ON SARMANOV MIXED ERLANG RISKS IN INSURANCE APPLICATIONS

BY

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ABSTRACT

In this paper we consider an extension to the aggregation of the FGM mixed Erlang risks, proposed by Cossette *et al.* (2013 *Insurance: Mathematics and Economics*, **52,** 560–572), in which we introduce the Sarmanov distribution to model the dependence structure. For our framework, we demonstrate that the aggregated risk belongs to the class of Erlang mixtures. Following results from S. C. K. Lee and X. S. Lin (2010 *North American Actuarial Journal*, **14**(1) 107–130), G. E. Willmot and X. S. Lin (2011 *Applied Stochastic Models in Business and Industry*, **27**(1) 8–22), analytical expressions of the contribution of each individual risk to the economic capital for the entire portfolio are derived under both the TVaR and the covariance capital allocation principle. By analysing the commonly used dependence measures, we also show that the dependence structure is wide and flexible. Numerical examples and simulation studies illustrate the tractability of our approach.

KEYWORDS

Risk aggregation, Sarmanov distribution, mixed Erlang distribution, dependence measures, capital allocation.

1. INTRODUCTION

Analysis of aggregated risk is important for insurance business, it allows the insurers to assess and to monitor their risks through the risk management framework. In the classical framework of independent and identically distributed risks, explicit analytical formulas for quantities of interest including Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) or Stop-loss premium formula for the aggregated risk can be derived explicitly for few tractable cases. For instance Willmot and Lin (2011), Lee and Lin (2010, 2012) and Cossette *et al.* (2013) have shown that this is the case if we choose the mixed Erlang distribution as a model for claim sizes. One reason for the tractability of the mixed Erlang distribution is the fact that the convolution of such risks is again mixed Erlang, see Klugman *et al.* (2008).

Since insurance data clearly shows that insurance risks are commonly dependent, in order to be able to get closed-form formulas for quantities of interest, an important task is the adequate choice of the dependence structure between the risks. Even for the simple case of the dependence specified by a log-normal framework with stochastic volatility, as shown in the recent contributions (Embrechts *et al.*, 2014; Hashorva and Kortschak, 2014; Hashorva and Li, 2015) only asymptotic results can be derived.

With motivation from Cossette *et al.* (2013) where the aggregation of FGM mixed Erlang risks is considered, in this contribution we shall investigate the Sarmanov mixed Erlang risks. The Sarmanov distribution includes the FGM distribution as a special case. One key advantage of the Sarmanov distribution is its flexibility; it also allows to model highly dependent risks, see e.g., Bairamov *et al.* (2001) and Lee (1996). The aim of this paper is to provide analytical results and properties of the aggregated dependent risks with mixed Erlang marginals by using the Sarmanov distribution as a model for the dependence structure. This model is promising in risk aggregation practice as it satisfies the four desirable properties of a multivariate parametric model mentioned in Joe (1997) p. 84, namely the interpretability property, the closure property, the flexibility and the wideness of the range of dependence, and the representation of the distribution function (df) and the probability density function (pdf) in analytical form.

The paper is organised as follows. In Section 2, we describe the background of the Sarmanov mixed Erlang distribution by exploring some definitions and properties of the Sarmanov distribution as a model for the dependence structure and the mixed Erlang distribution with a common scale parameter as a model for claim size distribution in insurance. In Section 3, we demonstrate that the distribution of the aggregated risk belongs to the class of Erlang mixtures; numerical illustrations and simulation studies are performed to show the robustness of the results. In Section 4, we derive explicit expressions for the allocated capital to each individual risk $X_i, i = 1, 2$ under the TVaR and the covariance capital allocation rules. We present some useful results and properties of the mixed Erlang distribution in Section 5. In Section 6, an extension of the results in the bivariate case to the multivariate framework is presented with numerical examples. All the proofs are relegated to Section 7. In the Appendix, the flexibility and the wideness of the dependence range of Sarmanov mixed Erlang distributions are discussed by calculating commonly used dependence measures, namely Pearson's correlation coefficient, Sperman's rho and Kendall's tau.

2. PRELIMINARIES

2.1. Sarmanov distribution

The Sarmanov distribution introduced in Sarmanov (1966) has proved valuable in numerous insurance applications. For instance Hernández-Bastida and Fernández-Sánchez (2012) used the multivariate Sarmanov distribution to model the dependence structure between risk profiles for the calculation of Bayes premiums in the collective risk model. The contribution (Sarabia and Gómez-Déniz, 2011) fitted multivariate insurance count data using the Sarmanov distribution with Poisson-Beta marginals. As shown in Yang and Hashorva (2013) and Yang and Wang (2013), the Sarmanov distribution allows for tractable asymptotic formulas in the context of ruin probabilities. Referring to Sarmanov (1966), a bivariate risk (X_1, X_2) has the Sarmanov distribution with joint pdf h given by

$$h(x_1, x_2) = f_1(x_1) f_2(x_2)(1 + \alpha_{12}\phi_1(x_1)\phi_2(x_2)), \quad \alpha_{12} \in \mathbb{R}, \tag{2.1}$$

where f_i is the pdf of X_i, $i = 1, 2$, and ϕ_1, ϕ_2 are two kernel functions, which are assumed to be bounded and non-constant such that

$$\mathbb{E}\left(\phi_1(X_1)\right) = \mathbb{E}\left(\phi_2(X_2)\right) = 0, \quad 1 + \alpha_{12}\phi_1(x_1)\phi_2(x_2) \geqslant 0 \tag{2.2}$$

is valid. If $\phi_i(x_i) = 1 - 2F_i(x_i)$ with F_i the df of X_i, then h is the joint pdf of the FGM distribution introduced by Morgenstern (1956) for Cauchy marginals and developed by Gumbel (1960) for exponential margins and generalized by Farlie (1960). Lee (1996) proposed some general methods for finding the kernel function $\phi_i(x_i)$ with different types of marginals. Yang and Hashorva (2013) considered $\phi_i(x_i) = g_i(x_i) - \mathbb{E}(g_i(X_i))$. When $g_i(x_i) = e^{-x_i}$ the corresponding kernel function coincides with the one explored by Lee (1996) for marginal distributions with support in $[0, \infty)$. We have

$$\phi_i(x_i) = e^{-x_i} - \mathbb{E}\left(e^{-X_i}\right) = e^{-x_i} - \mathcal{L}_i(1), \tag{2.3}$$

where $\mathcal{L}_i(t) = \mathbb{E}\left(e^{-tX_i}\right)$, $t > 0$ is the Laplace transform of X_i. In the rest of the paper, we set

$$\mathcal{L}_i := \mathcal{L}_i(1), \quad \mathcal{L}_i^{'} := \mathcal{L}_i^{'}(1).$$

The joint pdf h is thus given by

$$\begin{aligned} h(x_1, x_2) &= f_1(x_1) f_2(x_2)((1+\gamma) + \alpha_{12}(e^{-x_1-x_2} - e^{-x_1}\mathcal{L}_2 - e^{-x_2}\mathcal{L}_1)), \\ \gamma &= \alpha_{12}\mathcal{L}_1\mathcal{L}_2. \end{aligned} \tag{2.4}$$

Remarks 2.1. *If* (X_1, X_2) *has a Sarmanov distribution with kernel functions given in* (2.3)*, additionally if* X_i, $i = 1, 2$ *follows a mixture of Gamma distributions where the mixture components share the same scale parameter* $\beta_i \in (0, \infty)$*, then the joint df of* (X_1, X_2) *follows easily from integrating the pdf in* (2.4)*. Specifically, we have for* H *the joint df of* (X_1, X_2)

$$\begin{aligned} H(x_1, x_2) = (1+\gamma) F_1(x_1, \beta_1) F_2(x_2, \beta_2) + \gamma F_1(x_1, \beta_1 + 1) F_2(x_2, \beta_2 + 1) \\ - \gamma F_1(x_1, \beta_1 + 1) F_2(x_2, \beta_2) - \gamma F_1(x_1, \beta_1) F_2(x_2, \beta_2 + 1), \end{aligned} \tag{2.5}$$

where $F_i(x_i, \beta_i) = \sum_{k=1}^{\infty} q_k W_k(x_i, \beta_i)$, $i = 1, 2$ *with* $W_k(x_i, \beta_i)$ *is the df of the Gamma distribution with scale parameter* β_i *and shape parameter* $k \in (0, \infty)$ *and* q_k *is the mixing weight such that* $\sum_{k=1}^{\infty} q_k = 1$.

Compared to the FGM distribution which has $[-1/3, 1/3]$ as the range of Pearson's correlation coefficient ρ_{12} the Sarmanov distribution has a wider range of ρ_{12}, which is useful in the aggregation of strongly dependent insurance risks. For the Sarmanov case we have the explicit formula for ρ_{12}, namely

$$\rho_{12}(X_1, X_2) = \frac{\alpha_{12}\nu_1\nu_2}{\sigma_1\sigma_2}, \quad \nu_i = \mathbb{E}\left(X_i\phi_i(X_i)\right), \quad \sigma_i = \sqrt{Var(X_i)}, \quad i = 1, 2. \quad (2.6)$$

In the particular case that the kernels are given by (2.3), for two positive Sarmanov risks with finite variances the range of α_{12} is (see Lee, 1996)

$$\frac{-1}{\max\{\mathcal{L}_1\mathcal{L}_2, (1-\mathcal{L}_1)(1-\mathcal{L}_2)\}} \leqslant \alpha_{12} \leqslant \frac{1}{\max\{\mathcal{L}_1(1-\mathcal{L}_2), (1-\mathcal{L}_1)\mathcal{L}_2\}}, \quad (2.7)$$

where $\nu_i = -\mathcal{L}_i' - \mathcal{L}_i\mu_i$ and $\mu_i = \mathbb{E}\left(X_i\right), i = 1, 2$. Lee (1996) extended the Sarmanov distribution to the multivariate case by defining the joint pdf h of $(X_1, \ldots, X_n)$ as

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)(1 + R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x})), \qquad \boldsymbol{x} := (x_1, \ldots, x_n), \qquad (2.8)$$

where

$$\begin{aligned} R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x}) = 1 &+ \sum_{j_1<}^{n-1}\sum_{j_2}^{n} \alpha_{j_1,j_2}\phi_{j_1}(x_{j_1})\phi_{j_2}(x_{j_2}) \\ &+ \sum_{j_1<}^{n-2}\sum_{j_2<}^{n-1}\sum_{j_3}^{n} \alpha_{j_1,j_2,j_3}\phi_{j_1}(x_{j_1})\phi_{j_2}(x_{j_2})\phi_{j_3}(x_{j_3}) + \cdots + \alpha_{1,2,\ldots,n} \\ &\times \prod_{i=1}^{n}\phi_i(x_i), \end{aligned}$$

such that

$$1 + R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x}) \geqslant 0 \qquad (2.9)$$

is fulfilled for all $x_i \in \mathbb{R}$ with $\Omega_n = \{\alpha_{j_1,j_2}, \alpha_{j_1,j_2,j_3}, \ldots, \alpha_{1,2,\ldots,n}\} \in \mathbb{R}$. If the kernel functions are specified by (2.3), then h is given by (set $\Delta(x_i) := e^{-x_i} - \mathcal{L}_i$)

$$\begin{aligned} h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)\Bigg(1 &+ \sum_{j_1<}^{n-1}\sum_{j_2}^{n} \alpha_{j_1,j_2}\Delta(x_{j_1})\Delta(x_{j_2}) \\ &+ \sum_{j_1<}^{n-2}\sum_{j_2<}^{n-1}\sum_{j_3}^{n} \alpha_{j_1,j_2,j_3}\Delta(x_{j_1})\Delta(x_{j_2})\Delta(x_{j_3}) + \cdots + \alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\Delta(x_i)\Bigg). \end{aligned} \qquad (2.10)$$

2.2. Mixed Erlang claim sizes

These last decades, modeling claim size in insurance with the mixed Erlang distribution with a common scale parameter has been well developed. In risk theory, Dickson and Willmot (2005) and Dickson (2008) have explored an analytical form of the finite time ruin probability, using the mixed Erlang distribution as a claim size model. Recently, using the EM algorithm Lee and Lin (2010) have fitted some common parametric distributions and catastrophic loss data in the United States with the mixed Erlang distribution. Moreover, Lee and Lin (2012) have developed the multivariate mixed Erlang distribution to overcome some drawbacks of the copula approach. Furthermore, Cossette *et al.* (2013) have introduced a risk aggregation in the multivariate setup with mixed Erlang marginals and the FGM copula to capture the dependence structure. As its name indicates, the mixed Erlang distribution is constructed from the Erlang distribution which has the pdf

$$w_k(x, \beta) = \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!}, \quad x > 0, \tag{2.11}$$

where $k \in \mathbb{N}^*$ is the shape parameter and $\beta > 0$ is the scale parameter. Hence, the pdf of the mixed Erlang distribution is defined as

$$f(x, \beta, \underset{\sim}{Q}) = \sum_{k=1}^{\infty} q_k w_k(x, \beta), \tag{2.12}$$

where $\underset{\sim}{Q} = (q_1, q_2, \ldots)$ is a vector of non-negative weights satisfying $\sum_{k=1}^{\infty} q_k = 1$. In the following we write $X \sim ME(\beta, \underset{\sim}{Q})$ if X has pdf given by (2.12). By integrating the pdf in (2.12) the df F corresponding to f is given by

$$F(x, \beta, \underset{\sim}{Q}) = 1 - e^{-\beta x} \sum_{k=1}^{\infty} q_k \sum_{j=0}^{k-1} \frac{(\beta x)^j}{j!}. \tag{2.13}$$

As discussed in Lee and Lin (2010, 2012), Willmot and Lin (2011) and Cossette *et al.* (2013) one of the important advantages of employing the mixed Erlang distribution in insurance loss modeling is the fact that many useful risk related quantities, such as moments and mean excess function can be calculated explicitly by simple formulas. For instance, the quantile function (or VaR) of the mixed Erlang distribution can be easily obtained given the tractable form of the df. From (2.13), at a confidence level $p \in (0, 1)$, the VaR of X, denoted by x_p, is the solution of

$$e^{-\beta x_p} \sum_{k=1}^{\infty} q_k \sum_{j=0}^{k-1} \frac{(\beta x_p)^j}{j!} = 1 - p, \tag{2.14}$$

which can be solved numerically. Further, since for the mean excess function of X, we have (see Willmot and Lin (2011), p. 7)

$$\mathbb{E}((X-d)|X>d) = \frac{\sum_{k=0}^{\infty} Q_k^* \frac{(\beta d)^k}{k!}}{\beta \sum_{j=1}^{\infty} Q_j \frac{(\beta d)^{j-1}}{(j-1)!}}, \quad d>0, \tag{2.15}$$

where $Q_k^* = \sum_{j=k+1}^{\infty} Q_j$ with $Q_j = \frac{\sum_{k=j}^{\infty} q_k}{\sum_{k=1}^{\infty} kq_k}$, then the TVaR of X at a confidence level $p \in (0,1)$ is given by the following explicit formula

$$TVaR_X(p) = \frac{\sum_{k=0}^{\infty} Q_k^* \frac{(\beta x_p)^k}{k!}}{\beta \sum_{j=1}^{\infty} Q_j \frac{(\beta x_p)^{j-1}}{(j-1)!}} + x_p. \tag{2.16}$$

Remark that above we assume that $\mathbb{E}(X) = \sum_{k=1}^{\infty} kq_k$ is finite. Additionally, the mixed Erlang distribution is a tractable marginal distribution for the Sarmanov distribution. Next we present a result for the 2-dimensional setup, see Section 6 for the same results in higher dimensions.

3. Aggregation of Sarmanov Mixed Erlang Risks

Let (X_1, X_2) have a bivariate Sarmanov risk with kernel functions $\phi_i(x) = e^{-x_i} - \mathcal{L}_i$ for $i = 1, 2$. We shall assume that both X_1 and X_2 follow a mixed Erlang distribution, i.e.,

$$X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), \quad i = 1, 2,$$

where β_i is the scale parameter, $\underset{\sim}{Q}_i = (q_{i,1}, q_{i,2}, \ldots)$ denotes the mixing probabilities. The joint distribution of the random vector (X_1, X_2) will be referred to as a bivariate Sarmanov mixed Erlang (SmE) distribution and we shall abbreviate this as $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$ where $\boldsymbol{\beta} = (\beta_1, \beta_2)$. The dependence structure of the bivariate random vector (X_1, X_2) can be analysed by calculating commonly used dependence measures such as Pearson's correlation coefficient or Kendall's tau, see Appendix A. For given vectors of the mixing probabilities $\underset{\sim}{V}_i = (v_{i1}, v_{i2}, \ldots), i = 1, 2$ we define in the following $\pi_1\{\underset{\sim}{V}_1, \underset{\sim}{V}_2\} = 0$ and for $k > 1$

$$\pi_k\{\underset{\sim}{V}_1, \underset{\sim}{V}_2\} = \sum_{j=1}^{k-1} v_{1,j} v_{2,k-j}$$

The main result in this section is the derivation of the distribution of the aggregated risk $S_2 = X_1 + X_2$.

Proposition 3.1. *If* $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$ *with* $\beta_1 \leqslant \beta_2$*, then* $S_2 \sim ME(\beta_2 + 1, \underset{\sim}{P})$ *where the mixing weights* p_k *are given by (set* $\gamma := \alpha_{12}\mathcal{L}_1\mathcal{L}_2$*,*

TABLE 3.1

CENTRAL MOMENTS OF X_1 AND X_2.

	Mean	Variance	Skewness	Kurtosis
X_1	2.33	4.44	1.38	5.49
X_2	2.11	3.10	1.49	6.12

$\overline{\beta_i} := \beta_i/(\beta_i + 1)$)

$$p_k = (1+\gamma)\pi_k\{\Psi_1(Q_1), \Psi_2(Q_2)\} + \gamma\pi_k\{\Psi_1(\Theta_1), \Psi_2(\Theta_2)\} - \gamma\pi_k\{\Psi_1(\Theta_1), \Psi_2(Q_2)\} - \gamma\pi_k\{\Psi_1(Q_1), \Psi_2(\Theta_2)\}, \quad (3.1)$$

where for $i = 1, 2$ *the components of* $\Theta_i = (\theta_{i,1}, \theta_{i,2}, \ldots)$ *are defined by* $\theta_{i,k} = \frac{q_{i,k}\overline{\beta_i}^k}{\sum_{j=1}^{\infty} q_{i,j}\overline{\beta_i}^j}$*, whereas the components of* $\Psi_i(Q_i) = (\psi_{i,1}, \psi_{i,2}, \ldots)$ *are* $\psi_{i,k} = \sum_{j=1}^{k} q_{i,j}\binom{k-1}{j-1}(\frac{\beta_i}{\beta_2+1})^j(1 - \frac{\beta_i}{\beta_2+1})^{k-j}$.

Example 3.2. *As an illustration, let*

$$(X_1, X_2) \sim SME_2\left(\boldsymbol{\beta} = \begin{pmatrix} 0.9 \\ 0.95 \end{pmatrix}; Q_1 = (0.4, 0.2, 0.3, 0.1); Q_2 = (0.3, 0.5, 0.1, 0.1); \alpha_{12} = 2.87\right).$$

According to (2.12)*, one can write the pdf of* X_1 *and* X_2 *as follows*

$$f_1(x_1) = 0.4w_1(x_1, 0.9) + 0.2w_2(x_1, 0.9) + 0.3w_3(x_1, 0.9) + 0.1w_4(x_1, 0.9)$$
$$f_2(x_2) = 0.3w_1(x_2, 0.95) + 0.5w_2(x_2, 0.95) + 0.1w_3(x_2, 0.95) + 0.1w_4(x_2, 0.95).$$

Following (2.4)*, the joint density of* (X_1, X_2) *is given by*

$$h(x_1, x_2) = f_1(x_1)f_2(x_2)(1.22 + 2.87e^{-x_1-x_2} - 0.81e^{-x_1} - 0.78e^{-x_2}).$$

Table 3.1 above presents the central moments of the marginals.

It follows that the distribution of S_2 *is a mixed Erlang distribution with scale parameter* $\beta_{S_2} = 1.95$ *and mixing probabilities partially shown in 3.2. We notice that the higher the value of k is, the smaller the value of* p_k.

In order to validate our results, SmE risks have been simulated (see in Appendix B the details about the simulation algorithm). In this respect, analytical and simulated results on the aggregated risk $S_2 = X_1 + X_2$ *are presented and analysed. As displayed in Table 3.3, based on the VaR and TVaR risk measures the comparison of the exact and the simulated values shows that our results are robust for different values of the tolerance level p. Furthermore, it can be seen that VaR is more sensitive to the change of the tolerance level than TVaR. Similarly, by*

TABLE 3.2

MIXING PROBABILITIES OF THE DISTRIBUTION OF $S_2 = X_1 + X_2$, WITH SCALE PARAMETER $\beta_{S_2} = 1.95$.

k	p_k	k	p_k	k	p_k	k	p_k	k	p_k
1	0.0000	11	0.0664	21	0.0046	31	8.963E-05	41	9.294E-07
2	0.0675	12	0.0564	22	0.0033	32	5.803E-05	42	5.751E-07
3	0.0839	13	0.0465	23	0.0023	33	3.737E-05	43	3.547E-07
4	0.0645	14	0.0373	24	0.0016	34	2.393E-05	44	2.180E-07
5	0.0700	15	0.0292	25	0.0011	35	1.525E-05	45	1.336E-07
6	0.0740	16	0.0223	26	0.0007	36	9.668E-06	46	8.159E-08
7	0.0811	17	0.0168	27	0.0005	37	6.103E-06	47	4.970E-08
8	0.0840	18	0.0125	28	0.0003	38	3.835E-06	48	3.02E-08
9	0.0816	19	0.0091	29	0.0002	39	2.400E-06	49	1.828E-08
10	0.0753	20	0.0065	30	0.0001	40	1.496E-06	50	1.105E-08

TABLE 3.3

EXACT AND SIMULATED VALUES OF VAR AND TVAR OF $S_2 = X_1 + X_2$.

	Analytical Formula		Simulated		Percentage Difference (%)	
p	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$
90.00 %	8.26	10.24	8.22	10.21	0.49	0.29
92.50 %	8.88	10.80	8.86	10.77	0.23	0.28
95.00 %	9.71	11.56	9.66	11.53	0.52	0.26
97.50 %	11.05	12.82	10.98	12.82	0.64	0.00
99.00 %	12.71	14.41	12.79	14.46	− 0.63	−0.35
99.50 %	13.92	15.56	13.87	15.43	0.36	0.84
99.90 %	16.57	18.13	16.61	17.86	−0.24	1.51
99.99 %	20.15	21.62	19.42	20.79	3.62	3.84

changing the level of the dependence between marginals which is described by α_{12} *and for a tolerance level of 99%, the comparison of the exact and the simulated values of VaR and TVaR is displayed in Table 3.4. Note in passing that the maximum attainable value of* α_{12}*, in our example, is* 4.87 *while the minimum is* -1.91.

4. CAPITAL ALLOCATION

In this section, we derive analytical expressions for the amount of capital allocated to each individual risk under the TVaR and the covariance principles. Evaluating the economic capital for the entire portfolio that an insurance company needs to absorb large unexpected losses is of importance in enterprise risk management. In this respect, the so-called capital allocation consists in determining the contribution of each individual risk to the aggregate economic capital. This allows the insurance company to identify and to monitor efficiently

TABLE 3.4

DEPENDENCE LEVEL AND SENSITIVENESS OF RISK MEASURES.

	Analytical formula		Simulated		Percentage difference (%)	
α_{12}	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$
−1.91	12.24	13.92	12.26	13.91	− 0.16	0.10
− 0.87	12.35	14.04	12.38	14.03	− 0.25	0.06
0	12.44	14.13	12.48	14.13	− 0.31	0.03
0.87	12.53	14.22	12.57	14.22	− 0.29	0.01
1.87	12.62	14.31	12.66	14.32	− 0.33	− 0.02
2.87	12.71	14.41	12.74	14.41	− 0.24	− 0.05
3.87	12.80	14.49	12.82	14.50	− 0.14	− 0.08
4.87	12.88	14.57	12.90	14.59	− 0.14	− 0.10

their risks. In the literature, many capital allocation techniques have been developed, see Cummins (2000), Dhaene *et al.* (2012), McNeil *et al.* (2005), and Tasche (2004) and references therein. In practice, the TVaR and the covariance allocation principle are commonly used, since they take into account the dependence structure between risks. More precisely, if $S_n = \sum_{i=1}^{n} X_i$ is the aggregate risk where X_i is a continuous rv with finite mean that represents the individual risk, the amount of capital T_i allocated to each risk X_i, for $i = 1, \ldots, n$, is defined as (for a tolerance level $p \in (0, 1)$, denote $T_i = TVaR_p(X_i, S_n)$ under the TVaR allocation principle, $T_i = K_p(X_i, S_n)$ under the covariance allocation principle)

$$TVaR_p(X_i, S_n) = \frac{\mathbb{E}(X_i \mathbb{1}_{\{S_n > VaR_{S_n}(p)\}})}{1-p}, \tag{4.1}$$

$$K_p(X_i, S_n) = \mathbb{E}(X_i) + \frac{Cov(X_i, S_n)}{Var(S_n)}(TVaR_{S_n}(p) - \mathbb{E}(S_n)), \tag{4.2}$$

where we assume that S_n has finite and positive variance. We have

$$\sum_{i=1}^{n} T_i = \sum_{i=1}^{n} TVaR_p(X_i, S_n) = \sum_{i=1}^{n} K_p(X_i, S_n) = TVaR_{S_n}(p),$$

which means that for both allocation principle, based on TVaR as a risk measure, the capital required for the entire portfolio is equal to the sum of the allocated capital of each risk within the portfolio. Given some vector $\underline{V} = (v_1, v_2, \ldots)$ with non-negative components such that $\sum_{j=1}^{\infty} j v_j < \infty$ we define the new vector $\underline{G}(\underline{V}) = (g_1, g_2, \ldots)$ where

$$g_k = \begin{cases} 0 & \text{for } k = 1 \\ \frac{(k-1)v_{k-1}}{\sum_{j=1}^{\infty} j v_j} & \text{for } k > 1. \end{cases}$$

For notational simplicity we shall also write in the following $\overline{\beta_i}$ instead of $\beta_i/(\beta_i+1)$. Furthermore hereafter the df of the pdf given in (2.11) will be denoted by $W_k(\cdot,\beta)$ with survival function $\overline{W}_k(\cdot,\beta)$.
We derive next an explicit form of $TVaR_p(X_i, S_2)$ and $K_p(X_i, S_2)$, $i=1,2$, in the case of SmE type risks.

Proposition 4.1. *Let $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, Q_1, Q_2)$ with $\beta_1 \leqslant \beta_2$, further let Θ_i and Ψ_i be defined as in Proposition 3.1. If for $i=1,2$ both $\mu_i := \frac{1}{\beta_i}\sum_{k=1}^{\infty} k q_{i,k}$ and $\tilde{\mu}_i := \frac{1}{\beta_i+1}\sum_{k=1}^{\infty} k\theta_{i,k}$ are finite, then for any $p\in(0,1)$ the amount of capital allocated to each risk X_i, $i=1,2$, under the TVaR principle is*

$$TVaR_p(X_i, S_2) = \frac{1}{1-p}\sum_{k=1}^{\infty} z_{ik}\overline{W}_k(VaR_{S_2}(p), \beta_2+1), \tag{4.3}$$

where $\gamma = \alpha_{12}\mathcal{L}_1\mathcal{L}_2$,

$$\begin{aligned} z_{i,k} &= (1+\gamma)\mu_i\pi_k\{\Psi_i(G_i(Q_i)), \Psi_j(Q_j)\} + \gamma\tilde{\mu}_i\pi_k\{\Psi_i(G_i(\Theta_i)), \Psi_j(\Theta_j)\} \\ &\quad - \gamma\tilde{\mu}_i\pi_k\{\Psi_i(G_i(\Theta_i)), \Psi_j(Q_j)\} - \gamma\mu_i\pi_k\{\Psi_i(G_i(Q_i)), \Psi_j(\Theta_j)\}, \quad i\neq j, \end{aligned}$$

and the contribution of each risk X_i, $i=1,2$ to the economic capital of the entire portfolio, under the covariance principle, is given by

$$K_p(X_i, S_2) = \sum_{k=1}^{\infty}\frac{L_{i,k}}{\beta_2+1},$$

where

$$L_{i,k} = k\psi_{i,k} + \varepsilon_{i,j}\left(\frac{P_k^*((\beta_2+1)\,VaR_{S_2}(p))^k}{\varphi k!} + (\beta_2+1)\,VaR_{S_2}(p) - kp_k\right), i\neq j,$$

with

$$\begin{aligned} \varepsilon_{i,j} &= \frac{\sum_{m=1}^{\infty}(m^2+m)\psi_{im} - (\sum_{m=1}^{\infty} m\psi_{im})^2}{\sum_{m=1}^{\infty}(m^2+m)p_m - (\sum_{m=1}^{\infty} mp_m)^2} \\ &\quad + \left(\frac{\alpha_{12}(\beta_2+1)^2}{\sum_{m=1}^{\infty}(m^2+m)p_m - (\sum_{m=1}^{\infty} mp_m)^2}\right) \\ &\quad \times\left(\frac{1}{\beta_i+1}\sum_{m=1}^{\infty} mq_{i,m}\overline{\beta_i}^m - \frac{1}{\beta_i}\sum_{m=1}^{\infty} q_{i,m}\overline{\beta_i}^m\sum_{m=1}^{\infty} mq_{i,m}\right) \\ &\quad \times\left(\frac{1}{\beta_j+1}\sum_{m=1}^{\infty} mq_{j,m}\overline{\beta_j}^m - \frac{1}{\beta_j}\sum_{m=1}^{\infty} q_{j,m}\overline{\beta_j}^m\sum_{m=1}^{\infty} mq_{j,m}\right), \end{aligned} \tag{4.4}$$

TABLE 4.1

ANALYTICAL FORMULA: DEPENDENCE LEVEL, TVAR AND ALLOCATED CAPITAL TO EACH RISK X_i, $i = 1, 2$, UNDER THE TVAR AND THE COVARIANCE CAPITAL ALLOCATION PRINCIPLE.

α_{12}	$TVaR_{S_2}(0.99)$	$TVaR_{0.99}(X_1, S_2)$	$TVaR_{0.99}(X_2, S_2)$	$K_{0.99}(X_1, S_2)$	$K_{0.99}(X_2, S_2)$
– 1.91	13.92	7.70	6.22	7.69	6.23
– 0.87	14.04	7.74	6.30	7.73	6.31
0	14.13	7.77	6.36	7.75	6.38
0.87	14.22	7.80	6.42	7.78	6.44
1.87	14.31	7.84	6.47	7.81	6.50
2.87	14.41	7.87	6.54	7.84	6.57
3.87	14.49	7.90	6.59	7.87	6.62
4.87	14.57	7.93	6.64	7.89	6.68

$$\varphi = \sum_{j=1}^{\infty} \frac{P_j((\beta_2+1)\,VaR_{S_2}(p))^{j-1}}{(j-1)!}, \qquad P_k^* = \sum_{j=k}^{\infty} P_j,$$

$$P_j = \frac{\sum_{k=j}^{\infty} p_k}{\sum_{k=1}^{\infty} k p_k} \quad \text{and } p_k \text{ is given in (3.1).}$$

Example 4.2. *In this example, we consider the same marginals and dependence parameters as in Example 3.2. For different level of the dependence between X_1 and X_2, which is described by α_{12}, TVaRs have been calculated on the aggregated risk $S_2 = X_1 + X_2$ at a tolerance level $p = 99\%$. Furthermore, the allocated capital to each risk $X_i, i = 1, 2$, under the TVaR and the covariance capital allocation principle are also evaluated. Table 4.1 demonstrates that risk measures on the aggregated risk are sensitive to the level of dependence between individual risks. Actually, due to the relationship between dependence level and the diversification effect, the more X_1 and X_2 are dependent, the more the portfolio is risky, hence more capital is needed to cover the risks. In this respect, more capital is allocated to risk X_1 compared to the amount allocated to risk X_2 under the TVaR and the covariance principle.*

5. AUXILIARY RESULTS

One of the main features of the mixed Erlang distribution is that its pdf can be used to derive some results in an analytical way. In this respect, this section presents some useful properties of the mixed Erlang distribution.

Lemma 5.1. *If X is a random variable from the mixed Erlang distribution with pdf $g(x, \beta, \underset{\sim}{Q})$, then $g^{\theta}(x, \beta+1, \underset{\sim}{\Theta}) = \frac{e^{-x} g(x,\beta,\underset{\sim}{Q})}{\mathcal{L}}$, with $\mathcal{L} = \mathbb{E}\left(e^{-X}\right)$, is again a pdf of the mixed Erlang distribution with mixing probabilities $\underset{\sim}{\Theta} = (\theta_1, \theta_2, \ldots)$ and* *scale parameter* $\beta + 1$ *and we have*

$$g^{\theta}(x, \beta + 1, \underset{\sim}{\Theta}) = \sum_{k=1}^{\infty} \theta_k w_k(x, \beta + 1),$$

where $\theta_k = \frac{q_k \overline{\beta}^k}{\sum_{j=1}^{\infty} q_j \overline{\beta}^j}$ *with* $\overline{\beta} = \frac{\beta}{\beta+1}$.

The result presented in the next two lemmas can be found in Section 2.2 of Willmot and Woo (2007), and Section 7.2 of Lee and Lin (2010), respectively.

Lemma 5.2. *If* $X \sim ME(\beta_1, \underset{\sim}{Q})$*, then for any positive constant* $\beta_2 \geq \beta_1$ *we have*

$$X \sim ME(\beta_2, \Psi(\underset{\sim}{Q})),$$

where the mixing probabilities $\Psi(\underset{\sim}{Q}) = (\psi_1, \psi_2, \ldots)$ *and its individual components are given by*

$$\psi_k = \sum_{i=1}^{k} q_i \binom{k-1}{i-1} \left(\frac{\beta_1}{\beta_2}\right)^i \left(1 - \frac{\beta_1}{\beta_2}\right)^{k-i}, \quad k \geq 1.$$

Lemma 5.3. *Let* X_1, X_2 *be two independent random variables. If* $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), i = 1, 2$*, then* $S_2 = X_1 + X_2 \sim ME(\beta, \underset{\sim}{\Pi}\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2\})$*, provided that* $\beta_1 = \beta_2 = \beta$ *with*

$$\pi_l\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2\} = \begin{cases} 0 & \text{for} \quad l = 1 \\ \sum_{j=1}^{l-1} q_{1,j} q_{2,l-j} & \text{for} \quad l > 1. \end{cases}$$

Remarks 5.4. *According to Cossette et al. (2012) (Remark 2.1), the results in Lemma 5.3 can be extended to* $S_n = \sum_{i=1}^{n} X_i$*, as long as* $X_i, \ldots, X_n$ *are independent,* $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i)$ *and* $\beta_i = \beta$ *for* $i = 1, \ldots, n$*. Specifically,* $S_n \sim ME(\beta, \underset{\sim}{\Pi}\{\underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n\})$ *where the individual mixing probabilities can be evaluated iteratively as follows*

$$\pi_l\{\underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_{n+1}\} = \begin{cases} 0 & \text{for } l = 1, \ldots, n \\ \sum_{j=n}^{l-1} \pi_j\{\underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n\} q_{n+1,l-j} & \text{for } l = n+1, n+2, \ldots. \end{cases}$$

6. Multivariate SmE Risks

In this section, we assume that the joint distribution of the random vector $(X_1, \ldots, X_n)$ will be referred to as a multivariate SmE distribution and we shall abbreviate this as $(X_1, \ldots, X_n) \sim SME_n(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n)$ where

$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ with $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), i = 1, \ldots, n$. Furthermore, we shall set

$$\tilde{f}_i(x_i) := e^{-x_i} f_i(x_i).$$

6.1. Distribution of S_n

By decomposing the joint pdf of $(X_1, \ldots, X_n)$ in (2.10) and using some rules of integration, we show in the next proposition that the distribution of $S_n = \sum_{i=1}^n X_i$ belongs to the class of Erlang mixtures.

Proposition 6.1. *If* $(X_1, \ldots, X_n) \sim SME_n(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n)$ *with* $\beta_i \leqslant \beta_n$*, for* $i = 1, \ldots, n-1$*, then* $S_n \sim ME(\beta_n + 1, \underset{\sim}{P})$*. The components of* $\underset{\sim}{P} = (p_1, p_2, \ldots)$ *are given by*

$$\begin{aligned} p_k = \Big(1 &+ \sum_{j_1}\sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3} \alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ &+ \cdots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i=1}^n \mathcal{L}_i\Big)\overline{\pi}^{(k)} + \sum_{j_1}\Big(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2} \\ &+ \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} + \cdots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1\}}\mathcal{L}_i\Big)\overline{\pi}^{(k)}_{j1} \\ &+ \sum_{j_1}\sum_{j_2}\Big(\alpha_{j_1,j_2} - \sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\ &+ \cdots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Big)\overline{\pi}^{(k)}_{j1,j_2} \\ &+ \sum_{j_1}\sum_{j_2}\sum_{j_3}\Big(\alpha_{j_1,j_2,j_3} - \sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4} + \sum_{j_4}\sum_{j_5}\alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5} \\ &+ \cdots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Big)\overline{\pi}^{(k)}_{j1,j_2,j_3} \\ &+ \cdots + \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{n-1}}(\alpha_{j_1,j_2,\ldots,j_{n-1}} - \alpha_{1,2,\ldots,n}\mathcal{L}_{j_n})\overline{\pi}^{(k)}_{j1,\ldots,j_{n-1}} + \alpha_{1,2,\ldots,n}\overline{\pi}^{(k)}_{1,\ldots,n}, \end{aligned} \tag{6.1}$$

where
$\overline{\pi}^{(k)} = \pi_k\{\Psi_1(\underset{\sim}{Q}_1), \ldots, \Psi_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1} = \mathcal{L}_{j_1}\pi_k\{\Psi_{j_1}(\underset{\sim}{\Theta}_{j_1}), \Psi_{j_2}(\underset{\sim}{Q}_{j_2}), \ldots, \Psi_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1,j_2} = \mathcal{L}_{j_1}\mathcal{L}_{j_2}\pi_k\{\Psi_{j_1}(\underset{\sim}{\Theta}_{j_1}), \Psi_{j_2}(\underset{\sim}{\Theta}_{j_2}), \Psi_{j_3}(\underset{\sim}{Q}_{j_3})\ldots, \Psi_n(\underset{\sim}{Q}_n)\}$,

$\overline{\pi}^{(k)}_{j_1,j_2,j_3} = \mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3}\pi_k\{\Psi_{j_1}(\Theta_{j_1}), \Psi_{j_2}(\Theta_{j_2}), \Psi_{j_3}(\Theta_{j_3}) \ldots, \Psi_n(Q_n)\}$,
$\overline{\pi}^{(k)}_{j_1,\ldots,j_{n-1}} = \mathcal{L}_{j_1}\cdots\mathcal{L}_{j_{n-1}}\pi_k\{\Psi_{j_1}(\Theta_{j_1}), \ldots, \Psi_{j_{n-1}}(\Theta_{j_{n-1}}), \Psi_{j_n}(Q_{j_n})\}$,
$\overline{\pi}^{(k)}_{1,\ldots,n} = \mathcal{L}_1\cdots\mathcal{L}_n\pi_k\{\Psi_1(\Theta_1), \ldots, \Psi_n(\Theta_n)\}$,
with $C = \{1,\ldots,n\}$*,* $j_1 \in C, j_2 \in C\backslash\{j_1\}, j_3 \in C\backslash\{j_1, j_2\}, \ldots, j_n \in C\backslash\{j_1,\ldots,j_{n-1}\}$.

Example 6.2. *Let* $(X_1, X_2, X_3) \sim SME_3(\boldsymbol{\beta}, Q_1, Q_2, Q_3)$ *with* $\beta_i \leqslant \beta_3, i = 1, 2$ *then* $S_3 \sim ME(\beta_3 + 1, P)$ *where the components of* $P = (p_1, p_2, \ldots)$ *are given by (with* $C = \{1, 2, 3\}$*)*

$$\begin{aligned}
p_k =& \left(1 + \sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \alpha_{1,2,3}\prod_{i=1}^{3}\mathcal{L}_i\right)\overline{\pi}^{(k)} \\
&+ \sum_{j_1}\left(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} + \alpha_{1,2,3}\prod_{i\in C\backslash\{j_1\}}\mathcal{L}_i\right)\overline{\pi}^{(k)}_{j1} \\
&+ \sum_{j_1}\sum_{j_2}\left(\alpha_{j_1,j_2} - \alpha_{1,2,3}\mathcal{L}_{j_3}\right)\overline{\pi}^{(k)}_{j_1,j_2} + \alpha_{1,2,3}\overline{\pi}^{(k)}_{1,2,3} \\
=& (1 + \alpha_{1,2}\mathcal{L}_1\mathcal{L}_2 + \alpha_{1,3}\mathcal{L}_1\mathcal{L}_3 + \alpha_{2,3}\mathcal{L}_2\mathcal{L}_3 - \alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3) \\
&\times \pi_k\{\Psi_1(Q_1), \Psi_2(Q_2), \Psi_3(Q_3)\} \\
&+ (-\alpha_{1,2}\mathcal{L}_2 - \alpha_{1,3}\mathcal{L}_3 + \alpha_{1,2,3}\mathcal{L}_2\mathcal{L}_3)\mathcal{L}_1\pi_k\{\Psi_1(\Theta_1), \Psi_2(Q_2), \Psi_3(Q_3)\} \\
&+ (-\alpha_{1,2}\mathcal{L}_1 - \alpha_{2,3}\mathcal{L}_3 + \alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_3)\mathcal{L}_2\pi_k\{\Psi_1(Q_1), \Psi_2(\Theta_2), \Psi_3(Q_3)\} \\
&+ (-\alpha_{1,3}\mathcal{L}_1 - \alpha_{2,3}\mathcal{L}_2 + \alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2)\mathcal{L}_3\pi_k\{\Psi_1(Q_1), \Psi_2(Q_2), \Psi_3(\Theta_3)\} \\
&+ (\alpha_{1,3} - \alpha_{1,2,3}\mathcal{L}_2)\mathcal{L}_1\mathcal{L}_3\pi_k\{\Psi_1(\Theta_1), \Psi_2(Q_2), \Psi_3(\Theta_3)\} \\
&+ (\alpha_{2,3} - \alpha_{1,2,3}\mathcal{L}_1)\mathcal{L}_2\mathcal{L}_3\pi_k\{\Psi_1(Q_1), \Psi_1(\Theta_1), \Psi_3(\Theta_3)\} \\
&+ (\alpha_{1,2} - \alpha_{1,2,3}\mathcal{L}_3)\mathcal{L}_1\mathcal{L}_2\pi_k\{\Psi_1(\Theta_1), \Psi_2(\Theta_2), \Psi_3(Q_3)\} \\
&+ \alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\pi_k\{\Psi_1(\Theta_1), \Psi_2(\Theta_2), \Psi_3(\Theta_3)\}.
\end{aligned}$$

6.2. Capital allocation

The following propositions provide analytical formulas for the allocated capital to each individual risk X_m, $m = 1,\ldots,n$, under the TVaR and the covariance rules.

Proposition 6.3. *Let* $(X_1,\ldots,X_n) \sim SME_n(\boldsymbol{\beta}, Q_1,\ldots,Q_n)$ *with* $\beta_m \leqslant \beta_n$*, for* $m = 1,\ldots,n-1$*. Provided that both* $\mu_m = \frac{1}{\beta_m}\sum_{k=1}^{\infty} kq_{m,k}$ *and* $\tilde{\mu}_m = \frac{1}{\beta_m+1}\sum_{k=1}^{\infty} k\theta_{mk}$*,* $m = 1,\ldots,n$ *are finite , then for* $m = 1,\ldots,n$ *and* $p \in (0, 1)$ *the amount of capital allocated to each risk* X_m *under the TVaR principle is given* *by (set* $C := \{1, \ldots, n\}$)

$$TVaR_p(X_m, S_n) = \frac{1}{1-p} \sum_{k=1}^{\infty} z_{m,k} \overline{W}_k(VaR_{S_n}(p), \beta_n + 1),$$

where

$$\begin{aligned} z_{m,k} = &\Bigg(1 + \sum_{j_1} \sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_1} \mathcal{L}_{j_2} - \sum_{j_1} \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_1} \mathcal{L}_{j_2} \mathcal{L}_{j_3} + \cdots \\ &+ (-1)^n \alpha_{1,2,\ldots,n} \prod_{i=1}^{n} \mathcal{L}_i \Bigg) \mu_m \tilde{\pi}^{(k)} + \sum_{j_1 \neq m} \Bigg(- \sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_2} + \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \\ &\times \mathcal{L}_{j_2} \mathcal{L}_{j_3} + \cdots + (-1)^{n+1} \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1\}} \mathcal{L}_i \Bigg) \mu_m \tilde{\pi}_{j_1}^{(k)} + \Bigg(- \sum_{j_2 \neq m} \alpha_{m,j_2} \mathcal{L}_{j_2} \\ &+ \sum_{j_2 \neq m} \sum_{j_3 \neq m} \alpha_{m,j_2,j_3} \mathcal{L}_{j_2} \mathcal{L}_{j_3} + \cdots + (-1)^{n+1} \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{m\}} \mathcal{L}_i \Bigg) \tilde{\mu}_m \tilde{\pi}_m^{(k)} \\ &+ \sum_{j_1 \neq m} \sum_{j_2} \Bigg(\alpha_{j_1,j_2} - \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_3} + \sum_{j_3} \sum_{j_4} \alpha_{j_1,j_2,j_3,j_4} \mathcal{L}_{j_3} \mathcal{L}_{j_4} + \cdots \\ &+ (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1,j_2\}} \mathcal{L}_i \Bigg) \mu_m \tilde{\pi}_{j_1,j_2}^{(k)} + \sum_{j_2 \neq m} \Bigg(\alpha_{m,j_2} - \sum_{j_3 \neq m} \alpha_{m,j_2,j_3} \mathcal{L}_{j_3} \\ &+ \sum_{j_3 \neq m} \sum_{j_4 \neq m} \alpha_{m,j_2,j_3,j_4} \mathcal{L}_{j_3} \mathcal{L}_{j_4} + \cdots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{m,j_2\}} \mathcal{L}_i \Bigg) \tilde{\mu}_m \tilde{\pi}_{m,j_2}^{(k)} \\ &+ \cdots + \sum_{j_1 \neq m} \sum_{j_2} \ldots \sum_{j_{n-1}} (\alpha_{j_1,j_2,\ldots,j_{n-1}} - \alpha_{1,2,\ldots,n} \mathcal{L}_m) \mu_m \tilde{\pi}_{j_1,\ldots,j_{n-1}}^{(k)} \\ &+ \sum_{j_2 \neq m} \ldots \sum_{j_{n-1} \neq m} (\alpha_{m,j_2,\ldots,j_{n-1}} - \alpha_{1,2,\ldots,n} \mathcal{L}_{j_n \neq m}) \tilde{\mu}_m \tilde{\pi}_{m,j_2,\ldots,j_{n-1}}^{(k)} + \alpha_{1,2,\ldots,n} \tilde{\pi}_{1,\ldots,n}^{(k)}, \end{aligned} \tag{6.2}$$

where

$\tilde{\pi}^{(k)} = \pi_k\{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_1(\mathcal{Q}_1), \ldots, \Psi_n(\mathcal{Q}_n)\}$,
$\tilde{\pi}_{j_1}^{(k)} = \mathcal{L}_{j_1} \pi_k\{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_{j_1}(\Theta_{j_1}), \ldots, \Psi_n(\mathcal{Q}_n)\}$,
$\tilde{\pi}_m^{(k)} = \mathcal{L}_m \pi_k\{\Psi_m(\mathcal{G}_m(\Theta_m)), \ldots, \Psi_n(\mathcal{Q}_n)\}$,
$\tilde{\pi}_{j_1,j_2}^{(k)} = \mathcal{L}_{j_1} \mathcal{L}_{j_2} \pi_k\{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_{j_1}(\Theta_{j_1}), \Psi_{j_2}(\Theta_{j_2}), \ldots, \Psi_n(\mathcal{Q}_n)\}$,
$\tilde{\pi}_{m,j_2}^{(k)} = \mathcal{L}_m \mathcal{L}_{j_2} \pi_k\{\Psi_m(\mathcal{G}_m(\Theta_m)), \Psi_{j_2}(\Theta_{j_2}), \ldots, \Psi_n(\mathcal{Q}_n)\}$,
$\tilde{\pi}_{j_1,\ldots,j_{n-1}}^{(k)} = \mathcal{L}_{j_1} \cdots \mathcal{L}_{j_{n-1}} \pi_k\{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_{j_1}(\Theta_{j_1}) \ldots, \Psi_n(\Theta_{j_{n-1}})\}$,

$$\tilde{\pi}^{(k)}_{m,j_2,\ldots,j_{n-1}} = \mathcal{L}_m \mathcal{L}_{j_2} \cdots \mathcal{L}_{j_{n-1}} \pi_k \{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_{j_2}(\mathcal{Q}_{j_2}) \ldots, \Psi_{j_{n-1}}(\mathcal{Q}_{j_{n-1}}), \Psi_{j_n}(\mathcal{Q}_{j_n})\}$$
$$\tilde{\pi}^{(k)}_{1,\ldots,n} = \mathcal{L}_m \mathcal{L}_1 \cdots \mathcal{L}_n \pi_k \{\Psi_m(\mathcal{G}_m(\mathcal{Q}_m)), \Psi_{j_2}(\mathcal{Q}_{j_2}), \ldots, \Psi_n(\mathcal{Q}_{j_n}\}.$$

Proposition 6.4. *Let $\beta_m \leqslant \beta_n, m \leq n-1$, and consider $(X_1, \ldots, X_n) \sim SME_n(\boldsymbol{\beta}, \mathcal{Q}_1, \ldots, \mathcal{Q}_n)$. If S_n has a finite and positive variance, then for any index $m \leq n$ and $p \in (0,1)$ we have*

$$K_p(X_m, S_n) = \sum_{k=1}^{\infty} \frac{L_{m,k}}{\beta_n + 1},$$

where K_p is defined in (4.2)*,*

$$L_{m,k} = k\psi_{m,k} + \varepsilon_{m,j} \left(\frac{P_k^*((\beta_n+1)\,VaR_{S_n}(p))^k}{\varphi k!} + (\beta_n+1)\,VaR_{S_n}(p) - kp_k \right), \; m \neq j,$$

with

$$\varepsilon_{m,j} = \frac{\sum_{s=1}^{\infty}(s^2+s)\psi_{m,s} - (\sum_{s=1}^{\infty} s\psi_{m,s}^2)}{\sum_{s=1}^{\infty}(s^2+s)p_s - (\sum_{s=1}^{\infty} sp_s)^2} + \sum_{j=1}^{n} \left(\frac{\alpha_{mj}(\beta_n+1)^2}{\sum_{s=1}^{\infty}(s^2+s)p_s - (\sum_{s=1}^{\infty} sp_s)^2} \right)$$
$$\times \left(\frac{1}{\beta_m+1} \sum_{s=1}^{\infty} sq_{m,s}\overline{\beta}_m^s - \sum_{s=1}^{\infty} q_{m,s}\overline{\beta}_m^s \frac{1}{\beta_m} \sum_{s=1}^{\infty} sq_{m,s} \right)$$
$$\times \left(\frac{1}{\beta_j+1} \sum_{s=1}^{\infty} sq_{j,s}\overline{\beta}_j^s - \sum_{s=1}^{\infty} q_{j,s}\overline{\beta}_j^s \frac{1}{\beta_j} \sum_{s=1}^{\infty} sq_{j,s} \right),$$

$$\varphi = \sum_{s=1}^{\infty} \frac{P_s((\beta_n+1)\,VaR_{S_n}(p))^{s-1}}{(s-1)!}, \qquad P_k^* = \sum_{s=k}^{\infty} P_s,$$

$$P_s = \frac{\sum_{k=s}^{\infty} p_s}{\sum_{s=1}^{\infty} sp_s} \qquad \textit{and } p_s \textit{ is given in } (6.1).$$

Proof. The proof is similar to the bivariate case and is therefore omitted.

6.3. Trivariate SmE risks: numerical illustrations

Let (X_1, X_2, X_3) have a trivariate SmE risk, with $\alpha_{12} = 2.03$, $\alpha_{13} = 3.62$, $\alpha_{23} = -1.54$ and $\alpha_{123} = -1.03$ the dependence parameters. The parameters have been chosen so that the condition in (2.9) is fullfilled. Assume $\boldsymbol{\beta} = (0.75, 0.9, 0.95)$, $\mathcal{Q}_1 = (0.2, 0.6, 0.2)$, $\mathcal{Q}_2 = (0.4, 0.3, 0.1, 0.2)$ and $\mathcal{Q}_3 = (0.6, 0.1, 0.2, 0.1)$. In view of (2.10) the joint pdf of (X_1, X_2, X_3) are given by

$$h(\boldsymbol{x}) = \prod_{i=1}^{3} f_i(x_i) \Big(2.03(e^{-x_1} - 0.21)(e^{-x_2} - 0.28) + 3.62(e^{-x_1} - 0.21)(e^{-x_3} - 0.34)$$
$$- 1.54(e^{-x_2} - 0.28)(e^{-x_3} - 0.34) - 1.03(e^{-x_1} - 0.21)(e^{-x_2} - 0.28)(e^{-x_3} - 0.34) \Big).$$

TABLE 6.1

MIXING PROBABILITIES OF THE DISTRIBUTION OF $S_3 = X_1 + X_2 + X_3$, WITH SCALE PARAMETER $\beta_{S_3} = 1.95$.

k	p_k	k	p_k	k	p_k	k	p_k	k	p_k	k	p_k
1	0.0000	11	0.0670	21	0.0256	31	0.0022	41	8.729E-05	51	2.150E-06
2	0.0000	12	0.0676	22	0.0211	32	0.0017	42	5.751E-05	52	1.458E-06
3	0.0121	13	0.0662	23	0.0172	33	0.0012	43	4.289E-05	53	9.857E-07
4	0.0295	14	0.0631	24	0.0138	34	0.0009	44	2.988E-05	54	6.648E-07
5	0.0366	15	0.0588	25	0.0109	35	0.0006	45	2.019E-05	55	4.472E-07
6	0.0409	16	0.0536	26	0.0086	36	0.0005	46	9.869E-06	56	3.001E-07
7	0.0466	17	0.0478	27	0.0067	37	0.0003	47	9.612E-06	57	2.010E-07
8	0.0533	18	0.0419	28	0.0051	38	0.0002	48	4.635E-06	58	1.343E-07
9	0.0596	19	0.0361	29	0.0039	39	0.0002	49	4.513E-06	59	8.950E-08
10	0.0643	20	0.0307	30	0.0030	40	0.0001	50	3.161E-06	60	5.795E-08

TABLE 6.2

EXACT VALUES: TVAR OF $S_3 = X_1 + X_2 + X_3$ AND ALLOCATED CAPITAL TO EACH RISK $X_i, i = 1, 2, 3$, UNDER THE TVAR AND THE COVARIANCE CAPITAL ALLOCATION PRINCIPLE.

p	$TVaR_{S_3}$ (p)	$TVaR_p$ (X_1, S_3)	$TVaR_p$ (X_2, S_3)	$TVaR_p$ (X_3, S_3)	K_p (X_1, S_3)	K_p (X_2, S_3)	K_p (X_1, S_3)
90.0 %	14.16	5.53	4.73	3.90	5.56	4.70	3.90
92.5 %	14.84	5.79	4.96	4.09	5.84	4.93	4.07
95.0 %	15.77	6.13	5.29	4.35	6.20	5.23	4.34
97.5 %	17.29	6.70	5.82	4.77	6.82	5.72	4.75
99.0 %	19.20	7.45	6.47	5.28	7.58	6.35	5.27
99.5 %	20.58	8.01	6.94	5.63	8.13	6.80	5.65

In light of Proposition 6.1, $S_3 = X_1 + X_2 + X_3$ follows the mixed Erlang distribution with scale parameter $\beta_{S_3} = 1.95$ and mixing probabilities $\underline{P} = (p_1, p_2, \ldots)$, the first 60 values of $\underline{P}$ are given in Table 6.1. For different tolerance level p, Table 6.2 shows the TVaR of $S_3 = X_1 + X_2 + X_3$ and the allocated capital to each risk under the covariance and the TVaR capital allocation rules.

7. PROOFS

PROOF OF PROPOSITION 3.1 The pdf f of S_2 is given in terms of the joint pdf of (X_1, X_2) as follows

$$f_{S_2}(s) = \int_0^s h(y, s-y)dy.$$

Taking (2.1) into account the pdf of S_2 becomes

$$f_{S_2}(s) = (1+\alpha_{12}\mathcal{L}_1\mathcal{L}_2)\int_0^s f_1(y)f_2(s-y)dy + \alpha_{12}\int_0^s e^{-y}f_1(y)e^{-(s-y)}f_2(s-y)dy$$
$$-\alpha_{12}\mathcal{L}_2\int_0^s e^{-y}f_1(y)f_2(s-y)dy - \alpha_{12}\mathcal{L}_1\int_0^s e^{-(s-y)}f_2(s-y)f_1(y)dy.$$

Let $A(s)$, $B(s)$, $C(s)$, $D(s)$ be the four terms of the expression of $f_{S_2}(s)$ respectively. According to Lemma 5.2,

$$A(s) = (1+\alpha_{12}\mathcal{L}_1\mathcal{L}_2)\int_0^s f_1^{\psi}(s, \beta_2+1, \Psi_1(\mathcal{Q}_1))f_2^{\psi}(s-y, \beta_2+1, \Psi_2(\mathcal{Q}_2))dy$$

and from Lemma 5.3, $A(s)$ can be expressed as a pdf of the mixed Erlang distribution as follows

$$A(s) = (1+\alpha_{12}\mathcal{L}_1\mathcal{L}_2)\sum_{k=1}^{\infty}\pi_k(\Psi_1(\mathcal{Q}_1), \Psi_2(\mathcal{Q}_2))w_k(s, \beta_2+1).$$

In view of Lemma 5.1 and Lemma 5.2, the expression of $B(s)$ becomes

$$B(s) = \alpha_{12}\int_0^s \mathcal{L}_1 f_1^{\theta}(s, \beta_1+1, \Theta_1)\mathcal{L}_2 f_2^{\theta}(s-y, \beta_2+1, \Theta_2)dy$$
$$= \alpha_{12}\mathcal{L}_1\mathcal{L}_2\int_0^s f_1^{\psi}(s, \beta_2+1, \Psi_1(\Theta_1))f_2^{\psi}(s-y, \beta_2+1, \Psi_2(\Theta_2))dy.$$

From Lemma 5.3 one can write $B(s)$ as

$$B(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\Psi_1(\Theta_1), \Psi_2(\Theta_2))w_k(s, \beta_2+1),$$

which is again a pdf of some mixed Erlang distribution. Similarly to $B(s)$, using Lemma 5.1, 5.2 and Lemma 5.3 one can express $C(s)$ and $D(s)$ as pdfs of mixed Erlang distribution as follows

$$C(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\Psi_1(\Theta_1), \Psi_2(\mathcal{Q}_2))w_k(s, \beta_2+1),$$

$$D(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\Psi_1(\mathcal{Q}_1), \Psi_2(\Theta_2)w_k(s, \beta_2+1),$$

hence the claim follows. □

PROOF OF PROPOSITION 4.1 For $j \neq i$, we have

$$
\begin{aligned}
\mathbb{E}(X_i \mathbb{1}_{\{S_2=s\}}) &= \int_0^s y h(y, s-y) dy \\
&= (1 + \alpha_{12}\mathcal{L}_i\mathcal{L}_j) \int_0^s y f_i(y) f_j(s-y) dy \\
&\quad + \alpha_{12} \int_0^s y e^{-y} f_i(y) e^{-(s-y)} f_j(s-y) dy - \alpha_{12}\mathcal{L}_j \int_0^s y e^{-y} f_i(y) \\
&\quad \times f_j(s-y) dy - \alpha_{12}\mathcal{L}_i \int_0^s y f_i(y) e^{-(s-y)} f_j(s-y) dy.
\end{aligned}
$$

Let $A(s)$, $B(s)$, $C(s)$, $D(s)$ be the four terms of the expression of $\mathbb{E}(X_i \mathbb{1}_{\{S_2=s\}})$ respectively. In light of Cossette *et al.* (2013), Lemma 2.5, if $X_i \sim ME(\beta_i, \mathcal{Q}_i)$ then $\frac{x_i f_i(x_i, \beta_i, \mathcal{Q}_i)}{\mathbb{E}(X_i)}$ can be expressed as a pdf of mixed Erlang distribution with mixing probabilities $\mathcal{G}_i(\mathcal{Q}_i) = (g_1, g_2, \ldots)$ where the k-th individual mixing probability is given by

$$
g_k = \begin{cases} 0 & \text{for} \quad k = 1 \\ \frac{(k-1)q_{i,k-1}}{\sum_{j=1}^{k-1} j q_{i,j}} & \text{for} \quad k > 1. \end{cases} \tag{7.1}
$$

If we set $\mu_i := \mathbb{E}(X_i) = \frac{1}{\beta_i} \sum_{k=1}^{\infty} k q_{ik}$, $\gamma := \alpha_{12}\mathcal{L}_1\mathcal{L}_2$, then using (7.1), Lemma 5.1, 5.2 and 5.3, one can write $A(s)$ as

$$
A(s) = (1+\gamma)\mu_i \sum_{k=1}^{\infty} \pi_k\{\Psi_i(\mathcal{G}_i(\mathcal{Q}_i)), \Psi_j(\mathcal{Q}_j)\} w_k(s, \beta_2 + 1).
$$

Setting $\tilde{\mu}_i := \frac{1}{\beta_i + 1} \sum_{k=1}^{\infty} k\theta_{ik}$, in light of (7.1), Lemma 5.1, 5.2 and 5.3, similarly to $A(s)$, we get the expression of the last three terms of $\mathbb{E}(X_i \mathbb{1}_{\{S_2=s\}})$ as follows

$$
B(s) = \gamma \tilde{\mu}_i \sum_{k=1}^{\infty} \pi_k\{\Psi_i(\mathcal{G}_i(\Theta_i)), \Psi_j(\Theta_j)\} w_k(s, \beta_2 + 1),
$$

$$
C(s) = -\gamma \tilde{\mu}_i \sum_{k=1}^{\infty} \pi_k\{\Psi_i(\mathcal{G}_i(\Theta_i)), \Psi_j(\mathcal{Q}_j)\} w_k(s, \beta_2 + 1),
$$

$$
D(s) = -\gamma \mu_i \sum_{k=1}^{\infty} \pi_k\{\Psi_i(\mathcal{G}_i(\mathcal{Q}_i)), \Psi_j(\Theta_j)\} w_k(s, \beta_2 + 1).
$$

Hence, in view of (4.1)

$$TVaR_p(X_i, S_2) = \frac{1}{1-p}\sum_{k=1}^{\infty} z_{i,k}\overline{W}_k(VaR_{S_2}(p), \beta_2+1),$$

where z_{ik} is given in (4.3). Next, by Lemma 5.2, since $\beta_1 \leqslant \beta_2$ we obtain

$$\mathbb{E}(X_i) = \frac{1}{\beta_2+1}\sum_{k=1}^{\infty} k\psi_{i,k},$$

$$Var(X_i) = \frac{1}{(\beta_2+1)^2}\left(\sum_{m=1}^{\infty}(m^2+m)\psi_{i,m} - \left(\sum_{m=1}^{\infty} m\psi_{i,m}\right)^2\right).$$

In light of (A.1), we know that for $i \neq j$

$$Cov(X_i, X_j) = \alpha_{12}\left(\frac{1}{\beta_i+1}\sum_{m=1}^{\infty} mq_{i,m}\overline{\beta_i}^m - \sum_{m=1}^{\infty} q_{i,m}\overline{\beta_i}^m \frac{1}{\beta_i}\sum_{m=1}^{\infty} mq_{i,m}\right)$$
$$\times\left(\frac{1}{\beta_j+1}\sum_{s=1}^{\infty} sq_{j,s}\overline{\beta_j}^s - \sum_{s=1}^{\infty} q_{j,s}\overline{\beta_j}^s \frac{1}{\beta_j}\sum_{s=1}^{\infty} sq_{j,s}\right),$$

Furthermore, Proposition 3.1 and (2.16) yield

$$\mathbb{E}(S_2) = \frac{1}{\beta_2+1}\sum_{k=1}^{\infty} kp_k, \quad Var(S_2) = \frac{1}{(\beta_2+1)^2}\left(\sum_{m=1}^{\infty}(m^2+m)p_m - \left(\sum_{m=1}^{\infty} mp_m\right)^2\right),$$

$$TVaR_{S_2}(p) = \frac{1}{(\beta_2+1)\varphi}\sum_{k=0}^{\infty}\frac{P_k^*((\beta_2+1)VaR_{S_2}(p))^k}{k!} + VaR_{S_2}(p).$$

where

$$\varphi = \sum_{j=1}^{\infty}\frac{P_j((\beta_2+1)VaR_{S_2}(l))^{j-1}}{(j-1)!}, \quad P_k^* = \sum_{j=k}^{\infty} P_j,$$

$$P_j = \frac{\sum_{k=j}^{\infty} p_k}{\sum_{k=1}^{\infty} kp_k}, \quad \textit{and } p_k \textit{ is given in } (3.1).$$

Setting $L_{i,k} := k\psi_{i,k} + \varepsilon_{i,j}(\frac{P_k^*((\beta_2+1)VaR_{S_2}(p))^k}{\varphi k!} + (\beta_2+1)VaR_{S_2}(p) - kp_k)$ and plugging the value of $\mathbb{E}(X_i)$, $Var(X_i)$, $Cov(X_i, X_j)$, $Var(S_2)$, $TVaR_{S_2}(p)$ and $\mathbb{E}(S_2)$ in (4.2), we obtain the desired result for $K_p(X_i, S_2)$ where $\varepsilon_{i,j}$ is given in (4.4). □

PROOF OF LEMMA 5.1 We have

$$\begin{aligned} g^{\theta}(x, \beta+1, \Theta) &= \frac{e^{-x} g(x, \beta, Q)}{\mathcal{L}} \\ &= \sum_{k=1}^{\infty} q_k \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!} \frac{e^{-x}}{\mathcal{L}} \\ &= \sum_{k=1}^{\infty} \frac{q_k \left(\frac{\beta}{\beta+1}\right)^k}{\sum_{j=1}^{\infty} q_j \left(\frac{\beta}{\beta+1}\right)^j} w_k(x, \beta+1) \\ &= \sum_{k=1}^{\infty} \theta_k w_k(x, \beta+1). \end{aligned}$$

□

PROOF OF PROPOSITION 6.1 By definition

$$f_{S_n}(s) = \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{n-2}} h(x_1, x_2, \ldots, s-x_1-\ldots-x_{n-1}) dx_{n-1} \ldots dx_2 dx_1. \quad (7.2)$$

For $C = \{1, \ldots, n\}$, if we decompose the pdf h in (2.10), we obtain

$$\begin{aligned} h(\boldsymbol{x}) = &\Bigg(1 + \sum_{j_1}\sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} + \cdots \\ &+ (-1)^n \alpha_{1,2,\ldots,n} \prod_{i=1}^n \mathcal{L}_i\Bigg) \prod_{i=1}^n f_i(x_i) + \sum_{j_1}\Bigg(-\sum_{j_2} \alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3} \\ &\times \mathcal{L}_{j_2}\mathcal{L}_{j_3} + \cdots + (-1)^{n+1}\alpha_{1,2,\ldots,n} \prod_{i \in C\setminus\{j_1\}} \mathcal{L}_i\Bigg) \tilde{f}_{j_1}(x_{j_1}) \prod_{i\in C\setminus\{j_1\}} f_i(x_i) \\ &+ \sum_{j_1}\sum_{j_2}\Bigg(\alpha_{j_1,j_2} - \sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} + \cdots \\ &+ (-1)^n \alpha_{1,2,\ldots,n} \prod_{i\in C\setminus\{j_1,j_2\}} \mathcal{L}_i\Bigg) \tilde{f}_{j_1}(x_{j_1})\tilde{f}_{j_2}(x_{j_2}) \prod_{i\in C\setminus\{j_1,j_2\}} f_i(x_i) \\ &+ \sum_{j_1}\sum_{j_2}\sum_{j_3}\Bigg(\alpha_{j_1,j_2,j_3} - \sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4} + \sum_{j_4}\sum_{j_5}\alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5} \\ &+ \cdots + (-1)^{n+1}\alpha_{1,2,\ldots,n} \prod_{i\in C\setminus\{j_1,j_2\}} \mathcal{L}_i\Bigg) \tilde{f}_{j_1}(x_{j_1})\tilde{f}_{j_2}(x_{j_2})\tilde{f}_{j_3}(x_{j_3}) \prod_{i\in C\setminus\{j_1,j_2,j_3\}} f_i(x_i) \end{aligned}$$

$$+\cdots+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{n-1}}\left(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n}\right)\tilde{f}_{j_1}(x_{j_1})\times\ldots$$
$$\times\tilde{f}_{j_{n-1}}(x_{j_{n-1}})f_{j_n}(x_{j_n})+\alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\tilde{f}_i(x_i), \tag{7.3}$$

where $j_1\in C$, $j_2\in C\backslash\{j_1\}$, $j_3\in C\backslash\{j_1,j_2\},\ldots,j_n\in C\backslash\{j_1,\ldots,j_{n-1}\}$. Hence, using (7.3), one can express (7.2) as follows

$$f_{S_n}(s)=\Bigg(1+\sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2}-\sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\cdots$$
$$+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\mathcal{L}_i\Bigg)\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}\prod_{i=1}^{n-1}f_i(x_i)$$
$$\times f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1+\sum_{j_1}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2}$$
$$+\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\cdots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{j_1\}}\mathcal{L}_i\Bigg)$$
$$\times\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}\tilde{f}_{j_1}(x_{j_1})\prod_{i\in C\backslash\{j_1\}}f_i(x_i)f_n(s-x_1-\ldots$$
$$-x_{n-1})dx_{n-1}\ldots dx_2dx_1+\sum_{j_1}\sum_{j_2}\Bigg(\alpha_{j_1,j_2}-\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3}$$
$$+\sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4}+\cdots+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{j_1,j_2\}}\mathcal{L}_i\Bigg)$$
$$\times\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}\tilde{f}_{j_1}(x_{j_1})\tilde{f}_{j_2}(x_{j_2})\prod_{i\in C\backslash\{j_1,j_2\}}f_i(x_i)$$
$$\times f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1+\sum_{j_1}\sum_{j_2}\sum_{j_3}\Bigg(\alpha_{j_1,j_2,j_3}$$
$$-\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4}+\sum_{j_4}\sum_{j_5}\alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5}+\cdots+(-1)^{n+1}\alpha_{1,2,\ldots,n}$$
$$\times\prod_{i\in C\backslash\{j_1,j_2\}}\mathcal{L}_i\Bigg)\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}\tilde{f}_{j_1}(x_{j_1})\tilde{f}_{j_2}(x_{j_2})\tilde{f}_{j_3}(x_{j_3})$$

$$\times \prod_{i\in C\setminus\{j_1,j_2,j_3\}} f_i(x_i) f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1+\cdots$$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{n-1}}\left(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n}\right)\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}$$

$$\times \tilde{f}_1(x_1)\times\ldots\times\tilde{f}_{j_{n-1}}(x_{jn-1})f_{jn}(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1$$

$$+\alpha_{1,2,\ldots,n}\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}\prod_{i=1}^{n-1}\tilde{f}_i(x_i)\tilde{f}_n(s-x_1-\ldots$$

$$-x_{n-1})dx_{n-1}\ldots dx_2dx_1.$$

It can be seen that the pdf of S_n is a sum of convolutions of mixed Erlang distributions. Thus, as in the case of S_2, S_n follows a mixed Erlang distribution with scale parameter β_n+1 and mixing probabilities $\underset{\sim}{P}=(p_1,p_2,\ldots)$, we write $S_n\sim ME(\beta_n+1,\underset{\sim}{P})$. For $k\in\mathbb{N}^*$, the k-th component p_k of $\underset{\sim}{P}$ is given in (6.1). □

PROOF OF PROPOSITION 6.3 In view of (4.1) we need to evaluate

$$\mathbb{E}(X_m\mathbb{1}_{\{S_n=s\}})=\int_0^s\int_0^{s-x_1}\cdots\int_0^{s-x_1-\ldots-x_{n-2}}$$
$$\times x_mh(x_1,x_2,\ldots,s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1. \quad (7.4)$$

If we decompose $x_mh(\boldsymbol{x})$, we have

$$x_mh(\boldsymbol{x})=\Bigg(1+\sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2}-\sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\cdots$$

$$+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^n\mathcal{L}_i\Bigg)\Bigg(x_mf_m(x_m)\prod_{i\neq m}f_i(x_i)\Bigg)+\sum_{j_1\neq m}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2}$$

$$+\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\cdots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1\}}\mathcal{L}_i\Bigg)$$

$$\times\Bigg(x_mf_m(x_m)\tilde{f}_{j_1}(x_{j_1})\prod_{i\in C\setminus\{m,j_1\}}f_i(x_i)\Bigg)+\Bigg(-\sum_{j_2\neq m}\alpha_{m,j_2}\mathcal{L}_{j_2}$$

$$+\sum_{j_2\neq m}\sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\cdots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{m\}}\mathcal{L}_i\Bigg)$$

$$\times\Bigg(x_m\tilde{f}_m(x_m)\prod_{i\in C\setminus\{m\}}f_i(x_i)\Bigg)+\sum_{j_1\neq m}\sum_{j_2}\Bigg(\alpha_{j_1,j_2}-\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3}$$

$$+\sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4}+\cdots+(-1)^n\alpha_{1,2,\dots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Bigg)$$

$$\times\left(x_m f_m(x_m)\tilde{f}_{j_1}(x_{j_1})\tilde{f}_{j_2}(x_{j_2})\prod_{i\in C\setminus\{j_1,j_2,m\}}f_i(x_i)\right)$$

$$+\sum_{j_2\neq m}\Bigg(\alpha_{m,j_2}-\sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_3}+\sum_{j_3\neq m}\sum_{j_4\neq m}\alpha_{m,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4}+\cdots$$

$$+(-1)^n\alpha_{1,2,\dots,n}\prod_{i\in C\setminus\{m,j_2\}}\mathcal{L}_i\Bigg)\left(x_m\tilde{f}_m(x_m)\tilde{f}_{j_2}(x_{j_2})\prod_{i\in C\setminus\{m,j_2\}}f_i(x_i)\right)+\cdots$$

$$+\sum_{j_1\neq m}\sum_{j_2}\cdots\sum_{j_{n-1}}(\alpha_{j_1,j_2,\dots,j_{n-1}}-\alpha_{1,2,\dots,n}\mathcal{L}_m)\left(x_m f_m(x_m)\prod_{k=1,\,j_k\neq m}^{n}\tilde{f}_{j_k}(x_{j_k})\right)$$

$$+\sum_{j_2\neq m}\cdots\sum_{j_{n-1}\neq m}(\alpha_{m,j_2,\dots,j_{n-1}}-\alpha_{1,2,\dots,n}\mathcal{L}_{j_n\neq m})$$

$$\times\left(x_m\tilde{f}_m(x_m)f_{j_n}(x_{j_n})\prod_{k=1,\,j_k\neq m}^{n-1}\tilde{f}_{j_k}(x_{j_k})\right)+\alpha_{1,2,\dots,n}x_m\tilde{f}_m(x_m)\prod_{i\neq m}\tilde{f}_i(x_i).\tag{7.5}$$

Plugging (7.5) in (7.4) and using (7.1), Lemma 5.1, 5.2, and 5.3, similarly to the bivariate case one may express (7.4) as follows

$$\mathbb{E}(X_m\mathbb{1}_{\{S_n=s\}})=\sum_{k=1}^{\infty}z_{m,k}\overline{W}_k(VaR_{S_n}(p),\beta_n+1),$$

where $z_{m,k}$ is given in (6.2). Hence, the proof follows easily. □

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APPENDIX A

A. DEPENDENCE MEASURES

Pearson's correlation coefficient has been widely used as a measure of the dependence between two random variables (rv) X_1 and X_2. In this respect, the concept of dependence is assumed to be the linear relationship between the two rv. However, in practice the dependence structure is not always linear hence is why the concept of concordance has been introduced, see e.g., Nelsen (1999), McNeil *et al.* 2005, or Denuit *et al.* (2005). By definition, a rv X_1 is concordant with a rv X_2 if they tend to vary together. The two measures of association of X_1 and X_2, namely Spearman's rho and Kendall's tau are based on this concept. Probabilistically speaking, if (Y_1, Y_2) and (Z_1, Z_2) are independent copies of the pair of continuous random variables (X_1, X_2), then Kendall's tau is defined as

$$\tau(X_1, X_2) = \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) > 0\} - \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) < 0\},$$

and Spearman's rho is defined as

$$\rho_S(X_1, X_2) = 3\{\mathbb{P}[(X_1 - Y_1)(X_2 - Z_2) > 0] - \mathbb{P}[(X_1 - Y_1)(X_2 - Z_2) < 0]\},$$

where Y_1 and Z_2 are independent. If $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \mathcal{Q}_1, \mathcal{Q}_2)$ and further $X_i, i = 1, 2$ has finite mean, then we have:

TABLE A.1

MIXTURE PARAMETERS OF MARGINALS.

X_1		X_2	
k	$q_{1,k}$	k	$q_{2,k}$
1	0.5270	1	0.5050
40	0.0005	8	0.0150
50	0.0020	30	0.0105
75	0.0010	50	0.0020
150	0.0015	70	0.0015
345	0.0005	95	0.0010
902	0.0050	850	0.0055
970	0.4375	995	0.1050
993	0.0250	1000	0.3545

1. **Pearson's correlation coefficient**:
If we set $\eta_{ik} := \frac{1}{\beta_i+1}\sum_{k=1}^{\infty} kq_{i,k}\overline{\beta}_i^k$ and $\Gamma_{ik} := \sum_{k=1}^{\infty} q_{i,k}\overline{\beta}_i^k \mu_i$ for $i = 1, 2$, then by (2.6) Pearson's correlation coefficient of the bivariate SmE risks has an explicit form as

$$\rho_{12}(X_1, X_2) = \frac{\alpha_{12}(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\sigma_1\sigma_2}, \tag{A.1}$$

where μ_i is the expected value of X_i, $i = 1, 2$ and σ_i is its standard deviation.

Remarks A.1. *According to* (2.7)*, the maximal value of Pearson's correlation coefficient of the bivariate SmE risks can be written as follows*

$$\rho_{12}^{\max}(X_1, X_2) = \frac{(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\max\{\mathcal{L}_1(1-\mathcal{L}_2), (1-\mathcal{L}_1)\mathcal{L}_2\}\sigma_1\sigma_2} \tag{A.2}$$

and its minimal value can be expressed as

$$\rho_{12}^{\min}(X_1, X_2) = \frac{-(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\max\{\mathcal{L}_1\mathcal{L}_2, (1-\mathcal{L}_1)(1-\mathcal{L}_2)\}\sigma_1\sigma_2}. \tag{A.3}$$

In the following example, we show that the SmE distribution is flexible as a model for dependent risks.

Example A.2. ***Extremal dependence***
In this example, we analyse the bounds of Pearson's correlation coefficient of a bivariate mixed Erlang distribution with marginals which share the same scale parameter and consist of 9 Erlang components. The mixture parameters are summarized in Table A.1. Figure A.1 presents the lower and the upper bound of Pearson's correlation coefficient as a function of the common scale parameter β. We can see that $\rho_{12}^{\max}$ and $\rho_{12}^{\min}$ tend to reach the extremal dependence case which correspond to values of 1 *and* −1 *respectively. The strongest negative correlation $\rho_{12}^{\min} = -0.87545$ is attained for $\beta = 21.5723$ while the value of $\beta = 153.0315$ yields the maximal positive correlation $\rho_{12}^{\max} = 0.96871$. Hence, not only is the range of the dependence flexible but also wide. Moreover, the simulated values of $\rho_{12}^{\max}$ and $\rho_{12}^{\min}$, presented in dotted red lines in Figure A.1, correspond well with the exact values, this demonstrates again the robustness of our results.*

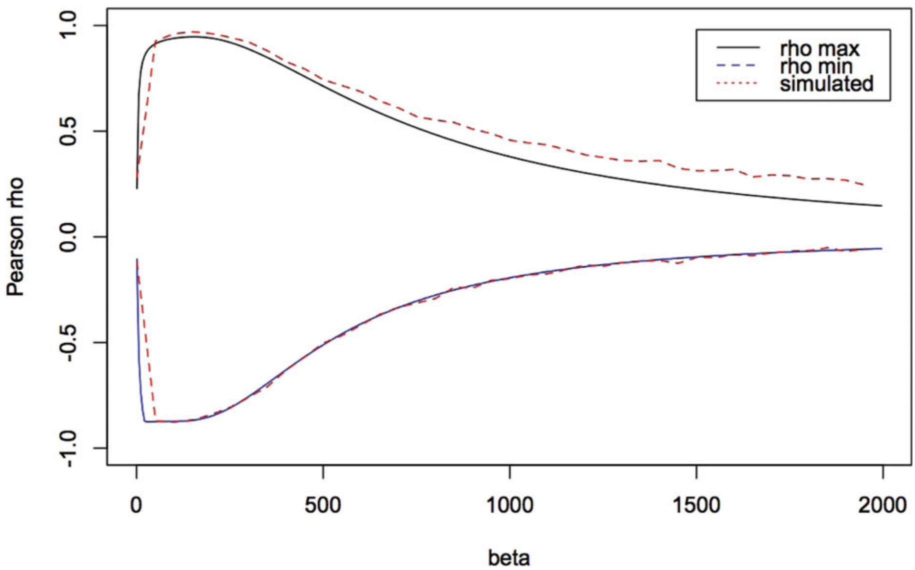

FIGURE A.1: $\rho_{12}^{\max}$ and $\rho_{12}^{\min}$ as a function the common scale parameter β. (Color online)

2. **Spearman's Rho**: Spearman's rho of the bivariate SmE risks can be expressed explicitly as follows

$$\rho_S(X_1, X_2) = 3(1+\gamma) + 6\alpha_{12}[2\zeta_1\zeta_2 - \mathcal{L}_1\zeta_2 - \mathcal{L}_2\zeta_1] - 3, \tag{A.4}$$

where $\zeta_i = \sum_{k=1}^{\infty} q_{i,k}\overline{\beta_i}^k \sum_{m=1}^{\infty}\sum_{j=0}^{k-1} q_{i,m}\binom{j+m-1}{m-1}\frac{\beta_i^m(\beta_i+1)^j}{(2\beta+1)^{m+j}}$, for $i = 1, 2$.

3. **Kendall's Tau**: Kendall's tau of the bivariate SmE is given by the following closed formula

$$\tau(X_1, X_2) = 4\,[(1+\gamma)12(\rho_S(X_1, X_2)+3) + \alpha_{12}\tau_1 - \alpha_{12}\mathcal{L}_2\tau_2 - \alpha_{12}\mathcal{L}_1\tau_3] - 1, \tag{A.5}$$

where $\rho_S(X_1, X_2)$ is Spearman's rho,

$$\tau_1 = (1+\gamma)Z_1Z_2 + \alpha_{12}T_1T_2 - \alpha_{12}\mathcal{L}_1Z_1T_2 - \alpha_{12}\mathcal{L}_2Z_2T_1,$$
$$\tau_2 = \frac{1}{2}(1+\gamma)Z_1 + \alpha_{12}T_1\zeta_2 - \alpha_{12}\mathcal{L}_1Z_1\zeta_2 - \frac{1}{2}\alpha_{12}\mathcal{L}_2T_1,$$
$$\tau_3 = \frac{1}{2}(1+\gamma)Z_2 + \alpha_{12}\zeta_1T_2 - \alpha_{12}\mathcal{L}_2\zeta_1Z_2 - \frac{1}{2}\alpha_{12}\mathcal{L}_1T_2,$$

with

$$Z_i = \sum_{k=1}^{\infty} q_{i,k}\sum_{m=1}^{\infty}\sum_{j=0}^{k-1} q_{i,m}\binom{j+m-1}{m-1}\left(\frac{\beta_i}{2\beta_i+1}\right)^{m+j}, \quad \text{for} \quad i = 1, 2,$$

$$T_i = \sum_{k=1}^{\infty} q_{i,k}\overline{\beta_i}^k\sum_{m=1}^{\infty}\sum_{j=0}^{k-1} q_{i,m}\binom{j+m-1}{m-1}\frac{\beta_i^m(\beta_i+2)^j}{(2\beta_i+2)^{m+j}}, \quad \text{for} \quad i = 1, 2.$$

B. SIMULATION OF SME RISKS

In simulation, in order to remove the dependence between two risks X_1 and X_2, the Rosenblatt transform introduced by Rosenblatt (1952) is widely used. In fact, to simulate X_2 this approach consists in using the conditional quantile function of X_2 given the value of X_1. Hence, the conditional df of X_2 is found accordingly. The following lemma yields how this can be done for the case of the bivariate SmE distribution.

Lemma B.1. *Let $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$, for a given value of X_1 the conditional df of X_2 is described as follows*

$$F_{2|1}(x_2|x_1) = \lambda F_2(x_2, \beta_2, \underset{\sim}{Q}_2) + \alpha_{12}\Delta_1 \sum_{k=1}^{\infty} q_{2,k}\overline{\beta}_2^k W_k(x_2, \beta_2 + 1), \tag{A.1}$$

where

$$\lambda = 1 + \alpha_{12}\mathcal{L}_2(\mathcal{L}_1 - e^{-x1}), \quad \Delta_1 = (e^{-x1} - \mathcal{L}_1).$$

Proof. For a given value of X_1, one can define the conditional distribution function of X_2 as

$$F_{2|1}(x_2|x_1) = \frac{\int_0^{x_2} h(x_1, s)ds}{f_1(x_1)}.$$

According to (2.1)

$$\begin{aligned} h(x_1, s) &= (1 + \alpha_{12}\mathcal{L}_1\mathcal{L}_2) f_1(x_1) f_2(s) + \alpha_{12} e^{-x_1} f_1(x_1) e^{-s} f_2(s) \\ &\quad - \alpha_{12}\mathcal{L}_2 e^{-x_1} f_1(x_1) f_2(s) - \alpha_{12}\mathcal{L}_1 e^{-s} f_2(s) f_1(x_1) \\ &= (1 + \alpha_{12}\mathcal{L}_1\mathcal{L}_2 - \alpha_{12}\mathcal{L}_2 e^{-x_1}) f_1(x_1) f_2(s) \\ &\quad + \alpha_{12}(e^{-x_1} - \mathcal{L}_1) f_1(x_1) e^{-s} f_2(s). \end{aligned}$$

Setting

$$\lambda := 1 + \alpha_{12}\mathcal{L}_2(\mathcal{L}_1 - e^{-x1}) \text{ and } \Delta_1 := e^{-x1} - \mathcal{L}_1,$$

the expression of $h(x_1, s)$ becomes

$$h(x_1, s) = \lambda f_1(x_1) f_2(s) + \alpha_{12}\Delta_1 f_1(x_1) e^{-s} f_2(s).$$

Hence

$$\begin{aligned} F_{2|1}(x_2|x_1) &= \frac{\int_0^{x_2} \lambda f_1(x_1) f_2(s) + \alpha_{12}\Delta_1 f_1(x_1) e^{-s} f_2(s) ds}{f_1(x_1)} \\ &= \lambda \int_0^{x_2} f_2(s)ds + \alpha_{12}\Delta_1 \int_0^{x_2} e^{-s} f_2(s)ds \\ &= \lambda F_2(x_2, \beta_2, \underset{\sim}{Q}_2) + \alpha_{12}\Delta_1 \int_0^{x_2} e^{-s} \sum_{k=1}^{\infty} q_{2k} \frac{\beta_2^k}{(k-1)!} s^{k-1} e^{-\beta_2 s} ds \\ &= \lambda F_2(x_2, \beta_2, \underset{\sim}{Q}_2) + \alpha_{12}\Delta_1 \sum_{k=1}^{\infty} q_{2k} \left(\frac{\beta_2}{\beta_2 + 1}\right)^k W_k(x_2, \beta_2 + 1). \end{aligned}$$

The inverse of $F_{2|1}$ can be computed numerically and as a result the Rosenblatt transform can be implemented efficiently. The simulation algorithm can be summarised as follows:

1. simulate two independent rv u_1 and u_2 uniformly distributed
2. simulate X_1 using the inverse transform: $x_1 = F_1^{-1}(u_1)$
3. simulate X_2 using the Rosenblatt transform: $x_2 = F_{2|1}^{-1}(u_2|x_1)$
4. simulate the aggregate rv $S_2 = X_1 + X_2$.

Remarks B.2. *The result in Lemma B.1 can be generalized for the multivariate case. Specifically, if $(X_1, \ldots, X_n)$ has a multivariate SmE distribution with $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i)$, $i = 1, \ldots, n$, for given values of $X_1, \ldots, X_{n-1}$ one can express the conditional distribution of X_n as follows (set $C := \{1, \ldots, n\}$)*

$$F_{n|1,\ldots,n-1}(x_n|x_1, \ldots, x_{n-1}) = \lambda F_n(x_n, \beta_n, \underset{\sim}{Q}_n) + \Delta \sum_{k=1}^{\infty} q_{n,k} \overline{\beta}_n^k W_k(x_n, \beta_n + 1),$$

where

$$\begin{aligned}\lambda = \frac{1}{D(x_1, \ldots, x_{n-1})} \Bigg\{ &(1+\gamma) + \sum_{j_1 \neq n} \Bigg(- \sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_2} + \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_2} \mathcal{L}_{j_3} \\ &+ \cdots + (-1)^{n+1} \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1\}} \mathcal{L}_i \Bigg) e^{-x_{j_1}} \\ &+ \sum_{j_1 \neq n} \sum_{j_2 \neq n} \Bigg(\alpha_{j_1,j_2} - \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_3} + \sum_{j_3} \sum_{j_4} \alpha_{j_1,j_2,j_3,j_4} \mathcal{L}_{j_3} \mathcal{L}_{j_4} \\ &+ \cdots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1,j_2\}} \mathcal{L}_i \Bigg) e^{-x_{j_1} - x_{j_2}} \\ &+ \cdots + (\alpha_{1,2,\ldots,n-1} - \alpha_{1,2,\ldots,n} \mathcal{L}_n) e^{-x_1 - \ldots - x_{n-1}} \Bigg\},\end{aligned}$$

$$\begin{aligned}\Delta = \frac{1}{D(x_1, \ldots, x_{n-1})} \Bigg\{ &\Bigg(- \sum_{j_2 \neq n} \alpha_{j_2,n} \mathcal{L}_{j_2} + \sum_{j_2 \neq n} \sum_{j_3 \neq n} \alpha_{j_2,j_3,n} \mathcal{L}_{j_2} \mathcal{L}_{j_3} + \cdots + (-1)^{n+1} \\ &\times \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{n\}} \mathcal{L}_i \Bigg) + \sum_{j_2 \neq n} \Bigg(\alpha_{j_2,n} - \sum_{j_3 \neq n} \alpha_{j_2,j_3,n} \mathcal{L}_{j_3} + \sum_{j_3 \neq n} \sum_{j_4 \neq n} \alpha_{j_2,j_3,j_4,n} \mathcal{L}_{j_3} \mathcal{L}_{j_4} + \cdots \\ &+ (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1,n\}} \mathcal{L}_i \Bigg) e^{-x_{j_2}} + \cdots + \sum_{j_1 \neq n} \sum_{j_2 \neq n} \cdots \sum_{j_{n-1} \neq n} (\alpha_{j_1,\ldots,j_{n-1}} - \alpha_{1,\ldots,n} \\ &\times \mathcal{L}_{l, l \in C \setminus \{j_1 \ldots, j_{n-1}\}}) e^{-x_{j_1} - \ldots - x_{j_{n-2}}} + \alpha_{1,2,\ldots,n} e^{-x_1} - \ldots - x_{n-1} \Bigg\},\end{aligned}$$

with

$$D(x_1,\ldots,x_{n-1}) = \Bigg(1 + \sum_{j_1\neq n}\sum_{j_2\neq n}\alpha_{j_1,j_2}(e^{-x_{j_1}} - \mathcal{L}_{j_1})(e^{-x_{j_2}} - \mathcal{L}_{j_2}) + \sum_{j_1\neq n}\sum_{j_2\neq n}\sum_{j_3\neq n}\alpha_{j_1,j_2,j_3} \times (e^{-x_{j_1}} - \mathcal{L}_{j_1})(e^{-x_{j_2}} - \mathcal{L}_{j_2})(e^{-x_{j_3}} - \mathcal{L}_{j_3}) + \cdots + \alpha_{1,2,\ldots,n-1}\prod_{i=1}^{n-1}(e^{-x_i} - \mathcal{L}_i)\Bigg),$$

$$\gamma = \sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} + \cdots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\mathcal{L}_i,$$

$j_1 \in C$, $j_2 \in C\backslash\{j_1\}$, $j_3 \in C\backslash\{j_1, j_2\}, \ldots, j_n \in C\backslash\{j_1,\ldots,j_{n-1}\}$.
Similarily to the simulation of two dependent SmE risks, one can simulate n dependent SmE risks iteratively.