with the population growth rate) is due to constant returns to scale in FO, i.e. $\sigma_L + \sigma_K = 1$.

with $\eta_L := \eta_L^p + \eta_L^e$, $\eta_K := \eta_K^p + \eta_K^e$, $\eta_A, \eta_L^p, \eta_K^p > 0$, $\eta_L^e, \eta_K^e < 0$, $0 < \eta_L, \eta_K < 1$ and $\eta_L^p + \eta_K^p = 1$.

To answer the first question raised above, we perform an adjustment of scale to obtain a convenient expression for the balanced growth path. To illustrate this procedure, consider a variable $X(t)$ which grows in the long run at constant rate g, i.e. $\lim_{t\to\infty} \dot{X}(t)/X(t) = g$. By defining a new variable $x(t)$, we can then perform an adjustment of scale yielding the scale-adjusted (or normalised) variable $x(t) := X(t)/e^{gt}$. By construction, $x(t)$ converges to its stationary value denoted as $\tilde{x}$ [and defined by $\dot{x}(t) = 0$] as time approaches infinity, i.e. $\lim_{t\to\infty} x(t) = \tilde{x}$. Using the definition above, the growth path of $X(t)$ can be expressed as $X(t) = x(t)\, e^{gt}$, while the balanced growth path is given by $\tilde{X}(t) = \tilde{x}\, e^{gt}$. The preceding expression for the balanced growth path is clearly instructive because it demonstrates that changes in $\tilde{x}$ affect the level of the balanced growth path. These changes might result from macroeconomic shocks, market distortions or policy measures.

Let us return to the hybrid non-scale growth model. Facing the balanced growth rates $\hat{K} = \hat{Y} = \hat{C} = \beta_K n$ and $\hat{A} = \beta_A n$ we choose the following scale-adjusted variables $y := Y/L^{\beta_K}$, $k := K/L^{\beta_K}$, $c := C/L^{\beta_K}$, $a := A/L^{\beta_A}$, $j := J/L^{\beta_A}$, $v_a := v/L^{\beta_K - \beta_A}$.[12] Furthermore, from (28) and (29) we can derive the production functions in scale-adjusted variables to read $y = \alpha_F a^{\sigma_A} \theta^{\sigma_L} (\phi k)^{\sigma_K}$ and $j = \alpha_J a^{\eta_A} (1-\theta)^{\eta_L} [(1-\phi)k]^{\eta_L}$.

The dynamic system in scale-adjusted variables possesses a stationary solution denoted by $\{\tilde{k}, \tilde{a}, \tilde{c}, \tilde{v}_a, \tilde{\theta}, \tilde{\phi}\}$. Scale-adjusted output can then readily be calculated from $\tilde{y} = \alpha_F\, \tilde{a}^{\sigma_A} \tilde{\theta}^{\sigma_L} (\tilde{\phi}\tilde{k})^{\sigma_K}$. Subsequently, the balanced growth path in terms of output can be expressed as $\tilde{Y}(t) = \tilde{y}\, e^{\beta_K n t}$. Unfortunately, due to the complexity of the underlying dynamic system, the stationary solution cannot be

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[12] $\beta_K := 1 - \eta_A + \eta_L/(1-\eta_A-\eta_K)$ and $\beta_A := \eta_L + \eta_K/(1-\eta_A-\eta_K)$. The scale-adjusted price ($v_a := v/L^{\beta_K - \beta_A}$) results from the following consideration: The growth rate of v may be expressed as $\hat{v} = (\varepsilon-1)\sigma_K Y q/(\varepsilon\phi K) - \delta - \sigma_K Y/\varepsilon A \phi K v$ (simplified version due to $\sigma_K = \eta_K^p$). Along the balanced growth path, the first term on the RHS is constant. Hence, v must grow at the rate $\hat{v} = \hat{Y} - \hat{A} = (\beta_K - \beta_A)n$ along the balanced growth path.

determined analytically. It can, nonetheless, be determined numerically using a benchmark set of parameters.

Table 2 shows the baseline set of parameters which underlies the numerical investigations. This set of parameters is very similar to those used in previous exercises (e.g. Ortigueira and Santos, 1997; Jones and Williams, 2000; Eicher and Turnovsky, 2001).

Table 2: Baseline set of parameters

FO technology (CG technology)*	$\sigma_A = 0.3$; $\sigma_L = 0.64$; $\sigma_K = 0.36$; $\delta = 0.04$
R&D technology	$\eta_A = 0.5$ $\eta_L^p = 0.64$; $\eta_L^e = -d\,\eta_L^p$ $\eta_K^p = 0.36$; $\eta_K^e = -d\,\eta_K^p$; $d = 0.2$
Preferences and population growth	$\rho = 0.05$; $\gamma = 1$; $n = 0.015$

* Note: FO: final output; CG: capital goods; R&D: research and development

Following Eicher and Turnovsky (2001, p. 100) both sectors are characterised by mildly increasing returns to scale: $\sigma_A + \sigma_L + \sigma_K = 1.20$ and $\eta_A + \eta_L + \eta_K = 1.24$. Moreover, Jones and Williams (2000, p. 74) argue that the social elasticity of the private inputs in R&D ($\eta_L + \eta_K$) should lie within the range of 0.5 and 1; the baseline set of parameters implies $\eta_L + \eta_K = 0.7$. The duplication externality parameter is set equal to $d = 0.2$.[13] This large value was chosen to additionally account for creative destruction effects.

The resulting balanced growth rate of FO is $\hat{Y} \cong 0.030$. The implied growth rate of total factor productivity (TFP) amounts to $\sigma_A\,\hat{A} \cong 0.010$. These values are in line with the empirical picture on growth in industrialised countries. For instance, the average growth rate of output in the U.S.A. from for the period 1945 until 2000 amounts to 3.1 %. Available evidence on the growth of TFP in the U.S.A. from 1947 until 1998 yields values of about 1.25 % (Jones and Williams, 2000, p. 73).[14]

[13] The duplication externalities η_L^e and η_K^e are defined as $\eta_L^e = -d\eta_L^p$ and $\eta_K^e = -d\eta_K^p$.

[14] Empirical TFP growth estimates are usually biased upwards due to efficiency changes and economies of scale. The empirical picture for the G7 from 1980 until 2000 is very similar. The growth rate of output amounts to 2.7 %, while the growth rate of TFP is about 0.9 % (Colecchia and Schreyer, 2002).

Table 3: Market economy: Growth rates and key economic ratios

$\hat{Y}$	$\sigma_A \hat{A}$ *	$\widetilde{Y/K}$	$\widetilde{C/Y}$	$\tilde{\theta}$	$\tilde{\phi}$
0.030	0.010	0.84	0.92	0.91	0.91

* Note: $\sigma_A \hat{A}$ equals the growth rate of TFP

By applying the same procedure as described above to the social solution (which has been provided by Eicher and Turnovsky, 1999), we can calculate the socially optimal level of the balanced growth path. At this stage, the gap between the socially optimal and the decentralised balanced growth path can be expressed by the ratio $\tilde{y}_S / \tilde{y}_M$. Based on the benchmark set of parameters, we obtain $\tilde{y}_S / \tilde{y}_M = 3.68$. Put differently, the socially optimal balanced growth path exceeds the decentralised balanced growth path by 268 %. This result indicates a huge loss in welfare due to the imperfections inherent in the market economy.

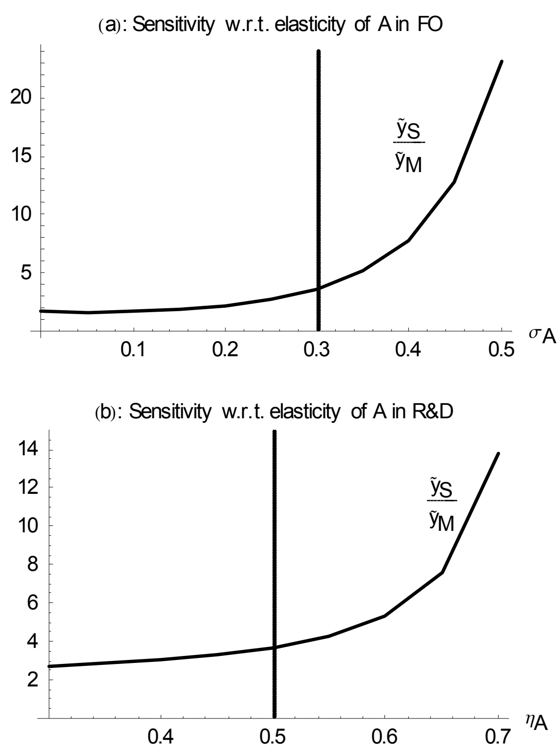

Figure 1: Relative balanced growth paths ($\tilde{y}_S / \tilde{y}_M$) in response to technology parameters

This tremendous gap is due to the imperfections inherent in the market economy, which comprise monopolistic competition in the CG sector, positive external effects associated with technological knowledge and negative externalities associated with the economy-wide averages of private inputs in R&D. As a result, the two fundamental allocation variables (investment rate in physical capital and R&D investment share) are biased. For standard model calibrations the market economy is found to slightly underinvest in R&D but at the same time heavily underinvests in physical capital accumulation; for details the reader is referred to Steger (2005).

Figure 1 provides a concise sensitivity analysis of the preceding result by varying the technology parameters σ_A and η_A, which are especially difficult to determine empirically. Notice that the vertical line represents the respective parameter value within the baseline set of parameters. It can be recognised that the result $\tilde{y}_S / \tilde{y}_M = 3.68$ is locally robust with respect to parameter variations. On the other hand, if the technology parameters under consideration increase substantially, then $\tilde{y}_S / \tilde{y}_M$ grows strongly.

We now turn to the stability properties and the speed of convergence of the hybrid non-scale R&D-based growth model. A rigorous stability proof for this general R&D-based non-scale model stands out.[15] Due to the complexity of the underlying model, this stability proof appears rather difficult. Therefore, we determine the stability properties for the underlying benchmark set of parameters numerically.

Figure 2 shows the characteristic polynomial of the Jacobian matrix evaluated at the respective stationary solutions. The characteristic polynomial is given by $|M - I\lambda| = 0$, where M denotes the Jacobian matrix, I the identity matrix and λ the vector of eigenvalues, respectively. The solid line applies to the market economy, whereas the dotted line applies to the social economy. In both cases, there are two (real-valued) positive and two (real-valued) negative eigenvalues. Since the dynamic model (18) to (23) possesses two jump variables (c and v_a) and two predetermined state variables (k and a), the dynamic system under consideration is saddle-point stable.

A further interesting aspect contained in Figure 2 concerns the speed of convergence. Let $\lambda_2 < \lambda_1 < 0$ denote the stable eigenvalues. The asymptotic speed of convergence is then given by λ_1, the smaller (in absolute terms) of the stable eigenvalues; which is labelled dominant eigenvalue. Figure 2 indicates that λ_1 resulting from the decentralised solution is nearly identical to λ_1 resulting from

[15] The determination of the dynamic stability properties is important to understand the dynamic properties of the model.

the centralised solution. This implies that both types of solution converge asymptotically at similar rates of convergence.[16]

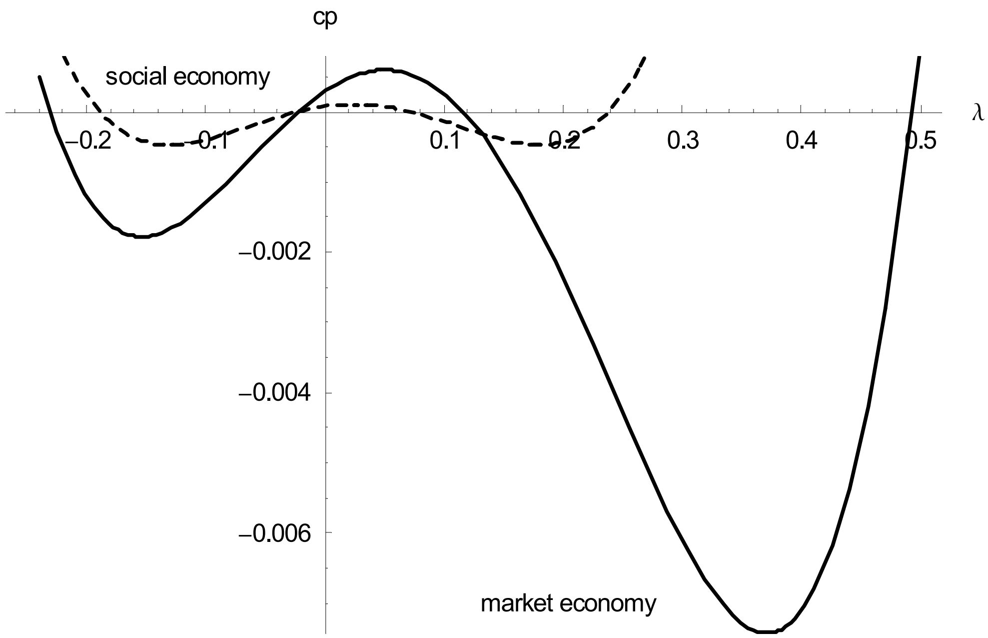

Figure 2: Characteristic polynomial (cp) of Jacobian matrix evaluated at stationary solution (market economy: solid line; socially controlled economy: dotted line)

The dominant eigenvalue of the social solution is $\lambda_1 = -0.024$, whereas the dominant eigenvalue of the market solution is $\lambda_1 = -0.022$. These values are in line with the empirical evidence on the speed of convergence (e.g. Barro and Sala-i-Martin, 1992). Moreover, these values indicate that transitional dynamics are important. This becomes evident by noting that a value of $\lambda_1 = -0.022$ is associated with a half life of around 31 years.

6. Summary and conclusion

The market solution for the class of R&D-based endogenous growth models of the increasing-variety type has been developed. The underlying growth model is general in the sense that technologies are not restricted to be of specific functional form unless this is required for a consistent microeconomic structure and that each factor of production is allowed to be productive in each sector. Since the general

[16] This observation is consistent with the fact that both types of solution grow at the same rate along the balanced growth rate.

R&D-based endogenous growth model developed in this paper encompasses a large number of specific models, it can be viewed as a summary of this strand of the literature.

The model developed in this paper is similar to the lab-equipment approach of Rivera-Batiz and Romer (1991) in that each input factor is allowed to be productive in each sector. However, the lab-equipment model is based on the simplifying assumption according to which the technologies used in final-output production and R&D must be identical. This assumption represents a severe restriction when the model is calibrated and applied to empirical data. The model developed here is more flexible since the production technologies between the final-output sector and the R&D sector are allowed to differ.

The above discussion yields the complete dynamic system in general form, which governs (together with appropriate endpoint conditions) the evolution of the market economy. This dynamic system is of independent interest since it serves as the basis for a number of important investigations including the analysis of the level of the balanced growth path as well as the analysis of transitional dynamics.

It is demonstrated that long-run growth crucially depends on the structural characteristics of the production side of the economy. Moreover, the balanced growth rates are proportional to the growth rate of the exogenously growing factor (labour). The balanced growth path is characterised by non-scale growth. Despite the fact that the balanced growth rates of the decentralised and the centralised solution coincide, the level of the balanced growth path differs between the two types of solution. It is shown numerically that the market economy and the socially controlled economy may grow at substantially different levels along the balanced growth path. This divergence causes a huge loss in welfare due to the market imperfections inherent in the decentralised economy.

As noted above, the social solution of the general R&D-based model has been analysed by Eicher and Turnovsky (1999). The authors derive a number of important results with respect to non-scale balanced growth. By developing the decentralised solution, this paper demonstrates that their general R&D-based model indeed possesses a sound microeconomic interpretation. This is far from being a trivial conclusion since general equilibrium requires a consistent interplay between production technology, market structure and profit maximisation.

Finally, the paper points to a number of interesting questions for future research. First, it would be clearly instructive to derive the stability conditions within this generalised set-up to fully understand the transitional dynamics implications. Second, the dynamic system developed in this paper can be used to investigate the qualitative and quantitative implications along the transition to the balanced growth path. Subsequently, the transitional processes of the market economy can be compared to those of the socially controlled economy.

Colophon

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7. Appendix

7.1. The household's optimisation problem

The dynamic problem together with its solution is summarised by the following set of equations; the Hamiltonian is given in current-value form.

$$\max_{\{C/L\}} \int_0^\infty \frac{(C/L)^{1-\gamma}-1}{1-\gamma} e^{-\rho t}\, dt \tag{A.1}$$

$$s.t.\ \ \dot{K}=(r-\delta)K+wL+A\pi-v\dot{A}-C,\ K(0)>0,\ A(0)>0 \tag{A.2}$$

$$H(C/L,K,\lambda):=\frac{(C/L)^{1-\gamma}-1}{1-\gamma}+\lambda\left[(r-\delta)K+wL+A\pi-v\dot{A}-C\right] \tag{A.3}$$

$$H_{C/L}=(C/L)^{-\gamma}-\lambda L=0 \quad \Leftrightarrow \quad C^{-\gamma}=\lambda L^{1-\gamma} \tag{A.4}$$

$$\dot{\lambda}=-H_K+\rho\lambda=-\lambda(r-\delta)+\rho\lambda \quad \Leftrightarrow \quad \frac{\dot{\lambda}}{\lambda}=\rho-(r-\delta) \tag{A.5}$$

$$\frac{\dot{C}}{C}=\frac{1}{\gamma}\left[r-\delta-\rho-(1-\gamma)n\right] \tag{A.6}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t)=0, \tag{A.7}$$

where π is given by (7), r given by (8), $w=F_{\theta L}(.)$, v is given by (21) together with $v(0)>0$, $\dot{A}$ by (19), A by (19) together with $A(0)>0$ and L by $\dot{L}=nL$ together with $L(0)>0$. Provided that the Hamiltonian is jointly concave in the control and the state variable (Mangasarian sufficiency conditions) or that the maximised Hamiltonian is concave in the state variable (Arrow sufficiency conditions), the necessary conditions are also sufficient.[17] The transversality condition demands for the following inequality constraint to be met $-\rho + \lim_{t\to\infty} \hat{\lambda} + \lim_{t\to\infty} \hat{K} < 0$.

7.2. A general R&D-based growth model: the social solution

7.2.1. Dynamic problem, first-order conditions and dynamic system

The social solution for the class of models under study is derived using a general formulation (see Eicher and Turnovsky, 1999). The social planner's problem may be expressed as follows.

$$\max_{\{C/L,\,\theta,\,\phi\}} \int_0^\infty \frac{(C/L)^{1-\gamma}-1}{1-\gamma} e^{-\rho t}\, dt \tag{A.8}$$

$$s.t.\;\; Y = F(A,\, \theta L,\, \phi K) \tag{A.9}$$

$$\dot{K} = Y - C - \delta K \tag{A.10}$$

$$\dot{A} = J\left[A,\, (1-\theta)L,\, (1-\phi)K\right] \tag{A.11}$$

$$K(0) > 0\,,\;\; A(0) > 0 \tag{A.12}$$

The current-value Hamiltonian together with the (necessary) optimality conditions are displayed below. The costate variables of capital and technology are denoted by μ_K and μ_A, respectively.

$$H\left(C/L,\, \theta,\, \phi,\, K,\, A,\, \mu_K,\, \mu_A\right) := \frac{(C/L)^{1-\gamma}-1}{1-\gamma} + \mu_K\left[F(.) - C - \delta K\right] + \mu_A J(.) \tag{A.13}$$

$$H_{C/L} = (C/L)^{-\gamma} - \mu_K L = 0 \;\;\Leftrightarrow\;\; C^{-\gamma} = \mu_K L^{1-\gamma} \quad \text{(maximum principle 1)} \tag{A.14}$$

$$H_\theta = \mu_K F_\theta(.) + \mu_A J_\theta(.) = \mu_K F_\theta(.) - \mu_A J_{1-\theta}(.) = 0 \quad \text{(maximum principle 2)} \tag{A.15}$$

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[17] For details on sufficiency conditions within optimal control theory see Kamien and Schwartz (1981, part II section 3 and section 15).

$$\Leftrightarrow \quad \mu_K F_\theta(.) = \mu_A J_{1-\theta}(.)$$

$$H_\phi = \mu_K F_\phi(.) + \mu_A J_\phi(.) = \mu_K F_\phi(.) - \mu_A J_{1-\phi}(.) = 0 \qquad \text{(maximum principle 3)} \qquad \text{(A.16)}$$

$$\Leftrightarrow \quad \mu_K F_\phi(.) = \mu_A J_{1-\phi}(.)$$

$$\dot{\mu}_K = -H_K + \rho\,\mu_K = -\left[\mu_K F_K(.) - \mu_K \delta_K + \mu_A J_K(.)\right] + \rho\,\mu_K \qquad \text{(costate equation 1)} \qquad \text{(A.17)}$$

$$\Leftrightarrow \quad \frac{\dot{\mu}_K}{\mu_K} = \rho + \delta_K - F_K(.) - \frac{\mu_A}{\mu_K} J_K(.)$$

$$\dot{\mu}_A = -H_A + \rho\,\mu_A = -\left[\mu_K F_A(.) + \mu_A J_A(.)\right] + \rho\,\mu_A \qquad \text{(costate equation 2)} \qquad \text{(A.18)}$$

$$\Leftrightarrow \quad \frac{\dot{\mu}_A}{\mu_A} = \rho - J_A(.) - \frac{\mu_K}{\mu_A} F_A(.)$$

$$\dot{K} = H_{\mu_K} = F(.) - C - \delta K \qquad \text{(state equation 1)} \qquad \text{(A.19)}$$

$$\dot{A} = H_{\mu_A} = J(.) \qquad \text{(state equation 2)} \qquad \text{(A.20)}$$

$$\lim_{t\to\infty} e^{-\rho t} K\,\mu_K = 0\,,\ \lim_{t\to\infty} e^{-\rho t} A\,\mu_A = 0 \qquad \text{(transvers. conditions)} \qquad \text{(A.21)}$$

With respect to (A.15) and (A.16) it should be noted that $F_\theta(.) = -F_{1-\theta}(.)$ and $J_\theta(.) = -J_{1-\theta}(.)$. From (A.14) together with (A.17) one can easily derive the differential equation in C: $\dot{C} = \frac{C}{\gamma}\left[F_K(.) - \delta - \rho - (1-\gamma)n + \frac{\mu_A}{\mu_K} J_K(.)\right]$; the last term in brackets, $\frac{\mu_A}{\mu_K} J_K(.)$, is the marginal product of capital in R&D multiplied by $1-\phi$ [i.e. $J_K(.) = J_{(1-\phi)K}(.)(1-\phi)$], in units of the final output good. The dynamic system can be summarised as follows.

$$\dot{K} = F(A, \theta L, \phi K) - \delta K - C \qquad \text{(A.22)}$$

$$\dot{A} = J\left[A, (1-\theta)L, (1-\phi)K\right] \qquad \text{(A.23)}$$

$$\dot{C} = \frac{C}{\gamma}\left[F_K(.) - \delta - \rho - (1-\gamma)n + \frac{\mu_A}{\mu_K} J_K(.)\right] \qquad \text{(A.24)}$$

$$\frac{\dot{\mu}_K}{\mu_K} = \rho + \delta - F_K(.) - \frac{\mu_A}{\mu_K} J_K(.) \tag{A.25}$$

$$\frac{\dot{\mu}_A}{\mu_A} = \rho - J_A(.) - \frac{\mu_K}{\mu_A} F_A(.) \tag{A.26}$$

$$\mu_K \, F_\theta(.) = \mu_A \, J_{1-\theta}(.) \tag{A.27}$$

$$\mu_K \, F_\phi(.) = \mu_A \, J_{1-\phi}(.) \tag{A.28}$$

7.2.2. Dynamic system in scale-adjusted variables

Provided that $n > 0$ the balanced growth rates are given by $\hat{Y} = \hat{K} = \hat{C} = \beta_K \, n$ and $\hat{A} = \hat{J} = \beta_A \, n$. The appropriate scale adjustments read as follows $y := Y / L^{\beta_K}$, $k := K / L^{\beta_K}$, $c := C / L^{\beta_K}$, $a := A / L^{\beta_A}$ and $j := J / L^{\beta_A}$. Furthermore, from (A.26) it follows that $\frac{\dot{\mu}_A}{\mu_A} = \rho - \eta_A \frac{J}{A} - \frac{\mu_K}{\mu_A} \sigma_A \frac{Y}{A}$. Along a balanced growth path, $\hat{\mu}_A$ must be constant. The second term on the RHS is a linear transform of $\hat{A}$ and, hence, constant along a balanced growth path. Accordingly, the last term on the RHS must be constant either, implying that $\hat{\mu}_K - \hat{\mu}_A = n(\beta_A - \beta_K) = \hat{A} - \hat{K}$. We can reduce the order of the system under study by taking the ratio of the two costate variables. The appropriate scale-adjustment for this ratio is given by $s := \frac{\mu_K}{\mu_A \, L^{\beta_A - \beta_K}}$. Differentiating this definition with respect to time yields $\hat{s} = \hat{\mu}_K - \hat{\mu}_A - n\left(\beta_A - \beta_K\right)$. The next step is to derive expressions for $\hat{\mu}_K$ and $\hat{\mu}_A$ in terms of scale-adjusted variables and insert these into the preceding equation. Taking the efficiency condition $s \frac{\sigma_K \, y}{\phi} = \frac{\eta_K \, j}{1-\phi}$ into account and noting that $\frac{\mu_A}{\mu_K} = L^{\beta_K - \beta_A} / s$, $\hat{\mu}_K$ can be expressed to read $\hat{\mu}_K = \rho + \delta_K - \frac{\sigma_K y}{k}\left(1 - \frac{1-\phi}{\phi}\right)$. Similarly, by noting that $s \frac{\sigma_L \, y}{\theta} = \frac{\eta_L \, j}{1-\theta}$ and $\frac{\mu_K}{\mu_A} = s \, L^{\beta_A - \beta_K}$, $\hat{\mu}_A$ can be written as $\hat{\mu}_A = \rho - \frac{j}{a}\left(\eta_A + \frac{\sigma_A \, \eta_L}{\sigma_L} \frac{\theta}{1-\theta}\right)$. The system in scale-adjusted variables may then be expressed as follows.

$$\dot{k} = y - c - \delta k - \beta_K n k \tag{A.29}$$

$$\dot{a} = j - \beta_A n a \tag{A.30}$$

$$\dot{c} = \frac{c}{\gamma}\left[\frac{\sigma_K y}{k} - \delta - \rho - (1-\gamma) n + \frac{\eta_K j}{s k}\right] - \beta_K n c \tag{A.31}$$

$$\dot{s} = s\left[\frac{j}{a}\left(\eta_A + \frac{\sigma_A \eta_L}{\sigma_L}\frac{\theta}{1-\theta}\right) - \frac{\sigma_K y}{k}\left(1 - \frac{1-\phi}{\phi}\right) - n(\beta_A - \beta_K) + \delta\right] \tag{A.32}$$

$$s\frac{\sigma_L y}{\theta} = \frac{\eta_L j}{1-\theta} \tag{A.33}$$

$$s\frac{\sigma_K y}{\phi} = \frac{\eta_K j}{1-\phi} \tag{A.34}$$

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