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# Counting Integral Points on Universal Torsors

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Manin's conjecture for the asymptotic behavior of the number of rational points of bounded height on del Pezzo surfaces can be approached through universal torsors. We prove several auxiliary results for the estimation of the number of integral points in certain regions on universal torsors. As an application, we prove Manin's conjecture for a singular quartic del Pezzo surface.

## 1 Introduction

The distribution of rational points on smooth and singular del Pezzo surfaces is predicted by a conjecture of Manin [10]. For a del Pezzo surface $S$ of degree $d \geq 3$ defined over the field $\mathbb{Q}$ of rational numbers, we consider a height function $H$ induced by an anticanonical embedding of $S$ into $\mathbb{P}^d$, where $H(\mathbf{x}) = \max\{|x_0|, \dots, |x_d|\}$ for $\mathbf{x} \in S(\mathbb{Q}) \subset \mathbb{P}^d(\mathbb{Q})$ represented by coprime integral coordinates $x_0, \dots, x_d$. Manin's conjecture makes the following prediction for the asymptotic behavior of the number of rational points of height at most $B$ on the complement $U$ of the lines on $S$. As $B \to \infty$,

$$N_{U,H}(B) = \#\{\mathbf{x} \in U(\mathbb{Q}) \mid H(\mathbf{x}) \leq B\} \sim cB(\log B)^{k-1},$$

where $k$ is the rank of the Picard group of $S$ (resp. of its minimal desingularization if $S$ is a singular del Pezzo surface), and the leading constant $c$ has a conjectural interpretation due to Peyre [13].

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One approach to Manin's conjecture for del Pezzo surfaces uses *universal torsors*. This approach was introduced by Salberger [14] in the case of toric varieties. It also led to the proof of Manin's conjecture for some nontoric del Pezzo surfaces that are *split*, i.e., all of whose lines are defined over $\mathbb{Q}$: quartic del Pezzo surfaces with a singularity of type $\mathbf{D}_5$ [1], $\mathbf{D}_4$ [9] resp. $\mathbf{A}_4$ [3], and a cubic surface with $\mathbf{E}_6$ singularity [2].

These proofs of Manin's conjecture for a split del Pezzo surface $S$ consist of three main steps.

(1) One constructs an explicit bijection between rational points of bounded height on $S$ and integral points in a region on a universal torsor $\mathcal{T}_S$.
(2) Using methods of analytic number theory, one estimates the number of integral points in this region on the torsor by its volume.
(3) One shows that the volume of this region grows asymptotically as predicted by Manin and Peyre.

Step 1 is the focus of joint work with Tschinkel [9, Section 4], giving a geometrically motivated approach to determine a parameterization of the rational points on $S$ by integral points on a universal torsor explicitly.

For step 2, we estimate the number of integral points on the $(k+2)$-dimensional variety $\mathcal{T}_S$ by performing $k+2$ summations over one *torsor variable* after the other; the remaining torsor variables are determined by the *torsor equations* defining $\mathcal{T}_S$ as an affine variety. In each summation, the main problem is to show that an error term summed over the remaining variables gives a negligible contribution (see Section 2 for the error term of the first summation in a certain setting).

For these summations, the previous papers rely on some auxiliary analytic results dealing with the average order of certain arithmetic functions over intervals that are proved in a specific setting. In this paper, we harmonize and generalize many of the analytic tools that have been brought to bear so far (see Figure 2 for an overview of the sets of arithmetic functions that we introduce). We expect that our results can be applied to many different del Pezzo surfaces, at least to cover the more standard bits of the argument. This will allow future work on Manin's conjecture for del Pezzo surfaces to concentrate on the essential difficulties in the estimation of some of the error terms, without having to reimplement the routine parts.

As an application of our general techniques, we prove Manin's conjecture in a new case: a quartic del Pezzo surface with singularity type $\mathbf{A}_3+\mathbf{A}_1$ (Section 8). This example also demonstrates how we can deal with a new geometric feature. In the final $k$ summations, the previous proofs of Manin's conjecture for split del Pezzo surfaces made crucial use of the fact that the nef cone (the dual of the effective cone with respect to the intersection form) is simplicial (in the quartic $\mathbf{D}_5$ and $\mathbf{D}_4$ cases and in the cubic $\mathbf{E}_6$ case) or at least the difference of two simplicial cones (in the quartic $\mathbf{A}_4$ case). The nef cone of the quartic surface treated here has neither of these shapes. However, the techniques introduced in Section 4 are not sensitive to the shape of the nef cone. In our example, they allow us to handle the final $k+1=7$ summations at the same time.

In fact, we expect that the techniques of Section 4 will cover the final $k$ summations for any del Pezzo surface. This would narrow down the main difficulty of the universal torsor strategy to the estimation of the error term in the first and second summations of step 2. For example, in recent joint work with Browning, a proof of Manin's conjecture for a cubic surface with $\mathbf{D}_5$ singularity [4], we make extensive use of the results in this paper to handle the final seven of nine summations, so that we can focus on the considerable additional technical effort that is needed to estimate the first two error terms.

Step 3 is mixed with the second step in the basic examples of the quartic $\mathbf{D}_5$ [1], $\mathbf{D}_4$ [9], and cubic $\mathbf{E}_6$ [2] surfaces. However, it seems more natural to treat the third step separately in more complicated cases, motivated by the shape of the polytope whose volume appears in the leading constant. First examples of this can be found in the treatment of the quartic $\mathbf{A}_4$ [3] and cubic $\mathbf{D}_5$ [4] surfaces, and we take the same approach in our example in Section 8.

## 2 The First Summation

Let $S \subset \mathbb{P}^d$ be an anticanonically embedded singular del Pezzo surface of degree $d \geq 3$, with minimal desingularization $\widetilde{S}$. The first step of the universal torsor approach is to translate the counting problem from rational points on $S$ to integral points on a universal torsor $\mathcal{T}_{\widetilde{S}}$. Then the number $N_{U,H}(B)$ of rational points of height at most $B$ on the complement $U$ of the lines on $S$ is the number of integral solutions to the equations defining $\mathcal{T}_{\widetilde{S}}$ that satisfy certain explicit coprimality conditions and height conditions.

In several cases (see Remark 2.1), the counting problem on $\mathcal{T}_{\widetilde{S}}$ has the following *special form*: $N_{U,H}(B)$ equals the number of $(\alpha_0, \beta_0, \gamma_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta)$ satisfying

- $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{Z}_* \times \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}_*$ is $\mathbb{Z}$ or $\mathbb{Z}_{\neq 0}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}_{>0}^r$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s) \in \mathbb{Z}_{>0}^s$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_t) \in \mathbb{Z}_{>0}^t$, $\delta \in \mathbb{Z}_{>0}$;
- one torsor equation of the form

$$\alpha_0^{a_0}\alpha_1^{a_1}\cdots\alpha_r^{a_r} + \beta_0^{b_0}\beta_1^{b_1}\cdots\beta_s^{b_s} + \gamma_0\gamma_1^{c_1}\cdots\gamma_t^{c_t} = 0, \tag{2.1}$$

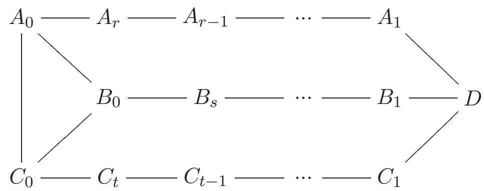


**Fig. 1.** Extended Dynkin diagram.

with $(a_0, \dots, a_r) \in \mathbb{Z}_{>0}^{r+1}$, $(b_0, \dots, b_s) \in \mathbb{Z}_{>0}^{s+1}$, $(c_1, \dots, c_t) \in \mathbb{Z}_{>0}^{t}$. In particular, $\gamma_0$ appears linearly in the torsor equation, while $\delta$ does not appear;

- height conditions that are written independently of $\gamma_0$ (which can be achieved using (2.1)) as

$$h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \leq 1, \tag{2.2}$$

  for some function $h : \mathbb{R}^{r+s+t+3} \times \mathbb{R}_{\geq 3} \to \mathbb{R}$. We assume that $h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \leq 1$ if and only if $\beta_0$ is in a union of finitely many intervals $I_1, \dots, I_n$ whose number $n = n(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B)$ is bounded independently of $\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta$, and $B$. By adding some empty intervals if necessary, we may assume that $n$ does not depend on $\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta$, and $B$. For $j = 1, \dots, n$, let $t_{0,j}, t_{1,j}$ be the start and end point of $I_j$;
- coprimality conditions that are described by Figure 1 in the following sense. Let $A_i$ (resp. $B_i, C_i, D$) correspond to $\alpha_i$ (resp. $\beta_i, \gamma_i, \delta$). Then two coordinates are required to be coprime if and only if the corresponding vertices in Figure 1 are not connected by an edge. For variables corresponding to triples of pairwise connected symbols (besides $A_0, B_0, C_0$, this happens for triples consisting of $D$ and two of $A_0, B_0, C_0$ if at least two of $r, s, t$ vanish), we assume that $\alpha_0, \beta_0, \gamma_0$ are allowed to have any common factor, while each prime dividing $\delta$ may divide at most one of $\alpha_0, \beta_0, \gamma_0$.

**Remark 2.1.** The geometric background of this special form is as follows. A natural realization of a universal torsor $\mathcal{T}_{\widetilde{S}}$ as an open subset of an affine variety is provided by

$$\mathcal{T}_{\widetilde{S}} \hookrightarrow \operatorname{Spec}(\operatorname{Cox}(\widetilde{S}))$$

[11, Theorem 5.6]. The coordinates of the affine variety $\operatorname{Spec}(\operatorname{Cox}(\widetilde{S}))$ correspond to generators of the Cox ring of $\widetilde{S}$.

**Table 1** Extended Dynkin diagrams in [6].

| Degree | Shape of Figure 1 | Different shape |
|---|---|---|
| 6 | $\mathbf{A}_1$, $\mathbf{A}_2$ | – |
| 5 | $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ | $\mathbf{A}_1$ |
| 4 | $\mathbf{A}_3$, $\mathbf{A}_3+\mathbf{A}_1$, $\mathbf{A}_4$, $\mathbf{D}_4$, $\mathbf{D}_5$ | $3\,\mathbf{A}_1$, $\mathbf{A}_2+\mathbf{A}_1$ |
| 3 | $\mathbf{A}_4+\mathbf{A}_1$, $\mathbf{A}_5+\mathbf{A}_1$, $\mathbf{D}_4$, $\mathbf{D}_5$, $\mathbf{E}_6$ | $\mathbf{A}_3+2\mathbf{A}_1$, $2\mathbf{A}_2+\mathbf{A}_1$ |

In [6], we have classified singular del Pezzo surfaces $S$ of degree $d \geq 3$ where $\operatorname{Spec}(\operatorname{Cox}(\widetilde{S}))$ is defined by precisely one *torsor equation*. It includes the *extended Dynkin diagrams* describing the configuration of the divisors on $\widetilde{S}$ that correspond to the generators of $\operatorname{Cox}(\widetilde{S})$. In many cases, the extended Dynkin diagram has the special shape of Figure 1 (see Table 1 for their singularity types). In all cases, besides one of the two isomorphy classes of cubic surfaces of type $\mathbf{D}_4$, the torsor equation has the form of equation (2.1).

If we construct the bijection between rational points on $S$ and integral points on $\mathcal{T}_{\widetilde{S}}$ using the geometrically motivated approach of [9, Section 4], then we expect to obtain coprimality conditions that are encoded in the extended Dynkin diagram.

Indeed, in the quartic $\mathbf{D}_4$ [9], $\mathbf{A}_4$ [3], and the cubic $\mathbf{D}_5$ [4] cases, both the extended Dynkin diagram and the counting problem have the special form. In the quartic $\mathbf{D}_5$ [1] and cubic $\mathbf{E}_6$ [2] cases, the extended Dynkin diagram has the shape of Figure 1, but the coprimality conditions are different. The reason is that the bijection between rational points on the del Pezzo surface and integral points on a universal torsor is constructed by ad hoc manipulations of the defining equations. If one uses the method of [9, Section 4] instead, the coprimality conditions turn out in the expected shape. □

Given a counting problem of the special form above, we show in the remainder of this section how to perform a first step toward estimating $N_{U,H}(B)$. This will result in Proposition 2.4.

Our first step can be described as follows, ignoring the coprimality conditions for the moment. We determine the number of $\beta_0, \gamma_0$ satisfying the torsor equation (2.1), while the other coordinates are fixed. For any $\beta_0$ satisfying

$$\alpha_0^{a_0}\alpha_1^{a_1}\cdots\alpha_r^{a_r} \equiv -\beta_0^{b_0}\beta_1^{b_1}\cdots\beta_s^{b_s} \pmod{\gamma_1^{c_1}\cdots\gamma_t^{c_t}},$$

there is a unique $\gamma_0$ such that (2.1) holds. Our assumption that the height conditions are written as $h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \leq 1$ (independently of $\gamma_0$) has the advantage that the number of $\beta_0, \gamma_0$ subject to (2.1) and (2.2) is the number of integers $\beta_0$ that lie in a certain subset $I$ of the real numbers described by this height condition and satisfy the congruence above. If $b_0 = 1$, one expects that this number is the measure of $I$ divided by the modulus $\gamma_1^{c_1} \cdots \gamma_t^{c_t}$, with an error of $O(1)$.

Before coming to the details of this argument, we reformulate the coprimality conditions.

**Definition 2.2.** Let

$$\Pi(\boldsymbol{\alpha}) = \alpha_1^{a_1} \cdots \alpha_r^{a_r}, \qquad \Pi'(\delta, \boldsymbol{\alpha}) = \begin{cases} \delta\alpha_1 \cdots \alpha_{r-1}, & r \geq 1, \\ 1, & r = 0, \end{cases}$$

and we define $\Pi(\boldsymbol{\beta}), \Pi'(\delta, \boldsymbol{\beta}), \Pi(\boldsymbol{\gamma}), \Pi'(\delta, \boldsymbol{\gamma})$ analogously. □

**Lemma 2.3.** Assume that $(\alpha_0, \beta_0, \gamma_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) \in \mathbb{Z}^{r+s+t+4}$ satisfies the torsor equation (2.1).

The coprimality conditions described by Figure 1 hold if and only if

$$\gcd(\alpha_0, \Pi'(\delta, \boldsymbol{\alpha})\Pi(\boldsymbol{\beta})\Pi(\boldsymbol{\gamma})) = 1, \tag{2.3}$$

$$\gcd(\beta_0, \Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha})) = 1, \tag{2.4}$$

$$\gcd(\gamma_0, \Pi'(\delta, \boldsymbol{\gamma})) = 1, \tag{2.5}$$

$$\text{coprimality conditions for } \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta \text{ as in Figure 1 hold.} \tag{2.6}$$

□

**Proof.** We must show that conditions (2.3)–(2.6) together with (2.1) imply $\gcd(\beta_0, \Pi(\boldsymbol{\gamma})) = 1$ and $\gcd(\gamma_0, \Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta})) = 1$.

Suppose a prime $p$ divides $\gamma_0, \Pi(\boldsymbol{\alpha})$, i.e., $p$ divides the first and third terms of (2.1). Then $p$ also divides the second term, $\beta_0^{b_0}\Pi(\boldsymbol{\beta})$. However, by (2.4) and (2.6), we have $\gcd(\beta_0^{b_0}\Pi(\boldsymbol{\beta}), \Pi(\boldsymbol{\alpha})) = 1$. The remaining statements are proved analogously. ■

For fixed $B \in \mathbb{R}_{\geq 3}$ and $(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) \in \mathbb{Z}_* \times \mathbb{Z}_{>0}^{r+s+t+1}$ subject to (2.3) and (2.6), let $N_1 = N_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B)$ be the number of $\beta_0, \gamma_0$ subject to the torsor equation (2.1), the coprimality conditions (2.4) and (2.5), and the height condition $h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \leq 1$. Then

$$N_{U,H}(B) = \sum_{\substack{(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) \in \mathbb{Z}_* \times \mathbb{Z}_{>0}^{r+s+t+1} \\ (2.3),\ (2.6) \text{ hold}}} N_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B).$$

Our goal is to find an estimation for $N_1$, with an error term whose sum over $\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta$ is small.

First, we remove (2.5) by a Möbius inversion to obtain that

$$N_1 = \sum_{k_c \mid \Pi'(\delta,\boldsymbol{\gamma})} \mu(k_c) \# \left\{ \beta_0, \gamma_0' \in \mathbb{Z} \;\middle|\; \begin{array}{l} \alpha_0^{a_0}\Pi(\boldsymbol{\alpha}) + \beta_0^{b_0}\Pi(\boldsymbol{\beta}) + k_c\gamma_0'\Pi(\boldsymbol{\gamma}) = 0, \\ (2.4),\ h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1 \end{array} \right\}.$$

The torsor equation determines $\gamma_0'$ uniquely if a congruence is fulfilled, so

$$N_1 = \sum_{k_c \mid \Pi'(\delta,\boldsymbol{\gamma})} \mu(k_c) \# \left\{ \beta_0 \in \mathbb{Z} \;\middle|\; \begin{array}{l} \alpha_0^{a_0}\Pi(\boldsymbol{\alpha}) \equiv -\beta_0^{b_0}\Pi(\boldsymbol{\beta}) \pmod{k_c\Pi(\boldsymbol{\gamma})}, \\ (2.4),\ h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1 \end{array} \right\}.$$

This congruence cannot be fulfilled unless $\gcd(k_c, \alpha_0\Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta})) = 1$. Indeed, if a prime $p$ divides $k_c$ and $\alpha_0^{a_0}\Pi(\boldsymbol{\alpha})$, then it divides also $\beta_0^{b_0}\Pi(\boldsymbol{\beta})$, but $\gcd(\Pi(\boldsymbol{\alpha}), \beta_0^{b_0}\Pi(\boldsymbol{\beta})) = 1$ by (2.4) and (2.6), while $\gcd(\alpha_0, \Pi(\boldsymbol{\beta})) = 1$ by (2.3), and $p \mid k_c, \alpha_0, \beta_0$ is impossible because of (2.3) and since $p \mid \delta, \alpha_0, \beta_0$ is not allowed by assumption; $p$ dividing $k_c$ and $\Pi(\boldsymbol{\beta})$ can be excluded similarly. Therefore, we may add the restriction $\gcd(k_c, \alpha_0\Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta})) = 1$ to the summation over $k_c$ without changing the result, so that

$$N_1 = \sum_{\substack{k_c \mid \Pi'(\delta,\boldsymbol{\gamma}) \\ \gcd(k_c, \alpha_0\Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta}))=1}} \mu(k_c) N_1(k_c),$$

where

$$N_1(k_c) = \# \left\{ \beta_0 \in \mathbb{Z} \;\middle|\; \begin{array}{l} \alpha_0^{a_0}\Pi(\boldsymbol{\alpha}) \equiv -\beta_0^{b_0}\Pi(\boldsymbol{\beta}) \pmod{k_c\Pi(\boldsymbol{\gamma})} \\ (2.4),\ h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1 \end{array} \right\}.$$

We note that both $\alpha_0^{a_0}\Pi(\boldsymbol{\alpha})$ and $\Pi(\boldsymbol{\beta})$ are coprime to $k_c\Pi(\boldsymbol{\gamma})$. Indeed, we have $\gcd(k_c, \alpha_0\Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta})) = 1$ by the restriction on $k_c$ just introduced, and $\gcd(\Pi(\gamma)),$ $\alpha_0\Pi(\alpha)\Pi(\beta)) = 1$ by (2.3) and (2.6).

We choose integers $A_1, A_2$ resp. $B_1, B_2$ depending only on $\alpha_0, \boldsymbol{\alpha}$ resp. $\boldsymbol{\beta}$, such that

$$A_1 A_2^{b_0} = \alpha_0^{a_0}\Pi(\boldsymbol{\alpha}), \quad B_1 B_2^{b_0} = \Pi(\boldsymbol{\beta}). \tag{2.7}$$

For example,

$$A_1 = \alpha_0^{a_0}\Pi(\boldsymbol{\alpha}), \quad A_2 = 1, \quad B_1 = \Pi(\boldsymbol{\beta}), \quad B_2 = 1$$

is one valid choice. Often it turns out to be convenient to move coordinates to $A_2$ that occur to a power of $b_0$ in $\alpha_0^{a_0}\Pi(\boldsymbol{\alpha})$; similarly for $B_2$.

Then $A_1, A_2, B_1, B_2$ are coprime to $k_c\Pi(\boldsymbol{\gamma})$. For each $\beta_0$ satisfying

$$\alpha_0^{a_0}\Pi(\boldsymbol{\alpha}) \equiv -\beta_0^{b_0}\Pi(\boldsymbol{\beta}) \pmod{k_c\Pi(\boldsymbol{\gamma})},$$

there is a unique $\varrho \in \{1, \dots, k_c\Pi(\boldsymbol{\gamma})\}$ satisfying

$$\gcd(\varrho, k_c\Pi(\boldsymbol{\gamma})) = 1, \quad A_1 \equiv -\varrho^{b_0} B_1 \pmod{k_c\Pi(\boldsymbol{\gamma})} \tag{2.8}$$

and

$$\beta_0 B_2 \equiv \varrho A_2 \pmod{k_c\Pi(\boldsymbol{\gamma})}.$$

This shows that

$$N_1(k_c) = \sum_{\substack{1\le\varrho\le k_c\Pi(\boldsymbol{\gamma})\\ (2.8) \text{ holds}}} \# \left\{ \beta_0 \in \mathbb{Z} \;\middle|\; \begin{array}{l} \beta_0 B_2 \equiv \varrho A_2 \pmod{k_c\Pi(\boldsymbol{\gamma})} \\ (2.4),\ h(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1 \end{array} \right\}.$$

We remove the coprimality condition (2.4) on $\beta_0$ by another Möbius inversion; writing $\beta_0 = k_b\beta_0'$, we get

$$N_1(k_c) = \sum_{\substack{1\le\varrho\le k_c\Pi(\boldsymbol{\gamma})\\ (2.8) \text{ holds}}} \sum_{k_b \mid \Pi'(\delta,\boldsymbol{\beta})\Pi(\boldsymbol{\alpha})} \mu(k_b) N_1(\varrho, k_b, k_c),$$

with

$$N_1(\varrho, k_b, k_c) = \# \left\{ \beta_0' \in \mathbb{Z} \;\middle|\; \begin{array}{l} k_b\beta_0' B_2 \equiv \varrho A_2 \pmod{k_c\Pi(\boldsymbol{\gamma})} \\ h(\alpha_0, k_b\beta_0', \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1 \end{array} \right\}.$$

Here, we may restrict to $k_b$ satisfying $\gcd(k_b, k_c\Pi(\boldsymbol{\gamma})) = 1$ because otherwise $\gcd(\varrho A_2, k_c\Pi(\boldsymbol{\gamma})) = 1$ implies that $N_1(\varrho, k_b, k_c) = 0$. We note that we have $\gcd(k_b B_2, k_c\Pi(\boldsymbol{\gamma})) = 1$ after this restriction.

We recall that $\{t \in \mathbb{R} \mid h(\alpha_0, t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1\}$ is assumed to consist of intervals $I_1, \dots, I_n$, with $I_j$ starting at $t_{0,j}$ and ending at $t_{1,j}$. Let $\psi(t) = \{t\} - 1/2$, where $\{t\}$ is the fractional part of $t \in \mathbb{R}$. For $j = 1, \dots, n$, by [1, Lemma 3],

$$\#\left\{\beta_0' \in \mathbb{Z} \,\middle|\, \begin{aligned} & k_b\beta_0'B_2 \equiv \varrho A_2 \pmod{k_c\Pi(\boldsymbol{\gamma})}, \\ & k_b\beta_0' \in I_j \end{aligned}\right\}$$
$$= \frac{t_{1,j} - t_{0,j}}{k_bk_c\Pi(\boldsymbol{\gamma})} + \psi\left(\frac{k_b^{-1}t_{0,j} - \varrho A_2\overline{k_bB_2}}{k_c\Pi(\boldsymbol{\gamma})}\right) - \psi\left(\frac{k_b^{-1}t_{1,j} - \varrho A_2\overline{k_bB_2}}{k_c\Pi(\boldsymbol{\gamma})}\right),$$

where $t_{0,j}, t_{1,j}$ (depending on $\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta$, and $B$) are the start and end points of $I_j$, and $\overline{x}$ is the multiplicative inverse modulo $k_c\Pi(\boldsymbol{\gamma})$ of an integer $x$ coprime to $k_c\Pi(\boldsymbol{\gamma})$.

We define

$$V_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) = \int_{h(\alpha_0, t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1} \frac{1}{\Pi(\boldsymbol{\gamma})}\, \mathrm{d}t. \tag{2.9}$$

The sum of the lengths of the intervals $I_1, \dots, I_n$ is $\Pi(\boldsymbol{\gamma})V_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B)$, so

$$N_1(\varrho, k_b, k_c) = \frac{1}{k_bk_c}V_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) + R_1(\varrho, k_b, k_c),$$

with

$$R_1(\varrho, k_b, k_c) = \sum_{j=1}^{n}\sum_{i \in \{0,1\}}(-1)^i\psi\left(\frac{k_b^{-1}t_{i,j} - \varrho A_2\overline{k_bB_2}}{k_c\Pi(\boldsymbol{\gamma})}\right).$$

Tracing through the argument gives the following estimation for $N_{U,H}(B)$, where, for any $n \in \mathbb{Z}_{>0}$, $\phi^*(n) = \frac{\phi(n)}{n} = \prod_{p \mid n}(1 - 1/p)$ and $\omega(n)$ is the number of distinct prime factors of $n$.

**Proposition 2.4.** If the counting problem has the special form described at the beginning of this section, then

$$N_{U,H}(B) = \sum_{\substack{(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) \in \mathbb{Z}_* \times \mathbb{Z}_{>0}^{r+s+t+1} \\ (2.3),\ (2.6)\text{ holds}}} N_1,$$

with

$$N_1 = \vartheta_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta)V_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) + R_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B),$$

where $V_1$ is defined by (2.9) and, with $A_1, A_2, B_1, B_2$ as in (2.7),

$$\vartheta_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) = \sum_{\substack{k_c \mid \Pi'(\delta, \boldsymbol{\gamma}) \\ \gcd(k_c, \alpha_0 \Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta}))=1}} \frac{\mu(k_c)\phi^*(\Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}))}{k_c \phi^*(\gcd(\Pi'(\delta, \boldsymbol{\beta}), k_c \Pi(\boldsymbol{\gamma})))} \sum_{\substack{1 \le \varrho \le k_c \Pi(\boldsymbol{\gamma}) \\ (2.8) \text{ holds}}} 1$$

and

$$R_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) = \sum_{\substack{k_c \mid \Pi'(\delta, \boldsymbol{\gamma}) \\ \gcd(k_c, \alpha_0 \Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta}))=1}} \mu(k_c) \sum_{\substack{k_b \mid \Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}) \\ \gcd(k_b, k_c \Pi(\boldsymbol{\gamma}))=1}} \mu(k_b) \times \sum_{\substack{1 \le \varrho \le k_c \Pi(\boldsymbol{\gamma}) \\ (2.8) \text{ holds}}} \sum_{j=1}^{n} \sum_{i \in \{0,1\}} (-1)^i \psi\left(\frac{k_b^{-1} t_{i,j} - \varrho A_2 \overline{k_b B_2}}{k_c \Pi(\boldsymbol{\gamma})}\right).$$

We have $R_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) = 0$ if $h(\alpha_0, t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) > 1$ for all $t \in \mathbb{R}$, while

$$R_1(\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \ll 2^{\omega(\Pi'(\delta, \boldsymbol{\gamma}))} 2^{\omega(\Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}))} b_0^{\omega(\delta \Pi(\boldsymbol{\gamma}))}$$

otherwise. □

**Proof.** For the main term, we note that $\vartheta_1$ is

$$\sum_{\substack{k_c \mid \Pi'(\delta, \boldsymbol{\gamma}) \\ \gcd(k_c, \alpha_0 \Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta}))=1}} \frac{\mu(k_c)}{k_c} \sum_{\substack{1 \le \varrho \le k_c \Pi(\boldsymbol{\gamma}) \\ (2.8) \text{ holds}}} \sum_{\substack{k_b \mid \Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}) \\ \gcd(k_b, k_c \Pi(\boldsymbol{\gamma}))=1}} \frac{\mu(k_b)}{k_b}$$
$$= \sum_{\substack{k_c \mid \Pi'(\delta, \boldsymbol{\gamma}) \\ \gcd(k_c, \alpha_0 \Pi(\boldsymbol{\alpha})\Pi(\boldsymbol{\beta}))=1}} \frac{\mu(k_c)\phi^*(\Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}))}{k_c \phi^*(\gcd(\Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}), k_c \Pi(\boldsymbol{\gamma})))} \sum_{\substack{1 \le \varrho \le k_c \Pi(\boldsymbol{\gamma}) \\ (2.8) \text{ holds}}} 1$$

and use $\gcd(\Pi(\boldsymbol{\alpha}), k_c \Pi(\boldsymbol{\gamma})) = 1$ by (2.6) and the assumption on $k_c$.

Our discussion before the statement of this result immediately gives the explicit formula for the error term $R_1$. Additionally, we note that both $N_1$ and $V_1$ vanish if $h(\alpha_0, t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta) > 1$ for all $t \in \mathbb{R}$. Otherwise, we estimate the inner sums over $j, i$ by $O(1)$. The total error is

$$\ll \sum_{k_c \mid \Pi'(\delta, \boldsymbol{\gamma})} |\mu(k_c)| \sum_{k_b \mid \Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha})} |\mu(k_b)| b_0^{\omega(k_c \Pi(\boldsymbol{\gamma}))}$$
$$\ll 2^{\omega(\Pi'(\delta, \boldsymbol{\gamma}))} 2^{\omega(\Pi'(\delta, \boldsymbol{\beta})\Pi(\boldsymbol{\alpha}))} b_0^{\omega(\delta \Pi(\boldsymbol{\gamma}))},$$

since (2.8) has at most $b_0^{\omega(k_c \Pi(\boldsymbol{\gamma}))}$ solutions $\varrho$ with $1 \le \varrho \le k_c \Pi(\boldsymbol{\gamma})$. ■

In this estimation of $N_1$, we expect that $\vartheta_1 V_1$ is the main term and $R_1$ is the error term. It is sometimes possible (see Lemma 8.4 for an example) to show that the crude bound for $R_1$ at the end of Proposition 2.4 summed over all $\alpha_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta$ for which there is a $t \in \mathbb{R}$ with $h(\alpha_0, t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \delta; B) \le 1$ gives a total contribution of $o(B(\log B)^{k-1})$. In other cases, this is impossible, and one has to show that there is additional cancellation when summing the precise expression for $R_1$ of Proposition 2.4 over the remaining variables (see [4], for example).

## 3 Another Summation

As the main result of this section, we show under certain conditions how to sum an expression such as the main term of Proposition 2.4 over another coordinate (Proposition 3.9 and Proposition 3.10).

In this section, we will start to define several sets $\Theta_i$ of real-valued functions in one variable and, for any $r \in \mathbb{Z}_{>0}$, several sets $\Theta_{j,r}$ and $\Theta'_{j,r}$ of real-valued functions in $r$ variables. We will be interested in the average order of these functions when summed over intervals.

Figure 2 gives an overview of the relations between these sets of functions, for appropriate constants $C, C', C'', C_1, C_2, C_3 \in \mathbb{R}_{\ge 0}$, and $b \in \mathbb{Z}_{>0}$, where each arrow denotes an inclusion. In case of an arrow from a set $\Theta_{j,r}$ to a set $\Theta_i$, we regard the functions in the first set as functions in one of the variables.

**Lemma 3.1.** Let $\vartheta : \mathbb{Z} \to \mathbb{R}$ be any function for which there exist $c \in \mathbb{R}_{\ge 0}$ and a function $E : \mathbb{R} \to \mathbb{R}$ such that, for all $t \in \mathbb{R}_{\ge 0}$,

$$\sum_{0<n\le t} \vartheta(n) = ct + E(t).$$

Let $t_1, t_2 \in \mathbb{R}_{\ge 0}$, with $t_1 \le t_2$. Let $g : [t_1, t_2] \to \mathbb{R}$ be a function that has a continuous derivative whose sign changes only $R(g)$ times on $[t_1, t_2]$. Then

$$\sum_{t_1<n\le t_2} \vartheta(n)g(n) = c\int_{t_1}^{t_2} g(t)\,\mathrm{d}t + O\left((R(g)+1)\left(\sup_{t_1\le t\le t_2} |E(t)|\right)\left(\sup_{t_1\le t\le t_2} |g(t)|\right)\right). \qquad \square$$

**Proof.** The proof is similar to [3, Lemma 2]. For any $t \in \mathbb{R}_{\ge 0}$, let

$$M(t) = \sum_{0<n\le t} \vartheta(n), \qquad S(t_1, t_2) = \sum_{t_1\le n\le t_2} \vartheta(n)g(n).$$

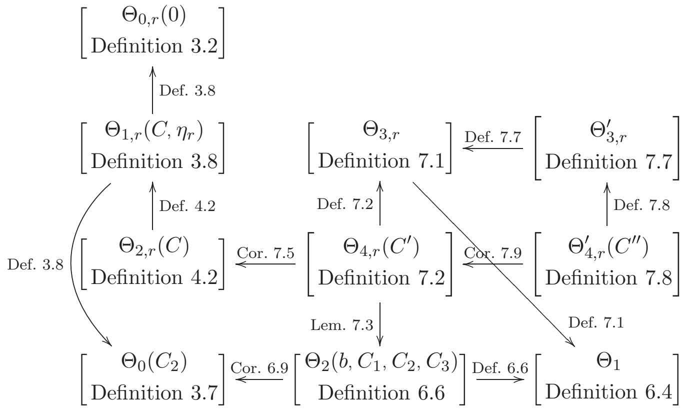


**Fig. 2.** Relations between our sets of functions.

Using partial summation, the estimate for $M(t)$ and integration by parts, $S(t_1, t_2)$ is

$$\begin{aligned}
&M(t_2)g(t_2) - M(t_1)g(t_1) - \int_{t_1}^{t_2} M(t)g'(t)\,\mathrm{d}t \\
&= c\int_{t_1}^{t_2} g(t)\,\mathrm{d}t + E(t_2)g(t_2) - E(t_1)g(t_1) - \int_{t_1}^{t_2} E(t)g'(t)\,\mathrm{d}t \\
&= c\int_{t_1}^{t_2} g(t)\,\mathrm{d}t + O\left(\left(\sup_{t_1\le t\le t_2} |E(t)|\right)\left(|g(t_1)| + |g(t_2)| + \int_{t_1}^{t_2} |g'(t)|\,\mathrm{d}t\right)\right).
\end{aligned}$$

The result follows once we split $[t_1, t_2]$ into $R(g) + 1$ intervals where the sign of $g'$ does not change. ■

**Definition 3.2.** Let $C \in \mathbb{R}_{\geq 0}$. Let $\Theta_{0,0}(C)$ be the set $\mathbb{R}$ of real numbers. For any $r \in \mathbb{Z}_{>0}$, we define $\Theta_{0,r}(C)$ recursively as the set of all nonnegative functions $\vartheta : \mathbb{Z}_{>0}^r \to \mathbb{R}$ with the following property. For any $i \in \{1, \ldots, r\}$, there is $\vartheta_i \in \Theta_{0,r-1}(C)$ such that, for any $t \in \mathbb{R}_{\geq 0}$,

$$\sum_{0<\eta_i\le t} \vartheta(\eta_1, \ldots, \eta_r) \le \vartheta_i(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_r) \cdot t(\log(t+2))^C.$$

For any $\vartheta \in \Theta_{0,r}(C)$ and $i = 1, \ldots, r$, we fix a function $\vartheta_i \in \Theta_{0,r-1}(C)$ as above and denote it by $\mathcal{M}(\vartheta(\eta_1, \ldots, \eta_r), \eta_i)$. For any pairwise distinct $i_1, \ldots, i_n \in \{1, \ldots, r\}$, let

$$\mathcal{M}\big(\vartheta(\eta_1, \ldots, \eta_r), \eta_{i_1}, \ldots, \eta_{i_n}\big) = \mathcal{M}\big(\cdots\mathcal{M}\big(\vartheta(\eta_1, \cdots, \eta_r), \eta_{i_1}\big)\cdots, \eta_{i_n}\big) \in \Theta_{0,r-n}(C).$$

For any $t \in \mathbb{R}_{\geq 0}$, we have

$$\sum_{0<\eta_{i_1},\dots,\eta_{i_n}\leq t} \vartheta(\eta_1,\dots,\eta_r) \leq \mathcal{M}\big(\vartheta(\eta_1,\dots,\eta_r),\eta_{i_1},\dots,\eta_{i_n}\big) t^n (\log(t+2))^{nC}. \qquad \square$$

**Example 3.3.** For any $n \in \mathbb{Z}_{>0}$, let

$$\phi^*(n) = \frac{\phi(n)}{n} = \prod_{p|n}\left(1-\frac{1}{p}\right), \qquad \phi^\dagger(n) = \prod_{p|n}\left(1+\frac{1}{p}\right).$$

Let $C \in \mathbb{Z}_{\geq 0}$. For any $t \in \mathbb{R}_{\geq 0}$, we have

$$\sum_{0<n\leq t} (\phi^*(n))^C \leq \sum_{0<n\leq t} (\phi^\dagger(n))^C \ll_C t,$$

(see [3, Equation 3.1]) and

$$\sum_{0<n\leq t} (1+C)^{\omega(n)} \ll_C t(\log(t+2))^C$$

(see [1, Section 5.1]).

Therefore, for any $C \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{>0}$,

$$\prod_{i=1}^r (\phi^*(\eta_i))^C \in \Theta_{0,r}(0), \quad \prod_{i=1}^r (\phi^\dagger(\eta_i))^C \in \Theta_{0,r}(0), \quad \prod_{i=1}^r (1+C)^{\omega(\eta_i)} \in \Theta_{0,r}(C). \qquad \square$$

**Lemma 3.4.** Let $C \in \mathbb{R}_{\geq 0}$. Let $\vartheta : \mathbb{Z} \to \mathbb{R}$ be a nonnegative function such that, for any $t \in \mathbb{R}_{\geq 0}$, we have $\sum_{0<n\leq t} \vartheta(n) \leq t(\log(t+2))^C$.

Let $t_1 \leq t_2 \in \mathbb{R}_{\geq 0}$, $\kappa \in \mathbb{R}$. Then

$$\sum_{t_1<n\leq t_2} \frac{\vartheta(n)}{n^\kappa} \ll_{C,\kappa} \begin{cases} t_2^{1-\kappa}(\log(t_2+2))^C, & \kappa < 1, \\ (\log(t_2+2))^{C+1}, & \kappa = 1, \\ \min\left(\frac{(\log(t_1+2))^C}{t_1^{\kappa-1}}, 1\right) \ll_{C,\kappa} 1, & \kappa > 1. \end{cases} \qquad \square$$

**Proof.** Let $S$ be the sum that we want to estimate. Let $M(t) = \sum_{0<n\leq t} \vartheta(n)$.

By partial summation,

$$\begin{aligned} S &= \frac{M(t_2)}{t_2^\kappa} - \frac{M(t_1)}{t_1^\kappa} - \int_{t_1}^{t_2} (-\kappa)\frac{M(t)}{t^{\kappa+1}}\,\mathrm{d}t \\ &\ll_\kappa \frac{(\log(t_2+2))^C}{t_2^{\kappa-1}} + \frac{(\log(t_1+2))^C}{t_1^{\kappa-1}} + \int_{t_1}^{t_2} \frac{(\log(t+2))^C}{t^\kappa}\,\mathrm{d}t. \end{aligned}$$

If $\kappa = 1$, the result follows from

$$\int_{t_1}^{t_2} \frac{(\log(t+2))^C}{t+2}\,\mathrm{d}t = \frac{(\log(t_2+2))^{C+1} - (\log(t_1+2))^{C+1}}{C+1}.$$

For $\kappa \neq 1$, the result follows by induction over $C$ from

$$\int_{t_1}^{t_2} \frac{(\log(t+2))^C}{(t+2)^\kappa}\,\mathrm{d}t \ll_{C,\kappa} \frac{(\log(t_2+2))^C}{(t_2+2)^{\kappa-1}} + \frac{(\log(t_1+2))^{C-1}}{(t_1+2)^{\kappa-1}} + \int_{t_1}^{t_2} \frac{(\log(t+2))^{C-1}}{(t+2)^\kappa}\,\mathrm{d}t,$$

which is obtained using integration by parts. Depending on whether $\kappa < 1$ or $\kappa > 1$, the first or second term gives the main contribution. ■

Now we come to the setup for the main result of this section. Let $r, s \in \mathbb{Z}_{\geq 0}$. We consider a nonnegative function $V : \mathbb{R}_{\geq 0}^{r+s+1} \times \mathbb{R}_{\geq 3} \to \mathbb{R}$ with the following properties. We assume that, for $j = 1, \dots, s$, there are

$$k_{0,j}, \dots, k_{r+j-1,j} \in \mathbb{R}, \quad k_{r+j,j} \in \mathbb{R}_{\neq 0}, \quad k_{r+j+1,j}, \dots, k_{r+s,j} = 0, \quad a_j \in \mathbb{R}_{>0},$$

such that

$$V(\eta_0, \dots, \eta_{r+s}; B) \ll \frac{B^{1-A}}{\eta_0^{1-A_0} \cdots \eta_{r+s}^{1-A_{r+s}}}, \tag{3.1}$$

where we define, for $i = 0, \dots, r+s$,

$$A = \sum_{j=1}^{s} a_j, \quad A_i = \sum_{j=1}^{s} a_j k_{i,j}.$$

We also assume that $V(\eta_0, \dots, \eta_{r+s}; B) = 0$ unless both

$$\eta_0^{k_{0,j}} \cdots \eta_{r+s}^{k_{r+s,j}} = \eta_0^{k_{0,j}} \cdots \eta_{r+j}^{k_{r+j,j}} \leq B, \tag{3.2}$$

for $j = 1, \dots, s$, and

$$1 \leq \eta_i \leq B, \tag{3.3}$$

for $i = 1, \dots, r+s$.

**Remark 3.5.** In (3.1) and for the remainder of this section, we use the convention that all implied constants (in the notation $\ll$ and $O(\cdots)$) are independent of $\eta_0, \dots, \eta_{r+s}$ and $B$, but may depend on all other parameters, in particular on $V$ and $\vartheta$. □

**Lemma 3.6.** In the situation described above, let $\vartheta \in \Theta_{0,r+s+1}(C)$ for some $C \in \mathbb{R}_{\geq 0}$. Then

$$\sum_{\eta_1,\dots,\eta_{r+s}} \vartheta(\eta_0,\dots,\eta_{r+s})V(\eta_0,\dots,\eta_{r+s};B) \ll \eta_0^{-1}\mathcal{M}(\vartheta(\eta_0,\dots,\eta_{r+s}),\eta_{r+s},\dots,\eta_1)B(\log B)^{r+(r+s)C}. \quad \square$$

**Proof.** For any $\ell \in \{0,\dots,r+s-1\}$, let

$$\vartheta_\ell(\eta_0,\dots,\eta_\ell) = \mathcal{M}(\vartheta(\eta_0,\dots,\eta_{r+s}),\eta_{r+s},\dots,\eta_{\ell+1}) \in \Theta_{0,\ell+1}(C).$$

For $\ell = s,\dots,0$, we claim that

$$\sum_{\eta_{r+\ell+1},\dots,\eta_{r+s}} \vartheta(\eta_0,\dots,\eta_{r+s})V(\eta_0,\dots,\eta_{r+s};B) \ll \frac{\vartheta_{r+\ell}(\eta_0,\dots,\eta_{r+\ell})B^{1-A^{(\ell)}}(\log B)^{(s-\ell)C}}{\eta_0^{1-A_0^{(\ell)}}\cdots\eta_{r+\ell}^{1-A_{r+\ell}^{(\ell)}}},$$

where

$$A^{(\ell)} = \sum_{j=1}^{\ell} a_j, \quad A_i^{(\ell)} = \sum_{j=1}^{\ell} a_j k_{i,j}.$$

For $\ell = s$, this is true by (3.1). To prove the claim in the other cases by induction, we must estimate

$$\sum_{\eta_{r+\ell}} \frac{\vartheta_{r+\ell}(\eta_0,\dots,\eta_{r+\ell})B^{1-A^{(\ell)}}(\log B)^{(s-\ell)C}}{\eta_0^{1-A_0^{(\ell)}}\cdots\eta_{r+\ell}^{1-A_{r+\ell}^{(\ell)}}}, \tag{3.4}$$

for $\ell = s,\dots,1$. Since $V(\eta_0,\dots,\eta_{r+s};B) = 0$ unless (3.2), the summation can be restricted to $\eta_{r+\ell}$ satisfying $\eta_{r+\ell} \leq T$ if $k_{r+\ell,\ell} > 0$ resp. $\eta_{r+\ell} \geq T$ if $k_{r+\ell,\ell} < 0$, with $T = (B/(\eta_0^{k_{0,\ell}}\cdots\eta_{r+\ell-1}^{k_{r+\ell-1,\ell}}))^{1/k_{r+\ell,\ell}}$. An application of Lemma 3.4 (with $\kappa = 1 - A_{r+\ell}^{(\ell)} = 1 - a_\ell k_{r+\ell,\ell}$) shows that (3.4) is

$$\ll \frac{\vartheta_{r+\ell-1}(\eta_0,\dots,\eta_{r+\ell-1})B^{1-A^{(\ell)}+a_\ell}(\log B)^{(s-(\ell-1))C}}{\eta_0^{1-A_0^{(\ell)}+a_\ell k_{0,\ell}}\cdots\eta_{r+\ell-1}^{1-A_{r+\ell-1}^{(\ell)}+a_\ell k_{r+\ell-1,\ell}}}.$$

The induction step is completed by observing $A^{(\ell)} - a_\ell = A^{(\ell-1)}$ and $A_i^{(\ell)} - a_\ell k_{i,\ell} = A_i^{(\ell-1)}$, for $i = 0,\dots,r+\ell-1$.

For $\ell = r, \ldots, 0$, we claim that

$$\sum_{\eta_{\ell+1},\ldots,\eta_{r+s}} \vartheta(\eta_0, \ldots, \eta_{r+s}) V(\eta_0, \ldots, \eta_{r+s}; B) \ll \frac{\vartheta_\ell(\eta_0, \ldots, \eta_\ell) B(\log B)^{r-\ell+(r+s-\ell)C}}{\eta_0 \cdots \eta_\ell}.$$

This is also proved by induction. The case $\ell = r$ is the ending of our first induction. From here, we apply Lemma 3.4 (with $\kappa = 1$) for the summation over $\eta_\ell$ subject to (3.3). ■

**Definition 3.7.** For any $C \in \mathbb{R}_{\geq 0}$, let $\Theta_0(C)$ be the set of all nonnegative functions $\vartheta : \mathbb{Z}_{>0} \to \mathbb{R}$ such that there is a $c_0 \in \mathbb{R}_{\geq 0}$ and a bounded function $E : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that, for any $t \in \mathbb{R}_{\geq 0}$,

$$\sum_{0<n\leq t} \vartheta(n) = c_0 t + E(t)(\log(t+2))^C.$$

If $\vartheta \in \Theta_0(C)$, the corresponding $c_0, E(t)$ are unique since $t$ grows faster than any power of $\log(t+2)$ for large $t$; we introduce the notation

$$\mathcal{A}(\vartheta(n), n) = c_0, \quad \mathcal{E}(\vartheta(n), n) = \sup_{t\in\mathbb{R}_{\geq 0}} \{|E(t)|\}. \qquad \square$$

**Definition 3.8.** For any $C \in \mathbb{R}_{\geq 0}$ and $r \in \mathbb{Z}_{>0}$, let $\Theta_{1,r}(C, \eta_r)$ be the set of all functions $\vartheta : \mathbb{Z}_{>0}^r \to \mathbb{R}$ in the variables $\eta_1, \ldots, \eta_r$ such that

(1) $\vartheta(\eta_1, \ldots, \eta_r)$ as a function in $\eta_1, \ldots, \eta_r$ lies in $\Theta_{0,r}(0)$.

(2) $\vartheta(\eta_1, \ldots, \eta_r)$ as a function in $\eta_r$ lies in $\Theta_0(C)$ for any $\eta_1, \ldots, \eta_{r-1} \in \mathbb{Z}$, so that we have corresponding

$$\mathcal{A}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r) : \mathbb{Z}_{>0}^{r-1} \to \mathbb{R}, \quad \mathcal{E}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r) : \mathbb{Z}_{>0}^{r-1} \to \mathbb{R}$$

as functions in $\eta_1, \ldots, \eta_{r-1}$.

(3) $\mathcal{A}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r)$ lies in $\Theta_{0,r-1}(0)$.

(4) $\mathcal{E}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r)$ lies in $\Theta_{0,r-1}(C)$.

We define $\Theta_{1,r}(C, \eta_i)$ for any other variable $\eta_i$ analogously. $\square$

We want to estimate

$$\sum_{\eta_0} \vartheta(\eta_0, \ldots, \eta_{r+s}) V(\eta_0, \ldots, \eta_{r+s}; B).$$

We assume that $V$ is as described before Lemma 3.6 with the additional property that $V$ as a function in the first variable $\eta_0$ has a continuous derivative whose sign changes only finitely often on the interval $[1, B]$ and vanishes outside this interval.

**Proposition 3.9.** Let $V$ be as above, and let $\vartheta \in \Theta_{1,r+s+1}(C, \eta_0)$ for some $C \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned}\sum_{\eta_0} &\vartheta(\eta_0, \dots, \eta_{r+s}) V(\eta_0, \dots, \eta_{r+s}; B) \\ &= \mathcal{A}(\vartheta(\eta_0, \dots, \eta_{r+s}), \eta_0) \int_{t_0 \geq 1} V(t_0, \eta_1, \dots, \eta_{r+s}; B)\, \mathrm{d}t_0 + R(\eta_1, \dots, \eta_{r+s}; B),\end{aligned}$$

where

$$\sum_{\eta_1, \dots, \eta_{r+s}} R(\eta_1, \dots, \eta_{r+s}; B) \ll B(\log B)^r (\log\log B)^{\max\{1,s\}}.$$

□

**Proof.** We note that we may always assume that $1 \leq \eta_0, \dots, \eta_r \leq B$ since all terms and error terms vanish otherwise. Let $\vartheta' \in \Theta_{0,r+s}(0)$ and $\vartheta'' \in \Theta_{0,r+s}(C)$ be defined as

$$\begin{aligned}\vartheta'(\eta_1, \dots, \eta_{r+s}) &= \mathcal{A}(\vartheta(\eta_0, \dots, \eta_{r+s}), \eta_0), \\ \vartheta''(\eta_1, \dots, \eta_{r+s}) &= \mathcal{E}(\vartheta(\eta_0, \dots, \eta_{r+s}), \eta_0).\end{aligned}$$

We proceed in three steps. Let $T = (\log B)^{s+(r+s+1)C}$.

(1) We show that

$$\sum_{\substack{\eta_0, \dots, \eta_{r+s} \\ \eta_0 < T}} \vartheta(\eta_0, \dots, \eta_{r+s}) V(\eta_0, \dots, \eta_{r+s}; B) \ll B(\log B)^r (\log\log B).$$

(2) Combining $\vartheta \in \Theta_0(C)$ as a function in $\eta_0$ with Lemma 3.1, we have

$$\begin{aligned}\sum_{\eta_0 \geq T} \vartheta(\eta_0, \dots, \eta_{r+s}) V(\eta_0, \dots, \eta_{r+s}; B) &= \vartheta'(\eta_1, \dots, \eta_{r+s}) \int_{t_0 \geq T} V(t_0, \eta_1, \dots, \eta_{r+s}; B)\, \mathrm{d}t_0 \\ &+ O\left(\vartheta''(\eta_1, \dots, \eta_{r+s})(\log B)^C \sup_{t_0 \geq T} V(t_0, \eta_1, \dots, \eta_{r+s}; B)\right).\end{aligned}$$

Here, we show that summing the error term over $\eta_1, \dots, \eta_{r+s}$ gives $O(B(\log B)^r)$.

(3) To complete the proof, we must estimate

$$\sum_{\eta_1, \dots, \eta_{r+s}} \vartheta'(\eta_1, \dots, \eta_{r+s}) \int_1^T V(t_0, \eta_1, \dots, \eta_{r+s}; B)\, \mathrm{d}t_0.$$

If $s = 1$ and $k_{0,1} > 0$, we consider the case $T^{k_{0,1}} \eta_1^{k_{1,1}} \cdots \eta_{r+1}^{k_{r+1,1}} \leq B$ and its opposite separately. If $s > 1$, we distinguish $2^s$ cases.

For (1), we use $\vartheta \in \Theta_{0,r+s+1}(0)$ and Lemma 3.6 for the summation over $\eta_1, \dots, \eta_{r+s}$ and Lemma 3.4 for the summation over $\eta_0$ to compute

$$\sum_{\eta_0,\dots,\eta_{r+s}} \vartheta(\eta_0, \dots, \eta_{r+s}) V(\eta_0, \dots, \eta_{r+s}; B) \ll \sum_{1 \leq \eta_0 < T} \eta_0^{-1} \mathcal{M}(\vartheta(\eta_0, \dots, \eta_{r+s}), \eta_{r+s}, \dots, \eta_1) B (\log B)^r$$
$$\ll B (\log B)^r (\log \log B).$$

For (2), we note that (3.1) and (3.2) imply

$$V(t_0, \eta_1, \dots, \eta_{r+s}; B) \ll \frac{B}{t_0 \eta_1 \cdots \eta_{r+s}}.$$

Combining $\vartheta'' \in \Theta_{0,r+s}(C)$ and (3.3) with Lemma 3.4 in the second step,

$$\begin{aligned}
&\sum_{\eta_1,\dots,\eta_{r+s}} \vartheta''(\eta_1, \dots, \eta_{r+s}) (\log B)^C \sup_{t_0 \geq T} V(t_0, \eta_1, \dots, \eta_{r+s}; B) \\
&\ll \sum_{\eta_1,\dots,\eta_{r+s}} \frac{\vartheta''(\eta_1, \dots, \eta_{r+s}) B (\log B)^C}{T \eta_1 \cdots \eta_{r+s}} \\
&\ll T^{-1} B (\log B)^{r+s+(r+s+1)C} \\
&\ll B (\log B)^r.
\end{aligned}$$

For (3), we assume $A_0 = 0$ first. We use $\vartheta' \in \Theta_{0,r+s}(0)$ and Lemma 3.6 (with $\eta_0 = 1$) to compute

$$\begin{aligned}
&\sum_{\eta_1,\dots,\eta_{r+s}} \vartheta'(\eta_1, \dots, \eta_{r+s}) \int_1^T V(t_0, \eta_1, \dots, \eta_{r+s}; B) \, \mathrm{d}t_0 \\
&\ll \sum_{\eta_1,\dots,\eta_{r+s}} \frac{\vartheta'(\eta_1, \dots, \eta_{r+s}) B^{1-A}}{\eta_1^{1-A_1} \cdots \eta_{r+s}^{1-A_{r+s}}} \int_1^T \frac{1}{t_0} \, \mathrm{d}t_0 \\
&\ll B (\log B)^r (\log \log B).
\end{aligned}$$

Now, we suppose $A_0 \neq 0$. Let

$$X_j = \eta_1^{k_{1,j}} \cdots \eta_{r+s}^{k_{r+s,j}} = \eta_1^{k_{1,j}} \cdots \eta_{r+j}^{k_{r+j,j}},$$

for $j = 1, \dots, s$. We distinguish $2^s$ cases, labeled by the subsets $J$ of $\{1, \dots, s\}$. In case $J$, we assume $X_j \le \min\{BT^{-k_{0,j}}, B\}$ for each $j \in J$, and $X_j > \min\{BT^{-k_{0,j}}, B\}$ for each $j \notin J$. By (3.2), $V(t_0, \eta_1, \dots, \eta_{r+s}; B) = 0$ unless $t_0^{k_{0,j}} X_j \le B$. Therefore, we may restrict to $X_j \le \max_{1 \le t_0 \le T}\{Bt_0^{-k_{0,j}}\}$.

In total, in case $J$, we may restrict the summation over $\eta_1, \dots, \eta_{r+s}$ to

$$X_j \in \begin{cases} [0, BT^{-k_{0,j}}], & j \in J,\ k_{0,j} \ge 0, \\ (BT^{-k_{0,j}}, B], & j \notin J,\ k_{0,j} \ge 0, \\ [0, B], & j \in J,\ k_{0,j} < 0, \\ (B, BT^{-k_{0,j}}], & j \notin J,\ k_{0,j} < 0; \end{cases}$$

in particular, the summation is trivial if $k_{0,j} = 0$ for some $j \notin J$, so we assume there is no such $j$. Furthermore, we may restrict the integration over $t_0$ to the interval $[T_1, T_2]$ where

$$T_1 = \max_{\substack{j \in \{1,\dots,s\},\\ k_{0,j} < 0}} \left\{1, \left(BX_j^{-1}\right)^{1/k_{0,j}}\right\}, \qquad T_2 = \min_{\substack{j \in \{1,\dots,s\}\\ k_{0,j} > 0}} \left\{T, \left(BX_j^{-1}\right)^{1/k_{0,j}}\right\};$$

we may assume that $T_1 \le T_2$ since the integral vanishes otherwise. We note that $1 \le (BX_j^{-1})^{1/k_{0,j}} \le T$ if and only if $j \notin J$.

We define

$$A' = \sum_{j \in J} a_j, \qquad A'_0 = \sum_{\substack{j \in J\\ k_{0,j} > 0}} a_j k_{0,j}, \qquad A'_i = \sum_{j \in J} a_j k_{i,j},$$

for $i = 1, \dots, r+s$.

Combining (3.1) with

$$\int_{T_1}^{T_2} \frac{1}{t_0^{1-A_0}} \,\mathrm{d}t_0 \ll T_1^{A_0} + T_2^{A_0} \ll \prod_{\substack{j \in J\\ k_{0,j} > 0}} T^{a_j k_{0,j}} \prod_{j \notin J} \left(BX_j^{-1}\right)^{a_j} = \frac{B^{A-A'} T^{A'_0}}{\eta_1^{A_1 - A'_1} \cdots \eta_{r+s}^{A_{r+s} - A'_{r+s}}},$$

we obtain as the contribution of case $J$ to the error term of (3)

$$\begin{aligned} \sum_{\eta_1,\dots,\eta_{r+s}} \vartheta'(\eta_1, \dots, \eta_{r+s}) \int_1^T V(t_0, \eta_1, \dots \eta_{r+s}; B) \,\mathrm{d}t_0 &\ll \sum_{\eta_1,\dots,\eta_{r+s}} \frac{\vartheta'(\eta_1, \dots, \eta_{r+s}) B^{1-A}}{\eta_1^{1-A_1} \cdots \eta_{r+s}^{1-A_{r+s}}} \int_{T_1}^{T_2} \frac{1}{t_0^{1-A_0}} \,\mathrm{d}t_0 \\ &\ll \sum_{\eta_1,\dots,\eta_{r+s}} \frac{\vartheta'(\eta_1, \dots, \eta_{r+s}) B^{1-A'} T^{A'_0}}{\eta_1^{1-A'_1} \cdots \eta_{r+s}^{1-A'_{r+s}}}. \end{aligned}$$

For $j = s, \dots, 1$, we handle the summation over $\eta_{r+j}$ using $\vartheta' \in \Theta_{0,r+s}(0)$ and Lemma 3.4. After the summations over $\eta_{r+s}, \dots, \eta_{r+j+1}$ are done, the exponent of $\eta_{r+j}$ in the denominator is $1 - a_j k_{r+j,j}$ if $j \in J$ and it is 1 otherwise. For $j \in J$ and $k_{0,j} \geq 0$, we use $X_j \leq BT^{-k_{0,j}}$, i.e.,

$$\eta_{r+j}^{a_j k_{r+j,j}} \leq \frac{B^{a_j} T^{-a_j k_{0,j}}}{\eta_1^{a_j k_{1,j}} \cdots \eta_{r+j-1}^{a_j k_{r+j-1,j}}}.$$

For $j \in J$ and $k_{0,j} < 0$, we use $X_j \leq B$, i.e.,

$$\eta_{r+j}^{a_j k_{r+j,j}} \leq \frac{B^{a_j}}{\eta_1^{a_j k_{1,j}} \cdots \eta_{r+j-1}^{a_j k_{r+j-1,j}}}.$$

For $j \notin J$, we use that $BT^{-k_{0,j}} < X_j \leq B$, for $k_{0,j} > 0$, resp. $B < X_j \leq BT^{-k_{0,j}}$, for $k_{0,j} < 0$, implies that, for $\eta_1, \dots, \eta_{r+j-1}$ fixed, there are $\ll T^{|k_{0,j}|}$ possibilities for $\eta_{r+j}$, which shows that we pick up a factor $(\log \log B)$.

It follows that we can continue our estimation as

$$\begin{aligned} &\ll \sum_{\eta_1, \dots, \eta_r} \frac{\mathcal{M}(\vartheta'(\eta_1, \dots, \eta_{r+s}), \eta_{r+s}, \dots, \eta_{r+1}) B (\log \log B)^{s-\#J}}{\eta_1 \cdots \eta_r} \\ &\ll B (\log B)^r (\log \log B)^s \end{aligned}$$

since $0 \leq \#J \leq s$. ■

The next result is concerned with a similar situation as in Proposition 3.9, with $r \in \mathbb{Z}_{>0}$ and $s = 1$.

Let $V : \mathbb{R}^{r+2} \times \mathbb{R}_{\geq 3} \to \mathbb{R}$ be a nonnegative function, and

$$k_0, \dots, k_r \in \mathbb{R}, \quad k_{r+1} \in \mathbb{R}_{\neq 0}, \quad a, b \in \mathbb{R}_{>0}$$

such that

$$V(\eta_0, \dots, \eta_{r+1}; B) \ll \min \left\{ \frac{B^{1-a}}{\eta_0^{1-ak_0} \cdots \eta_{r+1}^{1-ak_{r+1}}}, \frac{B^{1+b}}{\eta_0^{1+bk_0} \cdots \eta_{r+1}^{1+bk_{r+1}}} \right\}. \tag{3.5}$$

We assume that $V(\eta_0, \dots, \eta_{r+1}; B) = 0$ unless, for $i = 0, \dots, r+1$,

$$1 \leq \eta_i \leq B. \tag{3.6}$$

We assume that $V$ as a function in the first variable $\eta_0$ has a continuous derivative whose sign changes only finitely often on the interval $[1, B]$.

**Proposition 3.10.** For some $C \in \mathbb{R}_{\geq 0}$, let $\vartheta \in \Theta_{1,r+2}(C, \eta_0)$. Let $V$ be as above. Then

$$\sum_{\eta_0} \vartheta(\eta_0, \dots, \eta_{r+1}) V(\eta_0, \dots, \eta_{r+1}; B)$$
$$= \mathcal{A}(\vartheta(\eta_0, \dots, \eta_{r+1}), \eta_0) \int_{t_0 \geq 1} V(t_0, \eta_1, \dots, \eta_{r+s}; B)\, \mathrm{d}t_0 + R(\eta_1, \dots, \eta_{r+1}; B),$$

where

$$\sum_{\eta_1, \dots, \eta_{r+1}} R(\eta_1, \dots, \eta_{r+1}; B) \ll B(\log B)^r (\log\log B). \qquad \square$$

**Proof.** We define $\vartheta' \in \Theta_{0,r+1}(0)$ and $\vartheta'' \in \Theta_{0,r+1}(C)$ as in the proof of Proposition 3.9. Let

$$M = M(\eta_0, \dots, \eta_{r+1}; B) = \vartheta(\eta_0, \dots, \eta_{r+1}) V(\eta_0, \dots, \eta_{r+1}; B)$$

and

$$M'(t) = M'(t, \eta_1 \dots, \eta_{r+1}; B) = \vartheta'(\eta_1, \dots, \eta_{r+1}) \int_{t_0 \geq t} V(t_0, \eta_1, \dots, \eta_{r+1}; B)\, \mathrm{d}t_0.$$

We want to show that $M$ summed over all $\eta_0 \in \mathbb{Z}_{>0}$ agrees with $M'(1)$ up to an acceptable error. We do this in three steps, where $T = (\log B)^{1+(r+2)C}$.

(1) We show that $M$ summed over all $\eta_0$ agrees with $M$ summed over $\eta_0 \geq T$ up to an acceptable error, by proving that

$$\sum_{\substack{\eta_0, \dots, \eta_{r+1} \\ \eta_0 < T}} M \ll B(\log B)^r (\log\log B).$$

(2) We show that $M$ summed over $\eta_0 \geq T$ gives $M'(T)$ up to an error of $R' = R'(\eta_1, \dots, \eta_{r+1}; B)$ with $\sum_{\eta_1, \dots, \eta_{r+1}} R' \ll B(\log B)^r$.

(3) We show that $M'(T)$ summed over $\eta_1, \dots, \eta_{r+1}$ agrees with $M'(1)$ up to an acceptable error, by proving that

$$\sum_{\eta_1, \dots, \eta_{r+1}} (M'(1) - M'(T)) \ll B(\log B)^r (\log\log B).$$

If $k_0 < 0$, we distinguish three cases, where $\eta_1^{k_1} \cdots \eta_{r+1}^{k_{r+1}}$ is at most $B$, or at least $BT^{-k_0}$, or between these two numbers.

For (1), we use (3.5), $\vartheta \in \Theta_{0,r+2}(0)$, and (3.6). For $\eta_0^{k_0} \cdots \eta_{r+1}^{k_{r+1}} \leq B$, we apply Lemma 3.6 to compute

$$\begin{aligned}\sum_{\eta_0,\dots,\eta_{r+1}} M &\ll \sum_{\eta_0,\dots,\eta_{r+1}} \frac{\vartheta(\eta_0,\dots,\eta_{r+1})B^{1-a}}{\eta_0^{1-ak_0} \cdots \eta_{r+1}^{1-ak_{r+1}}} \\ &\ll \sum_{\eta_0} \eta_0^{-1} \mathcal{M}(\vartheta(\eta_0,\dots,\eta_{r+1}), \eta_{r+1},\dots,\eta_1) B (\log B)^r \\ &\ll B(\log B)^r (\log\log B).\end{aligned}$$

In the opposite case, by Lemma 3.4, we have

$$\begin{aligned}\sum_{\eta_0,\dots,\eta_{r+1}} M &\ll \sum_{\eta_0,\dots,\eta_{r+1}} \frac{\vartheta(\eta_0,\dots,\eta_{r+1})B^{1+b}}{\eta_0^{1+bk_0} \cdots \eta_{r+1}^{1+bk_{r+1}}} \\ &\ll \sum_{\eta_0,\dots,\eta_r} \frac{\mathcal{M}(\vartheta(\eta_0,\dots,\eta_{r+1}), \eta_{r+1})B}{\eta_0 \cdots \eta_r} \\ &\ll B(\log B)^r (\log\log B).\end{aligned}$$

For (2), we combine $\vartheta \in \Theta_0(C)$ as a function in $\eta_0$ with Lemma 3.1. This shows that $M$ summed over $\eta_0 \geq T$ gives the main term $M'(T)$ as above and an error term which can be estimated (using $V(\eta_0,\dots,\eta_{r+1}; B) \ll \frac{B}{\eta_0\cdots\eta_{r+1}}$ by (3.5), $\vartheta'' \in \Theta_{0,r+1}(C)$, (3.6), and Lemma 3.4) as

$$\begin{aligned}&\ll \sum_{\eta_1,\dots,\eta_{r+1}} (\log B)^C \vartheta''(\eta_1,\dots,\eta_{r+1}) \sup_{t_0 \geq T} V(t_0,\eta_1,\dots,\eta_{r+1}; B) \\ &\ll \sum_{\eta_1,\dots,\eta_{r+1}} \frac{(\log B)^C \vartheta''(\eta_1,\dots,\eta_{r+1})B}{T\eta_1\cdots\eta_{r+1}} \\ &\ll T^{-1}B(\log B)^{r+1+(r+2)C} = B(\log B)^r.\end{aligned}$$

For (3), we suppose $k_{r+1} > 0$; the case $k_{r+1} < 0$ is similar. In the following computations, we use (3.5), $\vartheta' \in \Theta_{0,r+1}(0)$, (3.6), and Lemma 3.4.

If $k_0 < 0$, we split the summation over $\eta_1,\dots,\eta_{r+1}$ and integration over $t_0$ into three parts, the first defined by the condition $\eta_1^{k_1} \cdots \eta_{r+1}^{k_{r+1}} \leq B$. We estimate using Lemma 3.6

(with $\eta_0 = 1$)

$$\begin{aligned}
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \vartheta'(\eta_1,\dots,\eta_{r+1}) \int_1^T V(t_0,\eta_1,\dots,\eta_{r+1};B)\,\mathrm{d}t_0\\
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \vartheta'(\eta_1,\dots,\eta_{r+1}) \int_1^T \frac{B^{1-a}}{t_0^{1-ak_0}\eta_1^{1-ak_1}\cdots\eta_{r+1}^{1-ak_{r+1}}}\,\mathrm{d}t_0\\
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \frac{\vartheta'(\eta_1,\dots,\eta_{r+1})B^{1-a}}{\eta_1^{1-ak_1}\cdots\eta_{r+1}^{1-ak_{r+1}}}\\
&\ll B(\log B)^r.
\end{aligned}$$

For the second subset defined by $B < \eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}} \le BT^{-k_0}$, we get

$$\begin{aligned}
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \vartheta'(\eta_1,\dots,\eta_{r+1}) \times \left(\int_{t_0\le\left(\eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}}/B\right)^{-1/k_0}} \frac{B^{1+b}}{t_0^{1+bk_0}\eta_1^{1+bk_1}\cdots\eta_{r+1}^{1+bk_{r+1}}}\,\mathrm{d}t_0\right.\\
&\qquad\qquad \left. + \int_{t_0\ge\left(\eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}}/B\right)^{-1/k_0}} \frac{B^{1-a}}{t_0^{1-ak_0}\eta_1^{1-ak_1}\cdots\eta_{r+1}^{1-ak_{r+1}}}\,\mathrm{d}t_0\right)\\
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \frac{\vartheta'(\eta_1,\dots,\eta_{r+1})B}{\eta_1\cdots\eta_{r+1}}\\
&\ll B(\log B)^r(\log\log B).
\end{aligned}$$

For the third subset defined by $\eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}} > BT^{-k_0}$, we get

$$\begin{aligned}
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \int_1^T \frac{\vartheta'(\eta_1,\dots,\eta_{r+1})B^{1+b}}{t_0^{1+bk_0}\eta_1^{1+bk_1}\cdots\eta_{r+1}^{1+bk_{r+1}}}\,\mathrm{d}t_0\\
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \frac{\vartheta'(\eta_1,\dots,\eta_{r+1})B^{1+b}T^{-bk_0}}{\eta_1^{1+bk_1}\cdots\eta_{r+1}^{1+bk_{r+1}}}\\
&\ll \sum_{\eta_1,\dots,\eta_r} \frac{\mathcal{M}(\vartheta'(\eta_1,\dots,\eta_{r+1}),\eta_{r+1})B}{\eta_1\cdots\eta_r}\\
&\ll B(\log B)^r.
\end{aligned}$$

If $k_0 > 0$, the computations are similar.

If $k_0 = 0$, we split the summation over $\eta_1,\dots,\eta_{r+1}$ into two subsets, the first defined by $\eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}} \le B$.

Here, we compute

$$
\begin{aligned}
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \vartheta'(\eta_1,\dots,\eta_{r+1}) \int_1^T \frac{B^{1-a}}{t_0 \eta_1^{1-ak_1}\cdots\eta_{r+1}^{1-ak_{r+1}}}\, \mathrm{d}t_0\\
&\ll \sum_{\eta_1,\dots,\eta_{r+1}} \frac{\vartheta'(\eta_1,\dots,\eta_{r+1}) B^{1-a}(\log\log B)}{\eta_1^{1-ak_1}\cdots\eta_{r+1}^{1-ak_{r+1}}}\\
&\ll B(\log B)^r(\log\log B).
\end{aligned}
$$

For the subset defined by $\eta_1^{k_1}\cdots\eta_{r+1}^{k_{r+1}} > B$, the computation is similar. ■

## 4 Completion of Summations

Let $r, s \in \mathbb{Z}_{\geq 0}$ with $r \geq s$. In this section, we consider functions

$$
\vartheta_{r+s} : \mathbb{Z}_{\geq 0}^{r+s} \to \mathbb{R}, \qquad V_{r+s} : \mathbb{R}_{\geq 0}^{r+s} \times \mathbb{R}_{\geq 3} \to \mathbb{R}.
$$

In the previous section, we summed the product of such functions over one variable; here, we sum over all variables and therefore want to estimate

$$
\sum_{\eta_1,\dots,\eta_{r+s}} \vartheta_{r+s}(\eta_1,\dots,\eta_{r+s}) V_{r+s}(\eta_1,\dots,\eta_{r+s}; B).
$$

This will be done in the case that $\vartheta_{r+s}$ and $V_{r+s}$ fulfill certain conditions described in the following that allow us to apply Proposition 3.9 repeatedly.

For the implied constants in this section, we use a similar convention as described in Remark 3.5, i.e., the implied constants are meant to be independent of $\eta_1,\dots,\eta_{r+s}$ and $B$, but may depend on everything else, in particular on $V_{r+s}$ and $\vartheta_{r+s}$.

For $V_{r+s} : \mathbb{R}_{\geq 0}^{r+s} \times \mathbb{R}_{\geq 3} \to \mathbb{R}$, a nonnegative function, we require the following, similar to Section 3. We assume that, for $j = 1,\dots,s$, we have $a_j \in \mathbb{R}_{>0}$ and

$$
\begin{aligned}
&k_{1,j},\dots,k_{r-s+j-1,j} \in \mathbb{R}, && k_{r-s+j,j} \in \mathbb{R}_{\neq 0}, && k_{r-s+j+1,j},\dots,k_{r,j} = 0,\\
&k_{r+1,j},\dots,k_{r+j-1,j} \in \mathbb{R}, && k_{r+j,j} \in \mathbb{R}_{\neq 0}, && k_{r+j+1,j},\dots,k_{r+s,j} = 0.
\end{aligned}
$$

For $\ell = 1, \dots, s$ and $i = 1, \dots, r+s$, we define

$$A^{(\ell)} = \sum_{j=1}^{\ell} a_j, \quad A_i^{(\ell)} = \sum_{j=1}^{\ell} a_j k_{i,j}.$$

We assume that

$$V_{r+s}(\eta_1, \dots, \eta_{r+s}; B) \ll \frac{B^{1-A^{(s)}}}{\eta_1^{1-A_1^{(s)}} \cdots \eta_{r+s}^{1-A_{r+s}^{(s)}}}, \tag{4.1}$$

and that $V_{r+s}(\eta_1, \dots, \eta_{r+s}; B) = 0$ unless both

$$\eta_1^{k_{1,j}} \cdots \eta_{r+s}^{k_{r+s,j}} = \eta_1^{k_{1,j}} \cdots \eta_{r+j}^{k_{r+j,j}} \leq B, \tag{4.2}$$

for $j = 1, \dots, s$, and

$$1 \leq \eta_i \leq B, \tag{4.3}$$

for $i = 1, \dots, r+s$.

For $\ell = r+s-1, \dots, 0$, we define recursively

$$\begin{aligned} V_\ell(\eta_1, \dots, \eta_\ell; B) &= \int_{\eta_{\ell+1}} V_{\ell+1}(\eta_1, \dots, \eta_{\ell+1}; B)\, \mathrm{d}\eta_{\ell+1} \\ &= \int_{\eta_{\ell+1}, \dots, \eta_{r+s}} V_{r+s}(\eta_1, \dots, \eta_{r+s})\, \mathrm{d}\eta_{r+s} \cdots \, \mathrm{d}\eta_{\ell+1}, \end{aligned} \tag{4.4}$$

and assume that $V_\ell$ as a function in $\eta_\ell$ has a continuous derivative whose sign changes only finitely often.

**Lemma 4.1.** In the situation described above, we have, for $\ell \in \{1, \dots, s\}$,

$$V_{r+\ell}(\eta_1, \dots, \eta_{r+\ell}; B) \ll \frac{B^{1-A^{(\ell)}}}{\eta_1^{1-A_1^{(\ell)}} \cdots \eta_{r+\ell}^{1-A_{r+\ell}^{(\ell)}}}$$

and, for $\ell \in \{1, \dots, r\}$,

$$V_\ell(\eta_1, \dots, \eta_\ell; B) \ll \frac{B(\log B)^{r-\ell}}{\eta_1 \cdots \eta_\ell}. \qquad \square$$

**Proof.** The proof is analogous to the proof of Lemma 3.6, skipping the step of replacing sums by integrals via Lemma 3.4. ■

Recall the notation of Definition 3.7 and Definition 3.8.

**Definition 4.2.** Let $C \in \mathbb{R}_{\geq 0}$. Let $\Theta_{2,0}(C)$ be the set $\mathbb{R}$ of real numbers. For any $r \in \mathbb{Z}_{>0}$, we define $\Theta_{2,r}(C)$ recursively as the set of all functions $\vartheta : \mathbb{Z}_{>0}^r \to \mathbb{R}$ in the variables $\eta_1, \dots, \eta_r$ such that $\vartheta \in \Theta_{1,r}(C, \eta_r)$ and $\vartheta' \in \Theta_{2,r-1}(C)$, where $\vartheta'(\eta_1, \dots, \eta_{r-1}) = \mathcal{A}(\vartheta(\eta_1, \dots, \eta_r), \eta_r)$.

For $\vartheta \in \Theta_{2,r}(C)$ and any pairwise distinct $i_1, \dots, i_n \in \{1, \dots, r\}$, we define

$$\mathcal{A}\big(\vartheta(\eta_1, \dots, \eta_r), \eta_{i_1}, \dots, \eta_{i_n}\big) = \mathcal{A}\big(\dots \mathcal{A}\big(\vartheta(\eta_1, \dots, \eta_r), \eta_{i_1}\big) \dots, \eta_{i_n}\big);$$

it is a function in $\Theta_{2,r-n}(C)$. □

**Proposition 4.3.** Let $V_{r+s}$ be as described before Lemma 4.1, and let $\vartheta_{r+s} \in \Theta_{2,r+s}(C)$ for some $C \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned}\sum_{\eta_1,\dots,\eta_{r+s}} \vartheta_{r+s}(\eta_1, \dots, \eta_{r+s}) V_{r+s}(\eta_1, \dots, \eta_{r+s}; B) &= c_0 \int_{\eta_1,\dots,\eta_{r+s}} V_{r+s}(\eta_1, \dots, \eta_{r+s}; B) \, \mathrm{d}\eta_{r+s} \cdots \, \mathrm{d}\eta_1 \\ &\quad + O\left(B(\log B)^{r-1}(\log\log B)^{\max\{1,s\}}\right),\end{aligned}$$

where $c_0 = \mathcal{A}(\vartheta_{r+s}(\eta_1, \dots, \eta_{r+s}), \eta_{r+s}, \dots, \eta_1)$. □

**Proof.** We proceed by induction as follows, for $\ell = r+s, \dots, 1$. Given $\vartheta_\ell \in \Theta_{2,\ell}(C)$, we define $\vartheta_{\ell-1} \in \Theta_{2,\ell-1}(C)$ by

$$\begin{aligned}\vartheta_{\ell-1}(\eta_1, \dots, \eta_{\ell-1}) &= \mathcal{A}(\vartheta_\ell(\eta_1, \dots, \eta_\ell), \eta_\ell) \\ &= \mathcal{A}(\vartheta_{r+s}(\eta_1, \dots, \eta_{r+s}), \eta_{r+s}, \dots, \eta_\ell).\end{aligned}$$

With $V_\ell, V_{\ell-1}$ as in (4.4), we apply Proposition 3.9 to show that

$$\sum_{\eta_\ell} \vartheta_\ell(\eta_1, \dots, \eta_\ell) V_\ell(\eta_1, \dots, \eta_\ell; B) = \vartheta_{\ell-1}(\eta_1, \dots, \eta_{\ell-1}) V_{\ell-1}(\eta_1, \dots, \eta_{\ell-1}; B) + R(\eta_1, \dots, \eta_{\ell-1}; B),$$

where

$$\sum_{\eta_1,\dots,\eta_{\ell-1}} R(\eta_1, \dots, \eta_{\ell-1}; B) \ll B(\log B)^{r-1}(\log\log B)^{\max\{1,\ell-r\}}.$$

**Table 2** Application of Proposition 3.9.

| Proposition 3.9 | $\ell \in \{1, \dots, r\}$ | $\ell \in \{r+1, \dots, r+s\}$ |
|---|---|---|
| (r,s) | $(\ell - 1, 0)$ | $(r-1, \ell - r)$ |
| $\eta_0$ | $\eta_\ell$ | $\eta_\ell$ |
| $\eta_1, \dots, \eta_r$ | $\eta_1, \dots, \eta_{\ell-1}$ | $\eta_1, \dots, \eta_{\ell-s-1}, \eta_{\ell-s+1}, \dots, \eta_r$ |
| $\eta_{r+s}, \dots, \eta_{r+s}$ | – | $\eta_{r+1}, \dots, \eta_{\ell-1}, \eta_{\ell-s}$ |
| $\vartheta \in \Theta_{1,r+s+1}(C, \eta_0)$ | $\vartheta_\ell \in \Theta_{2,\ell}(C)$ | $\vartheta_\ell \in \Theta_{2,\ell}(C)$ |
| $\mathcal{A}(\vartheta(\eta_0, \dots, \eta_{r+s}), \eta_0)$ | $\vartheta_{\ell-1} \in \Theta_{2,\ell-1}(C)$ | $\vartheta_{\ell-1} \in \Theta_{2,\ell-1}(C)$ |
| $V$ | $V_\ell/(\log B)^{r-\ell}$ | $V_\ell$ |
| $V'$ | $V_{\ell-1}/(\log B)^{r-\ell}$ | $V_{\ell-1}$ |
| $k_{0,j}, k_{1,j}, \dots, k_{r+s,j}$ | – | $k_{1,j}, \dots, k_{\ell,j}$ arranged as $\eta_1, \dots, \eta_\ell$ |
| $A; A_0, A_1, \dots, A_{r+s}$ | – | $A^{(\ell-r)}; A_1^{(\ell-r)}, \dots, A_\ell^{(\ell-r)}$ arranged as $\eta_1, \dots, \eta_\ell$ |
| (3.1) | Lemma 4.1 | Lemma 4.1 |
| (3.2) | – | (4.2) |
| (3.3) | (4.3) | (4.3) |

How to apply Proposition 3.9 (especially with respect to the order of the variables $\eta_1, \dots, \eta_\ell$) depends on whether $1 \le \ell \le r$ or $r+1 \le \ell \le r+s$; furthermore, there are many prerequisites to check. Therefore, we have listed the details for the application of Proposition 3.9 in Table 2. ■

**Remark 4.4.** An analogous result to Proposition 4.3 holds if we want to estimate $\vartheta_{r+1}(\eta_1, \dots, \eta_{r+1}) V_{r+1}(\eta_1, \dots, \eta_{r+1}; B)$ summed over $\eta_1, \dots, \eta_{r+1}$, but with (4.1) and (4.2) replaced by a bound analogous to (3.5). In the proof, we apply Proposition 3.10 instead of Proposition 3.9 in the first summation over $\eta_{r+1}$. □

## 5 Real-Valued Functions

The following result is often useful to derive bounds such as (3.1), (3.5), and (4.1) for real-valued functions defined through certain integrals; for example, we recover the bounds of [3, Lemma 8].

**Lemma 5.1.** Let $a, b \in \mathbb{R}_{\neq 0}$. Then we have the following bounds:

(1) $\int_{|at^2+b|\leq 1} \mathrm{d}t \ll \min\{|a|^{-1/2}, |ab|^{-1/2}\}$.
(2) $\int_{|at^2u+bu^k|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll |ab^{1/k}|^{-1/2}$, for $k > 1$.
(3) $\int_{|at^2+bu^k|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll |a|^{-1/2}|b|^{-1/k}$, for $k > 2$.
(4) $\int_{|at^2+bt|\leq 1} \mathrm{d}t \ll \min\{|a|^{-1/2}, |b|^{-1}\}$.
(5) $\int_{|at^2u+btu^2|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll |ab|^{-1/3}$.
(6) $\int_{|at^2+btu^k|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll |a|^{-(k-1)/(2k)}|b|^{-1/k}$, for $k > 1$. □

**Proof.** We treat only the case $a > 0$; its opposite is essentially the same.

For (1), we consider $t$ such that $|at^2 + b| \leq 1$; if there is no such $t$, the claim is obvious. Otherwise, suppose first $|b| \leq 2$. Then $|at^2 + b| \leq 1$ implies $|at^2| \leq 3$, i.e., $t \ll |a|^{-1/2} \ll |ab|^{-1/2}$. Next, suppose $|b| > 2$. Obviously $b > 2$ is impossible, so we assume $b < -2$. Then $|at^2 + b| \leq 1$ implies

$$\sqrt{\frac{-b-1}{a}} \leq t \leq \sqrt{\frac{-b+1}{a}}.$$

We note that the condition $\sqrt{x} \leq t \leq \sqrt{x+y}$ for $x, y > 0$ describes an interval of length $\ll x^{-1/2}y$. Here, $x = (|b|-1)/a > |b|/(2a)$ and $y = 2/a$, so the interval for $t$ has length $\ll |ab|^{-1/2} \ll |a|^{-1/2}$.

For (2), we apply (1) and obtain

$$\begin{aligned}\int_{|at^2u+bu^2|\leq 1} \mathrm{d}t\ \mathrm{d}u &\ll \int_0^\infty \min\left\{|au|^{-1/2}, |abu^{k+1}|^{-1/2}\right\} \mathrm{d}u \\ &\ll \int_0^{|b|^{-1/k}} |au|^{-1/2}\ \mathrm{d}u + \int_{|b|^{-1/k}}^\infty |abu^{k+1}|^{-1/2}\ \mathrm{d}u \ll \frac{1}{|ab^{1/k}|^{1/2}}.\end{aligned}$$

Similarly, for (3), we get

$$\begin{aligned}\int_{|at^2+bu^k|\leq 1} \mathrm{d}t\ \mathrm{d}u &\ll \int_0^\infty \min\left\{|a|^{-1/2}, |abu^k|^{-1/2}\right\} \mathrm{d}u \\ &\ll \int_0^{|b|^{-1/k}} |a|^{-1/2}\ \mathrm{d}u + \int_{|b|^{-1/k}}^\infty |abu^k|^{-1/2}\ \mathrm{d}u \ll \frac{1}{|a|^{1/2}|b|^{1/k}}.\end{aligned}$$

For (4), we transform $|at^2 + bt| \leq 1$ to

$$\sqrt{\max\left\{0, \frac{b^2-4a}{4a^2}\right\}} \leq |t + b/(2a)| \leq \sqrt{\frac{b^2+4a}{4a^2}}.$$

If $b^2 \leq 8a$ then $((b^2+4a)/(4a^2))^{1/2} \ll |a|^{-1/2} \ll |b|^{-1}$, which is also a bound for the length of the interval of allowed values of $t$. If $b^2 > 8a$, then we apply the above bound for $x = (b^2-4a)/(4a^2) > b^2/(8a^2)$ and $y = 2/a$ to conclude that the interval for $t$ has length $\ll |b|^{-1} \ll |a|^{-1/2}$.

For (5), we apply (4) to conclude

$$\int_{|at^2u+btu^2|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll \int_0^\infty \min\{|au|^{-1/2}, |bu^2|^{-1}\}\ \mathrm{d}u \\ \ll \int_0^{|a/b^2|^{1/3}} |au|^{-1/2}\ \mathrm{d}u + \int_{|a/b^2|^{1/3}}^\infty |bu^2|^{-1}\ \mathrm{d}u \ll \frac{1}{|ab|^{1/3}}.$$

For (6), we have

$$\int_{|at^2+btu^k|\leq 1} \mathrm{d}t\ \mathrm{d}u \ll \int_0^\infty \min\{|a|^{-1/2}, |bu^k|^{-1}\}\ \mathrm{d}u \\ \ll \int_0^{|a^{1/2}/b|^{1/k}} |a|^{-1/2}\ \mathrm{d}u + \int_{|a^{1/2}/b|^{1/k}}^\infty |bu^k|^{-1}\ \mathrm{d}u \ll \frac{1}{|a|^{(k-1)/(2k)}|b|^{1/k}}.$$

This completes the proof. ■

## 6 Arithmetic Functions in One Variable

In Sections 3 and 4, we were interested in the average size of arithmetic functions on intervals, with certain bounds on the error term.

In this section, we describe a set of functions in one variable (Definition 6.6) for which this information is computable explicitly (by Corollary 6.9). This includes the functions $f_{a,b}$ treated in [3, Lemma 1] (see Example 6.10).

**Lemma 6.1.** Let $\vartheta : \mathbb{Z}_{>0} \to \mathbb{R}$ be a function, and let $t, y \in \mathbb{R}_{\geq 0}$, with $y \leq t$. Let $a, q \in \mathbb{Z}_{>0}$, with $\gcd(a,q) = 1$. If the infinite sum

$$\sum_{\substack{d>0 \\ \gcd(d,q)=1}} \frac{(\vartheta * \mu)(d)}{d}$$

converges to $c_0 \in \mathbb{R}$, we have

$$\sum_{\substack{0<n\le t\\ n\equiv a \pmod q}} \vartheta(n) = \frac{c_0 t}{q} + O\left( \sum_{\substack{0<d\le y\\ \gcd(d,q)=1}} |(\vartheta * \mu)(d)| + \frac{t}{q}\cdot \left| \sum_{\substack{d>y\\ \gcd(d,q)=1}} \frac{(\vartheta*\mu)(d)}{d} \right| \right.$$
$$\left. + \sum_{\substack{0<n<t/y\\ \gcd(n,q)=1}} \left| \sum_{\substack{y<d\le t/n\\ nd\equiv a \pmod q}} (\vartheta*\mu)(d) \right| \right). \qquad \square$$

**Proof.** Since $\vartheta = (\vartheta * \mu) * 1$, we have

$$\sum_{\substack{0<n\le t\\ n\equiv a \pmod q}} \vartheta(n) = \sum_{\substack{0<n\le t\\ n\equiv a \pmod q}} \sum_{d\mid n} (\vartheta*\mu)(d) = \sum_{\substack{0<d\le t\\ \gcd(d,q)=1}} \sum_{\substack{0<n'\le t/d\\ n'd\equiv a \pmod q}} (\vartheta*\mu)(d).$$

Splitting this sum into the cases $d \le y$ and its opposite, we get

$$= \sum_{\substack{0<d\le y\\ \gcd(d,q)=1}} (\vartheta*\mu)(d)\cdot\left(\frac{t}{qd} + O(1)\right) + \sum_{\substack{0<n'<t/y\\ \gcd(n',q)=1}} \sum_{\substack{y<d\le t/n'\\ n'd\equiv a \pmod q}} (\vartheta*\mu)(d),$$

and the result follows. ■

**Lemma 6.2.** Let $C \in \mathbb{R}_{\ge 1}$. Let $\vartheta : \mathbb{Z}_{>0} \to \mathbb{R}$ be such that, for any $t \in \mathbb{R}_{\ge 0}$,

$$\sum_{0<n\le t} |(\vartheta*\mu)(n)| \cdot n \le t(\log(t+2))^{C-1}.$$

Then, for any $q \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$ with $\gcd(a,q) = 1$, the real number $c_0$ as in Lemma 6.1 exists, and

$$\sum_{\substack{0<n\le t\\ n\equiv a \pmod q}} \vartheta(n) = \frac{c_0 t}{q} + O_C((\log(t+2))^C). \qquad \square$$

**Proof.** We apply Lemma 6.1, with $y = t$. It remains to handle the error term, whose third part clearly vanishes. By Lemma 3.4 and our assumption on $\vartheta$, the first part of the error term is

$$\sum_{0<n\le t} |(\vartheta*\mu)(n)| \ll_C (\log(t+2))^C,$$

and the second part of the error term is

$$\frac{t}{q}\sum_{n>t}\frac{|(\vartheta*\mu)(n)|}{n} \ll_C q^{-1}(\log(t+2))^{C-1}.$$

This completes the proof. ■

**Remark 6.3.** For infinite products, we use the following convention. We require that the partial products of all nonvanishing factors of an infinite product converge to a nonzero number. If there are any vanishing factors, the value of the infinite product is zero. Otherwise, the infinite product cannot converge to zero. □

Let $\mathcal{P}$ denote the set of all primes.

**Definition 6.4.** Let $\Theta_1$ be the set of all nonnegative functions $\vartheta : \mathbb{Z}_{>0} \to \mathbb{R}$ such that there is a $c \in \mathbb{R}$ and a system of nonnegative functions $A_p : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ for $p \in \mathcal{P}$ satisfying

$$\vartheta(n) = c\prod_{p^\nu \| n} A_p(\nu)\prod_{p\nmid n} A_p(0)$$

for all $n \in \mathbb{Z}$ (where the first product is over all $p \in \mathcal{P}$ and $\nu \in \mathbb{Z}_{>0}$ such that $p^\nu \mid n$ but $p^{\nu+1} \nmid n$). In this situation, we say that $\vartheta \in \Theta_1$ *corresponds to* $c, A_p$. □

**Lemma 6.5.** Suppose $\vartheta \in \Theta_1$ is not identically zero and corresponds to $c, A_p$ and $c', A'_p$. Then there are unique $b_p \in \mathbb{R}_{>0}$, for $p \in \mathcal{P}$, such that $\prod_p b_p$ converges to a number $b_0 \in \mathbb{R}_{>0}$, $A'_p(\nu) = b_p A_p(\nu)$ for all $p \in \mathcal{P}$, $\nu \in \mathbb{Z}_{\geq 0}$, and $c' = c/b_0$.

Conversely, given $\vartheta \in \Theta_1$ corresponding to $c, A_p$, and $b_p \in \mathbb{R}_{>0}$, for $p \in \mathcal{P}$, such that $b_0 = \prod_p b_p \in \mathbb{R}_{>0}$ exists. Then $\vartheta$ also corresponds to $c', A'_p$ defined as $c' = c/b_0$ and $A'_p(\nu) = b_p A_p(\nu)$ for all $p \in \mathcal{P}$, $\nu \geq 0$. □

**Proof.** Fix $n = \prod_p p^{k(p)} \in \mathbb{Z}_{>0}$ such that $\vartheta(n) \neq 0$. Then $A_p(k(p))$ and $A'_p(k(p))$ are nonzero, so $b_p \in \mathbb{R}_{>0}$ is uniquely defined as $A'_p(k(p))/A_p(k(p))$. Since

$$\frac{A_p(\nu)}{A_p(k(p))} = \frac{\vartheta\left(p^{\nu-k(p)}n\right)}{\vartheta(n)} = \frac{A'_p(\nu)}{A'_p(k(p))},$$

we have $A'_p(\nu) = b_p A_p(\nu)$ for all $\nu \in \mathbb{Z}_{\geq 0}$.

Since $\prod_{p\nmid n} A_p(0)$ and $\prod_{p\nmid n} A'_p(0)$ are well-defined nonzero numbers, also $\prod_{p\nmid n} b_p \in \mathbb{R}_{>0}$ and therefore $b_0 \in \mathbb{R}_{>0}$ exist. Since

$$\vartheta(n) = c' \prod_{p^\nu \| n} A'_p(\nu) \prod_{p\nmid n} A'_p(0) = c' b_0 \prod_{p^\nu \| n} A_p(\nu) \prod_{p\nmid n} A_p(0),$$

we conclude that $c = c' b_0$.

It is straightforward to check the converse statement. ■

**Definition 6.6.** For any $b \in \mathbb{Z}_{>0}$, $C_1, C_2, C_3 \in \mathbb{R}_{\geq 1}$, let $\Theta_2(b, C_1, C_2, C_3)$ be the set of all functions $\vartheta \in \Theta_1$ for which there exist corresponding $c, A_p$ satisfying the following conditions:

(1) For all $p \in \mathcal{P}$ and $\nu \geq 1$,

$$|A_p(\nu) - A_p(\nu - 1)| \leq \begin{cases} C_1, & p^\nu \mid b, \\ C_2 p^{-\nu}, & p^\nu \nmid b. \end{cases}$$

(2) For all $k \in \mathbb{Z}_{>0}$, we have $|c \prod_{p\nmid k} A_p(0)| \leq C_3$.

Given $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$, we will see in Proposition 6.8 that, for any $q \in \mathbb{Z}_{>0}$, the infinite product

$$c \prod_{p\nmid q} \left( \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{A_p(\nu)}{p^\nu} \right) \prod_{p\mid q} A_p(0)$$

converges to a real number, which we denote by $\mathcal{A}(\vartheta(n), n, q)$. □

If $A_p(\nu) = A_p(\nu + 1)$ for all primes $p$ and all $\nu \geq 1$, then the formula is simplified to

$$A(\vartheta(n), n, q) = c \prod_{p\nmid q} \left( \left(1 - \frac{1}{p}\right) A_p(0) + \frac{1}{p} A_p(1) \right) \prod_{p\mid q} A_p(0).$$

We will see in Corollary 6.9 how the notation $\mathcal{A}(\vartheta(n), n, q)$ of Definition 6.6 is related to the notation $\mathcal{A}(\vartheta(n), n)$ of Definition 3.7.

**Remark 6.7.** If $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$ corresponds to $c, A_p$ and $c', A'_p$, where $c, A_p$ satisfy conditions (1) and (2) of Definition 6.6, then $c', A'_p$ do not necessarily satisfy these conditions. However, with $b_p \in \mathbb{R}_{>0}$ as in Lemma 6.5, if we replace $C_1, C_2, C_3$ by

$$C_1 \max_{p|b}\{b_p\}, \qquad C_2 \sup_p\{b_p\}, \qquad C_3 \prod_{\substack{p \\ |b_p|>1}} b_p,$$

then $c'$, $A'_p$ satisfy conditions (1) and (2). □

In all statements regarding $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$, we will mark explicitly by subscripts if an implied constant in the notation $\ll$ and $O(\cdots)$ depends on any of $b, C_1, C_2, C_3$, or $\vartheta$. The reason is that we will apply the results of this section in Section 7 to functions in several variables $\eta_1, \ldots, \eta_r$. As functions in $\eta_r$, they will lie in $\Theta_2(b, C_1, C_2, C_3)$, but (some of) $b, C_1, C_2, C_3$ will depend on $\eta_1, \ldots, \eta_{r-1}$.

**Proposition 6.8.** Let $\vartheta \in \Theta_1$ be nontrivial, with corresponding $c$, $A_p$.

(1) For any $n \in \mathbb{Z}_{>0}$,

$$(\vartheta * \mu)(n) = c \prod_{p \nmid n} A_p(0) \prod_{p^\nu \| n} (A_p(\nu) - A_p(\nu - 1)).$$

(2) We assume $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$. For any $t \in \mathbb{R}_{\geq 0}$,

$$\sum_{0<n\leq t} |(\vartheta * \mu)(n)| \cdot n \ll_{C_2} \tau(b)(C_1C_2)^{\omega(b)} C_3 t (\log(t+2))^{C_2 - 1},$$

where $\tau(n) = \sum_{d|n} 1$ is the divisor function.

(3) We assume $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$. For any $q \in \mathbb{Z}_{>0}$, the infinite sum and the infinite product

$$\sum_{\substack{n>0 \\ \gcd(n,q)=1}} \frac{(\vartheta * \mu)(n)}{n}, \qquad c \prod_{p \nmid q} \left( \left(1 - \frac{1}{p}\right) \sum_{\nu=0}^{\infty} \frac{A_p(\nu)}{p^\nu} \right) \prod_{p|q} A_p(0)$$

converge to the same real number. □

**Proof.** Up to the converging product $\prod_{p\nmid n} A_p(0)$, claim (1) is an identity of finite algebraic expressions

$$c\prod_{p\nmid n} A_p(0)\prod_{p^\nu\|n}(A_p(\nu)-A_p(\nu-1)) = \sum_{\substack{d\mid n\\ |\mu(d)=1|}} c\prod_{p\nmid n} A_p(0)\prod_{\substack{p^\nu\|n\\ p\nmid d}} A_p(\nu)\prod_{\substack{p^\nu\|n\\ p\mid d}}(-A_p(\nu-1))$$
$$= \sum_{d\mid n}\mu(d)c\prod_{p\nmid \frac{n}{d}} A_p(0)\prod_{p^\nu\|\frac{n}{d}} A_p(\nu)$$
$$= \sum_{d\mid n}\mu(d)\vartheta(n/d)$$
$$= (\vartheta*\mu)(n).$$

For (2), it follows from (1) that

$$|(\vartheta*\mu)(n)| \le C_1^{\omega(\gcd(b,n))}C_2^{\omega(n)}C_3\gcd(b,n)n^{-1}.$$

Therefore,

$$\sum_{0<n\le t}|(\vartheta*\mu)(n)|\cdot n \ll \sum_{0<n\le t} C_1^{\omega(\gcd(n,b))}C_2^{\omega(n)}C_3\gcd(n,b)$$
$$\ll \sum_{d\mid b}\sum_{\substack{0<n'\le t/d\\ \gcd(n',b/d)=1}} C_1^{\omega(d)}C_2^{\omega(dn')}C_3 d$$
$$\ll_{C_2} \sum_{d\mid b}(C_1C_2)^{\omega(d)}C_3 t(\log(t+2))^{C_2-1}$$
$$\ll \tau(b)(C_1C_2)^{\omega(b)}C_3 t(\log(t+2))^{C_2-1},$$

using Example 3.3.

For (3), for $p\in\mathcal{P}$, let $\nu_p=\min\{\nu\in\mathbb{Z}_{\ge 0}\mid A_p(\nu)\ne 0\}$. Since $\vartheta$ is nontrivial, $\nu_p=0$ for all but finitely many $p$, so $a=\prod_p p^{\nu_p}$ defines a positive integer. If $a\nmid n$, then $\vartheta(n)=0$ and $(\vartheta*\mu)(n)=0$.

We define the multiplicative function $B:\mathbb{Z}_{>0}\to\mathbb{R}$ by

$$B(p^\nu)=\frac{A_p(\nu+\nu_p)-A_p(\nu+\nu_p-1)}{A_p(\nu_p)},$$

for any $p \in \mathcal{P}$ and $\nu \in \mathbb{Z}_{>0}$, and

$$c' = c \prod_p A_p(\nu_p) \in \mathbb{R}.$$

If $n = an'$ for some $n' \in \mathbb{Z}_{>0}$, then, by (1),

$$(\vartheta * \mu)(n) = c \prod_{p \nmid an'} A_p(0) \prod_{p^\nu \| an'} (A_p(\nu) - A_p(\nu - 1)) = c' B(n').$$

We assume that $\gcd(a, q) = 1$. By (2) and Lemma 3.4, the following sum converges absolutely, so that we may form the Euler product in the second step.

$$\begin{aligned}
\sum_{\substack{n=1 \\ \gcd(n,q)=1}}^{\infty} \frac{(\vartheta * \mu)(n)}{n} &= \sum_{\substack{n'=1 \\ \gcd(n',q)=1}}^{\infty} \frac{c' B(n')}{an'} = \frac{c'}{a} \prod_{p \nmid q} \left( \sum_{\nu=0}^{\infty} \frac{B(p^\nu)}{p^\nu} \right) \\
&= c \prod_p \frac{A_p(\nu_p)}{p^{\nu_p}} \prod_{p \nmid q} \left( 1 + \sum_{\nu=1}^{\infty} \frac{A_p(\nu + \nu_p) - A_p(\nu + \nu_p - 1)}{p^\nu A_p(\nu_p)} \right) \\
&= c \prod_{p \mid q} \frac{A_p(\nu_p)}{p^{\nu_p}} \prod_{p \nmid q} \left( \left(1 - \frac{1}{p}\right) \sum_{\nu=\nu_p}^{\infty} \frac{A_p(\nu)}{p^\nu} \right).
\end{aligned}$$

Since $A_p(\nu) = 0$ for any $\nu < \nu_p$, and $\nu_p = 0$ for any $p \mid q$, this proves the claim in the case $\gcd(a, q) = 1$.

If $\gcd(a, q) > 1$, then $(\vartheta * \mu)(n) = 0$ for all $n$ satisfying $\gcd(n, q) = 1$, so that (3) is trivially true. ■

Because of the following result, $\mathcal{A}(\vartheta(n), n, q)$ should be viewed as the average size of $\vartheta(n)$ when summed over all $n$ in a residue class modulo $q$ in a sufficiently long interval.

**Corollary 6.9.** Let $\vartheta \in \Theta_2(b, C_1, C_2, C_3)$ be nontrivial. If $q \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$, then

$$\sum_{\substack{0 < n \leq t \\ n \equiv a \pmod q}} \vartheta(n) = \frac{t}{q} \mathcal{A}(\vartheta(n), n, q) + O_{C_2}\left(\tau(b)(C_1 C_2)^{\omega(b)} C_3 (\log(t+2))^{C_2}\right),$$

for any $t \in \mathbb{R}_{\geq 0}$. In particular, in the notation of Definition 3.7, $\vartheta \in \Theta_0(C_2)$, with $\mathcal{A}(\vartheta(n), n) = \mathcal{A}(\vartheta(n), n, 1)$ and $\mathcal{E}(\vartheta(n), n) = O_{C_2}(\tau(b)(C_1 C_2)^{\omega(b)} C_3)$. □

**Proof.** Let $C_4 = \tau(b)(C_1 C_2)^{\omega(b)} C_3$. By Proposition 6.8(2), Lemma 6.2 applies to $C_4^{-1}\vartheta$, with $c_0 = C_4^{-1} \mathcal{A}(\vartheta(n), n, q)$ by Proposition 6.8(3). ■

**Example 6.10.** For $a, b \in \mathbb{Z}_{>0}$, we consider $f_{a,b}$ as in [3, (3.2)]. Then $f_{a,b} \in \Theta_1$, corresponding to $c$, $A_p$, where $c = 1$ and $A_p(0) = 1$ for any prime $p$, while

$$A_p(\nu) = \begin{cases} 0, & p \mid b, \\ 1, & p \nmid b,\ p \mid a, \\ 1 - \frac{1}{p}, & p \nmid ab, \end{cases}$$

for any $\nu > 0$. Clearly, $f_{a,b} \in \Theta_2(\prod_{p|b} p, 1, 1, 1)$, and we compute

$$\mathcal{A}(f_{a,b}(n), n, q) = \prod_{\substack{p|b \\ p\nmid q}} \left(1 - \frac{1}{p}\right) \prod_{p\nmid abq} \left(1 - \frac{1}{p^2}\right),$$

for any $q \in \mathbb{Z}_{>0}$. Since $\tau(\prod_{p|b} p) = 2^{\omega(b)}$, Corollary 6.9 gives another proof of [3, Lemma 1]. □

## 7 Arithmetic Functions in Several Variables

Here, we are interested in the average size of certain arithmetic functions in several variables when summing them over some or all of these variables. Our goal is to characterize functions explicitly that typically appear in proofs of Manin's conjecture and to show that they lie in $\Theta_{2,r}(C)$ (see Definition 4.2), so that we can apply Proposition 4.3.

**Definition 7.1.** Let $r \in \mathbb{Z}_{\geq 0}$. For any $\eta_1, \dots, \eta_r \in \mathbb{Z}_{>0}$ and any prime $p$, we define

$$\mathbf{k}_p(\eta_1, \dots, \eta_r) = (k_1, \dots, k_r),$$

where $p^{k_i} \parallel \eta_i$ for $i = 1, \dots, r$.

Let $\Theta_{3,0} = \mathbb{R}$. For $r \in \mathbb{Z}_{>0}$, let $\Theta_{3,r}$ be the set of all nonnegative functions $\vartheta : \mathbb{Z}_{>0}^r \to \mathbb{R}$ for which there are nonnegative functions $\vartheta_p : \mathbb{Z}_{\geq 0}^r \to \mathbb{R}$ for any prime $p$ such that

$$\vartheta(\eta_1, \dots, \eta_r) = \prod_p \vartheta_p(\mathbf{k}_p(\eta_1, \dots, \eta_r)),$$

for all $\eta_1, \dots, \eta_r \in \mathbb{Z}_{>0}$. We call the functions $\vartheta_p$ *local factors* of $\vartheta$.

For $\mathbf{k} \in \mathbb{Z}^r$, we define

$$\mathrm{supp}(\mathbf{k}) = \{i \in \{1, \dots, r\} \mid k_i \neq 0\}, \qquad \Sigma(\mathbf{k}) = k_1 + \dots + k_r.$$ □

**Definition 7.2.** Let $C \in \mathbb{R}_{\geq 1}$. Let $\Theta_{4,0}(C) = \mathbb{R}$. For any $r \in \mathbb{Z}_{>0}$, let $\Theta_{4,r}(C)$ be the set of all functions $\vartheta \in \Theta_{3,r}$ whose local factors $\vartheta_p$ fulfill the following conditions for any prime $p$:

(1) For any $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^r_{\geq 0}$ with $\mathrm{supp}(\mathbf{k} - \mathbf{k}') = \{i\}$ and $\Sigma(\mathbf{k} - \mathbf{k}') = 1$ (i.e., $\mathbf{k}, \mathbf{k}'$ differ by 1 at the $i$th coordinate $k_i, k_i'$ and coincide at all other coordinates),

$$|\vartheta_p(\mathbf{k}) - \vartheta_p(\mathbf{k}')| \leq \begin{cases} C, & k_i = 1,\ \#\mathrm{supp}(\mathbf{k}) \geq 2, \\ Cp^{-k_i}, & \text{otherwise.} \end{cases}$$

(2) For any $\mathbf{k} \in \mathbb{Z}^r_{\geq 0}$,

$$\vartheta_p(\mathbf{k}) \leq \begin{cases} 1 + Cp^{-2}, & \mathbf{k} = (0, \dots, 0), \\ 1 + \#\mathrm{supp}(\mathbf{k}) \cdot Cp^{-1}, & \text{otherwise.} \end{cases}$$ □

We recall Definition 6.6 of $\Theta_2$.

**Lemma 7.3.** For $r \in \mathbb{Z}_{>0}$, $C \in \mathbb{R}_{\geq 1}$, let $\vartheta \in \Theta_{4,r}(C)$, with local factors $\vartheta_p$. As a function in $\eta_r$,

$$\vartheta \in \Theta_2\left(\prod_{p \mid \eta_1 \cdots \eta_{r-1}} p, C, C, (3rC)^{\omega(\eta_1 \cdots \eta_{r-1})} \prod_p \left(1 + \frac{C}{p^2}\right)\right).$$

The function $\vartheta' : \mathbb{Z}^{r-1}_{>0} \to \mathbb{R}$ defined by

$$\vartheta'(\eta_1, \dots, \eta_{r-1}) = \mathcal{A}(\vartheta(\eta_1, \dots, \eta_r), \eta_r, 1),$$

has local factors

$$\vartheta'_p(\mathbf{k}) = \left(1 - \frac{1}{p}\right) \sum_{k_r = 0}^{\infty} \frac{\vartheta_p(\mathbf{k}, k_r)}{p^{k_r}}.$$ □

**Proof.** We have

$$\vartheta(\eta_1, \dots, \eta_r) = \prod_{p^{k_r} \| \eta_r} \vartheta_p(\mathbf{k}_p(\eta_1, \dots, \eta_{r-1}), k_r) \prod_{p \nmid \eta_r} \vartheta_p(\mathbf{k}_p(\eta_1, \dots, \eta_{r-1}), 0).$$

Therefore, $\vartheta$ as a function in $\eta_r$ lies in $\Theta_1$, with corresponding $c = 1$ and $A_p(\nu) = \vartheta_p(\mathbf{k}_p(\eta_1, \ldots, \eta_{r-1}), \nu)$ for any $\nu \in \mathbb{Z}_{\geq 0}$ and $p \in \mathcal{P}$.

Now we check that $c$, $A_p$ fulfill the conditions of Definition 6.6. For any $\mathbf{k} \in \mathbb{Z}_{\geq 0}^r$, $\vartheta_p(\mathbf{k})$ is at most

$$\begin{aligned}
&\vartheta_p((0, \ldots, 0)) + \sum_{i=1}^{r} \sum_{n=1}^{k_i} |\vartheta_p(k_1, \ldots, k_{i-1}, n, 0, \ldots, 0) - \vartheta_p(k_1, \ldots, k_{i-1}, n-1, 0, \ldots, 0)| \\
&\leq (1 + Cp^{-2}) + \sum_{i=1}^{r} \left( C + \sum_{n=2}^{k_i} Cp^{-n} \right) \\
&\leq 1 + Cp^{-2} + r \left( C + \frac{C}{p^2(1 - p^{-1})} \right) \\
&\leq 3rC.
\end{aligned}$$

Therefore,

$$|A_p(0)| \leq \begin{cases} 3rC, & p \mid \eta_1 \cdots \eta_{r-1}, \\ 1 + Cp^{-2}, & p \nmid \eta_1 \cdots \eta_{r-1}, \end{cases}$$

so that, for any $k \in \mathbb{Z}_{>0}$,

$$\left| c \prod_{p \nmid k} A_p(0) \right| \leq (3rC)^{\omega(\eta_1 \cdots \eta_{r-1})} \prod_p \left( 1 + \frac{C}{p^2} \right).$$

Furthermore, for any prime $p$ and $\nu \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
|A_p(\nu) - A_p(\nu - 1)| &= |\vartheta_p(\mathbf{k}_p(\eta_1, \ldots, \eta_{r-1}), \nu) - \vartheta_p(\mathbf{k}_p(\eta_1, \ldots, \eta_{r-1}), \nu - 1)| \\
&\leq \begin{cases} C, & \nu = 1,\ \#\operatorname{supp}(\mathbf{k}_p(\eta_1, \ldots, \eta_{r-1})) > 0, \\ Cp^{-\nu}, & \text{otherwise}, \end{cases}
\end{aligned}$$

where the first case applies if and only if $p^\nu \mid \prod_{p \mid \eta_1 \cdots \eta_{r-1}} p$.

Therefore, we may define $\vartheta'$ as in the statement of the lemma. By definition,

$$\vartheta'(\eta_1, \ldots, \eta_{r-1}) = \prod_p \left( \left( 1 - \frac{1}{p} \right) \sum_{k_r=0}^{\infty} \frac{\vartheta_p(\mathbf{k}_p(\eta_1, \ldots, \eta_{r-1}), k_r)}{p^{k_r}} \right),$$

for any $\eta_1, \ldots, \eta_{r-1}$. Here, we can read off local factors for $\vartheta'$ as claimed. ■

**Lemma 7.4.** Let $r, C, \vartheta, \vartheta'$ be as in Lemma 7.3. Then $\vartheta' \in \Theta_{4,r-1}(3C)$. □

**Proof.** By Lemma 7.3, local factors of $\vartheta'$ are

$$\vartheta'_p(\mathbf{k}) = \left(1 - \frac{1}{p}\right) \sum_{k_r=0}^{\infty} \frac{\vartheta_p(\mathbf{k}, k_r)}{p^{k_r}}.$$

For $k_r \in \mathbb{Z}_{>0}$, we have

$$|\vartheta_p(0, \ldots, 0, k_r) - \vartheta_p(0, \ldots, 0, 0)| \leq \sum_{n=1}^{k_r} \frac{C}{p^n} \leq \frac{2C}{p}.$$

Therefore,

$$|\vartheta'_p(0, \ldots, 0) - \vartheta_p(0, \ldots, 0, 0)| \leq \left(1 - \frac{1}{p}\right) \sum_{k_r=1}^{\infty} \frac{|\vartheta_p(0, \ldots, 0, k_r) - \vartheta_p(0, \ldots, 0, 0)|}{p^{k_r}} \leq \frac{2C}{p^2}.$$

By the assumption on $\vartheta_p(0, \ldots, 0)$, this implies $\vartheta'_p(0, \ldots, 0) \leq 1 + 3Cp^{-2}$.

For $\mathbf{k} \in \mathbb{Z}_{\geq 0}^{r-1} \setminus \{(0, \ldots, 0)\}$, so that $\#\operatorname{supp}(\mathbf{k}) + 1 \leq 2\#\operatorname{supp}(\mathbf{k})$, we have

$$\vartheta'_p(\mathbf{k}) \leq \left(1 - \frac{1}{p}\right) \sum_{k_r=0}^{\infty} \frac{1 + (1 + \#\operatorname{supp}(\mathbf{k}))Cp^{-1}}{p^{k_r}} \leq 1 + \frac{\#\operatorname{supp}(\mathbf{k}) \cdot 2C}{p}.$$

Now we consider $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}_{\geq 0}^{r-1}$ with $\operatorname{supp}(\mathbf{k} - \mathbf{k}') = \{i\}$ and $\Sigma(\mathbf{k} - \mathbf{k}') = 1$, so that we have $k_i = k'_i + 1$ for the $i$th coordinates $k_i, k'_i$ of $\mathbf{k}, \mathbf{k}'$. We have

$$|\vartheta'_p(\mathbf{k}) - \vartheta'_p(\mathbf{k}')| \leq \left(1 - \frac{1}{p}\right) \sum_{k_r=0}^{\infty} \frac{|\vartheta_p(\mathbf{k}, k_r) - \vartheta_p(\mathbf{k}', k_r)|}{p^{k_r}}.$$

If $k_i \geq 2$, then

$$|\vartheta'_p(\mathbf{k}) - \vartheta'_p(\mathbf{k}')| \leq \frac{C}{p^{k_i}}.$$

If $k_i = 1$ and $\#\operatorname{supp}(\mathbf{k}) = 1$, then

$$|\vartheta'_p(\mathbf{k}) - \vartheta'_p(\mathbf{k}')| \leq \left(1 - \frac{1}{p}\right) \left(\frac{C}{p} + \sum_{k_r=1}^{\infty} \frac{C}{p^{k_r}}\right) \leq \frac{2C}{p}.$$

If $k_i = 1$ and $\#\operatorname{supp}(\mathbf{k}) \geq 2$, then

$$|\vartheta'_p(\mathbf{k}) - \vartheta'_p(\mathbf{k}')| \leq C.$$

This completes the proof. ■

Recall Definition 3.2 of $\Theta_{0,r}(C)$, Definition 3.8 of $\Theta_{1,r}(C, \eta_r)$, and Definition 4.2 of $\Theta_{2,r}(C)$.

**Corollary 7.5.** For any $r \in \mathbb{Z}_{\geq 0}$, $C \in \mathbb{Z}_{\geq 0}$, we have

$$\Theta_{4,r}(C) \subset \Theta_{0,r}(0) \cap \Theta_{1,r}(6rC^3, \eta_r) \cap \Theta_{2,r}(6r(3^r C)^3).$$ □

**Proof.** We prove the results by induction on $r$. The case $r = 0$ is trivial. Let $r \in \mathbb{Z}_{>0}$ and $\vartheta \in \Theta_{4,r}(C)$.

Since

$$\vartheta(\eta_1, \ldots, \eta_r) \leq \prod_{i=1}^{r} (\phi^\dagger(\eta_i))^C \prod_p \left(1 + \frac{C}{p^2}\right),$$

for any $\eta_1, \ldots, \eta_r \in \mathbb{Z}_{>0}$, we have $\vartheta \in \Theta_{0,r}(0)$ (see Example 3.3).

By Lemma 7.3 and Corollary 6.9, $\vartheta \in \Theta_0(C)$ as a function in $\eta_r$. We define

$$\vartheta'(\eta_1, \ldots, \eta_{r-1}) = \mathcal{A}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r),$$
$$\vartheta''(\eta_1, \ldots, \eta_{r-1}) = \mathcal{E}(\vartheta(\eta_1, \ldots, \eta_r), \eta_r).$$

By Lemma 7.4, we have $\vartheta' \in \Theta_{4,r-1}(3C)$. By induction, $\vartheta' \in \Theta_{0,r-1}(0)$. By Corollary 6.9,

$$\vartheta''(\eta_1, \ldots, \eta_{r-1}) = O_C\left((6rC^3)^{\omega(\eta_1 \cdots \eta_{r-1})}\right)$$

since $\tau(\prod_{p|n} p) = 2^{\omega(n)}$ for any $n \in \mathbb{Z}_{>0}$. By Example 3.3, $\vartheta'' \in \Theta_{0,r-1}(6rC^3)$. Therefore, $\vartheta \in \Theta_{1,r}(6rC^3, \eta_r)$.

Since $\vartheta' \in \Theta_{2,r-1}(6(r-1)(3^{r-1}(3C))^3)$ by induction, this implies $\vartheta \in \Theta_{2,r}(6r(3^r C)^3)$. ■

**Lemma 7.6.** Let $r \in \mathbb{Z}_{>0}$ and $\vartheta_r \in \Theta_{4,r}(C)$, with local factors $\vartheta_{r,p}$. Let $\ell \in \{0, \dots, r-1\}$. Local factors of $\vartheta_\ell = \mathcal{A}(\vartheta_r(\eta_1, \dots, \eta_r), \eta_r, \dots, \eta_{\ell+1})$ are given by

$$\vartheta_{\ell,p}(\mathbf{k}) = \left(1 - \frac{1}{p}\right)^{r-\ell} \sum_{\mathbf{k}' \in \mathbb{Z}_{\geq 0}^{r-\ell}} \frac{\vartheta_{r,p}(\mathbf{k}, \mathbf{k}')}{p^{\Sigma(\mathbf{k}')}}.$$

In particular, for $\vartheta_0 = \mathcal{A}(\vartheta_r(\eta_1, \dots, \eta_r), \eta_r, \dots, \eta_1) \in \mathbb{R}$, we have

$$\vartheta_0 = \prod_p \left(\left(1 - \frac{1}{p}\right)^r \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^r} \frac{\vartheta_{r,p}(\mathbf{k})}{p^{\Sigma(\mathbf{k})}}\right). \qquad \square$$

**Proof.** We prove the claim by induction on $\ell$. Local factors of $\vartheta_{r-1}$ are given by Lemma 7.3. By an application of Lemma 7.3 to $\vartheta_\ell \in \Theta_{4,\ell}(3^{r-\ell}C)$ (Lemma 7.4) and the induction hypothesis, local factors of $\vartheta_{\ell-1}$ are

$$\begin{aligned}
\vartheta_{\ell-1,p}(\mathbf{k}) &= \left(1 - \frac{1}{p}\right) \sum_{k_\ell=0}^{\infty} \frac{\vartheta_{\ell,p}(\mathbf{k}, k_\ell)}{p^{k_\ell}} \\
&= \left(1 - \frac{1}{p}\right)^{r-(\ell-1)} \sum_{k_\ell=0}^{\infty} \frac{1}{p^{k_\ell}} \sum_{\mathbf{k}' \in \mathbb{Z}_{\geq 0}^{r-\ell}} \frac{\vartheta_{r,p}(\mathbf{k}, k_\ell, \mathbf{k}')}{p^{\Sigma(\mathbf{k}')}} \\
&= \left(1 - \frac{1}{p}\right)^{r-(\ell-1)} \sum_{\mathbf{k}'' \in \mathbb{Z}_{\geq 0}^{r-(\ell-1)}} \frac{\vartheta_{r,p}(\mathbf{k}, \mathbf{k}'')}{p^{\Sigma(\mathbf{k}'')}}.
\end{aligned}$$

This completes the induction step. ■

In many applications, we are concerned with a function $\vartheta \in \Theta_{3,r}$ whose local factors $\vartheta_p(\mathbf{k})$ only depend on $\operatorname{supp}(\mathbf{k})$. In this case, the notation and results can be simplified as follows.

**Definition 7.7.** Let $\Theta'_{3,0} = \mathbb{R}$. For $r \in \mathbb{Z}_{>0}$, let $\Theta'_{3,r}$ be the set of all $\vartheta \in \Theta_{3,r}$, with local factors $\vartheta_p$, such that, for any $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}_{\geq 0}^r$ with $\operatorname{supp}(\mathbf{k}) = \operatorname{supp}(\mathbf{k}')$, we have $\vartheta_p(\mathbf{k}) = \vartheta_p(\mathbf{k}')$.

Let $\vartheta \in \Theta'_{3,r}$ with local factors $\vartheta_p$. For any $I \subset \{1, \dots, r\}$, we define $\vartheta_p(I)$ as $\vartheta_p(\mathbf{k}_I)$ for any $\mathbf{k}_I \in \mathbb{Z}_{\geq 0}^r$ with $\operatorname{supp}(\mathbf{k}_I) = I$.

For any $\eta_1, \dots, \eta_\ell \in \mathbb{Z}_{>0}$, let

$$I_p(\eta_1, \dots, \eta_r) = \operatorname{supp}(\mathbf{k}_p(\eta_1, \dots, \eta_r)) = \{i \in \{1, \dots, r\} : p \mid \eta_i\},$$

so that

$$\vartheta(\eta_1, \ldots, \eta_r) = \prod_p \vartheta_p(I_p(\eta_1, \ldots, \eta_r)). \qquad \square$$

**Definition 7.8.** Let $r \in \mathbb{Z}_{>0}$ and $C \in \mathbb{R}_{\geq 1}$. Let $\Theta'_{4,r}(C)$ be the set of all $\vartheta \in \Theta'_{3,r}$ such that, for any $I \subset \{1, \ldots, r\}$ and $p \in \mathcal{P}$,

$$|\vartheta_p(I) - 1| \leq \begin{cases} Cp^{-2}, & \#I = 0, \\ Cp^{-1}, & \#I = 1, \\ C, & \#I \geq 2 \end{cases}$$

and $\vartheta_p(I) \leq 1 + \#I \cdot Cp^{-1}$ if $\#I > 0$. $\square$

**Corollary 7.9.** For any $r \in \mathbb{Z}_{>0}$ and $C \in \mathbb{R}_{\geq 1}$, we have

$$\Theta'_{4,r}(C) \subset \Theta_{4,r}(2C) \subset \Theta_{0,r}(0) \cap \Theta_{1,r}(48rC^3, \eta_r) \cap \Theta_{2,r}(48r(3^rC)^3). \qquad \square$$

**Proof.** Let $\vartheta \in \Theta'_{4,r}(C)$. Let $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^r_{\geq 0}$ with $\mathrm{supp}(\mathbf{k} - \mathbf{k}') = \{i\}$ and $\Sigma(\mathbf{k} - \mathbf{k}') = 1$. If $k_i \geq 2$, then $\mathrm{supp}(\mathbf{k}) = \mathrm{supp}(\mathbf{k}')$, so that $\vartheta_p(\mathbf{k}) = \vartheta_p(\mathbf{k}')$. If $k_i = 1$, then $\#\,\mathrm{supp}(\mathbf{k}) = \#\,\mathrm{supp}(\mathbf{k}') + 1$, so that

$$|\vartheta_p(\mathbf{k}) - \vartheta_p(\mathbf{k}')| = |\vartheta_p(\mathrm{supp}(\mathbf{k})) - \vartheta_p(\mathrm{supp}(\mathbf{k}'))| \leq \begin{cases} 2C, & \#\,\mathrm{supp}(\mathbf{k}) \geq 2, \\ 2Cp^{-1}, & \#\,\mathrm{supp}(\mathbf{k}) = 1. \end{cases}$$

Furthermore, for any $\mathbf{k} \in \mathbb{Z}^r_{\geq 0}$,

$$\vartheta_p(\mathbf{k}) = \vartheta_p(\mathrm{supp}(\mathbf{k})) \leq \begin{cases} 1 + Cp^{-2}, & \mathbf{k} = (0, \ldots, 0), \\ 1 + \#\,\mathrm{supp}(\mathbf{k}) \cdot Cp^{-1}, & \text{otherwise}. \end{cases}$$

This shows that $\vartheta \in \Theta_{4,r}(2C)$, and the result follows from Corollary 7.5. ■

**Corollary 7.10.** Let $r \in \mathbb{Z}_{>0}$ and $\vartheta_r \in \Theta'_{4,r}(C)$. Let $\ell \in \{0, \dots, r-1\}$. The function $\vartheta_\ell$ defined by $\vartheta_\ell(\eta_1, \dots, \eta_\ell) = \mathcal{A}(\vartheta_r(\eta_1, \dots, \eta_r), \eta_r, \dots, \eta_{\ell+1})$ has local factors $\vartheta_{\ell,p}$ given by

$$\vartheta_{\ell,p}(I) = \sum_{J \subset \{\ell+1,\dots,r\}} \left(1 - \frac{1}{p}\right)^{r-\ell-\#J} \left(\frac{1}{p}\right)^{\#J} \vartheta_{r,p}(I \cup J),$$

for any $I \subset \{1, \dots, \ell\}$. In particular,

$$\vartheta_0 = \prod_p \sum_{J \subset \{1,\dots,r\}} \left(1 - \frac{1}{p}\right)^{r-\#J} \left(\frac{1}{p}\right)^{\#J} \vartheta_{r,p}(J),$$

while $\mathcal{A}(\vartheta_r(\eta_1, \dots, \eta_r), \eta_r)$ has local factors

$$\vartheta_{r-1,p}(I) = \left(1 - \frac{1}{p}\right) \vartheta_{r,p}(I) + \frac{1}{p} \vartheta_{r,p}(I \cup \{r\}).$$ □

**Proof.** This is a special case of Lemma 7.6, which we may apply because of Corollary 7.9. ■

## 8 Application to a Quartic del Pezzo Surface

Let $S \subset \mathbb{P}^4$ be the quartic del Pezzo surface defined by

$$x_0^2 + x_0 x_3 + x_2 x_4 = x_1 x_3 - x_2^2 = 0.$$

It contains exactly two singularities, namely $(0:0:0:0:1)$ of type $\mathbf{A}_3$ and $(0:1:0:0:0)$ of type $\mathbf{A}_1$, and three lines,

$$\{x_0 = x_1 = x_2 = 0\}, \quad \{x_0 + x_3 = x_1 = x_2 = 0\}, \quad \{x_0 = x_2 = x_3 = 0\}.$$

**Theorem 8.1.** We have

$$N_{U,H}(B) = \alpha(\widetilde{S}) \left(\prod_p \omega_p\right) \omega_\infty B(\log B)^5 + O(B(\log B)^4 (\log\log B)^2),$$

for $B \geq 3$, where

$$\alpha(\widetilde{S}) = \frac{1}{8640},$$
$$\omega_p = \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right),$$
$$\omega_\infty = \int_{|x_0|,|x_2|,|x_2^2/x_1|,|(x_0^2 x_1 + x_0 x_2^2)/(x_1 x_2)| \leq 1,\ 0 \leq x_1 \leq 1} \frac{1}{x_1 x_2}\, \mathrm{d}x_0\, \mathrm{d}x_1\, \mathrm{d}x_2. \qquad \square$$

**Remark 8.2.** We note that $S$ is not an equivariant compactification of the additive group $\mathbb{G}_a^2$, so that Theorem 8.1 does not follow from the general results of [5].

Indeed, the projection $S \dashrightarrow \mathbb{P}^2$ from the line $\{x_0 = x_1 = x_2 = 0\}$ is an isomorphism between the complement $U$ of the three lines in $S$ and the complement of two lines in $\mathbb{P}^2$. If $S$ were an equivariant compactification of $\mathbb{G}_a^2$, then there would be a $\mathbb{G}_a^2$-structure on $\mathbb{P}^2$ fixing two lines, contradicting [12, Proposition 3.2]. $\square$

Since all lines on $S$ are defined over $\mathbb{Q}$, the minimal desingularization $\widetilde{S}$ of $S$ is the blowup of $\mathbb{P}^2$ in five rational points, so that $\mathrm{Pic}(\widetilde{S}) \cong \mathbb{Z}^6$. The effective cone in $\mathrm{Pic}(\widetilde{S})_{\mathbb{R}} = \mathrm{Pic}(\widetilde{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^6$ of $\widetilde{S}$ has seven generators. The investigation of the geometry of $\widetilde{S}$ in [6, Section 7] shows the intersection of its dual (with respect to the intersection form $(\cdot,\cdot)$ on $\mathrm{Pic}(\widetilde{S})_{\mathbb{R}}$) with the hyperplane $\{\mathbf{t} \in \mathrm{Pic}(\widetilde{S})_{\mathbb{R}} \mid (\mathbf{t}, -K_{\widetilde{S}}) = 1\}$ is the polytope

$$P = \left\{ (t_1, \ldots, t_6) \in \mathbb{R}^6_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 + t_3 - 2t_5 - t_6 \geq 0, \\ 2t_1 + 2t_2 + 3t_3 + 2t_4 + t_6 = 1 \end{array} \right\}$$
$$\cong P' = \left\{ (t_1, \ldots, t_5) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} 2t_1 + 2t_2 + 3t_3 + 2t_4 \leq 1, \\ 3t_1 + 3t_2 + 4t_3 + 2t_4 - 2t_5 \geq 1 \end{array} \right\}. \tag{8.1}$$

We check that Theorem 8.1 agrees with the conjectures of Manin [10] and Peyre [13] that predict an asymptotic formula with main term $cB(\log B)^k$, where $k = \mathrm{rk}\,\mathrm{Pic}(\widetilde{S}) - 1$ and $c$ is the product of local densities and $\mathrm{Vol}(P)$. Indeed, $\mathrm{rk}\,\mathrm{Pic}(\widetilde{S}) = 6$ since $S$ is split. By a computation as in [1, Lemma 1], $\omega_p$ resp. $\omega_\infty$ as in the statement of Theorem 8.1 agree with the density of $p$-adic resp. real points on $S$. Finally,

$$\mathrm{Vol}(P) = \mathrm{Vol}(P') = \alpha(\widetilde{S}) = \frac{1/180}{\#W(\mathbf{A}_1) \cdot \#W(\mathbf{A}_3)} = \frac{1}{8640}$$

by [7, Theorem 4] and [8, Theorem 1.3], where $W(\mathbf{A}_i)$ is the Weyl group of the root system $\mathbf{A}_i$.

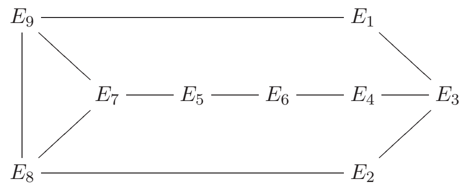


**Fig. 3.** Configuration of curves on $\widetilde{S}$.

### 8.1 Passage to a universal torsor

We carry out step (1) of the strategy described in Section 1. Let

$$\boldsymbol{\eta} = (\eta_1, \dots, \eta_7), \quad \boldsymbol{\eta}' = (\eta_1, \dots, \eta_8), \quad \boldsymbol{\eta}'' = (\eta_1, \dots, \eta_9), \quad \boldsymbol{\eta}^{\mathbf{k}} = \eta_1^{k_1} \cdots \eta_7^{k_7},$$

for any $\mathbf{k} = (k_1, \dots, k_7) \in \mathbb{R}^7$. For $i = 1, \dots, 9$, let

$$(\mathbb{Z}_i, J_i, J_i') = \begin{cases} (\mathbb{Z}_{>0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{\geq 1}), & i \in \{1, \dots, 5\}, \\ (\mathbb{Z}_{>0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{\geq 0}), & i = 6, \\ (\mathbb{Z}_{\neq 0}, \mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1}, \mathbb{R}), & i = 7, \\ (\mathbb{Z}, \mathbb{R}, \mathbb{R}), & i \in \{8, 9\}. \end{cases} \tag{8.2}$$

The following result is based on our investigation [6, Section 7] of

$$\operatorname{Cox}(\widetilde{S}) = \mathbb{Q}[\eta_1, \dots, \eta_9]/\big(\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7\big),$$

where $\mathcal{T}_{\widetilde{S}}$ is an open subset of Spec(Cox($\widetilde{S}$)). It is derived using the method developed in [9, Section 4]. Figure 3 shows the configuration of curves $E_1, \dots, E_9$ on $\widetilde{S}$ that correspond to the generators $\eta_1, \dots, \eta_9$ of Cox($\widetilde{S}$), with edges between pairs of intersecting curves. Here, $E_1, E_2, E_5$ are strict transforms of the three lines $\{x_0 + x_3 = x_1 = x_2 = 0\}$, $\{x_0 = x_1 = x_2 = 0\}$, $\{x_0 = x_2 = x_3 = 0\}$, while $E_3, E_4, E_6$, and $E_7$ are the exceptional divisors obtained by blowing up the $\mathbf{A}_3$ and $\mathbf{A}_1$ singularities.

**Lemma 8.3.** The map $\psi : \mathcal{T}_{\widetilde{S}} \to S$ defined by

$$\boldsymbol{\eta}'' \mapsto \big(\boldsymbol{\eta}^{(0,1,1,1,1,1,1)}\eta_8, \boldsymbol{\eta}^{(2,2,3,2,0,1,0)}, \boldsymbol{\eta}^{(1,1,2,2,2,2,1)}, \boldsymbol{\eta}^{(0,0,1,2,4,3,2)}, \eta_7\eta_8\eta_9\big)$$

induces a bijection $\Psi$ between

$$T_0(B) = \{\boldsymbol{\eta}'' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_9 \mid (8.3), (8.4), (8.5) \text{ hold}\}$$

and $\{\mathbf{x} \in U(\mathbb{Q}) \mid H(\mathbf{x}) \leq B\}$, where

$$\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7 = 0, \tag{8.3}$$

$$\max_{i\in\{0,\dots,4\}} |\Psi(\boldsymbol{\eta}'')_i| \leq B, \tag{8.4}$$

$$\eta_1, \dots, \eta_9 \text{ fulfill coprimality conditions as in Figure 3.} \tag{8.5}$$

□

Using (8.3) to eliminate $\eta_9$, the height condition (8.4) is equivalent to $h(\boldsymbol{\eta}'; B) \leq 1$, where

$$h(\boldsymbol{\eta}'; B) = B^{-1} \max \left\{ \begin{array}{l} \left|\boldsymbol{\eta}^{(0,1,1,1,1,1,1)}\eta_8\right|, \left|\boldsymbol{\eta}^{(2,2,3,2,0,1,0)}\right|, \left|\boldsymbol{\eta}^{(1,1,2,2,2,2,1)}\right|, \\ \left|\boldsymbol{\eta}^{(0,0,1,2,4,3,2)}\right|, \left|\eta_1^{-1}(\eta_2\eta_7\eta_8^2 + \eta_4\eta_5^3\eta_6^2\eta_7^2\eta_8)\right| \end{array} \right\}.$$

### 8.2 Counting points

We come to step (2) of our strategy. We recall the definition (8.2) of $J_1, \dots, J_8$ and define

$$\mathcal{R}(B) = \{\boldsymbol{\eta}' \in J_1 \times \cdots \times J_8 \mid h(\boldsymbol{\eta}'; B) \leq 1\}.$$

Using the results of Sections 2, 4, and 7, we show (Lemma 8.5) that the number of integral points in the region $\mathcal{R}(B)$ on $\mathcal{T}_{\widetilde{S}}$ that satisfy the coprimality conditions (8.5) can be approximated by the product of the volume of $\mathcal{R}(B)$ and $p$-adic densities coming from the coprimality conditions.

**Lemma 8.4.** We have

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta}\in\mathbb{Z}_1\times\cdots\times\mathbb{Z}_7} \vartheta_1(\boldsymbol{\eta}) V_1(\boldsymbol{\eta}; B) + O(B(\log B)^2),$$

**Table 3** Application of Proposition 2.4.

| | | | |
|---|---|---|---|
| $(r,s,t)$ | $(3,1,1)$ | $\delta$ | $\eta_3$ |
| $(\alpha_0;\alpha_1,\dots,\alpha_r)$ | $(\eta_7;\eta_4,\eta_6,\eta_5)$ | $(a_0;a_1,\dots,a_r)$ | $(1;1,2,3)$ |
| $(\beta_0;\beta_1,\dots,\beta_s)$ | $(\eta_8;\eta_2)$ | $(b_0;b_1,\dots,b_s)$ | $(1;1)$ |
| $(\gamma_0;\gamma_1,\dots,\gamma_t)$ | $(\eta_9;\eta_1)$ | $(c_1,\dots,c_t)$ | $(1)$ |
| $\Pi(\boldsymbol{\alpha})$ | $\eta_4\eta_5^3\eta_6^2$ | $\Pi'(\delta,\boldsymbol{\alpha})$ | $\eta_3\eta_4\eta_6$ |
| $\Pi(\boldsymbol{\beta})$ | $\eta_2$ | $\Pi'(\delta,\boldsymbol{\beta})$ | $\eta_3$ |
| $\Pi(\boldsymbol{\gamma})$ | $\eta_1$ | $\Pi'(\delta,\boldsymbol{\gamma})$ | $\eta_3$ |

where

$$V_1(\boldsymbol{\eta};B)=\int_{\boldsymbol{\eta}'\in\mathcal{R}(B)}\eta_1^{-1}\,\mathrm{d}\eta_8$$

and, in the notation of Definition 7.7,

$$\vartheta_1(\boldsymbol{\eta})=\prod_p\vartheta_{1,p}(I_p(\boldsymbol{\eta})),$$

with $I_p(\boldsymbol{\eta})=\{i\in\{1,\dots,7\}:p\mid\eta_i\}$ and

$$\vartheta_{1,p}(I)=\begin{cases}1, & I=\emptyset,\{1\},\{2\},\{7\},\\ 1-\frac{1}{p}, & I=\{4\},\{5\},\{6\},\{1,3\},\{2,3\},\{3,4\},\{4,6\},\{5,6\},\{5,7\},\\ 1-\frac{2}{p}, & I=\{3\},\\ 0, & \text{all other } I\subset\{1,\dots,7\}.\end{cases}$$ □

**Proof.** By Lemma 8.3, our counting problem has the special form of Section 2. Table 3 provides a dictionary between the notation of Section 2 and the present situation.

By Proposition 2.4,

$$N_{U,H}(B)=\sum_{\boldsymbol{\eta}\in\mathbb{Z}_1\times\cdots\times\mathbb{Z}_7}(\vartheta_1(\boldsymbol{\eta})V_1(\boldsymbol{\eta};B)+R_1(\boldsymbol{\eta};B)),$$

where local factors of $\vartheta_1$ as in the statement of Proposition 2.4 are easily computed to be the ones in the statement of this lemma, and

$$R_1(\boldsymbol{\eta};B)\ll 2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)}.$$

Both $N_1$ and $V_1$ and therefore also $R_1$ vanish unless $|\boldsymbol{\eta}^{(1,1,2,2,2,2,1)}| \leq B$, so

$$\begin{aligned}\sum_{\boldsymbol{\eta}} R_1(\boldsymbol{\eta}; B) &\ll \sum_{\boldsymbol{\eta}} 2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)} \\ &\ll \sum_{\eta_1,\ldots,\eta_6} \frac{2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6))} B}{\boldsymbol{\eta}^{(1,1,2,2,2,2,0)}} \\ &\ll B(\log B)^2.\end{aligned}$$

This completes the proof. ■

**Lemma 8.5.** We have

$$N_{U,H}(B) = \left(\prod_p \omega_p\right) V_0(B) + O(B(\log B)^4(\log\log B)^2),$$

where

$$V_0(B) = \int_{\boldsymbol{\eta}} V_1(\boldsymbol{\eta}; B)\, \mathrm{d}\boldsymbol{\eta} = \int_{\boldsymbol{\eta}' \in \mathcal{R}(B)} \eta_1^{-1}\, \mathrm{d}\boldsymbol{\eta}'.$$ □

**Proof.** Clearly, $\vartheta_1 \in \Theta'_{4,7}(2)$, so $\vartheta_1 \in \Theta_{2,7}(C)$ for some $C \in \mathbb{Z}_{>0}$ by Corollary 7.9. By Lemma 5.1(4),

$$V_1(\boldsymbol{\eta}; B) \ll \frac{B^{1/2}}{\eta_1^{1/2}\eta_2^{1/2}|\eta_7|^{1/2}} = \frac{B}{\left|\boldsymbol{\eta}^{(1,1,1,1,1,1,1)}\right|} \cdot \left(\frac{B}{\left|\boldsymbol{\eta}^{(2,2,3,2,0,1,0)}\right|}\right)^{-1/4} \left(\frac{B}{\left|\boldsymbol{\eta}^{(0,0,1,2,4,3,2)}\right|}\right)^{-1/4}.$$

As $V_1(\boldsymbol{\eta}; B) = 0$ unless $1 \leq \eta_1, \ldots, \eta_6, |\eta_7| \leq B$ and $|\boldsymbol{\eta}^{(2,2,3,2,0,1,0)}| \leq B$ and $|\boldsymbol{\eta}^{(0,0,1,2,4,3,2)}| \leq B$, we can apply Proposition 4.3 with $(r, s) = (5, 2)$, $a_1 = a_2 = 1/4$,

$$(k_{i,j})_{\substack{1\leq i\leq 7\\1\leq j\leq 2}} = \begin{pmatrix} 2 & 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 4 & 3 & 2 \end{pmatrix}.$$

We compute

$$\mathcal{A}(\vartheta_1(\boldsymbol{\eta}), \eta_7, \ldots, \eta_1) = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right) = \prod_p \omega_p$$

using Corollary 7.10. ■

### 8.3 The expected leading constant

We carry out step (3) of our strategy. This step is necessary as Lemma 8.6 shows that the main term in Theorem 8.1 is obtained by replacing the integral over $\mathcal{R}(B)$ by an integral over a region $\mathcal{R}'(B)$ that is closely related to the shape of the polytope $P'$ (8.1). Recalling (8.2), we define

$$\mathcal{R}'_1(B) = \left\{(\eta_1, \dots, \eta_5) \in J'_1 \times \dots \times J'_5 \mid \eta_1^2\eta_2^2\eta_3^3\eta_4^2 \le B,\ \eta_1^3\eta_2^3\eta_3^4\eta_4^2\eta_5^{-2} \ge B\right\},$$
$$\mathcal{R}'_2(\eta_1, \dots, \eta_5; B) = \{(\eta_6, \eta_7, \eta_8) \in J'_6 \times J'_7 \times J'_8 \mid h(\eta_1, \dots, \eta_8; B) \le 1\},$$
$$\mathcal{R}'(B) = \{(\eta_1, \dots, \eta_8) \in \mathbb{R}^8 \mid (\eta_1, \dots, \eta_5) \in \mathcal{R}'_1(B), (\eta_6, \eta_7, \eta_8) \in \mathcal{R}'_2(\eta_1, \dots, \eta_5; B)\},$$

and

$$V'_0(B) = \int_{\boldsymbol{\eta}' \in \mathcal{R}'(B)} \eta_1^{-1}\, \mathrm{d}\boldsymbol{\eta}'.$$

**Lemma 8.6.** We have

$$V'_0(B) = \alpha(\widetilde{S})\omega_\infty B(\log B)^5. \qquad \square$$

**Proof.** By substituting

$$x_1 = B^{-1}\boldsymbol{\eta}^{(2,2,3,2,0,1,0)},\ x_2 = B^{-1}\boldsymbol{\eta}^{(1,1,2,2,2,2,1)},\ x_0 = B^{-1}\boldsymbol{\eta}^{(0,1,1,1,1,1,1)}\eta_8$$

into the expression for $\omega_\infty$ given in the statement of Theorem 8.1, we prove

$$\frac{B\omega_\infty}{\eta_1 \cdots \eta_5} = \int_{(\eta_6,\eta_7,\eta_8) \in \mathcal{R}'_2(\eta_1,\dots,\eta_5;B)} \eta_1^{-1}\, \mathrm{d}\eta_6\, \mathrm{d}\eta_7\, \mathrm{d}\eta_8.$$

Substituting $t_i = \frac{\log \eta_i}{\log B}$ into $\alpha(\widetilde{S}) = \mathrm{Vol}(P') = \int_{\mathbf{t} \in P'} \mathrm{d}\mathbf{t}$ shows

$$\alpha(\widetilde{S})(\log B)^5 = \int_{\mathcal{R}'_1(B)} \frac{1}{\eta_1 \cdots \eta_5}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_5.$$

This completes the proof. ■

**Lemma 8.7.** We have

$$V_0(B) = V'_0(B) + O(B(\log B)^4). \qquad \square$$

**Proof.** We define

$$V^{(i)}(B) = \int_{h(\boldsymbol{\eta}';B)\le 1,\ \boldsymbol{\eta}\in\mathcal{R}_i(B)} \eta_1^{-1}\, \mathrm{d}\boldsymbol{\eta}',$$

where

$$\mathcal{R}_0(B) = \{\boldsymbol{\eta}' \in J_1' \times \cdots \times J_8' \mid \eta_6, |\eta_7| \ge 1\},$$
$$\mathcal{R}_1(B) = \left\{\boldsymbol{\eta}' \in J_1' \times \cdots \times J_8' \mid \eta_6, |\eta_7| \ge 1,\ \boldsymbol{\eta}^{(2,2,3,2,0,0,0)} \le B\right\},$$
$$\mathcal{R}_2(B) = \left\{\boldsymbol{\eta}' \in J_1' \times \cdots \times J_8' \;\middle|\; \begin{array}{l} \eta_6, |\eta_7| \ge 1, \\ \boldsymbol{\eta}^{(2,2,3,2,0,0,0)} \le B,\ \boldsymbol{\eta}^{(3,3,4,2,-2,0,0)} \ge B \end{array}\right\},$$
$$\mathcal{R}_3(B) = \left\{\boldsymbol{\eta}' \in J_1' \times \cdots \times J_8' \mid \eta_6 \ge 1,\ \boldsymbol{\eta}^{(2,2,3,2,0,0,0)} \le B,\ \boldsymbol{\eta}^{(3,3,4,2,-2,0,0)} \ge B\right\},$$
$$\mathcal{R}_4(B) = \left\{\boldsymbol{\eta}' \in J_1' \times \cdots \times J_8' \mid \boldsymbol{\eta}^{(2,2,3,2,0,0,0)} \le B,\ \boldsymbol{\eta}^{(3,3,4,2,-2,0,0)} \ge B\right\}.$$

For $i \in \{0, \dots, 3\}$, we will show that

$$|V^{(i)}(B) - V^{(i+1)}(B)| \le \int_{\boldsymbol{\eta}'\in(\mathcal{R}_i(B)\cup\mathcal{R}_{i+1}(B))\setminus(\mathcal{R}_i(B)\cap\mathcal{R}_{i+1}(B)),\ h(\boldsymbol{\eta}';B)\le 1} \eta_1^{-1}\, \mathrm{d}\boldsymbol{\eta}'$$

is $O(B(\log B)^4)$. Since $V_0(B) = V^{(0)}(B)$ and $V_0'(B) = V^{(4)}(B)$, this proves the result.

For $i = 0$, we note that $h(\boldsymbol{\eta}'; B) \le 1$ and $\eta_6 \ge 1$ imply $\boldsymbol{\eta}^{(2,2,3,2,0,0,0)} \le B$. Therefore, $V^{(0)}(B) = V^{(1)}(B)$.

For $i = 1$, we note that $\boldsymbol{\eta}' \in \mathcal{R}_1(B) \setminus \mathcal{R}_2(B)$ implies $\eta_5^2 > \boldsymbol{\eta}^{(3,3,4,2,0,0,0)}/B$ and $1 \le \eta_1, \eta_2, \eta_3, \eta_4 \le B$ and $|\eta_7| \ge 1$. Combining these bounds for the integration over $\eta_1, \dots, \eta_5, \eta_7$ with

$$\int_{h(\boldsymbol{\eta}';B)\le 1} \eta_1^{-1}\, \mathrm{d}\eta_6\, \mathrm{d}\eta_8 \ll \left(\frac{B^3}{|\boldsymbol{\eta}^{(1,1,0,2,6,0,5)}|}\right)^{1/4}$$

by Lemma 5.1(6) leads to the estimation

$$\begin{aligned} V^{(1)}(B) - V^{(2)}(B) &\ll \int \left(\frac{B^3}{|\boldsymbol{\eta}^{(1,1,0,2,6,0,5)}|}\right)^{1/4} \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_5\, \mathrm{d}\eta_7 \\ &\ll \int \frac{B}{\eta_1\eta_2\eta_3\eta_4|\eta_7|^{5/4}}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_4\, \mathrm{d}\eta_7 \\ &\ll B(\log B)^4. \end{aligned}$$

For $i = 2$, we note that $\boldsymbol{\eta}' \in \mathcal{R}_3(B) \setminus \mathcal{R}_2(B)$ implies $|\eta_7| \le 1$, $0 \le \eta_6 \le B/(\boldsymbol{\eta}^{(2,2,3,2,0,0,0)})$, $\eta_5^2 \le \boldsymbol{\eta}^{(3,3,4,2,0,0,0)}/B$, and $1 \le \eta_1, \dots, \eta_4 \le B$. We combine these bounds for the integration over $\eta_1, \dots, \eta_7$ with

$$\int_{h(\boldsymbol{\eta}';B)\leq 1} \eta_1^{-1}\, \mathrm{d}\eta_8 \ll \frac{B^{1/2}}{\eta_1^{1/2}\eta_2^{1/2}|\eta_7|^{1/2}}$$

by Lemma 5.1(4) for the integration over $\eta_8$ to obtain

$$\begin{aligned} V^{(3)}(B) - V^{(2)}(B) &\ll \int \frac{B^{1/2}}{\eta_1^{1/2}\eta_2^{1/2}}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_6 \\ &\ll \int \frac{B^{3/2}}{\boldsymbol{\eta}^{(5/2,5/2,3,2,0,0,0)}}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_5 \\ &\ll \int \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,0,0,0)}}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_4 \\ &\ll B(\log B)^4. \end{aligned}$$

For $i = 3$, we note that $\boldsymbol{\eta}' \in \mathcal{R}_4(B) \setminus \mathcal{R}_3(B)$ implies $|\eta_6| \leq 1$, $\eta_4^2 \leq B/(\boldsymbol{\eta}^{(2,2,3,0,0,0,0)})$, and $1 \leq \eta_1, \eta_2, \eta_3, \eta_5 \leq B$. We combine these bounds for the integration over $\eta_1, \dots, \eta_6$ with

$$\int_{h(\boldsymbol{\eta}';B)\leq 1} \eta_1^{-1}\, \mathrm{d}\eta_8\, \mathrm{d}\eta_7 \ll \frac{B^{2/3}}{\boldsymbol{\eta}^{(1/3,1/3,0,1/3,1,2/3,0)}}$$

by Lemma 5.1(5) to show that

$$\begin{aligned} V^{(4)}(B) - V^{(3)}(B) &\ll \int \frac{B^{2/3}}{\boldsymbol{\eta}^{(1/3,1/3,0,1/3,1,0,0)}}\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_5 \\ &\ll \int \frac{B}{\boldsymbol{\eta}^{(1,1,1,0,1,0,0)}}\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}\eta_3\, \mathrm{d}\eta_5 \\ &\ll B(\log B)^4. \end{aligned}$$

This completes the proof. ■

Theorem 8.1 follows from Lemma 8.5, Lemma 8.6, and Lemma 8.7.

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